The Spectral Geometry of the Standard Model

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References

- A.H. Chamseddine, A. Connes, M. Marcolli, *Gravity and the Standard Model with Neutrino Mixing*, Adv. Theor. Math. Phys., Vol.11 (2007) 991–1090

Building a particle physics model

Minimal input ansatz:

- left-right symmetric algebra

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

- involution $(\lambda, q_L, q_R, m) \mapsto (\bar{\lambda}, \bar{q}_L, \bar{q}_R, m^*)$
- subalgebra $\mathbb{C} \oplus M_3(\mathbb{C})$ integer spin $\mathbb{C}$-alg
- subalgebra $\mathbb{H}_L \oplus \mathbb{H}_R$ half-integer spin $\mathbb{R}$-alg

More general choices of initial ansatz:

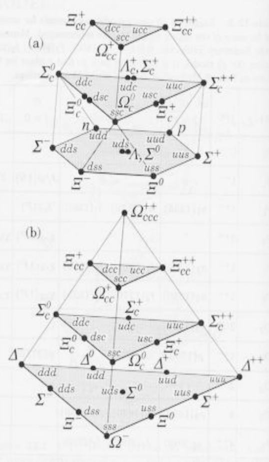
- A.Chamseddine, A.Connes, *Why the Standard Model*, J.Geom.Phys. 58 (2008) 38–47

Slogan: algebras better than Lie algebras, more constraints on reps

Comment: associative algebras versus Lie algebras

- In geometry of gauge theories: bundle over spacetime, connections and sections, automorphisms gauge group: Lie group
- Decomposing composite particles into elementary particles: Lie group representations (hadrons and quarks)
- If want only elementary particles: associative algebras have very few representations (very constrained choice)
- Get gauge groups later from inner automorphisms

Example Composite particles (baryons)



Classification in terms of elementary particles (quarks) using representations of Lie groups: $SU(3)$

Adjoint action: $\mathcal{M}$ bimodule over $\mathcal{A}$, $u \in \mathcal{U}(\mathcal{A})$ unitary

$$\mathrm{Ad}(u)\xi = u\xi u^* \quad \forall \xi \in \mathcal{M}$$

Odd bimodule: $\mathcal{M}$ bimodule for $\mathcal{A}_{LR}$ *odd* iff
$s = (1, -1, -1, 1)$ acts by $Ad(s) = -1$

$\Leftrightarrow$ Rep of $\mathcal{B} = (\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}_{LR}^{op})_p$ as $\mathbb{C}$-algebra
$p = \frac{1}{2}(1 - s \otimes s^0)$, with $\mathcal{A}^0 = \mathcal{A}^{op}$

$$\mathcal{B} = \oplus^{4-times} M_2(\mathbb{C}) \oplus M_6(\mathbb{C})$$

Contragredient bimodule of $\mathcal{M}$

$$\mathcal{M}^0 = \{\bar{\xi}\,;\ \xi \in \mathcal{M}\}, \quad a\,\bar{\xi}\,b = \overline{b^*\xi\, a^*}$$

The bimodule $\mathcal{M}_F$

$\mathcal{M}_F =$ sum of all inequivalent irreducible odd $\mathcal{A}_{LR}$-bimodules

- $\dim_{\mathbb{C}} \mathcal{M}_F = 32$
- $\mathcal{M}_F = \mathcal{E} \oplus \mathcal{E}^0$

$$\mathcal{E} = \mathbf{2}_L \otimes \mathbf{1}^0 \oplus \mathbf{2}_R \otimes \mathbf{1}^0 \oplus \mathbf{2}_L \otimes \mathbf{3}^0 \oplus \mathbf{2}_R \otimes \mathbf{3}^0$$

- $\mathcal{M}_F \cong \mathcal{M}_F^0$ by antilinear $J_F(\xi, \bar{\eta}) = (\eta, \bar{\xi})$ for $\xi\,,\ \eta \in \mathcal{E}$

$$J_F^2 = 1\,, \quad \xi\, b = J_F\, b^* J_F\, \xi \qquad \xi \in \mathcal{M}_F\,,\ b \in \mathcal{A}_{LR}$$

- Sum irreducible representations of $\mathcal{B}$

$$\mathbf{2}_L \otimes \mathbf{1}^0 \oplus \mathbf{2}_R \otimes \mathbf{1}^0 \oplus \mathbf{2}_L \otimes \mathbf{3}^0 \oplus \mathbf{2}_R \otimes \mathbf{3}^0$$

$$\oplus \mathbf{1} \otimes \mathbf{2}_L^0 \oplus \mathbf{1} \otimes \mathbf{2}_R^0 \oplus \mathbf{3} \otimes \mathbf{2}_L^0 \oplus \mathbf{3} \otimes \mathbf{2}_R^0$$

- Grading: $\gamma_F = c - J_F\, c\, J_F$ with $c = (0, 1, -1, 0) \in \mathcal{A}_{LR}$

$$J_F^2 = 1\,, \quad J_F\, \gamma_F = -\,\gamma_F\, J_F$$

Grading and KO-dimension: commutations $\Rightarrow$ KO-dim 6 mod 8

KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$
antilinear isometry $J : \mathcal{H} \to \mathcal{H}$

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J$$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

In particular, $J^2 = 1$ and $J\gamma = -\gamma J$ gives KO-dim $= 6$
So in this case *metric dimension* is zero but *KO-dimension* is 6

Interpretation as particles (Fermions)

$$q(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \qquad q(\lambda)|\uparrow\rangle = \lambda|\uparrow\rangle, \qquad q(\lambda)|\downarrow\rangle = \bar{\lambda}|\downarrow\rangle$$

- $\mathbf{2}_L \otimes \mathbf{1}^0$: neutrinos $\nu_L \in |\uparrow\rangle_L \otimes \mathbf{1}^0$ and charged leptons $e_L \in |\downarrow\rangle_L \otimes \mathbf{1}^0$
- $\mathbf{2}_R \otimes \mathbf{1}^0$: right-handed neutrinos $\nu_R \in |\uparrow\rangle_R \otimes \mathbf{1}^0$ and charged leptons $e_R \in |\downarrow\rangle_R \otimes \mathbf{1}^0$
- $\mathbf{2}_L \otimes \mathbf{3}^0$ (color indices): u/c/t quarks $u_L \in |\uparrow\rangle_L \otimes \mathbf{3}^0$ and d/s/b quarks $d_L \in |\downarrow\rangle_L \otimes \mathbf{3}^0$
- $\mathbf{2}_R \otimes \mathbf{3}^0$ (color indices): u/c/t quarks $u_R \in |\uparrow\rangle_R \otimes \mathbf{3}^0$ and d/s/b quarks $d_R \in |\downarrow\rangle_R \otimes \mathbf{3}^0$
- $\mathbf{1} \otimes \mathbf{2}^0_{L,R}$: antineutrinos $\bar{\nu}_{L,R} \in \mathbf{1} \otimes \uparrow\rangle^0_{L,R}$, and charged antileptons $\bar{e}_{L,R} \in \mathbf{1} \otimes |\downarrow\rangle^0_{L,R}$
- $\mathbf{3} \otimes \mathbf{2}^0_{L,R}$ (color indices): antiquarks $\bar{u}_{L,R} \in \mathbf{3} \otimes |\uparrow\rangle^0_{L,R}$ and $\bar{d}_{L,R} \in \mathbf{3} \otimes |\downarrow\rangle^0_{L,R}$

Subalgebra and order one condition:

$N = 3$ generations (input): $\mathcal{H}_F = \mathcal{M}_F \oplus \mathcal{M}_F \oplus \mathcal{M}_F$

Left action of $\mathcal{A}_{LR}$ sum of representations $\pi|_{\mathcal{H}_f} \oplus \pi'|_{\mathcal{H}_{\bar{f}}}$ with $\mathcal{H}_f = \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{E}$ and $\mathcal{H}_{\bar{f}} = \mathcal{E}^0 \oplus \mathcal{E}^0 \oplus \mathcal{E}^0$ and with no equivalent subrepresentations (disjoint)

If D mixes $\mathcal{H}_f$ and $\mathcal{H}_{\bar{f}} \Rightarrow$ no order one condition for $\mathcal{A}_{LR}$

Problem for coupled pair: $\mathcal{A} \subset \mathcal{A}_{LR}$ and D with off diagonal terms

maximal $\mathcal{A}$ where order one condition holds

$$\mathcal{A}_F = \{(\lambda, q_L, \lambda, m) \mid \lambda \in \mathbb{C}, \; q_L \in \mathbb{H}, \; m \in M_3(\mathbb{C})\}$$

$$\sim \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

unique up to $\mathrm{Aut}(\mathcal{A}_{LR})$

$\Rightarrow$ spontaneous breaking of LR symmetry

Subalgebras with off diagonal Dirac and order one condition

Operator $T : \mathcal{H}_f \to \mathcal{H}_{\bar{f}}$

$$\mathcal{A}(T) = \{b \in \mathcal{A}_{LR} \mid \pi'(b)T = T\pi(b),$$

$$\pi'(b^*)T = T\pi(b^*)\}$$

involutive unital subalgebra of $\mathcal{A}_{LR}$

$\mathcal{A} \subset \mathcal{A}_{LR}$ involutive unital subalgebra of $\mathcal{A}_{LR}$

- restriction of π and π' to $\mathcal{A}$ disjoint $\Rightarrow$ no off diag D for $\mathcal{A}$
- $\exists$ off diag D for $\mathcal{A} \Rightarrow$ pair e, e' min proj in commutants of $\pi(\mathcal{A}_{LR})$ and $\pi'(\mathcal{A}_{LR})$ and operator T

$$e' T e = T \neq 0 \quad \text{and} \quad \mathcal{A} \subset \mathcal{A}(T)$$

- Then case by case analysis to identify max dimensional

Symmetries

Up to a finite abelian group

$$\mathrm{SU}(\mathcal{A}_F) \sim \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

Adjoint action of $\mathrm{U}(1)$ (in powers of $\lambda \in \mathrm{U}(1)$)

	$\uparrow \otimes \mathbf{1}^0$	$\downarrow \otimes \mathbf{1}^0$	$\uparrow \otimes \mathbf{3}^0$	$\downarrow \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

$\Rightarrow$ correct hypercharges of fermions (confirms identification of $\mathcal{H}_F$ basis with fermions)

Classifying Dirac operators for $(\mathcal{A}_F, \mathcal{H}_F, \gamma_F, J_F)$ all possible D_F self adjoint on $\mathcal{H}_F$, commuting with J_F, anticommuting with γ_F and $[[D,a],b^0]=0$, $\forall a,b\in\mathcal{A}_F$

Input conditions (massless photon): commuting with subalgebra

$$\mathbb{C}_F \subset \mathcal{A}_F\,, \quad \mathbb{C}_F = \{(\lambda,\lambda,0)\,,\lambda\in\mathbb{C}\}$$

$$\text{then} \quad D(Y) = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix} \quad \text{with} \quad S = S_1 \oplus (S_3 \otimes 1_3)$$

$$S_1 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 1)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 1)} \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{pmatrix}$$

same for S_3, with $Y_{(\downarrow 1)}$, $Y_{(\uparrow 1)}$, $Y_{(\downarrow 3)}$, $Y_{(\uparrow 3)} \in \mathrm{GL}_3(\mathbb{C})$ and Y_R symmetric:

$$T : E_R = \uparrow_R \otimes \mathbf{1}^0 \to J_F\, E_R$$

Moduli space $\mathcal{C}_3 \times \mathcal{C}_1$ $\mathcal{C}_3 =$ pairs $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ modulo

$$Y'_{(\downarrow 3)} = W_1 \, Y_{(\downarrow 3)} \, W_3^*, \qquad Y'_{(\uparrow 3)} = W_2 \, Y_{(\uparrow 3)} \, W_3^*$$

W_j unitary matrices

$$\mathcal{C}_3 = (K \times K) \backslash (G \times G)/K$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$ $\dim_{\mathbb{R}} \mathcal{C}_3 = 10 = 3 + 3 + 4$
$(3 + 3$ eigenvalues, 3 angles, 1 phase$)$

$\mathcal{C}_1 =$ triplets $(Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_R)$ with Y_R symmetric modulo

$$Y'_{(\downarrow 1)} = V_1 \, Y_{(\downarrow 1)} V_3^*, \quad Y'_{(\uparrow 1)} = V_2 \, Y_{(\uparrow 1)} \, V_3^*, \quad Y'_R = V_2 \, Y_R \, \bar{V}_2^*$$

$\pi : \mathcal{C}_1 \to \mathcal{C}_3$ surjection forgets Y_R fiber symmetric matrices mod
$Y_R \mapsto \lambda^2 Y_R$ $\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$ (dim fiber 12-1=11)

Physical interpretation: Yukawa parameters and Majorana masses
Representatives in $\mathcal{C}_3 \times \mathcal{C}_1$:

$$Y_{(\uparrow 3)} = \delta_{(\uparrow 3)} \qquad Y_{(\downarrow 3)} = U_{CKM}\, \delta_{(\downarrow 3)}\, U^*_{CKM}$$

$$Y_{(\uparrow 1)} = U^*_{PMNS}\, \delta_{(\uparrow 1)}\, U_{PMNS} \qquad Y_{(\downarrow 1)} = \delta_{(\downarrow 1)}$$

$\delta_\uparrow$, $\delta_\downarrow$ diagonal: Dirac masses

$$U = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e_\delta & c_1 c_2 s_3 + s_2 c_3 e_\delta \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e_\delta & c_1 s_2 s_3 - c_2 c_3 e_\delta \end{pmatrix}$$

angles and phase $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, $e_\delta = \exp(i\delta)$
U_{CKM} = Cabibbo–Kobayashi–Maskawa
U_{PMNS} = Pontecorvo–Maki–Nakagawa–Sakata
$\Rightarrow$ neutrino mixing
Y_R = Majorana mass terms for right-handed neutrinos

Geometric point of view:

- CKM and PMNS matrices data: coordinates on moduli space of Dirac operators
- Experimental constraints define subvarieties in the moduli space
- Symmetric spaces $(K \times K)\backslash(G \times G)/K$ interesting geometry
- Get parameter relations from "interesting subvarieties"?

Summary: matter content of the NCG model

νMSM: Minimal Standard Model with additional right handed neutrinos with Majorana mass terms

Free parameters in the model:

- 3 coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses, 3 lepton mixing angles, 1 complex phase
- 3 neutrino masses
- 11 Majorana mass matrix parameters
- 1 QCD vacuum angle

Moduli space of Dirac operators on the finite NC space F: all masses, mixing angles, phases, Majorana mass terms

Other parameters:

- coupling constants: product geometry and action functional
- vacuum angle not there (but quantum corrections...?)

Symmetries and NCG

Symmetries of gravity coupled to matter:
$G = U(1) \times SU(2) \times SU(3)$

$$\mathcal{G} = \mathrm{Map}(M, G) \rtimes \mathrm{Diff}(M)$$

Is it $\mathcal{G} = \mathrm{Diff}(X)$? Not for a manifold, yes for an NC space

Example: $\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$ $\quad G = PSU(n)$

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1$$

$$1 \to \mathrm{Map}(M, G) \to \mathcal{G} \to \mathrm{Diff}(M) \to 1.$$

- Symmetries viewpoint: can think of $X = M \times F$ noncommutative with $\mathcal{G} = \mathrm{Diff}(X)$ pure gravity symmetries for X combining gravity and gauge symmetries together (no a priori distinction between "base" and "fiber" directions)
- Want same with action functional for pure gravity on NC space $X = M \times F$ giving gravity coupled to matter on M

Product geometry $M \times F$

Two spectral triples $(\mathcal{A}_i, \mathcal{H}_i, D_i, \gamma_i, J_i)$ of KO-dim 4 and 6:

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2$$

$$\gamma = \gamma_1 \otimes \gamma_2 \qquad J = J_1 \otimes J_2$$

Case of 4-dimensional spin manifold M and finite NC geometry F:

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F = C^\infty(M, \mathcal{A}_F)$$

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F)$$

$$D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$$

D_F chosen in the moduli space described last time

Dimension of NC spaces: different notions of dimension for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$

- Metric dimension: growth of eigenvalues of Dirac operator
- KO-dimension (mod 8): sign commutation relations of J, γ, D
- Dimension spectrum: poles of zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$

For manifolds first two agree and third contains usual dim; for NC spaces not same: DimSp $\subset \mathbb{C}$ can have non-integer and non-real points, KO not always metric dim mod 8, see F case

$X = M \times F$ metrically four dim $4 = 4 + 0$; KO-dim is $10 = 4 + 6$ (equal 2 mod 8); DimSp $k \in \mathbb{Z}_{\geq 0}$ with $k \leq 4$

Variant: almost commutative geometries

$$(C^\infty(M,\mathcal{E}), L^2(M,\mathcal{E}\otimes S), \mathcal{D}_{\mathcal{E}})$$

- M smooth manifold, $\mathcal{E}$ algebra bundle: fiber $\mathcal{E}_x$ finite dimensional algebra $\mathcal{A}_F$
- $C^\infty(M,\mathcal{E})$ smooth sections of a algebra bundle $\mathcal{E}$
- Dirac operator $\mathcal{D}_{\mathcal{E}} = c \circ (\nabla^{\mathcal{E}} \otimes 1 + 1 \otimes \nabla^S)$ with spin connection ∇^S and hermitian connection on bundle
- Compatible grading and real structure

An equivalent intrinsic (abstract) characterization in:

- Branimir Ćaćić, *A reconstruction theorem for almost-commutative spectral triples*, arXiv:1101.5908

Here we will assume for simplicity just a product $M \times F$

Inner fluctuations and gauge fields

Setup:

- Right $\mathcal{A}$-module structure on $\mathcal{H}$

$$\xi\, b = \, b^0\, \xi, \quad \xi \in \mathcal{H}\,, \quad b \in \mathcal{A}$$

- Unitary group, adjoint representation:

$$\xi \in \mathcal{H} \to \mathrm{Ad}(u)\, \xi = u\, \xi\, u^* \quad \xi \in \mathcal{H}$$

Inner fluctuations:

$$D \to D_A = D + A + \varepsilon'\, J\, A\, J^{-1}$$

with $A = A^*$ self-adjoint operator of the form

$$A = \sum a_j [D, b_j]\,, \quad a_j, b_j \in \mathcal{A}$$

Note: not an equivalence relation (finite geometry, can fluctuate D to zero) but like "self Morita equivalences"

Properties of inner fluctuations $(\mathcal{A}, \mathcal{H}, D, J)$

- Gauge potential $A \in \Omega^1_D$, $A = A^*$
- Unitary $u \in \mathcal{A}$, then

$$\mathrm{Ad}(u)(D + A + \varepsilon' \, J \, A \, J^{-1})\mathrm{Ad}(u^*) =$$

$$D + \gamma_u(A) + \varepsilon' \, J \, \gamma_u(A) \, J^{-1}$$

where $\gamma_u(A) = u\,[D, u^*] + u\,A\,u^*$

- $D' = D + A$ (with $A \in \Omega^1_D$, $A = A^*$) then

$$D' + B = D + A' , \quad A' = A + B \in \Omega^1_D$$

$\forall B \in \Omega^1_{D'}$, $B = B^*$

- $D' = D + A + \varepsilon' \, J \, A \, J^{-1}$ then

$$D' + B + \varepsilon' \, J \, B \, J^{-1} = D + A' + \varepsilon' \, J \, A' \, J^{-1} \quad A' = A + B \in \Omega^1_D$$

$\forall B \in \Omega^1_{D'}$, $B = B^*$

Gauge bosons and Higgs boson

- Unitary $\mathrm{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^* u = 1\}$
- Special unitary

$$\mathrm{SU}(\mathcal{A}_F) = \{u \in \mathrm{U}(\mathcal{A}_F) \mid \det(u) = 1\}$$

det of action of u on $\mathcal{H}_F$

- Up to a finite abelian group

$$\mathrm{SU}(\mathcal{A}_F) \sim \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

- Unimod subgr of $\mathcal{U}(\mathcal{A})$ adjoint rep $\mathrm{Ad}(u)$ on $\mathcal{H}$ is gauge group of SM
- Unimodular inner fluctuations (in M directions) $\Rightarrow$ gauge bosons of SM: $U(1)$, $SU(2)$ and $SU(3)$ gauge bosons
- Inner fluctuations in F direction $\Rightarrow$ Higgs field

More on Gauge bosons

Inner fluctuations $A^{(1,0)} = \sum_i a_i[\not\partial_M \otimes 1, a'_i]$ with with $a_i = (\lambda_i, q_i, m_i)$, $a'_i = (\lambda'_i, q'_i, m'_i)$ in $\mathcal{A} = C^\infty(M, \mathcal{A}_F)$

- $U(1)$ gauge field $\Lambda = \sum_i \lambda_i \, d\lambda'_i = \sum_i \lambda_i[\not\partial_M \otimes 1, \lambda'_i]$
- $SU(2)$ gauge field $Q = \sum_i q_i \, dq'_i$, with $q = f_0 + \sum_\alpha if_\alpha \sigma^\alpha$ and $Q = \sum_\alpha f_\alpha[\not\partial_M \otimes 1, if'_\alpha \sigma^\alpha]$
- $U(3)$ gauge field $V' = \sum_i m_i \, dm'_i = \sum_i m_i[\not\partial_M \otimes 1, m'_i]$
- reduce the gauge field V' to $SU(3)$ passing to unimodular subgroup $\mathrm{SU}(\mathcal{A}_F)$ and unimodular gauge potential $\mathrm{Tr}(A) = 0$

$$V' = -V - \frac{1}{3}\begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix} = -V - \frac{1}{3}\Lambda 1_3$$

Gauge bosons and hypercharges

The $(1,0)$ part of $A + JAJ^{-1}$ acts on quarks and leptons by

$$\begin{pmatrix} \frac{4}{3}\Lambda + V & 0 & 0 & 0 \\ 0 & -\frac{2}{3}\Lambda + V & 0 & 0 \\ 0 & 0 & Q_{11} + \frac{1}{3}\Lambda + V & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} + \frac{1}{3}\Lambda + V \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\Lambda & 0 & 0 \\ 0 & 0 & Q_{11} - \Lambda & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} - \Lambda \end{pmatrix}$$

$\Rightarrow$ correct hypercharges!

More on Higgs boson

Inner fluctuations $A^{(0,1)}$ in the F-space direction

$$\sum_i a_i[\gamma_5 \otimes D_F, a'_i](x)|_{\mathcal{H}_f} = \gamma_5 \otimes (A_q^{(0,1)} + A_\ell^{(0,1)})$$

$$A_q^{(0,1)} = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix} \otimes 1_3 \quad A_1^{(0,1)} = \begin{pmatrix} 0 & Y \\ Y' & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} \Upsilon_u^* \varphi_1 & \Upsilon_u^* \varphi_2 \\ -\Upsilon_d^* \bar\varphi_2 & \Upsilon_d^* \bar\varphi_1 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} \Upsilon_u \varphi'_1 & \Upsilon_d \varphi'_2 \\ -\Upsilon_u \bar\varphi'_2 & \Upsilon_d \bar\varphi'_1 \end{pmatrix}$$

$$Y = \begin{pmatrix} \Upsilon_\nu^* \varphi_1 & \Upsilon_\nu^* \varphi_2 \\ -\Upsilon_e^* \bar\varphi_2 & \Upsilon_e^* \bar\varphi_1 \end{pmatrix} \quad \text{and} \quad Y' = \begin{pmatrix} \Upsilon_\nu \varphi'_1 & \Upsilon_e \varphi'_2 \\ -\Upsilon_\nu \bar\varphi'_2 & \Upsilon_e \bar\varphi'_1 \end{pmatrix}$$

$\varphi_1 = \sum \lambda_i(\alpha'_i - \lambda'_i)$, $\varphi_2 = \sum \lambda_i \beta'_i$ $\varphi'_1 = \sum \alpha_i(\lambda'_i - \alpha'_i) + \beta_i \bar\beta'_i$ and $\varphi'_2 = \sum(-\alpha_i \beta'_i + \beta_i(\bar\lambda'_i - \bar\alpha'_i))$, for $a_i(x) = (\lambda_i, q_i, m_i)$ and

$a'_i(x) = (\lambda'_i, q'_i, m'_i)$ and $q = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$

More on Higgs boson

Discrete part of inner fluctuations: quaternion valued function $H = \varphi_1 + \varphi_2 j$ or $\varphi = (\varphi_1, \varphi_2)$

$$D_A^2 = (D^{1,0})^2 + 1_4 \otimes (D^{0,1})^2 - \gamma_5 \, [D^{1,0}, 1_4 \otimes D^{0,1}]$$

$$[D^{1,0}, 1_4 \otimes D^{0,1}] = \sqrt{-1} \, \gamma^\mu \, [(\nabla^s_\mu + \mathbb{A}_\mu), 1_4 \otimes D^{0,1}]$$

This gives $D_A^2 = \nabla^* \nabla - E$ where $\nabla^* \nabla$ Laplacian of $\nabla = \nabla^s + \mathbb{A}$

$$-E = \frac{1}{4} \, s \otimes \mathrm{id} + \sum_{\mu<\nu} \gamma^\mu \gamma^\nu \otimes \mathbb{F}_{\mu\nu} - i \, \gamma_5 \, \gamma^\mu \otimes \mathbb{M}(D_\mu \varphi) + 1_4 \otimes (D^{0,1})^2$$

with $s = -R$ scalar curvature and $\mathbb{F}_{\mu\nu}$ curvature of $\mathbb{A}$

$$D_\mu \varphi = \partial_\mu \varphi + \frac{i}{2} g_2 W^\alpha_\mu \varphi \, \sigma^\alpha - \frac{i}{2} g_1 B_\mu \, \varphi$$

$SU(2)$ and $U(1)$ gauge potentials