Chapter 3

Feynman integrals and algebraic varieties

3.1 The parametric Feynman integrals

Using the Feynman parameters introduced in §1.7 above, we show how to reformulate the Feynman integral $U(\Gamma, p_1, \ldots, p_N)$ of (1.35) in the form known as *Feynman parametric representation.*

The first step is to rewrite the denominator $q_1 \cdots q_n$ of (1.35) in the form of an integration on the topological simplex σ_n as in (1.49), in terms of the Feynman parameters $t = (t_1, \ldots, t_n) \in \sigma_n$.

In writing the integral (1.35) we have made a choice of an orientation of the graph Γ, since the matrix $\epsilon_{v,i}$ involved in writing the conservation laws at vertices in (1.35) depends on the orientation given to the edges of the graph. Now we also make a choice of a set of generators for the first homology group $H_1(\Gamma, \mathbb{Z})$, *i.e.* a choice of a maximal set of independent loops in the graph, $\{l_1, \ldots, l_\ell\}$ with $\ell = b_1(\Gamma)$ the first Betti number.

We define then another matrix associated to the graph Γ, the *circuit matrix* $\eta = (\eta_{ik})$, with $i \in E(\Gamma)$ and $k = 1, \ldots, \ell$ ranging over the chosen basis of loops, given by

$$\eta_{kr} = \begin{cases} +1 \text{ if edge } e_i \in \text{ loop } l_k, \text{ same orientation} \\ -1 \text{ if edge } e_i \in \text{ loop } l_k, \text{ reverse orientation} \\ \;\;0 \text{ if edge } e_i \notin \text{ loop } l_k. \end{cases} \tag{3.1}$$

There is a relation between the circuit and the incidence matrix of the graph, which is given as follows.

Lemma 3.1.1. *The incidence matrix $\epsilon = (\epsilon_{v,i})$ and the circuit matrix $\eta = (\eta_{ik})$ of a graph Γ satisfy the relation $\epsilon\eta = 0$. This holds independently of the choice of the orientation of the graph and the basis of $H_1(\Gamma, \mathbb{Z})$.*

Proof. *(To be added later)* □

We then define the Kirchhoff matrix of the graph, also known as the Symanzik matrix.

Definition 3.1.2. The Kirchhoff–Symanzik matrix $M_\Gamma(t)$ of the graph Γ is the $\ell \times \ell$-matrix given by

$$(M_\Gamma(t))_{kr} = \sum_{i=1}^{n} t_i \eta_{ik} \eta_{ir}. \tag{3.2}$$

Equivalently, it can be written as

$$M_\Gamma(t) = \eta^\dagger \Lambda(t) \eta,$$

where $\dagger$ is the transpose and $\Lambda(t)$ is the diagonal matrix with entries $(t_1, \ldots, t_n)$. We think of M_Γ as a function

$$M_\Gamma : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad t = (t_1, \ldots, t_n) \mapsto M_\Gamma(t) = (M_\Gamma(t))_{kr} \tag{3.3}$$

where $\mathbb{A}$ denotes the affine line over a field (here mostly $\mathbb{C}$ or $\mathbb{R}$ or $\mathbb{Q}$).

Definition 3.1.3. The Kirchhoff–Symanzik polynomial $\Psi_\Gamma(t)$ of the graph Γ is defined as

$$\Psi_\Gamma(t) = \det(M_\Gamma(t)). \tag{3.4}$$

Notice that, while the construction of the matrix $M_\Gamma(t)$ depends on the choice of an orientation on the graph Γ and of a basis of $H_1(\Gamma, \mathbb{Z})$, the graph polynomial is independent of these choices.

Lemma 3.1.4. *The Kirchhoff–Symanzik polynomial $\Psi_\Gamma(t)$ is independent of the choice of edge orientation and of the choice of generators for $H_1(\Gamma, \mathbb{Z})$.*

Proof. A change of orientation in a given edge results in a change of sign in one of the columns of $\eta = \eta_{ik}$. The change of sign in the corresponding row of $\eta^\dagger$ leaves the determinant of $M_\Gamma(t) = \eta^\dagger \Lambda(t) \eta$ unaffected. A change of basis for $H_1(\Gamma, \mathbb{Z})$ changes $M_\Gamma(t) \mapsto A M_\Gamma(t) A^{-1}$, where $A \in \mathrm{GL}_\ell(\mathbb{Z})$ is the matrix that gives the change of basis. The determinant is once again unchanged. □

We view it as a function $\Psi_\Gamma : \mathbb{A}^n \to \mathbb{A}$. We define the affine graph hypersurface $\hat{X}_\Gamma$ to be the locus of zeros of the graph polynomial

$$\hat{X}_\Gamma = \{t \in \mathbb{A}^n \mid \Psi_\Gamma(t) = 0\}. \tag{3.5}$$

The polynomial Ψ_Γ is by construction a homogeneous polynomial of degree $\ell = b_1(\Gamma)$, hence we can view it as defining a hypersurface in projective space $\mathbb{P}^{n-1} = (\mathbb{A}^n \smallsetminus \{0\})/\mathbb{G}_m$,

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} \mid \Psi_\Gamma(t) = 0\}, \tag{3.6}$$

of which $\hat{X}_\Gamma$ is the affine cone $\hat{X}_\Gamma = C(X_\Gamma)$.

After rewriting the denominator of the integrand in (1.35) in terms of an integration on σ_n using the Feynman parameters, we want to replace in the Feynman integral $U(\Gamma, p_1, \dots, p_N)$ the variables k_i associated to the internal edges, and the integration in these variables, by variables x_r associated to the independent loops in the graph and an integration only on these variables, using the linear constraints at the vertices. We set

$$k_i = u_i + \sum_{r=1}^{\ell} \eta_{ir} x_r, \tag{3.7}$$

with the constraint

$$\sum_{i=1}^{n} t_i u_i \eta_{ir} = 0, \quad \forall r = 1, \dots, \ell, \tag{3.8}$$

that is, we require that the column vector $\Lambda(t)u$ is orthogonal to the rows of the circuit matrix η.

The momentum conservation conditions in the delta function in the numerator of (1.35) gives

$$\sum_{i=1}^{n} \epsilon_{v,i} k_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j = 0. \tag{3.9}$$

Lemma 3.1.5. *Using the change of variables* (3.7) *and the constraint* (3.8) *one finds the conservation condition*

$$\sum_{i=1}^{n} \epsilon_{v,i} u_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j = 0. \tag{3.10}$$

Proof. This follows immediately from the orthogonality relation between the incidence and circuit matrix of Lemma 3.1.1. □

The two equations (3.8) and (3.10) constitute the Krichhoff laws of circuits applied to the flow of momentum through the Feynman graph. In particular they determine uniquely the $u_i = u_i(p)$ as functions of the external momenta. We see the explicit form of the solution in Proposition 3.1.6 below.

Proposition 3.1.6. *The term* $\sum_i t_i u_i^2$ *is of the form* $\sum_i t_i u_i^2 = p^\dagger R_\Gamma(t) p$, *where* $R_\Gamma(t)$ *is an* $N \times N$*-matrix, with* $N = \# E_{ext}(\Gamma)$ *with*

$$p^\dagger R_\Gamma(t) p = \sum_{v,v' \in V(\Gamma)} P_v (D_\Gamma(t)^{-1})_{v,v'} P_{v'},$$

with

$$(D_\Gamma(t))_{v,v'} = \sum_{i=1}^n \epsilon_{v,i} \epsilon_{v',i} t_i^{-1}$$

and

$$P_v = \sum_{e \in E_{ext}(\Gamma), t(e)=v} p_e.$$

Proof. *(To be added later)* □

We set

$$V_\Gamma(t,p) = p^\dagger R_\Gamma(t) p + m^2. \tag{3.11}$$

In the massless case ($m = 0$), we will see below that this is a ratio of two homogeneous polynomials in t,

$$V_\Gamma(t,p)|_{m=0} = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t,p)}, \tag{3.12}$$

of which the denominator is the graph polynomial (3.4) and $P_\Gamma(t,p)$ is a homogeneous polynomial of degree $b_1(\Gamma) + 1$.

We can now rewrite the Feynman integral in its parametric form as follows, see [Bjorken and Drell (1964)] §8 and [Bjorken and Drell (1965)] §18.

Theorem 3.1.7. *Up to a multiplicative constant* $C_{n,\ell}$*,the Feynman integral* $U(\Gamma, p_1, \ldots, p_N)$ *can be equivalently written in the form*

$$U(\Gamma, p_1, \ldots, p_N) = \frac{\Gamma(n - \frac{D\ell}{2})}{(4\pi)^{D\ell/2}} \int_{\sigma_n} \frac{\omega_n}{\psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n - D\ell/2}}, \tag{3.13}$$

where ω_n *is the volume form on the simplex* σ_n.

Proof. We first show that we have

$$\int \frac{d^D x_1 \cdots d^D x_\ell}{(\sum_{i=0}^n t_i q_i)^n} = C_{\ell,n} \det(M_\Gamma(t))^{-D/2} (\sum_{i=0}^n t_i (u_i^2 + m^2))^{-n + D\ell/2}, \tag{3.14}$$

where $u_i = u_i(p)$ as above. In fact, after the change of variables (3.7), the left hand side of (3.14) reads

$$\int \frac{d^D x_1 \cdots d^D x_\ell}{(\sum_{i=0}^n t_i(u_i^2 + m^2) + \sum_{kr} (M_\Gamma)_{kr} x_k x_r)^n} \ .$$

The integral can then be reduced by a further change of variables that diagonalizes the matrix M_Γ to an integral of the form

$$\int \frac{d^D y_1 \cdots d^D y_\ell}{(a + \sum_k \lambda_k y_k^2)^n} = C_{\ell,n}\, a^{-n+D\ell/2} \prod_{k=1}^{\ell} \lambda_k^{-D/2},$$

with

$$C_{\ell,n} = \int \frac{d^D x_1 \cdots d^D x_\ell}{(1 + \sum_k x_k^2)^n}.$$

We then write $\det M_\Gamma(t) = \Psi_\Gamma(t)$ and we use the expression of Proposition 3.1.6 to express the term $(\sum_i t_i(u_i^2 + m^2))^{-n+D\ell/2}$ in terms of

$$\sum_i t_i(u_i^2 + m^2) = \sum_i t_i u_i^2 + m^2 = V_\Gamma(t,p),$$

with $V_\Gamma(t,p)$ as in (3.11). □

The graph polynomial $\Psi_\Gamma(t)$ has a more explicit combinatorial description in terms of the graph Γ, as follows.

Proposition 3.1.8. *The Kirchhoff–Symanzik polynomial* $\Psi_\Gamma(t)$ *of* (3.4) *is given by*

$$\Psi_\Gamma(t) = \sum_{T \subset \Gamma} \prod_{e \notin E(T)} t_e, \tag{3.15}$$

where the sum is over all the spanning trees T *of the graph* Γ *and for each spanning tree the product is over all edges of* Γ *that are not in that spanning tree.*

Proof. *(To be added later)* □