States as "measures" on NC spaces

Algebras of measurable functions (rather than continuous functions)

Von Neumann algebras (instead of C*-algebras)

\[ \text{e.g. difference between } C_{0}(\mathcal{O}) \text{ and } L^{0}(\mathcal{O}, \mu) \text{ separable \text{ } not \text{ } separable} \]

Also typically C*-alg's contain "few" projections

Von Neumann algs usually have lots of projs (class function of measurable sets in commut. case)

\[ B(H) \text{ algebra of bounded operators on a } \text{ Hilbert space} \]

Weak \& strong topologies

\[ X \text{ topd. vector space} \quad x \xrightarrow{w} x \quad \forall \phi \in X^* \quad \phi(x_n) \rightarrow \phi(x) \]

\[ T_n \rightarrow T \text{ strongly iff } T_n(\xi) \rightarrow T(\xi) \quad \in \text{ norm.} \quad \forall \xi \in H \]

\[ T_n \rightarrow T \text{ weakly iff } T_n(\xi) \rightharpoonup T(\xi) \quad \forall \xi \in H \quad \text{weakly in } H \]

\[ T_n \rightarrow T \Rightarrow T_n \rightharpoonup T \Rightarrow T_n \mathcal{U} T \text{ (not converges)} \]

\[ \begin{bmatrix} \text{A von Neumann algebra is a unital } \ast \text{-subalgebra of } B(H) \end{bmatrix} \]

which is weakly closed

Example: \( (X, \mathcal{B}, \mu) \) a measure space

\( \mathcal{B} \) algebra of measurable sets (full to measure)

\( L^{\infty}(X, \mathcal{B}, \mu) \) acting as multiplication

operators on \( L^{2}(X, \mu) \) subalg of \( B(c_{0}(X, \mu)) \)

is a von Neumann algebra

Other way to think of von Neumann algebras:

algebras generated by symmetries

\[ M = \{ T \in B(H) : UTU^* = T \quad \forall U \in \mathcal{U} \} \]
Commutant: $\mathcal{S} \subset B(\mathcal{H})$ subset

$\mathcal{F} = \{ T \in B(\mathcal{H}) : TS = ST \ \forall S \in \mathcal{F} \}$

$\mathcal{F}$' unitral subalgebra closed in both weak & strong top.

$\mathcal{F}''$ double commutant $(\mathcal{F}''')'$

$A(\mathcal{F}) = \text{algebra generated by } \mathcal{F}$ (algebraically no topological condition)

**Double commutant theorem:**

If $\mathcal{F} \subset B(\mathcal{H})$ closed under $\ast$ ($S^\ast \in \mathcal{F}$ for all $S \in \mathcal{F}$) nondegenerate

$(S_S^2 = 0 \ \forall S \in \mathcal{F} \Rightarrow \xi = 0 \ \in \mathcal{H})$

then $A(\mathcal{F})$ is dense (strongly $\Rightarrow$ weakly)
in $\mathcal{F}''$

**Proof (sketch):** $A(\mathcal{F}) \subset \mathcal{F}''$ easy; need dense $T \in \mathcal{F}''$ want $T_k \in A(\mathcal{F}) \rightarrow T_k \rightarrow T$ i.e.

given $\xi_1, ..., \xi_n \in \mathcal{H}$ and $\varepsilon > 0 \ \exists S \in A(\mathcal{F})$ st.

$\| T_k \xi_i - S \xi_i \| < \varepsilon$

$M = \text{closure of } A(\mathcal{F}) S_0 \quad (\text{want to show } = T \xi_0)$

$S M \subset M \ \forall S \in \mathcal{F} \quad P_M \text{ projection onto } M$

$P S P = S P \ \& \ P S^\ast P = S^\ast P \Rightarrow P \in \mathcal{F}'$

Since $T \in (\mathcal{F}''')'$ $TP = PT \Rightarrow TM \subset M \quad \text{also } S_0 \in \mathcal{M}$ since nondegen.

$S(1-P)S = (1-P)SS = (1-P)P\xi_0 = 0 \Rightarrow (1-P)\xi_0 = 0$ $\xi_0$
A non-$\pi$-algebra, $A'' = \text{strong (weak) closure of } A$

$A'' = \text{enveloping von Neumann algebra generated by } A$

von Neumann adj. closure of $A$

$A = \text{von Neumann alg. if } A'' = A''''$

Note: one can show a von Neumann alg is in partic a
C*-alg is possibly confusing since measure space weaker than top.
but can always make a measure space topological with a
"local" (non separable alg.) topology -- all proj should correspond
open and closed sets
only "separable" C*-alg are "nice" top. spaces

A quick overview of some von Neumann algebras proper,
later direct sum

Two extreme cases

abelian $M = \bigoplus Z_n$

up to (unitary equiv.)

Z_n n-fold sum

$(T \mapsto T\otimes 1_T)$

of a maximal abelian subalg. of $B(H)$

each $= L^\infty([0,1])$ or

$L^\infty([0,1])$, $L^\infty([0,1])$ or sum

$L^\infty([0,1]) \oplus L^\infty([0,1])$

in between cases

$(X,\mu) \times \mapsto R_x$ $R_x \subset B(H)$

$M = \int_X R_x \text{ alg. acting on } L^2(X,\mu, H)$

$M \subset L^\infty(X,\mu, B(H))$

$\{ F : F_x \in R_x \}$

every vN alg direct integral of factors

abelian = measure spaces

Murray-von Neumann subdivision into types

according to "dimension" (real valued)
Equivalence relation on projections \( p \sim q \) in \( \mathcal{M} \) \( (4) \)

if \( \exists s \) partial isometry \( p = ss^* \) \( q = s^*s \)
with \( s \in \mathcal{M} \)

\[ p \preceq q \text{ if } p = p' \text{ for some } p \leq q \text{ (inclusion of subspaces)} \]

* if \( p \preceq q \) and \( q \preceq r \) then \( p \preceq r \)
* any \( p, q \) proj. \( \in \mathcal{M} \), proj. \( p \preceq q \) if \( q = q^+ \)

\[ \sim \text{ partial ordering on the set of projections } = \mathcal{D}(\mathcal{M}) \]

\[ \Rightarrow \exists \text{ partial ordering on the set of projections } \mathcal{D}(\mathcal{M}) \]

Murray-von Neumann : \( \mathcal{M} \) factor

\[ \mathcal{D}(\mathcal{M}) = \begin{cases} \{0, 1, 2, \ldots, n\} & \text{type In} \\ \{0, 1, 2, \ldots, \infty\} & \text{type I_\infty} \\ \{0, 1\} & \text{type II_1} \\ \{0, \infty\} & \text{type II_\infty} \\ \{0, 0^{**}\} & \text{type III} \end{cases} \]

all possibilities (dimensions)

\[ (d: \mathcal{P}(\mathcal{M}) \rightarrow [0, 1]) \]
\[ d(1) = 1 \text{ if } M \text{ type II} \]
\[ d(p \sim q) = d(p) = d(q) \]
\[ d(1) = d(p + q) = d(p) + d(q) \]
\[ p \perp q \Rightarrow d(p + q) = d(p) + d(q) \]
\[ p \perp q \Rightarrow d(p + q) = d(p) + d(q) \]
\[ d(0) = d(p) = d(q) \]
\[ d(1) = 0 \Rightarrow p = 0 \]

Traces : \( \mathcal{M} \) \( \varphi: \mathcal{M} \rightarrow [0, \infty] \) additive, homogeneous for \( \lambda > 1 \)

\[ \varphi(\lambda p) = \lambda \varphi(p) \quad \text{for } \lambda > 1 \]

- finite if \( \varphi(p) \) finite valued \( \Rightarrow \varphi(ab) = \varphi(ba) \) and positivity

- \( \varphi(p) = \varphi(q) \) for \( p \sim q \) proj.

- semi-finite if elements of fin trace dense (weak top)

- faithful \( \varphi(a) \neq 0 \) for \( a \neq 0 \)

- normal \( \varphi(\text{sup} a) = \text{sup} \varphi(a) \) \( a, b \in \mathcal{M} \) \( (\text{con}) \)

\[ \varphi(\text{finite, same as weak-top continuity}) \]

\[ \varphi(\text{family of seminorms } + \|T_0\|) \]
Abelian case: (additional remark)

\[ T \in \mathcal{B}(H) \text{ normal } T^* T = T T^* \]

\[ C^*(T) \text{ abelian C}^* \text{ alg. } = C(\sigma(T)) \]

\[ M = W^*(T) \text{ von Neumann alg. gen by } T \text{ id = } \text{strong span of } C^*(T) \]

(suppose cyclic vector i.e., \( x \in H \) st. \( M x \) dense in \( H \))

polynomial

\[ p(t) = \sum_{k=0}^{n} a_k t^k \]

operator

\[ p(T, T^*) = \sum_{k, l} a_{k,l} T^l T^k \text{ in } W^*(T) \]

approximate function in \( X = \sigma(T) \) by these

all Borel measurable functions \( W^*(T) = L^\infty(X, \mu) \) or spectrum.

This condition is the "max commutative subalg. condition stated before (implies)

--- Traces ---

**Type II** factors: dimensional

- **finite**: defines a faithful finite trace
  - unique & normal
  - \( T(1) = 1 \)

- **infinite**: non-zero semi-finite trace
  - unique up to scale factor
  - faithful, normal

**Type II\(_\infty\)**: usual trace semi-finite, unique normal

but 3 non-normal cases

Dixmier trace (trace class ops have \( T_{\infty} = 0 \))

\[ T \in \mathcal{B}(H) \text{ } T \text{ compact op} \]

\[ T T^* \text{ has discrete spectrum } \mu_i \]

(characteristic values of \( T \))

\[ \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \mu_k(T)}{\log n} \]

\[ Tr_w(T) = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \mu_k(T)}{\log n} \]

\[ \lim_w \text{ extension of lim to bounded sequences} \]

\[ \lim_w (a_n) = 0 \text{ if all } a_n \to 0, \text{ lim}_w (a_n) = \text{lim}_w \text{ exists} \]

\[ \lim_w (a, a, 0, 0, \ldots) = \text{lim}_w (a_n) \]

scale \( \in \mathbb{R} \).
Takes & Weights
probability measures

Examples arise from crossed product constructions

\[ M \rtimes_{\alpha} G \]  (as in \( \star \)-case but weak instead of strong completions)

\[ M (B(H)) \text{ von Neumann algebra w/ cyclic separating vector} \]
\[ \text{not necessarily cyclic} \quad \overline{M \xi_0} = \overline{M^* \xi_0} = H \]

then set \( S(a \xi_0) = a^* \xi_0 \) polar decomposition \( S = JA_{\xi_0}^2 \)

\[ J : H \to H \text{ anti-linear isometry} \]
\[ J^2 = 1 \text{ involution} \]

Set \[ \theta \mapsto \theta T \theta \]
\[ T \in B(H) \text{ autom. of } B(H) \]

then \[ \theta(M) = M' \]

\[ \theta(a) b \xi_0 = J a b^* \xi_0 = b a^* \xi_0 \]

apply twice:

\[ a, \theta(a_2) \theta b \xi_0 = \theta(a_2) a b \xi_0 \]

\[ \Rightarrow \theta(M) \subset M' \]

exchange \( M' \) and \( M = M'' \) get reverse

\[ \Delta: \text{modular op. self-adjoint (unbounded)} \]

\[ T \mapsto \Delta T \Delta^* \text{ maps } M \to M \]

"natural time evol." on \( M \)
\[ \phi(a) = \langle \xi_0, a \xi_0 \rangle \]

1. \( \Delta t M = M \)
2. \( \Delta^a M \Delta^a = M \)
3. \( \phi \) is KMS, stable for this \( \Delta^a \)

\( \xi_0, \xi_1 \) Tomita-Takesaki

Connes' 73: the class of \( \Delta^a \)
in \( \text{Out}(M) = \text{Aut}(M)/\text{Inn}(M) \)

\( \text{Out}(M) \) is independent of \( \phi \)

\( \Delta^a \) is a von Neumann algebra \( M \) has a "canonical" time-evolution modulo inner automorphisms

Often interesting \( \Delta^a \)'s in \( C^* \)-algebras are restrictions of modular automorphism group from an enveloping von Neumann alg. when \( \Delta^a \) preserves the \( C^* \)-subalg.

\[ \text{Connes': from this } \delta: \mathbb{R} \to \text{Out}(M) \]
construction of invariants of type III factors

\[ \Rightarrow \) further refinement of classification

\[ \text{III}_0 \cup \text{III}_1 \cup \text{III}_2 \]

\[ \text{Connes' invariants:} \]

\[ S(M) = \bigcup \{ \sigma(\Delta^\gamma) : \gamma \text{ faithful, normal, semi-finite weight} \} \]

\[ T(M) = \{ t \in \mathbb{R} : \sigma_t \text{ inner} \} \quad \text{M semifinite} \quad \text{iff } T(M) = \mathbb{R} \]

\[ S(M) = \begin{cases} \{ 0, \infty \} & \text{III}_0 \\ \{ \lambda^n : n \in \mathbb{Z} \cup \{ 0 \} \} & \text{III}_0 \cup \text{III}_1 \cup \text{III}_2 \\ \{ 0, 1 \} & \text{III}_1 \\ \{ 0, 1 \} & \text{III}_2 \end{cases} \]

\[ \text{Connes':} \]

\( \sigma(\Delta^\gamma) \) faithfull, normal, semi-finite weight

\( T(M) = \{ t \in \mathbb{R} : \sigma_t \text{ inner} \} \quad \text{M semifinite} \quad \text{iff } T(M) = \mathbb{R} \)

\[ S(M) = \begin{cases} \{ 0, \infty \} & \text{III}_0 \\ \{ \lambda^n : n \in \mathbb{Z} \cup \{ 0 \} \} & \text{III}_1 \cup \text{III}_2 \\ \{ 0, 1 \} & \text{III}_2 \end{cases} \]