

The Spectral Action and the Standard Model

Matilde Marcolli

MAT1314HS Winter 2019, University of Toronto
T 12-2 and W 12 BA6180

References

- A.H. Chamseddine, A. Connes, M. Marcolli, *Gravity and the Standard Model with Neutrino Mixing*, Adv. Theor. Math. Phys., Vol.11 (2007) 991–1090

The spectral action functional

- Ali Chamseddine, Alain Connes, *The spectral action principle*, Comm. Math. Phys. 186 (1997), no. 3, 731–750.

A good action functional for noncommutative geometries

$$\mathrm{Tr}(f(D/\Lambda))$$

D Dirac, Λ mass scale, $f > 0$ even smooth function (cutoff approx)
Simple dimension spectrum \Rightarrow expansion for $\Lambda \rightarrow \infty$

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_k f_k \Lambda^k \int |D|^{-k} + f(0) \zeta_D(0) + o(1),$$

with $f_k = \int_0^\infty f(v) v^{k-1} dv$ momenta of f

where $\mathrm{DimSp}(\mathcal{A}, \mathcal{H}, D) =$ poles of $\zeta_{b,D}(s) = \mathrm{Tr}(b|D|^{-s})$

Asymptotic expansion of the spectral action

$$\mathrm{Tr}(e^{-t\Delta}) \sim \sum a_\alpha t^\alpha \quad (t \rightarrow 0)$$

and the ζ function

$$\zeta_D(s) = \mathrm{Tr}(\Delta^{-s/2})$$

- Non-zero term a_α with $\alpha < 0 \Rightarrow$ pole of ζ_D at -2α with

$$\mathrm{Res}_{s=-2\alpha} \zeta_D(s) = \frac{2 a_\alpha}{\Gamma(-\alpha)}$$

- No $\log t$ terms \Rightarrow regularity at 0 for ζ_D with $\zeta_D(0) = a_0$

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- Get first statement from

$$|D|^{-s} = \Delta^{-s/2} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{-t\Delta} t^{s/2-1} dt$$

with $\int_0^1 t^{\alpha+s/2-1} dt = (\alpha + s/2)^{-1}$.

- Second statement from

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)} \sim \frac{s}{2} \quad \text{as } s \rightarrow 0$$

contrib to $\zeta_D(0)$ from pole part at $s = 0$ of

$$\int_0^\infty \text{Tr}(e^{-t\Delta}) t^{s/2-1} dt$$

given by $a_0 \int_0^1 t^{s/2-1} dt = a_0 \frac{2}{s}$

Spectral action with fermionic terms

$$S = \text{Tr}(f(D_A/\Lambda)) + \frac{1}{2} \langle J\tilde{\xi}, D_A\tilde{\xi} \rangle, \quad \tilde{\xi} \in \mathcal{H}_{cl}^+,$$

D_A = Dirac with unimodular inner fluctuations, J = real structure,
 \mathcal{H}_{cl}^+ = classical spinors, Grassmann variables

Fermionic terms

$$\frac{1}{2} \langle J\tilde{\xi}, D_A\tilde{\xi} \rangle$$

antisymmetric bilinear form $\mathfrak{A}(\tilde{\xi})$ on

$$\mathcal{H}_{cl}^+ = \{\xi \in \mathcal{H}_{cl} \mid \gamma\xi = \xi\}$$

\Rightarrow nonzero on Grassmann variables

Euclidean functional integral \Rightarrow Pfaffian

$$\text{Pf}(\mathfrak{A}) = \int e^{-\frac{1}{2}\mathfrak{A}(\tilde{\xi})} D[\tilde{\xi}]$$

avoids Fermion doubling problem of previous models based on symmetric $\langle \xi, D_A\xi \rangle$ for NC space with $\text{KO-dim}=0$

Grassmann variables

Anticommuting variables with basic integration rule

$$\int \xi d\xi = 1$$

An antisymmetric bilinear form $\mathfrak{A}(\xi_1, \xi_2)$: if ordinary commuting variables $\mathfrak{A}(\xi, \xi) = 0$ but not on Grassmann variables

Example: 2-dim case $\mathfrak{A}(\xi', \xi) = a(\xi'_1 \xi_2 - \xi'_2 \xi_1)$, if ξ_1 and ξ_2 anticommute, with integration rule as above

$$\int e^{-\frac{1}{2}\mathfrak{A}(\xi, \xi)} D[\xi] = \int e^{-a\xi_1 \xi_2} d\xi_1 d\xi_2 = a$$

Pfaffian as functional integral: antisymmetric quadratic form

$$Pf(\mathfrak{A}) = \int e^{-\frac{1}{2}\mathfrak{A}(\xi, \xi)} D[\xi]$$

Method to treat Majorana fermions in the Euclidean setting

Fermionic part of SM Lagrangian

Explicit computation of

$$\frac{1}{2} \langle J_{\tilde{\xi}}, D_A \tilde{\xi} \rangle$$

gives part of SM Lagrangian with

- \mathcal{L}_{Hf} = coupling of Higgs to fermions
- \mathcal{L}_{gf} = coupling of gauge bosons to fermions
- \mathcal{L}_f = fermion terms

Bosonic part of the spectral action

$$\begin{aligned} S = & \frac{1}{\pi^2} (48 f_4 \Lambda^4 - f_2 \Lambda^2 c + \frac{f_0}{4} \mathfrak{d}) \int \sqrt{g} d^4 x \\ & + \frac{96 f_2 \Lambda^2 - f_0 c}{24 \pi^2} \int R \sqrt{g} d^4 x \\ & + \frac{f_0}{10 \pi^2} \int \left(\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) \sqrt{g} d^4 x \\ & + \frac{(-2 a f_2 \Lambda^2 + e f_0)}{\pi^2} \int |\varphi|^2 \sqrt{g} d^4 x \\ & + \frac{f_0 a}{2 \pi^2} \int |D_\mu \varphi|^2 \sqrt{g} d^4 x \\ & - \frac{f_0 a}{12 \pi^2} \int R |\varphi|^2 \sqrt{g} d^4 x \\ & + \frac{f_0 b}{2 \pi^2} \int |\varphi|^4 \sqrt{g} d^4 x \\ & + \frac{f_0}{2 \pi^2} \int \left(g_3^2 G_{\mu\nu}^i G^{\mu\nu i} + g_2^2 F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + \frac{5}{3} g_1^2 B_{\mu\nu} B^{\mu\nu} \right) \sqrt{g} d^4 x, \end{aligned}$$

Parameters:

- f_0, f_2, f_4 free parameters, $f_0 = f(0)$ and, for $k > 0$,

$$f_k = \int_0^\infty f(v) v^{k-1} dv.$$

- $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ functions of Yukawa parameters of ν MSM

$$\mathfrak{a} = \text{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d))$$

$$\mathfrak{b} = \text{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2)$$

$$\mathfrak{c} = \text{Tr}(MM^\dagger)$$

$$\mathfrak{d} = \text{Tr}((MM^\dagger)^2)$$

$$\mathfrak{e} = \text{Tr}(MM^\dagger Y_\nu^\dagger Y_\nu).$$

Gilkey's theorem using $D_A^2 = \nabla^* \nabla - E$

Differential operator $P = -(g^{\mu\nu} \partial_\mu \partial_\nu + A^\mu \partial_\mu + B)$ with A, B bundle endomorphisms, $m = \dim M$

$$\mathrm{Tr} e^{-tP} \sim \sum_{n \geq 0} t^{\frac{n-m}{2}} \int_M a_n(x, P) dv(x)$$

$P = \nabla^* \nabla - E$ and $E_{;\mu}{}^\mu := \nabla_\mu \nabla^\mu E$

$$\nabla_\mu = \partial_\mu + \omega'_\mu, \quad \omega'_\mu = \frac{1}{2} g_{\mu\nu} (A^\nu + \Gamma^\nu \cdot \mathrm{id})$$

$$E = B - g^{\mu\nu} (\partial_\mu \omega'_\nu + \omega'_\mu \omega'_\nu - \Gamma_{\mu\nu}^\rho \omega'_\rho)$$

$$\Omega_{\mu\nu} = \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu + [\omega'_\mu, \omega'_\nu]$$

Seeley-DeWitt coefficients

$$a_0(x, P) = (4\pi)^{-m/2} \mathrm{Tr}(\mathrm{id})$$

$$a_2(x, P) = (4\pi)^{-m/2} \mathrm{Tr} \left(-\frac{R}{6} \mathrm{id} + E \right)$$

$$\begin{aligned} a_4(x, P) &= (4\pi)^{-m/2} \frac{1}{360} \mathrm{Tr} \left(-12 R_{;\mu}{}^\mu + 5R^2 - 2R_{\mu\nu} R^{\mu\nu} \right. \\ &\quad \left. + 2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 60 R E + 180 E^2 + 60 E_{;\mu}{}^\mu \right. \\ &\quad \left. + 30 \Omega_{\mu\nu} \Omega^{\mu\nu} \right) \end{aligned}$$

Normalization and coefficients

- Rescale Higgs field $H = \frac{\sqrt{af_0}}{\pi} \varphi$ to normalize kinetic term

$$\int \frac{1}{2} |D_\mu \mathbf{H}|^2 \sqrt{g} d^4x$$

- Normalize Yang-Mills terms

$$\frac{1}{4} G_{\mu\nu}^i \overline{G}^{\mu\nu i} + \frac{1}{4} F_{\mu\nu}^\alpha \overline{F}^{\mu\nu\alpha} + \frac{1}{4} B_{\mu\nu} \overline{B}^{\mu\nu}$$

Normalized form:

$$\begin{aligned} S = & \frac{1}{2\kappa_0^2} \int R \sqrt{g} d^4x + \gamma_0 \int \sqrt{g} d^4x \\ & + \alpha_0 \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g} d^4x + \tau_0 \int R^* R^* \sqrt{g} d^4x \\ & + \frac{1}{2} \int |DH|^2 \sqrt{g} d^4x - \mu_0^2 \int |H|^2 \sqrt{g} d^4x \\ & - \xi_0 \int R |H|^2 \sqrt{g} d^4x + \lambda_0 \int |H|^4 \sqrt{g} d^4x \\ & + \frac{1}{4} \int (G_{\mu\nu}^i G^{\mu\nu i} + F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + B_{\mu\nu} B^{\mu\nu}) \sqrt{g} d^4x \end{aligned}$$

where $R^* R^* = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\mu\nu}^{\alpha\beta} R_{\rho\sigma}^{\gamma\delta}$ integrates to the Euler characteristic $\chi(M)$ and $C^{\mu\nu\rho\sigma}$ Weyl curvature

Coefficients

$$\frac{1}{2\kappa_0^2} = \frac{96f_2\Lambda^2 - f_0c}{24\pi^2} \quad \gamma_0 = \frac{1}{\pi^2}(48f_4\Lambda^4 - f_2\Lambda^2c + \frac{f_0}{4}d)$$

$$\alpha_0 = -\frac{3f_0}{10\pi^2} \quad \tau_0 = \frac{11f_0}{60\pi^2}$$

$$\mu_0^2 = 2\frac{f_2\Lambda^2}{f_0} - \frac{e}{a} \quad \xi_0 = \frac{1}{12}$$

$$\lambda_0 = \frac{\pi^2 b}{2f_0 a^2}$$

Energy scale: Unification ($10^{15} - 10^{17}$ GeV)

$$\frac{g^2 f_0}{2\pi^2} = \frac{1}{4}$$

Preferred energy scale, unification of coupling constants

1-loop RGE equations for ν MSN

$$\partial_t x_i(t) = \beta_{x_i}(x(t))$$

variable $t = \log(\Lambda/M_Z)$

- **Coupling constants:**

$$\beta_1 = \frac{41}{96\pi^2} g_1^3, \quad \beta_2 = -\frac{19}{96\pi^2} g_2^3, \quad \beta_3 = -\frac{7}{16\pi^2} g_3^3$$

at 1-loop decoupled from other equations (Notation: $\tilde{g}_1^2 = \frac{5}{3}g_1^2$)

- **Yukawa parameters:**

$$16\pi^2 \beta_{Y_u} = Y_u \left(\frac{3}{2} Y_u^\dagger Y_u - \frac{3}{2} Y_d^\dagger Y_d + \mathbf{a} - \frac{17}{20} \tilde{g}_1^2 - \frac{9}{4} g_2^2 - 8g_3^2 \right)$$

$$16\pi^2 \beta_{Y_d} = Y_d \left(\frac{3}{2} Y_d^\dagger Y_d - \frac{3}{2} Y_u^\dagger Y_u + \mathbf{a} - \frac{1}{4} \tilde{g}_1^2 - \frac{9}{4} g_2^2 - 8g_3^2 \right)$$

$$16\pi^2 \beta_{Y_\nu} = Y_\nu \left(\frac{3}{2} Y_\nu^\dagger Y_\nu - \frac{3}{2} Y_e^\dagger Y_e + \mathbf{a} - \frac{9}{20} \tilde{g}_1^2 - \frac{9}{4} g_2^2 \right)$$

$$16\pi^2 \beta_{Y_e} = Y_e \left(\frac{3}{2} Y_e^\dagger Y_e - \frac{3}{2} Y_\nu^\dagger Y_\nu + \mathbf{a} - \frac{9}{4} \tilde{g}_1^2 - \frac{9}{4} g_2^2 \right)$$

- **Majorana mass terms:**

$$16\pi^2 \beta_M = Y_\nu Y_\nu^\dagger M + M(Y_\nu Y_\nu^\dagger)^T$$

- **Higgs self coupling:**

$$16\pi^2 \beta_\lambda = 6\lambda^2 - 3\lambda(3g_2^2 + g_1^2) + 3g_2^4 + \frac{3}{2}(g_1^2 + g_2^2)^2 + 4\lambda a - 8b$$

- **MSM approximation:** top quark Yukawa parameter dominant term: in λ running neglect all terms except coupling constants g_i and Yukawa parameter of top quark y_t

$$\beta_\lambda = \frac{1}{16\pi^2} \left(24\lambda^2 + 12\lambda y^2 - 9\lambda(g_2^2 + \frac{1}{3}g_1^2) - 6y^4 + \frac{9}{8}g_2^4 + \frac{3}{8}g_1^4 + \frac{3}{4}g_2^2 g_1^2 \right)$$

where Yukawa parameter for the top quark runs by

$$\beta_y = \frac{1}{16\pi^2} \left(\frac{9}{2}y^3 - 8g_3^2 y - \frac{9}{4}g_2^2 y - \frac{17}{12}g_1^2 y \right)$$

Renormalization group flow

- The coefficients α, b, c, d, e (depend on Yukawa parameters) run with the RGE flow
- Initial conditions at unification energy: compatibility with physics at low energies

RGE in the MSM case

Running of coupling constants at one loop: $\alpha_i = g_i^2/(4\pi)$

$$\beta_{g_i} = (4\pi)^{-2} b_i g_i^3, \quad \text{with} \quad b_i = \left(\frac{41}{6}, -\frac{19}{6}, -7\right),$$

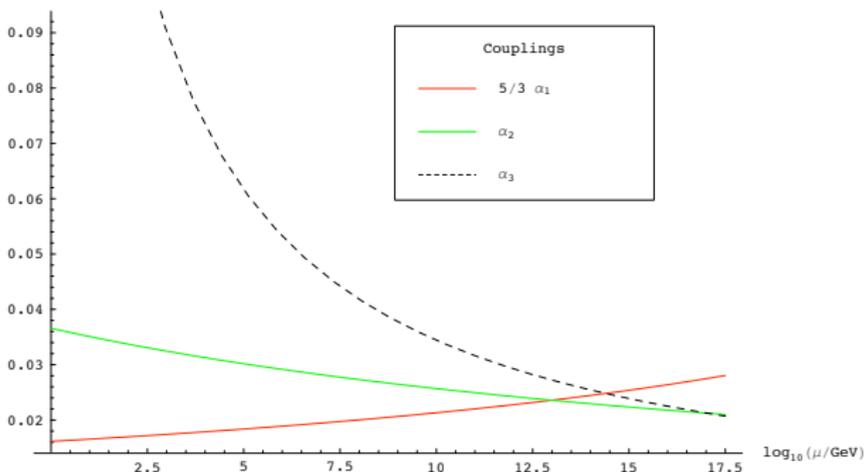
$$\alpha_1^{-1}(\Lambda) = \alpha_1^{-1}(M_Z) - \frac{41}{12\pi} \log \frac{\Lambda}{M_Z}$$

$$\alpha_2^{-1}(\Lambda) = \alpha_2^{-1}(M_Z) + \frac{19}{12\pi} \log \frac{\Lambda}{M_Z}$$

$$\alpha_3^{-1}(\Lambda) = \alpha_3^{-1}(M_Z) + \frac{42}{12\pi} \log \frac{\Lambda}{M_Z}$$

$M_Z \sim 91.188$ GeV mass of Z^0 boson

At one loop RGE for coupling constants decouples from Yukawa parameters (not at 2 loops!)



Well known triangle problem: with known low energy values constants don't meet at unification $g_3^2 = g_2^2 = 5g_1^2/3$

Geometry point of view

- At one loop coupling constants decouple from Yukawa parameters
- Solving for coupling constants, RGE flow defines a vector field on moduli space $\mathcal{C}_3 \times \mathcal{C}_1$ of Dirac operators on the finite NC space F
- Subvarieties invariant under flow are relations between the SM parameters that hold at all energies
- At two loops or higher, RGE flow on a rank three vector bundle (fiber = coupling constants) over the moduli space $\mathcal{C}_3 \times \mathcal{C}_1$
- Geometric problem: studying the flow and the geometry of invariant subvarieties on the moduli space

Constraints at unification

The geometry of the model imposes conditions at unification energy: specific to this NCG model

- λ parameter constraint

$$\lambda(\Lambda_{unif}) = \frac{\pi^2}{2f_0} \frac{b(\Lambda_{unif})}{a(\Lambda_{unif})^2}$$

- Higgs vacuum constraint

$$\frac{\sqrt{af_0}}{\pi} = \frac{2M_W}{g}$$

- See-saw mechanism and c constraint

$$\frac{2f_2\Lambda_{unif}^2}{f_0} \leq c(\Lambda_{unif}) \leq \frac{6f_2\Lambda_{unif}^2}{f_0}$$

- Mass relation at unification

$$\sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2)|_{\Lambda=\Lambda_{unif}} = 8M_W^2|_{\Lambda=\Lambda_{unif}}$$

Need to have compatibility with low energy behavior

Mass relation at unification $Y_2(S) = 4g^2$

$$Y_2 = \sum_{\sigma} (y_{\nu}^{\sigma})^2 + (y_e^{\sigma})^2 + 3(y_u^{\sigma})^2 + 3(y_d^{\sigma})^2$$

$$(k_{(\uparrow 3)})_{\sigma\kappa} = \frac{g}{2M} m_u^{\sigma} \delta_{\sigma}^{\kappa}$$

$$(k_{(\downarrow 3)})_{\sigma\kappa} = \frac{g}{2M} m_d^{\mu} C_{\sigma\mu} \delta_{\mu}^{\rho} C_{\rho\kappa}^{\dagger}$$

$$(k_{(\uparrow 1)})_{\sigma\kappa} = \frac{g}{2M} m_{\nu}^{\sigma} \delta_{\sigma}^{\kappa}$$

$$(k_{(\downarrow 1)})_{\sigma\kappa} = \frac{g}{2M} m_e^{\mu} U^{lep}{}_{\sigma\mu} \delta_{\mu}^{\rho} U^{lep\dagger}{}_{\rho\kappa}$$

$\delta_i^j =$ Kronecker delta, then **constraint**:

$$\text{Tr}(k_{(\uparrow 1)}^* k_{(\uparrow 1)} + k_{(\downarrow 1)}^* k_{(\downarrow 1)} + 3(k_{(\uparrow 3)}^* k_{(\uparrow 3)} + k_{(\downarrow 3)}^* k_{(\downarrow 3)})) = 2g^2$$

\Rightarrow mass matrices satisfy

$$\sum_{\sigma} (m_{\nu}^{\sigma})^2 + (m_e^{\sigma})^2 + 3(m_u^{\sigma})^2 + 3(m_d^{\sigma})^2 = 8M^2$$

See-saw mechanism: $D = D(Y)$ Dirac

$$\begin{pmatrix} 0 & M_\nu^* & M_R^* & 0 \\ M_\nu & 0 & 0 & 0 \\ M_R & 0 & 0 & \bar{M}_\nu^* \\ 0 & 0 & \bar{M}_\nu & 0 \end{pmatrix}$$

on subspace $(\nu_R, \nu_L, \bar{\nu}_R, \bar{\nu}_L)$: largest eigenvalue of $M_R \sim \Lambda$
unification scale. Take $M_R = x k_R$ in flat space, Higgs vacuum v
small (w/resp to unif scale) $\partial_u \text{Tr}(f(D_A/\Lambda)) = 0 \quad u = x^2$

$$x^2 = \frac{2 f_2 \Lambda^2 \text{Tr}(k_R^* k_R)}{f_0 \text{Tr}((k_R^* k_R)^2)}$$

Dirac mass $M_\nu \sim$ Fermi energy v

$$\frac{1}{2} (\pm m_R \pm \sqrt{m_R^2 + 4 v^2})$$

two eigenvalues $\sim \pm m_R$ and two $\sim \pm \frac{v^2}{m_R}$

Compare with estimates

$$(m_R)_1 \geq 10^7 \text{ GeV}, \quad (m_R)_2 \geq 10^{12} \text{ GeV}, \quad (m_R)_3 \geq 10^{16} \text{ GeV}$$

Low energy limit: compatibilities and predictions
Running of top Yukawa coupling (dominant term):

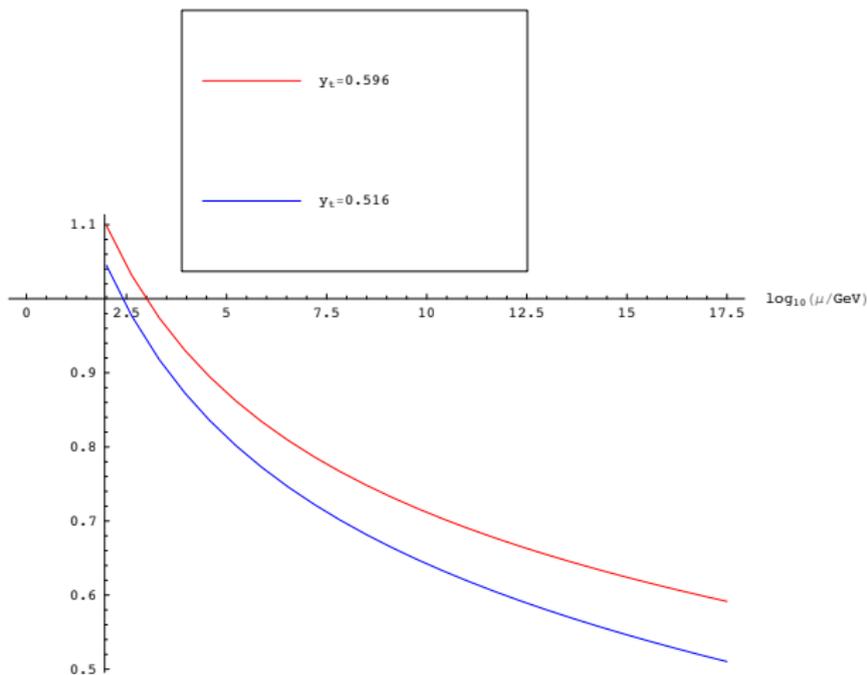
$$\frac{v}{\sqrt{2}}(y^\sigma) = (m^\sigma),$$

$$\frac{dy_t}{dt} = \frac{1}{16\pi^2} \left[\frac{9}{2}y_t^3 - (a g_1^2 + b g_2^2 + c g_3^2) y_t \right],$$

$$(a, b, c) = \left(\frac{17}{12}, \frac{9}{4}, 8 \right)$$

\Rightarrow value of top quark mass agrees with known (1.04 times if neglect other Yukawa couplings)

Top quark running using mass relation at unification



correction to MSM flow by y_ν^σ for τ neutrino (allowed to be comparably large by see-saw) lowers value

Higgs mass prediction using RGE for MSM

Higgs scattering parameter:

$$\frac{f_0}{2\pi^2} \int b |\varphi|^4 \sqrt{g} d^4x = \frac{\pi^2}{2f_0} \frac{b}{a^2} \int |\mathbf{H}|^4 \sqrt{g} d^4x$$

\Rightarrow relation at unification ($\tilde{\lambda}$ is $|\mathbf{H}|^4$ coupling)

$$\tilde{\lambda}(\Lambda) = g_3^2 \frac{b}{a^2}$$

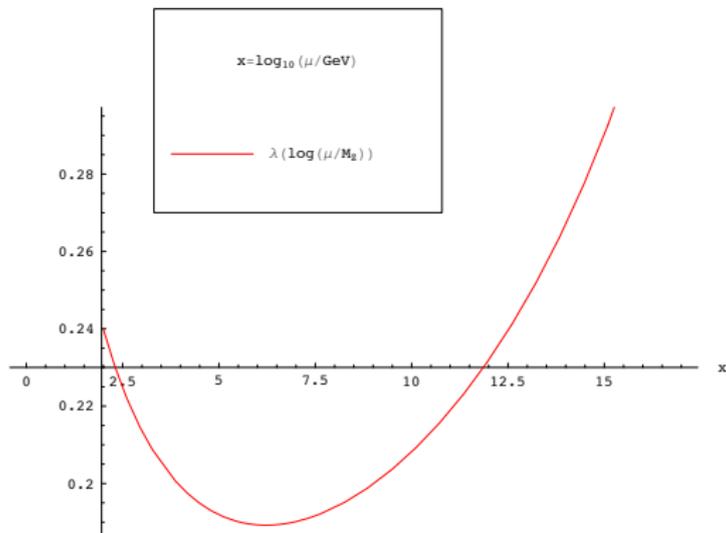
Running of Higgs scattering parameter:

$$\frac{d\lambda}{dt} = \lambda\gamma + \frac{1}{8\pi^2}(12\lambda^2 + B)$$

$$\gamma = \frac{1}{16\pi^2}(12y_t^2 - 9g_2^2 - 3g_1^2) \quad B = \frac{3}{16}(3g_2^4 + 2g_1^2g_2^2 + g_1^4) - 3y_t^4$$

Higgs estimate (in MSM approximation for RGE flow)

$$m_H^2 = 8\lambda \frac{M^2}{g^2}, \quad m_H = \sqrt{2\lambda} \frac{2M}{g}$$



$\lambda(M_Z) \sim 0.241$ and Higgs mass ~ 170 GeV (w/correction from see-saw ~ 168 GeV) ... **Problem: wrong Higgs mass! too heavy**