Chapter 1

Perturbative quantum field theory and Feynman diagrams

1.1 A calculus exercise in Feynman integrals

To understand the role of Feynman graphs in perturbative quantum field theory, it is convenient to first see how graphs arise in the more familiar setting of finite dimensional integrals, as a convenient way of parameterizing the terms in the integration by parts of polynomials with respect to a Gaussian measure. It all starts with the simplest Gaussian integral

$$\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx = \left(\frac{2\pi}{a}\right)^{1/2}, \tag{1.1}$$

for $a > 0$, which follows from the usual polar coordinates calculation

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2} dx \int_{-\infty}^{\infty} e^{-\frac{1}{2}ay^2} dy = 2\pi \int_0^{\infty} e^{-\frac{1}{2}ar^2}\, r\, dr = \frac{2\pi}{a}\int_0^{\infty} e^{-u}\, du.$$

Similarly, the Gaussian integral with source term is given by

$$\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2+Jx}\, dx = \left(\frac{2\pi}{a}\right)^{1/2} e^{\frac{J^2}{2a}}. \tag{1.2}$$

This also follows easily from (1.1), by completing the square

$$-\frac{ax^2}{2} + Jx = -\frac{a}{2}(x^2 - \frac{2Jx}{a}) = -\frac{a}{2}(x - \frac{J}{a})^2 + \frac{J^2}{2a}$$

and then changing coordinates in the integral to $y = x + \frac{J}{a}$. In this one-dimensional setting a first example of computation of an expectation value can be given in the form

$$\langle x^{2n} \rangle := \frac{\int_{\mathbb{R}} x^{2n}\, e^{-\frac{1}{2}ax^2} dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx} = \frac{(2n-1)!!}{a^n}, \tag{1.3}$$

where $(2n-1)!! = (2n-1)\cdot(2n-3)\cdots 5\cdot 3\cdot 1$. One obtains (1.3) inductively from (1.1) by repeatedly applying the operator $-2\frac{d}{da}$ to (1.1). It is worth pointing out that the factor $(2n-1)!!$ has a combinatorial meaning, namely it counts all the different ways of connecting in pairs the $2n$ linear terms x in the monomial $x^{2n} = x \cdot x \cdots x$ in the integral (1.3). In physics one refers to such pairings as Wick contractions. As we discuss below, the analog of the Gaussian integrals in the infinite dimensional setting of quantum field theory will be the free field case, where only the quadratic terms are present in the Lagrangian. The one-dimensional analog of Lagrangians that include interaction terms will be integrals of the form

$$Z(J) = \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + P(x) + Jx}\, dx, \tag{1.4}$$

where $P(x)$ is a polynomial in x of degree $\deg P \geq 3$. The main idea in such cases, which we'll see applied similarly to the infinite dimensional case, is to treat the additional term $P(x)$ as a perturbation of the original Gaussian integral and expand it out in Taylor series, reducing the problem in this way to a series of terms, each given by the integral of a polynomial under a Gaussian measure. Namely, one writes

$$Z(J) = \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} \frac{P(x)^n}{n!} \right) e^{-\frac{1}{2}ax^2 + Jx}\, dx. \tag{1.5}$$

The perturbative expansion of the integral (1.4) is defined to be the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} P(x)^n \, e^{-\frac{1}{2}ax^2 + Jx}\, dx. \tag{1.6}$$

Notice then that, for a monomial x^k, the integral above satisfies

$$\int_{\mathbb{R}} x^k \, e^{-\frac{1}{2}ax^2 + Jx}\, dx = \left(\frac{d}{dJ} \right)^k \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + Jx}\, dx. \tag{1.7}$$

Using (1.2), this gives

$$\int_{\mathbb{R}} x^k \, e^{-\frac{1}{2}ax^2 + Jx}\, dx = \left(\frac{2\pi}{a} \right)^{1/2} \left(\frac{d}{dJ} \right)^k e^{\frac{J^2}{2a}}.$$

Thus, in the case where the polynomial $P(x)$ consists of a single term

$$P(x) = \frac{\lambda}{k!} x^k,$$

one can rewrite each term in the perturbative expansion using (1.7), so that one obtains

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \left(\frac{\lambda}{k!} x^k \right)^n e^{-\frac{1}{2}ax^2 + Jx}\, dx =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\lambda}{k!} \left(\frac{d}{dJ} \right)^k \right)^n \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + Jx} \, dx.$$

Thus, the perturbative expansion can be written in the form

$$Z(J) = \left(\frac{2\pi}{a} \right)^{1/2} \exp\left(\frac{\lambda}{k!} \left(\frac{d}{dJ} \right)^k \right) \exp\left(\frac{J^2}{2a} \right). \tag{1.8}$$

Two examples of this kind that will reappear frequently in the infinite dimensional version are the cubic case with $P(x) = \frac{g}{6} x^3$ and the quartic case with $P(x) = \frac{\lambda}{4!} x^4$.

To see then how the combinatorics of graphs can be used as a convenient device to label the terms of different order in λ and J in the perturbative series of $Z(J)$, first observe that the term of order λ^α and J^β in $Z(J)$ is produced by the combination of the term of order α in the Taylor expansion of the exponential $\exp(\frac{\lambda}{k!}(\frac{d}{dJ})^k)$ and the term of order $\beta + k\alpha$ in J in the Taylor expansion of the other exponential $\exp(\frac{J^2}{2a})$ in (1.8). All the resulting terms will be of a similar form, consisting of a combinatorial factor given by a ratio of two products of factorials, a power of J, a power of λ and a power of $2a$ in the denominator. The graphs are introduced as a visual way to keep track of the power counting in these terms, which are associated the the vertices and the internal and external edges of the graph. The combinatorial factor can then also be described in terms of symmetries of the graphs.

Here as in general in perturbative quantum field theory, one thinks of graphs as being constructed out of a set of vertices and a set of half edges. Each half edge has an end that is connected to a vertex and another end that may pair to another half edge or remain unpaired. An internal edge of the graph consists of a pair of half edges, hence it is an edge in the usual graph theoretic sense, connecting two vertices. An external edge is an unpaired half edge attached to a vertex of the graph. The graphs we consider will not necessarily be connected. We adopt here the convention that a connected component of a graph which contains a single line should be thought of as consisting of an internal edge and two external edges.

The way one assigns graphs to monomials of the form $\frac{\lambda^\alpha J^\beta}{a^\kappa}$ is by the following rules.

- To each factor of λ one associates a vertex of valence equal to the degree of the monomial $P(x) = \frac{\lambda}{k!} x^k$. This means a vertex with k half edges attached.
- To each factor J one associates an external edge.

- The power of a^{-1} is then determined by the resulting number of internal edge obtained by *pairing* all the half to form a graph.

Notice that the procedure described here produces not one but a finite collection of graphs associated to a given monomial $\frac{\lambda^\alpha J^\beta}{a^\kappa}$, depending on all the different possible pairings between the half edges. This collection of graphs can in turn be subdivided into isomorphism types, each occurring with a given multiplicity, which corresponds to the number of different pairings that produce equivalent graphs. These combinatorial factors are the *symmetry factors* of graphs. To see more precisely how these factors can be computed, we can introduce the analog, in this 1-dimensional toy model, of the Green functions in quantum field theory. The function $Z(J)$ of (1.4) can be thought of as a generating function for the Green functions

$$Z(J) = \sum_{N=0}^{\infty} \frac{J^N}{N!} \int_{\mathbb{R}} x^N \, e^{-\frac{1}{2}ax^2+P(x)} \, dx = Z \cdot \sum_{N=0}^{\infty} J^N G_N, \tag{1.9}$$

where $Z = \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2+P(x)} \, dx$ and the Green functions are

$$G_N = \frac{\int_{\mathbb{R}} \frac{x^N}{N!} \, e^{-\frac{1}{2}ax^2+P(x)} \, dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2+P(x)} \, dx}. \tag{1.10}$$

Upon expanding out the interaction term $\exp(P(x))$, with $P(x) = \frac{\lambda}{k!}x^k$, one obtains

$$G_N = \frac{\sum_{n=0}^{\infty} \frac{x^N}{N!} \frac{(\lambda x^k)^n}{(k!)^n \, n!} e^{-\frac{1}{2}ax^2} \, dx}{\sum_{n=0}^{\infty} \frac{(\lambda x^k)^n}{(k!)^n \, n!} e^{-\frac{1}{2}ax^2} \, dx}. \tag{1.11}$$

Using (1.9), we then see that one way of computing the coefficient of a term in $\frac{\lambda^\alpha J^\beta}{a^\kappa}$ in the asymptotic expansion of $Z(J)$ is to count all the pairings (the Wick contractions) that occur in the integration

$$\int_{\mathbb{R}} x^N x^{kn} e^{-\frac{1}{2}ax^2} \, dx. \tag{1.12}$$

As we have seen in (1.3), these are $(N + kn - 1)!!$. Taking into account the other coefficients that appear in (1.9) and (1.11), one obtains the factor

$$\frac{(N + kn - 1)!!}{N! \, n! \, (k!)^n}.$$

The meaning of this factor in terms of symmetries of graphs can be explained, by identifying $(N + kn - 1)!!$ with the number of all the possible pairings of half edges, from which one factors out $N!$ permutations of the external edges, $k!$ permutations of the half edges attached to each valence

k vertex and $n!$ permutations of the n valence k vertices along with their star of half edges, leaving all the different pairings of half edges. These then correspond to the sum over all the possible topologically distinct graphs obtained via these pairings, each divided by its own symmetry factor. Thus, in terms of graphs, the terms of the asymptotic series become of the form

$$\sum_{\Gamma \in \text{graphs}} \frac{\lambda^{\#V(\Gamma)} J^{\#E_{ext}(\Gamma)}}{\#\text{Aut}(\Gamma)\, a^{\#E_{int}(\Gamma)}}.$$

Notice also how, when computing the terms of the asymptotic series using either the Taylor series of the exponentials of (1.8) or by first using the expansion in Green functions and then the terms (1.12), one is implicitly using the combinatorial identity

$$(2n-1)!! = \frac{(2n)!}{2^n n!}.$$

Passing from the 1-dimensional case to a finite dimensional case in many variables is notationally more complicated but conceptually very similar. One replaces an integral of the form

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}x^t A x + Jx} dx_1 \cdots dx_N = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}} e^{\frac{1}{2} J A^{-1} J^t}, \tag{1.13}$$

where the positive real number $a > 0$ of the 1-dimensional case is now replaced by an $N \times N$-real matrix A with $A^t = A$ and $\det(A) > 0$. The real number J is here an N-vector, with Jx the inner product. The form of (1.13) is obtained by diagonalizing the matrix and reducing it back to the 1-dimensional case. One can again compute the asymptotic series for the integral

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}x^t A x + P(x) + Jx} dx_1 \cdots dx_N,$$

where the interaction term here will be a polynomial in the coordinates x_i of x, such as $P(x) = \frac{\lambda}{4!}(\sum_{i=1}^N x_i^4)$. One can use the same method of labeling the terms in the asymptotic series by graphs, where now instead of attaching a factor a^{-1} to the internal edges one finds factors $(A^{-1})_{ij}$ for edges corresponding to a Wick contraction pairing an x_i and an x_j.

The conceptually more difficult step is to adapt this computational procedure for finite dimensional integral to a recipe that is used to make sense of "analogous" computations of functional integrals in quantum field theory.

1.2 From Lagrangian to effective action

In the case of a scalar field theory, one replaces the expression $\frac{1}{2}x^2 + P(x)$ of the one-dimensional toy model we saw in the previous section with a non-linear functional, the Lagrangian density, defined on a configuration space of classical fields. Here we give only a very brief account of the basics of perturbative quantum field theory. A more detailed presentation, aimed at giving a self contained introduction to mathematicians, can be found in the book [Connes and Marcolli (2008)].

In the scalar case the classical fields are (smooth) functions on a spacetime manifolds, say $\phi \in \mathcal{C}^\infty(\mathbb{R}^D, \mathbb{R})$, and the Lagrangian density is given by an expression of the form

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{P}(\phi), \tag{1.14}$$

where $(\partial\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ for $g^{\mu\nu}$ the Lorentzian metric of signature $(1, -1, -1, \ldots, -1)$ on $\mathbb{R}^D$ and a summation over repeated indices understood. The *interaction term* $\mathcal{P}(\phi)$ in the Lagrangian is a polynomial in the field ϕ of degree $\deg \mathcal{P} \geq 3$. Thus, when one talks about a scalar field theory one means the choice of the data of the Lagrangian density and the spacetime dimension D. We can assume for simplicity that $\mathcal{P}(\phi) = \frac{\lambda}{k!}\phi^k$. We will give explicit examples using the special case of the ϕ^3 theory in dimension $D = 6$: while this is not a physically significant example because of the unstable equilibrium point of the potential at $\phi = 0$, it is both sufficiently simple and sufficiently generic with respect to the renormalization properties (*i.e.* non superrenormalizable, unlike the more physical ϕ^4 in dimension $D = 4$) .

To the Lagrangian density one associates a classical action functional

$$S_L(\phi) = \int_{\mathbb{R}^D} \mathcal{L}(\phi) d^D x. \tag{1.15}$$

The subscript L here stays for the Lorentzian signature of the metric and we'll drop it when we pass to the Euclidean version. This classical action is written as the sum of two terms $S_L(\phi) = S_{free,L}(\phi) + S_{int,L}(\phi)$, where the free field part is

$$S_{free,L}(\phi) = \int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 \right) d^D x$$

and the interaction part is given by

$$S_{int,L}(\phi) = -\int_{\mathbb{R}^D} P(\phi) d^D x.$$

The *probability amplitude* associated to the classical action is the expression

$$e^{i\frac{S_L(\phi)}{\hbar}}, \tag{1.16}$$

where $\hbar = h/2\pi$ is Planck's constant. In the following we follow the convention of taking units where $\hbar = 1$ so that we do not have to write explicitly the powers of $\hbar$ in the terms of the expansions. An observable of a scalar field theory is a functional on the configuration space of the classical fields, which we write as $\mathcal{O}(\phi)$. The *expectation value* of an observable is defined to be the functional integral

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int \mathcal{O}(\phi) e^{iS_L(\phi)}\, \mathcal{D}[\phi]}{\int e^{iS_L(\phi)}\, \mathcal{D}[\phi]}, \tag{1.17}$$

where the integration is supposed to take place on the configuration space of all classical fields. In particular, one has the N-points Green functions, defined here as

$$G_N(x_1, \ldots, x_N) = \frac{\int \phi(x_1) \cdots \phi(x_N)\, e^{iS_L(\phi)}\, \mathcal{D}[\phi]}{\int e^{iS_L(\phi)}\, \mathcal{D}[\phi]}, \tag{1.18}$$

for which the generating function is given again by a functional integral with source term

$$\int e^{iS_L(\phi) + \langle J, \phi \rangle}\, \mathcal{D}[\phi], \tag{1.19}$$

where J is a linear functional (a distribution) on the space of classical fields and $\langle J, \phi \rangle = J(\phi)$ is the pairing of the space of fields and its dual. If $J = J(x)$ is itself a smooth function then $\langle J, \phi \rangle = \int_{\mathbb{R}^D} J(x)\phi(x) d^D x$.

Although the notation of (1.17) and (1.18) is suggestive of what the computation of expectation values should be, there are in fact formidable obstacles in trying to make sense rigorously of the functional integral involved. Despite the successes of constructive quantum field theory in several important cases, in general the integral is ill defined mathematically. This is, in itself, not an obstacle to doing quantum field theory, as long as one regards the expression (1.17) as a shorthand for a corresponding asymptotic expansion, obtained by analogy to the finite dimensional case we have seen previously.

A closer similarity between (1.19) and (1.4) appears when one passes to Euclidean signature by a Wick rotation to imaginary time $t \mapsto it$. This has the effect of switching the signature of the metric to $(1, 1, \ldots, 1)$, after collecting a minus sign, which turns the probability amplitude into the Euclidean version

$$e^{iS_L(\phi)} \mapsto e^{-S(\phi)}, \tag{1.20}$$

with the Euclidean action

$$S(\phi) = \int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{P}(\phi) \right) d^D x. \tag{1.21}$$

Thus, in the Euclidean version we are computing functional integrals of the form

$$\frac{\int \phi(x_1) \cdots \phi(x_N)\, e^{-S(\phi)}\, \mathcal{D}[\phi]}{\int e^{-S(\phi)}\, \mathcal{D}[\phi]}, \tag{1.22}$$

for which the generating function resembles (1.4) in the form

$$Z[J] = \int e^{-\int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{P}(\phi) + J(x)\phi(x) \right) d^D x}\, \mathcal{D}[\phi]. \tag{1.23}$$

In order to make sense of this functional integral, one uses an analog of the asymptotic expansion (1.6), where one expands out the exponential of the interaction term $S_{int}(\phi) = \int_{\mathbb{R}^D} \mathcal{P}(x)\, d^D x$ of the Euclidean action and one follows the same formal rules about integration by parts of the final dimensional case to write the label the terms of the expansion by graphs. What is needed in order to write the contribution of a given graph to the asymptotic series is to specify the rules that associate the analogs of the powers of λ, J and a^{-1} to the vertices, external and internal edges of the graph. These are provided by the *Feynman rules* of the theory.