- States as "measures" on NC spaces
- Algebras of <u>measurable</u> functions (rather than continuous functions)

## Von Neumann algebras

(instead of $C^*$-algebras)

( different functional analytic properties ),

e.g. difference between $C([0,1])$ and $L^\infty([0,1])$
separable — not separable

Also typically $C^*$-alg's contain "few" projections
Von Neumann alg's usually have lots of proj's (char. function of measurable sets in commut. case)

$\mathcal{B}(\mathcal{H})$ algebra of bounded operators on a (sep.) Hilbert space

weak & strong topologies

$X$ topol. vector space $\quad x_n \xrightarrow{w} x$ iff $\phi(x_n) \to \phi(x)$ $\forall \phi \in X^*$ dual space

$T_n \to T$ strongly iff $T_n(\xi) \to T(\xi)$ in norm $\forall \xi \in \mathcal{H}$

$T_n \to T$ weakly iff $T_n(\xi) \to T(\xi)$ weakly in $\mathcal{H}$ $\forall \xi \in \mathcal{H}$

$$T_n \xrightarrow{\|\cdot\|} T \Rightarrow T_n \xrightarrow{s} T \Rightarrow T_n \xrightarrow{w} T \quad \text{(not converses)}$$

[ A von Neumann algebra is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is weakly closed ]

<u>example</u> $(X, \mathcal{B}, \mu)$ a measure space $\quad \mathcal{B}$ $\sigma$-algebra of measurable sets (finite meas.)

$L^\infty(X, \mathcal{B}, \mu)$ acting as multiplication operators on $L^2(X, \mu)$ $\quad$ subalg of $\mathcal{B}(L^2(X,\mu))$

is a von Neumann algebra

Other way to think of von Neumann algebras:
algebras generated by symmetries

$$M = \{ T \in \mathcal{B}(\mathcal{H}) : UTU^* = T \quad \forall U \in \mathcal{U} \}$$

some group of unitary transformations

Commutant: $\mathcal{S} \subset B(\mathcal{H})$ subset

$\mathcal{S}' = \{ T \in B(\mathcal{H}) : TS = ST \ \ \forall S \in \mathcal{S} \}$

$\mathcal{S}'$ unital subalgebra closed in both weak & strong top.

$\mathcal{S}''$ double commutant $(\mathcal{S}')'$

$\mathcal{A}(\mathcal{S})$ = algebra generated by $\mathcal{S}$ (algebraically. no topol. condition)

Double commutant theorem:

if $\mathcal{S} \subset B(\mathcal{H})$ closed under $*$
($S^* \in \mathcal{S}$ for all $S \in \mathcal{S}$)
nondegenerate
($S\xi = 0 \ \ \forall S \in \mathcal{S} \Rightarrow \xi = 0 \in \mathcal{H}$)

then $\mathcal{A}(\mathcal{S})$ is dense (strongly $\Rightarrow$ weekly)
in $\mathcal{S}''$

Pf (sketch): $\mathcal{A}(\mathcal{S}) \subset \mathcal{S}''$ easy; need dense

$T \in \mathcal{S}''$ want $T_\alpha \in \mathcal{A}(\mathcal{S})$ $T_\alpha \xrightarrow{s} T$ i.e.

given $\xi_1, \dots, \xi_n \in \mathcal{H}$ and $\varepsilon > 0$ $\exists\, S \in \mathcal{A}(\mathcal{S})$ s.t.

$\| T\xi_i - S\xi_i \| < \varepsilon$

$\mathcal{M}$ = closure of $\mathcal{A}(\mathcal{S})\xi_0$ (want to show $= T\xi_0$)

$S\mathcal{M} \subset \mathcal{M}$ $\forall S \in \mathcal{S}$ $P_{\mathcal{M}}$ projection onto $\mathcal{M}$

$\Downarrow$

$PSP = SP$ & $PS^*P = S^*P$ $\Rightarrow$ $P \in \mathcal{S}'$

since $T \in (\mathcal{S}')'$ $TP = PT$ $\Rightarrow T\mathcal{M} \subset \mathcal{M}$ also $\xi_0 \in \mathcal{M}$ since nondegen.:

$S(1-P)\xi_0 = (1-P)S\xi_0 = (1-P)PS\xi_0 = 0$ $\forall S$

$\Rightarrow (1-P)\xi_0 = 0$

$\Rightarrow$ $\mathcal{A} \subset B(\mathcal{H})$ $*$-subalgebra

$\mathcal{A}''$ = strong (weak) closure of $\mathcal{A}$

$\mathcal{A}''$ = (enveloping) von Neumann algebra generated by $\mathcal{A}$

von Neumann alg. closure of $\mathcal{A}$

if $\mathcal{A}$ = vonN. alg $\Rightarrow \mathcal{A} = \mathcal{A}''$ (so $\mathcal{A}'$ also vonN alg.)

$\mathcal{A}$ v.N alg.
$\mathcal{A} \cap \mathcal{A}' = Z(\mathcal{A}) = Z(\mathcal{A}')$ center

Note: one can show a von Neumann alg is in partic. a $C^*$-alg ; possibly confusing since measure space weaker than topo
but can always make a measure space topological with a "bad" (non separable alg.) topology $\leftarrow$ all proj. should corresp to open & closed sets

only "separable" $C^*$-alg's are "nice" topol. spaces

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A quick overview of some von Neumann algebras properties

Two extreme cases

abelian $\mathcal{M} = \bigoplus Z_n$ ($\ell^\infty$-direct sum) (up to unitary equiv.)

$Z_n$ $n$-fold sum ($T \mapsto T \oplus \cdots \oplus T$) of a maximal abelian subalg. of $B(H)$

each $\simeq$ $L^\infty[0,1]$ or $\ell^\infty(n)$, $\ell^\infty(\mathbb{N})$ or sum $L^\infty[0,1] \oplus \ell^\infty$

factors (trivial center) $= \mathbb{C}\cdot 1$

cannot be decomposed as direct sum

in between cases

$(X, \mu)$ $\quad x \longmapsto R_x \quad R_x \subset B(\mathcal{H})$

$\mathcal{M} = \int_X R_x \, d\mu$ alg. acting on $L^2(X, \mu; \mathcal{H})$ $\quad \mathcal{M} \subset L^\infty(X, \mu, B(\mathcal{H}))$
$= \{F : F_x \in R_x\}$

every vN alg direct integral of factors

abelian = measure spaces $\leftarrow$ see p. 5

remains to understand factors

Murray-von Neumann subdivision into types according to "dimension" (real valued)

Equivalence relation on projections $p \sim q$ in $\mathcal{M}$
(Murray von-Neumann equivalence)

if $\exists\, s$ partial isometry $p = ss^*$ $q = s^*s$
with $s \in \mathcal{M}$

$p \precsim q$ if $p \sim p'$ for some $p' \leq q$ (inclusion of subspaces)

* if $p \precsim q$ and $q \precsim p$ then $p \sim q$
* any $p, q$ proj.'s $\exists\, z$ proj. $pz \succsim qz$ and $qz^\perp \precsim pz^\perp$

$\Rightarrow$ $\precsim$ partial ordering on the set of projections$/\sim$ $= \mathcal{D}(\mathcal{M})$

Murray-von Neumann: $\mathcal{M}$ factor

$$\mathcal{D}(\mathcal{M}) = \begin{cases} \{0,1,2,\dots,n\} & \text{type } I_n \leftarrow \oplus M_{n_i}(\mathbb{C}) \\ \{0,1,2,\dots,\infty\} & \text{type } I_\infty \leftarrow B(\mathcal{H}) \\ [0,1] & \text{type } II_1 \\ [0,\infty] & \text{type } II_\infty \\ \{0,\infty\} & \text{type } III \end{cases}$$

all possibilities (dimension)
$d : P(\mathcal{M}) \to [0,1]$ if $\mathcal{M}$ type II
$d(1) = 1$
$p \sim q \Rightarrow d(p) = d(q)$
$p \perp q \Rightarrow d(p+q) = d(p) + d(q)$
$p \precsim q$ $d(p) \leq d(q)$
$d(p) = 0 \Rightarrow p = 0$

Traces: $\mathcal{M}$ $\varphi : \mathcal{M}_+ \to [0,\infty]$ additive, homogeneous for $\lambda \geq 0$
$\varphi(uau^*) = \varphi(a)$
if finite values $\Rightarrow$ $\varphi(ab) = \varphi(ba)$ and positivity
($\varphi(p) = \varphi(q)$ for $p \sim q$ proj.)

- semifinite if elements of fin trace dense (weak top)
- faithful $\varphi(a) \neq 0$ for $a \neq 0$
- normal $\varphi(\sup a_i) = \sup \varphi(a_i)$ $a_i \in \mathcal{M}_+$ (cone)
  (finite: same as weak-continuity)
  family of seminorms $T \mapsto |Tr(ST)|$

Abelian case: (additional remark)

$T \in B(\mathcal{H})$ normal $T^*T = TT^*$

$C^*(T)$ abelian $C^*$-alg. $= C(\sigma(T))$

$\mathcal{M} = W^*(T)$ von Neumann alg. gen by $T$ & id $=$ ~~(weak)~~ strong closure of $C^*(T)$

( suppose $\exists$ cyclic vector i.e. $\xi \in \mathcal{H}$ s.t. $\mathcal{M}\xi$ dense in $\mathcal{H}$ )

polynomial $p(z, \bar{z}) = \sum a_{ij} z^i \bar{z}^j$

operator $p(T, T^*) = \sum a_{ij} T^i T^{*j}$ in $W^*(T)$

approximate functions on $X = \sigma(T)$ by these

$\rightarrow$ all Borel measurable functions $W^*(T) \cong L^\infty(X, \mu)$

spectral measure

spectrum

this condition is the "max commutative subalg." condition stated before (implies)

--- Traces [redacted] [redacted]

type II factors: dimension

finite: defines a faithful finite trace w/ $\tau(1) = 1$ unique & normal

infinite: non-zero semifinite trace unique up to scale factor faithful, normal

type $I_\infty$: usual trace semifinite unique normal one but $\exists$ non-normal ones Dixmier trace

( trace class ops have $Tr_\omega = 0$ )

$T \in \mathcal{B}(\mathcal{H})$ $T$ compact op

$T^*T$ has discrete spectrum $\mu_i$ (characteristic values of $T$)

$$\lim_{n\,\omega} \frac{\sum_{k=1}^n \mu_k(T)}{\log n}$$

$$Tr_\omega(T) = \lim_\omega \left( \frac{\sum_{k=1}^n \mu_k(T)}{\log n} \right)$$

$\lim_\omega$ extension of lim to bounded sequences

$\lim_\omega(a_n) \geq 0$ if all $a_n \geq 0$, $\lim_\omega(a_n) = \lim(a_n)$ when lim exists

$\lim_\omega(a_1, a_1, a_2, a_2, a_3, a_3, \ldots) = \lim_\omega(a_n)$ scale inv.

type III algebras don't have nontrivial traces

Tomita theory

States & weights — replace the role of traces for these algebras

probability measures → States; measures w/ $\infty$ total mass → weights

Examples arise from crossed product constructions
$\mathcal{M} \rtimes_\alpha G$ (as in $C^*$-case but weak instead of norm completions)

$\mathcal{M} \subset B(\mathcal{H})$ von Neumann algebra w/ cyclic separating vector (cyclic for $\mathcal{M}'$)

$$\overline{\mathcal{M}\xi_0} = \overline{\mathcal{M}'\xi_0} = \mathcal{H}$$

$S$ not nec. isometry (in fact not bounded in general)

then set $S(a\xi_0) = a^*\xi_0$ polar decomp $S = J\Delta^{1/2}$

$\Rightarrow$ anti-linear ~~isometry~~ $J : \mathcal{H} \to \mathcal{H}$ isometry
$J^2 = 1$ involution

Set $T \xmapsto{\theta} JTJ$ $T \in B(\mathcal{H})$ autom. of $B(\mathcal{H})$

then $\theta(\mathcal{M}) = \mathcal{M}'$

$$\theta(a)b\xi_0 = Jab^*\xi_0 = ba^*\xi_0$$

apply twice:

$$a_1\theta(a_2)b\xi_0 = \theta(a_2)a_1b\xi_0$$

$\Rightarrow \theta(\mathcal{M}) \subset \mathcal{M}'$

exchange $\mathcal{M}'$ and $\mathcal{M} = \mathcal{M}''$ get reverse

$\Delta$ = modular op. self-adjoint (unbounded)

$T \mapsto \Delta^{it} T \Delta^{-it}$ maps $\mathcal{M}$ to $\mathcal{M}$ modular autom. group "natural time evol." on $\mathcal{M}$

state $\varphi(a) = \langle \xi_0, a\xi_0 \rangle$

$\sigma_t^\varphi(a) = \Delta^{it} a \Delta^{-it}$

① $JMJ = M'$

② $\Delta^{-it} M \Delta^{it} = M$

③ $\varphi$ is $KMS_1$ - state for this $\sigma_t^\varphi$

① + ② + ③ Tomita-Takesaki

Connes '73 : the class of $\sigma_t^\varphi$ in $Out(M) = Aut(M)/Inn(M)$ indep. of $\varphi$

i.e. a von Neumann algebra $M$ has a "canonical" time evolution modulo inner automorphisms

Often interesting $\sigma_t$'s on C*-algs are restriction of modular autom. group from an enveloping von Neumann alg. when $\sigma_t^\varphi$ preserves the C*-subalg.

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Connes: from this $\delta : \mathbb{R} \to Out(M)$
construction of invariants of type III factors

$\Rightarrow$ further refinement of classification
$III_0$ ; $III_\lambda$ $0<\lambda<1$ ; $III_1$

Connes' invariants:

$S(M) = \bigcap \{ \sigma(\Delta_\varphi) : \varphi$ faithful, normal, semifinite weight $\}$

$T(M) = \{ t \in \mathbb{R} : \sigma_t$ inner $\}$ — $M$ semifinite iff $T(M) = \mathbb{R}$

$$S(M) = \begin{cases} [0,\infty) & III_0 \\ \{\lambda^n : n \in \mathbb{Z}\} \cup \{0\} & \text{some } \lambda \in (0,1) \quad III_\lambda \\ \{0,1\} & III_1 \end{cases}$$