Models based on Finite Spectral Triple

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References

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Gauge networks

• using finite spectral triple for a model combining gauge theory on a lattice (or graph) and spin networks approach to gravity

• an action functional (in terms of Dirac operator) that recovers the Wilson action (which in continuum limit gives Yang–Mills) will additional terms for a Higgs field in adjoint representation

• build a category of finite spectral triples with morphisms built from algebra morphisms and unitary operators

• representations of quivers (oriented graphs) in this category of finite spectral triples

• configuration space (of such representation) modulo gauge action

• morphisms between gauge networks by correspondences (bimodules); Hamiltonian and time evolution

• discretized Dirac operator and continuum limit

$\mathcal{C}_0$ Category of finite spectral triples with trivial Dirac $D = 0$

• objects $(\mathcal{A}, \pi, \mathcal{H})$, fin. dim. algebra $\mathcal{A}$ and fin. Hilbert space rep. $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$

• morphisms $\Phi : (\mathcal{A}_1, \pi_1, \mathcal{H}_1) \to (\mathcal{A}_2, \pi_2, \mathcal{H}_2)$ pair $\Phi = (\phi, L)$
$\phi : \mathcal{A}_1 \to \mathcal{A}_2$ morphism of unital $\star$-algebras, $L : \mathcal{H}_1 \to \mathcal{H}_2$ unitary

$$L\pi_1(a)L^* = \pi_2(\phi(a))$$

$\mathcal{C}$ Category of finite spectral triples

• objects $(\mathcal{A}, \pi, \mathcal{H}, D)$ fin spectral triples

• morphisms $\Phi : (\mathcal{A}_1, \pi_1, \mathcal{H}_1, D_1) \to (\mathcal{A}_2, \pi_2, \mathcal{H}_2, D_2)$ as above with also $LD_1L^* = D_2$

Bratteli diagrams

- Wedderburn theorem:

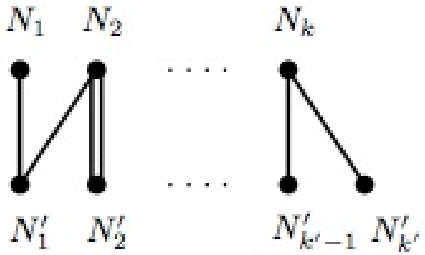
$$\mathcal{A}_1 = \bigoplus_{i=1}^{k} M_{N_i}(\mathbb{C}), \quad \mathcal{A}_2 = \bigoplus_{j=1}^{k'} M_{N'_j}(\mathbb{C})$$

- unital $*$-algebra morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ direct sum

$$\phi_j : \bigoplus_{i=1}^{k} M_{N_i}(\mathbb{C}) \to M_{N'_j}(\mathbb{C})$$

ϕ_j splits as a direct sum of representation $\phi_{ij} : M_{N_i}(\mathbb{C}) \to M_{N'_j}(\mathbb{C})$ with multiplicity $d_{ij} \geq 0$, with $N'_j = \sum_i d_{ij} N_i$

- Bratteli diagrams: two rows of vertices: top k vertices labeled $N_1, \ldots, N_k$, bottom k' vertices labeled by $N'_1, \ldots, N'_{k'}$; d_{ij} edges between vertex i (top row) and j (bottom row)



$\phi : \mathcal{A}_1 \to \mathcal{A}_2$ unital, so all vertices in bottom row reached by an edge, but top row can have vacant vertices

Example

- $\mathcal{A}_1 = \mathbb{C} \oplus M_2(\mathbb{C})$, $\mathcal{H}_1 = \mathbb{C} \oplus \mathbb{C}^2$, $\mathcal{A}_2 = M_3(\mathbb{C})$, $\mathcal{H}_2 = \mathbb{C}^3$
- unital $*$-algebra map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ two possibilities

$$(z, a) \in \mathbb{C} \oplus M_2(\mathbb{C}) \quad \mapsto u \begin{pmatrix} z & \\ & a \end{pmatrix} u^* \in M_3(\mathbb{C})$$

with $u \in U(3)$ or

$$(z, a) \in \mathbb{C} \oplus M_2(\mathbb{C}) \quad \mapsto z 1_3 \in M_3(\mathbb{C})$$

with kernel $M_2(\mathbb{C})$

- unitary map of $\mathcal{H}_1$ to $\mathcal{H}_2$

$$(x, y) \in \mathbb{C} \oplus \mathbb{C}^2 \mapsto U \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^3$$

with $U \in U(3)$

- compatibility of ϕ and L: first case OK with $u = U$

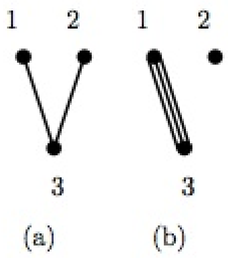
$$u \begin{pmatrix} z & \\ & a \end{pmatrix} u^* = U \begin{pmatrix} z & \\ & a \end{pmatrix} U^*.$$

but in second case

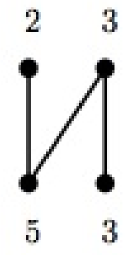
$$z 1_3 = U \begin{pmatrix} z & \\ & a \end{pmatrix} U^*.$$

cannot be satisfied for arbitrary $(z, a) \in \mathcal{A}_1$

- so get $\mathrm{Hom}((A_1, H_1), (A_2, H_2)) \simeq U(3)$ and Bratteli diagram

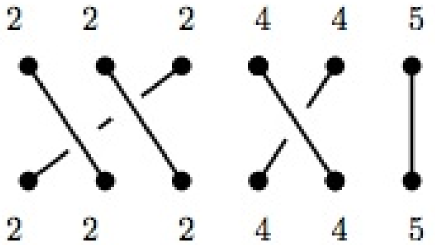


Example



Bratteli diagram for the only unital $*$-algebra map
$M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ given $(a, b) \mapsto (a \oplus b, b)$

• to better take care also of permutations of matrix blocks of the same dimension: braid Bratteli diagrams



braid Bratteli diagram with permutations of matrix blocks of same dim in $M_2(\mathbb{C})^{\oplus 3} \oplus M_4(\mathbb{C})^{\oplus 2} \oplus M_5(\mathbb{C})$

• any Bratteli diagram $\mathbb{B}$ for a pair (A_1, A_2) gives homomorphism $\phi_{\mathbb{B}} : A_1 \to A_2$ embedding matrix blocks of A_1 into those of A_2 following lines in $\mathbb{B}$

• any other unital $*$-algebra morphisms $\phi : A_1 \to A_2$ can be obtained from $\phi_{\mathbb{B}}$ by unitary change of basis
$\phi(\cdot) = U\phi_{\mathbb{B}}(\cdot)U^* =: \mathrm{Ad}\, U\phi_{\mathbb{B}}(\cdot)$ some unitary U in A_2

• representation λ of A on finite dim Hilbert space H, two-sided ideal $\mathrm{Ker}(\lambda)$ with $A = \widetilde{A} \oplus \mathrm{Ker}(\lambda)$ and $\widetilde{(}A) \simeq \lambda(A)$

• morphisms (ϕ, L) with $\star$-homomorphisms $\phi : A_1 \to A_2$ and unitary $L : H_1 \to H_2$ with $L\lambda_1(a)L^* = \lambda_2(\pi(a))$

• decompose as $\phi = \widetilde{\phi} + \phi_0$ with $\widetilde{\phi} : \widetilde{A}_1 \to \widetilde{A}_2$ with $\widetilde{\phi}(\widetilde{a}) = L\widetilde{a}L^*$ and $\phi_0 : A_1 \to \mathrm{Ker}(\lambda_2)$

• identify $\mathrm{Aut}((A, \lambda, H)) \simeq \mathcal{U}(\widetilde{A}) \rtimes S(\widetilde{A}; H) \times P\mathcal{U}(\ker \lambda) \rtimes S(\ker \lambda)$ with $S(\widetilde{A}; H)$ and $S(\ker \lambda)$ groups of permutations of matrix blocks of equal dimension in $\widetilde{A}$ and H and $\ker \lambda$ (proj unitary group because adjoint action of center of $\mathcal{U}(\ker \lambda)$ on $\ker \lambda$ trivial)

- for algebras and Hilbert spaces

$$A_1 = \bigoplus_{i=1}^{k+l} M_{N_i}(\mathbb{C}) \qquad A_2 = \bigoplus_{j=1}^{k'+l'} M_{N'_j}(\mathbb{C})$$

$$H_1 = \bigoplus_{i=1}^{k} n_i \mathbb{C}^{N_i} \qquad H_2 = \bigoplus_{j=1}^{k'} n'_j \mathbb{C}^{N'_j}$$

- any morphism (ϕ, L) given by

$$\phi = \mathrm{Ad}\, U \phi_{\widetilde{\mathbb{B}}} + \mathrm{Ad}\, V \phi_{\mathbb{B}_0} \qquad L = U L_{\widetilde{\mathbb{B}}}$$

  - unitary $U \in \mathrm{Aut}_{\widetilde{A}_2}(H_2) \simeq \prod_{j=1}^{k'} U(n_j N_j)$
  - unitary $V \in \mathcal{U}(\ker \lambda_2) \simeq \prod_{j=k'+1}^{k'+l'} U(N_j)$
  - Bratteli diagrams $\widetilde{\mathbb{B}}, \mathbb{B}_0$ of $*$-algebra maps $\widetilde{A}_1 \hookrightarrow \widetilde{A}_2$ and $A_1 \to \ker \lambda_2$
  - unitary map $L_{\widetilde{\mathbb{B}}} : H_1 \to H_2$ implements $*$-algebra map $\phi_{\widetilde{\mathbb{B}}} : \widetilde{A}_1 \to \widetilde{A}_2$

$$L_{\widetilde{\mathbb{B}}} \widetilde{a} L^*_{\widetilde{\mathbb{B}}} = \phi_{\widetilde{\mathbb{B}}}(\widetilde{a}) \qquad \forall \widetilde{a} \in \widetilde{A}_1$$

Quiver representations in categories

- Quiver Γ directed graph

- representation π of a quiver Γ in a category $\mathcal{C}$:

- object π_v for each vertex v
- morphism π_e in $\mathrm{Hom}(\pi_{s(e)}, \pi_{t(e)})$ for each directed edge e.

- two representations π, π' of Γ in same category equivalent if $\pi_v = \pi'_v$, for all $v \in V(\Gamma)$ and $\exists$ family of invertible morphisms $\phi_v \in \mathrm{Hom}(\pi(v), \pi(v))$ for $v \in V(\Gamma)$ such that

$$\pi_e = \phi_{t(e)} \circ \pi'_e \circ \phi^{-1}_{s(e)}$$

- For categories $\mathcal{C}$ (or $\mathcal{C}_0$) of finite spectral triples, representation π of a quiver Γ assigns

- spectral triples $(\mathcal{A}_v, \mathcal{H}_v, D_v)$ ($D_v = 0$ for $\mathcal{C}_0$) to vertices $v \in V(\Gamma)$
- pairs $(\phi, L) \in \mathrm{Hom}((\mathcal{A}_{s(e)}, \mathcal{H}_{s(e)}, D_{s(e)}), (\mathcal{A}_{t(e)}, \mathcal{H}_{t(e)}, D_{t(e)}))$ to edges $e \in E(\Gamma)$

Equivalence of quiver representations

• two representations π, π' of Γ in the same category are equivalent if

- $\pi_v = \pi'_v$ for all $v \in \Gamma^{(0)}$
- there exists a family of invertible morphisms $\phi_v \in \mathrm{Hom}(\pi(v), \pi(v))$ indexed by the vertices v such that

$$\pi_e = \phi_{t(e)} \circ \pi'_e \circ \phi_{s(e)}^{-1}$$

• if we view a quiver Γ itself as a category, a representation is a functor π from Γ to a category; equivalent representations coincide on objects and are related via invertible natural transformations

Example $U(N)$ spin networks (John Baez)

• If $(\mathcal{A}_v, \mathcal{H}_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ and $D = 0$, unitary $u_e \in U(N)$ along each edge and gauge action $g_v \in U(N)$ at each vertex with

$$u_e \mapsto g_{t(e)} u_e g^*_{s(e)}$$

• only possible Bratteli diagram in this case for $\phi : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ is single edge between one upper row vertex and one lower row vertex

• J.C. Baez, *Spin network states in gauge theory*, Adv. Math. 117 (1996) 253–272

General case: gauge networks

$$\{\Gamma, (A_v, \lambda_v, H_v; \iota_v)_v, (\rho_e, \mathbb{B}_e)_e\}$$

- Γ directed graph
- (A_v, λ_v, H_v) is an object in the category $\mathcal{C}_0^s$ for each vertex $v \in V(\Gamma)$
- Edge $e \in E(\Gamma)$: representation ρ_e of unitary group $G_e = \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)}) \times \mathcal{U}(\ker \lambda_{t(e)})$
- Edge $e \in E(\Gamma)$: Bratteli diagram $\mathbb{B}_e$ for $*$-algebra maps $A_{s(e)} \to A_{t(e)}$
- subdiagrams $\widetilde{\mathbb{B}}$ for $\widetilde{A}_{s(e)} \to \widetilde{A}_{t(e)}$ and $\mathbb{B}_0$ for $A_{s(e)} \to \ker \lambda_{t(e)}$
- "intertwiners at vertices" between representations ρ_e associated to edges (more on this later)

• space of representations of Γ in $\mathcal{C}_0^s$

$$\mathcal{X} = \coprod_{\{A_v,H_v\}_v} \prod_{e\in E(\Gamma)} \mathcal{X}_e$$

$\mathcal{X}_e = \mathrm{Hom}((A_{s(e)}, \lambda_{s(e)}, H_{s(e)}), (A_{t(e)}, \lambda_{t(e)}, H_{t(e)}))$

• elements $(\phi_e, L_e) \in \mathcal{X}_e$

$$\phi_e = \mathrm{Ad}\, U \phi_{\widetilde{\mathbb{B}}_e} + \mathrm{Ad}\, V \phi_{\mathbb{B}_{e0}}; \qquad L_e = U L_{\widetilde{\mathbb{B}}_e}$$

unitaries $U \in \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)})$, $V \in \mathcal{U}(\ker \lambda_{t(e)})$ and a Bratteli diagram $\mathbb{B}_e$ (with subdiagrams $\widetilde{\mathbb{B}}_e, \mathbb{B}_{e0}$) for each edge e

• this means unitary group $\mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)})$ together with all $\mathcal{U}(\ker \lambda_{t(e)})$-orbits of $\phi_{\mathbb{B}_{e0}}$ for all such $\mathbb{B}_{e0}$ gives all of $\mathcal{X}_e$

• Orbit-stabilizer: isotropy subgroup $\mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$ of $\phi_{\mathbb{B}_{e0}}$

$$\mathcal{X}_e = \coprod_{\mathbb{B}_e} \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)}) \times \mathcal{U}(\ker \lambda_{t(e)})/\mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$$

• elements in $\mathcal{X}$ by $(U_e, [V_e], \mathbb{B}_e)_e$ with $U_e \in \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)})$ and $V_e \in \mathcal{U}(\ker \lambda_{t(e)})$

• equivalence of quiver representations: collection of unitaries

$$(g_v, \sigma_v) := (\widetilde{g}_v, \widetilde{\sigma}_v; g_{v0}, \sigma_{v0}) \in \mathcal{G}_v$$

$$\mathcal{G}_v := \mathrm{Aut}_{\widetilde{A}_v}(H_v) \rtimes S(\widetilde{A}_v; H_v) \times P\mathcal{U}(\ker \lambda_v) \rtimes S(\ker \lambda_v)$$

mapping $(U_e, [V_e], \mathbb{B}_e) \in \mathcal{X}_e$ to

$$(\widetilde{g}_{t(e)} U_e \phi_{\widetilde{\mathbb{B}}_e}(\widetilde{g}^*_{s(e)}), [g_{t(e)0} V_e \phi_{\mathbb{B}_{e0}}(g^*_{s(e)})], \sigma_{t(e)} \circ \mathbb{B}_e \circ \sigma_{s(e)})$$

• Peter-Weyl theorem for compact Lie groups G

$$L^2(G) \simeq \bigoplus_{\rho\in\widehat{G}} \rho \otimes \rho^*$$

with $\widehat{G}$ irreducible unitary reps, isomorphism of $G \times G$-representations with

$$((g_1, g_2)f)(x) = f(g_1^{-1} x g_2) \qquad \forall f \in L^2(G)$$

$$(g_1, g_2)(y_1 \otimes y_2) = \rho(g_1)y_1 \otimes \rho^*(g_2)(y_2) \qquad \forall y_1 \in \rho, y_2 \in \rho^*$$

• this means orthonormal basis for $L^2(G)$ (Haar measure) constructed using matrix coefficients $\langle \pi(g)e_i, e_j\rangle$ for $g \in G$, over representatives π of isomorphism classes of irreducible unitary representations

• G compact Lie group, K and H mutually commuting closed subgroups

$$L^2(G/K) \simeq L^2(G)^K \simeq \bigoplus_{\rho\in\widehat{G}} \rho \otimes (\rho^*)^K$$

isomorphism of $G \times H$-representations, with ρ^K the K-invariant subspace of the G-representation ρ

- then get for the space of representations of Γ in $\mathcal{C}_0^s$

$$L^2(\mathcal{X}) \simeq \bigoplus_{\{A_v,H_v\}} \bigotimes_e \bigoplus_{\mathbb{B}_e} L^2(G_e/K_{\mathbb{B}_e})$$

$$G_e := \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)}) \times \mathcal{U}(\ker \lambda_{t(e)}) \qquad K_{\mathbb{B}_e} := \{e\} \times \mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$$

by Peter-Weyl theorem

$$L^2(\mathcal{X}) \simeq \bigoplus_{\{A_v,H_v\}} \bigotimes_e \bigoplus_{\mathbb{B}_e} \bigoplus_{\rho_e \in \widehat{G_e}} \rho_e \otimes (\rho_e^*)^{K_{\mathbb{B}_e}}$$

- action of $\mathcal{G}$

$$\bigoplus_{\{A_v,H_v\}} \bigotimes_e \bigoplus_{\mathbb{B}_e} \bigoplus_{\rho_e \in \widehat{G_e}} \rho_e(g_{t(e)}) \otimes \rho_e^* \circ \phi_{\mathbb{B}}(g_{s(e)}),$$

- rewrite $L^2(\mathcal{X})$ in the form

$$L^2(\mathcal{X}) \simeq \bigoplus_{\substack{\{A_v,H_v\}\\ \{\rho_e,\mathbb{B}_e\}}} \bigotimes_v \left(\bigotimes_{e\in T(v)} \rho_e \otimes \bigotimes_{e\in S(v)} (\rho_e^*)^{K_{\mathbb{B}_e}} \right)$$

with $S(v)$, $T(v)$) sets of edges with v as a source, target

- group $\mathcal{G}$ acts by

$$\bigoplus_{\substack{\{A_v,H_v\}\\ \{\rho_e,\mathbb{B}_e\}}} \bigotimes_v \left(\bigotimes_{e\in T(v)} \rho_e(g_v) \otimes \bigotimes_{e\in S(v)} \rho_e^* \circ \phi_{\mathbb{B}}(g_v) \right)$$

• orthonormal basis decomposition of $L^2(\mathcal{X}/\mathcal{G}) \equiv L^2(\mathcal{X})^{\mathcal{G}}$

$$L^2(\mathcal{X}/\mathcal{G}) \simeq \bigoplus_{\substack{\{A_v,H_v\} \\ \{\rho_e,\mathbb{B}_e\}}} \bigotimes_v \mathrm{Inv}(v,\rho),$$

where $\mathrm{Inv}(v,\rho)$ are intertwining operators ι_v on each vertex v, *i.e.*

$$\iota_v : \bigotimes_{e\in T(v)} \rho_e \to \bigotimes_{e\in S(v)} (\rho_e)^{K_{\mathbb{B}_e}} \circ \phi_{\mathbb{B}}$$

as representations of the group $U(A_v)$
(with ρ_e a representation of $U(A_{t(e)})$)

Intertwiners at vertices of gauge networks

additional data for gauge networks

- Vertex v with $e'_1, \dots, e'_k$ incoming edges and $e_1, \dots, e_l$ outgoing edges at v
- the intertwiners ι_v for the group $\mathcal{G}_v = U(\mathcal{A}_v) \rtimes S(\mathcal{A}_v)$:

$$\iota_v : \rho_{e'_1} \otimes \cdots \otimes \rho_{e'_k} \to \rho_{e_1}^{K_{\mathbb{B}_{e_1}}} \circ \phi_{\mathbb{B}} \otimes \cdots \otimes \rho_{e_l}^{K_{\mathbb{B}_{e_l}}} \circ \phi_{\mathbb{B}}$$

isotropy group $K_{\mathbb{B}_e} = \mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$

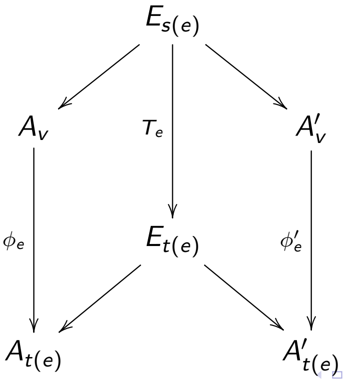
Correspondences between gauge networks

- two π, π' quiver reps of Γ
- $\mathcal{A}_v - \mathcal{A}'_v$ Bimodules $\mathcal{E}_v$

$$\mathcal{H}_v = \mathcal{E} \otimes_{\mathcal{A}'_v} \mathcal{H}'_v$$

- morphisms $T_e : \mathcal{E}_{s(e)} \to \mathcal{E}_{t(e)}$ compatible with alg maps ϕ_e, ϕ'_e

$$T_e(a\eta b) = \phi_e(a) T_e(\eta) \phi'_e(b), \quad a \in A_{s(e)}, \eta \in E_{s(e)}, b \in A'_{s(e)}$$



Algebra of gauge networks and correspondences

- given gauge networks

$$\psi = (\Gamma, (A_v, H_v, \iota_v)_v, (\rho_e, \mathbb{B}_e)_e), \qquad \psi' = (\Gamma, (A'_v, H'_v, \iota_v)_v, (\rho'_e, \mathbb{B}'_e)_e)$$

and correspondences ${}_{\psi}\Psi_{\psi'}$

$$\Psi = \{\Gamma, ({}_{A_v}E_{A'_v}, \iota_v \otimes \iota'_v)_v, (\rho_e \otimes \rho'_e, \mathbb{B}_e \times \mathbb{B}'_e)_e\}$$

- composition of correspondences (tensor product of bimodules)

$$\Psi_1 = \{\Gamma, ({}_{A_v}E_{A'_v}, \iota_v \otimes \iota'_v)_v, (\rho_e \otimes \rho'_e, \mathbb{B}_e \times \mathbb{B}'_e)_e\}$$

$$\Psi_2 = \{\Gamma, ({}_{A'_v}F_{A''_v}, \iota'_v \otimes \iota''_v)_v, (\rho'_e \otimes \rho''_e, \mathbb{B}'_e \times \mathbb{B}''_e)_e\}$$

$$\Psi_1 \circ \Psi_2 = \{\Gamma, ({}_{A_v}E \otimes_{A'_v} F_{A''_v}, \iota_v \otimes \iota''_v)_v, (\rho_e \otimes \rho''_e, \mathbb{B}_e \times \mathbb{B}''_e)_e\}$$

• $\mathcal{S}$ = category of gauge networks with correspondences as morphisms

• algebra $\mathbb{C}[\mathcal{S}]$ elements $a = \sum_\Psi a_\Psi \Psi$ convolution product

$$(a * b)_\Psi = \sum_{\Psi = \Psi_1 \circ \Psi_2} a_{\Psi_1} b_{\Psi_2}.$$

• can be completed to a C^*-algebra represented on a Hilbert space

• dynamical: Hamiltonian and time evolution, built using quadratic Casimir (kind of Lie group Laplacian) on $\mathcal{U}(A_{t(e)})$

Spectral action and lattice field theory

- Γ embedded in a Riemannian spin manifold M: pullback spin geometry of M to Γ
- $\mathcal{S}$ fiber of spinor bundle on M; take $\mathcal{S}^{V(\Gamma)}$ space of spinors on Γ
- holonomy $\mathrm{Hol}(e, \nabla^S)$ of spin connection along edges e of Γ

$$\mathrm{Hol}(e, \nabla^S) = \mathcal{P} e^{\int_e \omega \cdot dx} \sim 1 + l_e \omega_e(s(e)) + \mathcal{O}(l_e^2)$$

$\omega_e(v)$ pairing of 1-form ω and vector $\dot{e}$ at vertex v

- Dirac operator on Γ:

$$(D_\Gamma \psi)_v = \sum_{t(e)=v} \frac{1}{2l_e} \gamma_e \, \mathrm{Hol}(e, \nabla^S) \psi_{s(e)} + \sum_{s(e)=v} \frac{1}{2l_{\bar{e}}} \gamma_{\bar{e}} \, \mathrm{Hol}(\bar{e}, \nabla^S) \psi_{t(e)};$$

$l_e =$ geodesic length of embedded edge e; $\bar{e} =$ opposite orientation

- gamma matrices γ_e defined so that (discretization/continuum)

$$\sum_{e \in S(v)} \gamma_e \omega_e = \gamma^\mu \omega_\mu$$

Continuum limit of Dirac operator

• lattice spacing l_e goes to zero; assume $l_e = l$ for all edges and square lattice

$$(D_\Gamma \psi)_v = \sum_{v_1, v_2} \frac{1}{2l} \gamma_e (\psi_{v_1} - \psi_{v_2}) + \frac{1}{2} \gamma_e \omega_e(v) (\psi_{v_1} + \psi_{v_2}) + \mathcal{O}(l).$$

sum over all collinear $\quad v_1 \xrightarrow[e']{} v \xrightarrow[e]{} v_2$

• formally, when $l \to 0$

$$(D_\Gamma \psi)_v \longrightarrow \gamma^\mu (\partial_\mu + \omega_\mu) \psi(v)$$

Dirac twisted with finite spectral triples

• if also quiver representation of Γ in the category of finite spectral triples

$$(D_{\Gamma,L}\psi)_v = \sum_{t(e)=v} \frac{1}{2l_e}\gamma_e \left(\mathrm{Hol}(e,\nabla^S)\otimes L_e\right)\psi_{s(e)}$$

$$+\sum_{s(e)=v} \frac{1}{2l_{\bar{e}}}\gamma_{\bar{e}} \left(\mathrm{Hol}(\bar{e},\nabla^S)\otimes L_{\bar{e}}\right)\psi_{t(e)} + \gamma D_v \psi_v$$

where $L_{\bar{e}} = L_e^*$ and γ grading on spinor bundle of M if even dimensional

• if $(A_v, H_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ at all vertices v, then morphism (ϕ, L) unitary in $U(N)$ holonomy of some gauge connection 1-form A_μ, then Dirac on Γ reduces to Dirac on M twisted by gauge field

Spectral action: finite spectral triples

$$S[\{L_e\},\{D_v\}] = \mathrm{Tr} f(D_{\Gamma,L})$$

some function f on the real line

- lattice gauge fields on $M = \mathbb{R}^4$, cutoff $\Lambda \propto l^{-1}$

$$S_\Lambda[\{L_e\},\{D_v\}] := \mathrm{Tr} f(D_{\Gamma,L}/\Lambda) \equiv l^4 \mathrm{Tr}((D_{\Gamma,L})^4)$$

- on square lattice $\mathbb{Z}^4$ find

$$S_\Lambda[\{L_e\},\{D_v\}] = -\frac{1}{4} \sum_{\partial p = e_4 \cdots e_1} (\mathrm{Tr}\,(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1}) + \mathrm{Tr}\,(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4})) + \text{const}$$

$$+ \sum_v l^4 \mathrm{Tr} D_v^4 + 4 l^2 \sum_e \left(\mathrm{Tr} D_{s(e)}^2 + \mathrm{Tr} D_{t(e)}^2 - \mathrm{Tr} L_e^* D_{t(e)} L_e D_v \right)$$

from counting contributions of different cycles in the lattice

- flat case: holonomy of spin connection trivial: $S_\Lambda[\{L_e\}]$ is

$$= 4l^4 \sum_{\partial p = \bar{e}_4 \bar{e}_3 e_2 e_1} \frac{1}{(2l)^4} \mathrm{Tr}(\gamma_\nu \gamma_\mu)^2 \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1} \right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4} \right) \right)$$

plus constant terms

$$= -\frac{1}{4} \sum_{\partial p = \bar{e}_4 \bar{e}_3 e_2 e_1} \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1} \right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4} \right) \right) + \text{const}$$

Similar argument for the other terms

Continuum limit and Wilson action

• μ direction of e and A_μ continuous gauge field at $s(e)$

$$L_e = \mathcal{P} e^{i \int_e A \cdot dx} \sim e^{i A_\mu l} \quad \text{for } l \to 0$$

• with $(A_v, H_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ at all vertices v, limit $l \to 0$ and $\Lambda \propto l^{-1}$ spectral action S_Λ becomes

$$\frac{1}{4} \int_M \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + 2 \int_M \mathrm{Tr}(\partial_\mu \Phi - [iA_\mu, \Phi])(\partial^\mu \Phi - [iA^\mu, \Phi])$$

$$+ 8\Lambda^2 \int_M \mathrm{Tr}\Phi^2 + \int_M \mathrm{Tr}\Phi^4.$$

Yang–Mills coupled to a Higgs field with quartic potential

- For a plaquette

$$\mathrm{Tr}\,(L_{\bar{e}_4}L_{\bar{e}_3}L_{e_2}L_{e_1}) = \mathrm{Tr}e^{-ilA_\nu(x)}e^{-ilA_\mu(x+l\hat{\nu})}e^{ilA_\nu(x+l\hat{\mu})}e^{ilA_\mu(x)}$$

$$\sim \mathrm{Tr}e^{il^2F_{\mu\nu}} \quad \text{for } l \to 0$$

and similarly for $\mathrm{Tr}\,(L_{\bar{e}_1}L_{\bar{e}_2}L_{e_3}L_{e_4})$

- so for $l \to 0$ (and $\Lambda \to \infty$)

$$S_\Lambda \sim \frac{1}{4}\int_M \mathrm{tr} F_{\mu\nu}F^{\mu\nu}$$

- Higgs terms: vertex v at position x

$$\mathrm{Tr}e^{-iA_\mu l}\Phi(x+l\hat{\mu})e^{iA_\mu l}\Phi(x) \sim$$

$$\mathrm{Tr}\Big(\Phi(x)\Phi(x+l\hat{\mu}) + l\Phi(x+l\hat{\mu})[iA_\mu,\Phi(x)]$$

$$-\frac{1}{2}l^2[iA_\mu,\Phi(x+l\hat{\mu})][iA_\mu,\Phi(x)]\Big) + \mathcal{O}(l^3)$$

$\Phi(x)$ continuous (hermitian) Higgs field corresponding to D_x and L_e is expanded in A_μ

- modulo $\mathcal{O}(l^3)$ find in S_Λ

$$S_\Lambda = -\frac{1}{4} \sum_{\partial p = \bar{e}_4 \bar{e}_3 e_2 e_1} \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1} \right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4} \right) \right)$$

$$+ \sum_v l^4 \mathrm{Tr} D_v^4 + 4 l^2 \sum_e \left(\mathrm{Tr} D_{s(e)}^2 + \mathrm{Tr} D_{t(e)}^2 - \mathrm{Tr} L_e^* D_{t(e)} L_e D_{s(e)} \right)$$

$$\sim \frac{1}{2} \mathrm{Tr} e^{i l^2 F_{\mu\nu}} + l^4 \mathrm{Tr} \Phi^4(x) + 2 l^2 \sum_\mu \mathrm{tr} \Phi^2(x) + \mathrm{tr} \Phi^2(x + l\hat{\mu})$$

$$+ 2 l^4 \sum_\mu \frac{1}{l^2} \mathrm{Tr}(\Phi(x + l\hat{\mu}) - \Phi(x))^2$$

$$- \frac{2}{l} \mathrm{Tr} \Phi(x + l\hat{\mu}) [i A_\mu(x), \Phi(x)] + \mathrm{Tr}([i A_\mu(x), \Phi(x)])^2$$

John Barret's Random noncommutative geometries

• a geometry: $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ finite spectral triple with real structure

• random geometry: fixed fermion space $(\mathcal{A}, \mathcal{H}, J, \gamma)$ and varying Dirac operator D up to unitary equivalences

• a random geometry is a "random" (in a suitable probability distribution) point in the moduli space of Dirac operators

• want measure to reflect some action functional, as in path integral:

$$e^{-S(D)}\, dD$$

• view this as a random matrix model where the matrices D are constrained by the properties of Dirac operators of finite spectral triples

• take action functional as a spectral action

$$S(D) = \mathrm{Tr}(f(D)) = \sum_{\lambda \in Spec(D)} f(\lambda)$$

• here want some function $f(x)$ with $f(x) \to \infty$ for $|x| \to \infty$ for convergence of

$$Z = \int_{\mathcal{M}} e^{-S(D)}\, dD$$

• simplest choice quartic polynomial: $g_4 > 0$ (or $g_4 = 0$, $g_2 > 0$)

$$f(D) = g_2 D^2 + g_4 D^4$$

• observables $\mathcal{O}(D)$ functions of D

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\mathcal{M}} \mathcal{O}(D) e^{-S(D)}\, dD$$

behavior in limit $N \to \infty$ of large matrices

• use only Dirac operators that resemble those on manifolds

• different possibilities for Dirac operators: action on $\mathcal{H} = V \otimes M_n(\mathbb{C})$ with $V = \mathbb{C}^k$ a Clifford module signature (p, q) (with $k = 2^{d/2}$ or $k = 2^{(d-1)/2}$)

• express all the possibilities for (p, q) writing Dirac operators in terms of gamma matrices and commutators $[L, \cdot]$ or anticommutators $\{L, \cdot\}$ with given hermitian matrices H and anti-hermitian L

• Example: $(1, 0)$ has $D = \{H, \cdot\}$ and $(0, 1)$ has $D = -i[L, \cdot]$

• Example: $(1, 1)$ has $(\gamma^1)^2 = 1$ and $(\gamma^2)^2 = -1$ and

$$D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot]$$

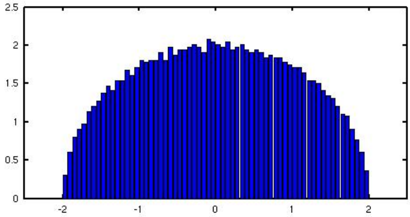
etc.

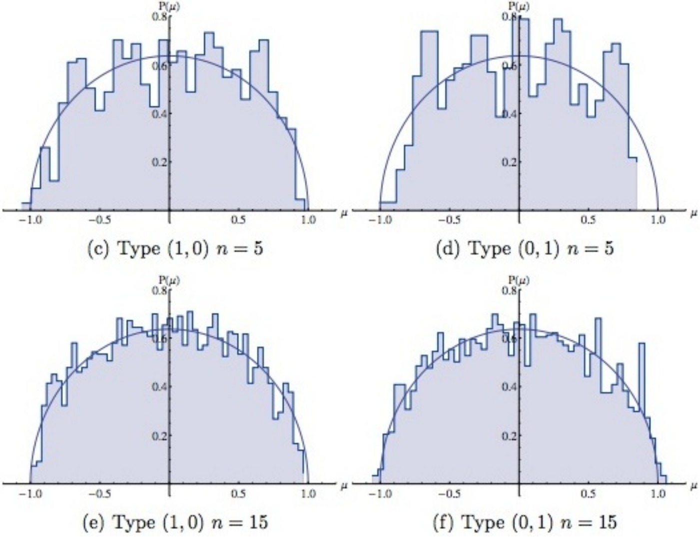
Monte Carlo simulation

• start with random D and construct $D + \delta D$ by δH_i and δL_i

• accept if $\Delta S(D) = S(D_{new}) - S(D_{old}) < 0$ or (to escape local minima) if $\exp(S(D_{old}) - S(D_{new})) > p$ uniformly distributed random number on $[0, 1]$ otherwise keep D_{old}

• compare results with Wigner's semicircle law for random matrix model with real symmetric matrices large order N

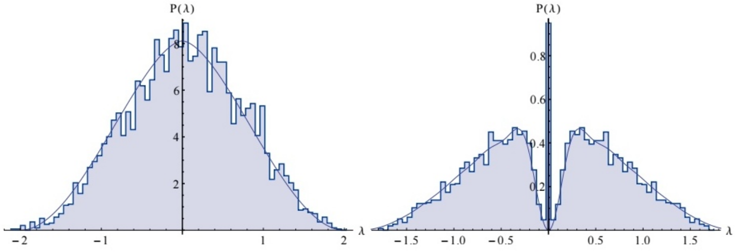




(c) Type $(1,0)$ $n = 5$ (d) Type $(0,1)$ $n = 5$

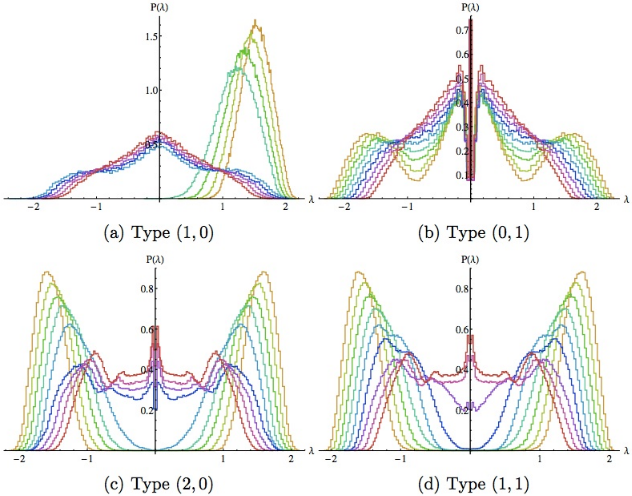
(e) Type $(1,0)$ $n = 15$ (f) Type $(0,1)$ $n = 15$

Density of states for H and L from Barrett and Glaser arXiv:1510.01377, Gaussian case, with $\mathrm{Tr}(D^2)$ action



(a) $(1,0)$ $n=5$ (b) $(0,1)$ $n=5$

Eigenvalue density distribution for the Dirac operator as a combination of H and L, Gaussian case, from Barrett and Glaser arXiv:1510.01377



quartic action $\mathrm{Tr}(g_2 D^2 + g_4 D^4)$, with g_2 ranging from -5 to -1 from Barrett and Glaser arXiv:1510.01377

- in three of four cases in last figure the graphs show a phase transition
- the eigenvalue distribution at the critical value of g_2 resembles the eigenvalue distribution on a manifold, power law $|\lambda|^{d-1}$ for dimension d
- finite spectral triples as an approximation to an "emergent" manifold-like spacetime?
- what is a good rigorous random matrix model for the phenomena observed in Barrett and Glaser? (recent work of Shahab Azarfar and Masoud Khalkhali)

Some Background on Random Matrix Theory

- H an $N \times N$ real matrix whose entries are independently sampled from a Gaussian probability distribution
- $H_s = (H + H^t)/2$ symmetrization
- GOE Gaussian Orthogonal Ensemble
- similarly with complex or quaternionic entries (and hermitianization)
- GUE Gaussian Unitary Ensemble and GSE Gaussian Symplectic Ensemble
- generate n such matrices and plot histogram of the N eigenvalues of these matrices
- what is the shape in the limit $N \to \infty$?
- there is a limiting shape (Wigner semicircle law)

- for randomly sampled matrix H independent Gaussian variables
$$\rho[H] = \prod_{i,j=1}^{N} \exp\left(-\frac{H_{ij}^2}{2}\right) / \sqrt{2\pi}$$
- for the symmetrization $H_{s,ij} = (H_{ij} + H_{ji})/2$
$$\rho[H_s] = \prod_{i=1}^{N} \left(\exp(-\frac{H_{ii}^2}{2})/\sqrt{2\pi}\right) \cdot \prod_{i<j} (\exp(-H_{s,ij}^2)/\sqrt{\pi})$$
variance of off-diagonal entries is half of variance of diagonal
- distribution of eigenvalues?

Coulomb Gas (Dyson, Wigner)

- 2D fluid of charges particles (electrostatic potential is logarithmic) confined on a 1D line
- probability distribution

$$\rho(x_1,\ldots,x_N) = \frac{1}{\mathcal{Z}_{N,\beta}} e^{-\frac{1}{2}\sum_{i=1}^N x_i^2} \prod_{j<k} |x_j - x_k|^\beta$$

- tension between exponential confinement and electrostatic repulsion
- rescaling $x_i \mapsto x_i\sqrt{\beta N}$ normalization factor

$$C_{N,\beta} = (\sqrt{\beta N})^{N+\beta N(N-1)/2}$$

- partition function

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} e^{-\frac{\beta N}{2}\sum_{i=1}^N x_i^2} \prod_{j<k} |x_j - x_k|^\beta \prod_{j=1}^N dx_j$$

- rewrite partition function in terms of an *energy* functional $\mathcal{E}[x]$

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} e^{-\beta N^2 \mathcal{E}[x]} \prod_{j=1}^{N} dx_j$$

$$\mathcal{E}[x] = \frac{1}{2N} \sum_i x_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \log |x_i - x_j|$$

- this describes a fluid of particles with positions $x_1, \dots, x_N$ on a line in equilibrium with Boltzmann–Gibbs distribution $e^{-\beta N^2 \mathcal{E}[x]}$ at inverse temperature β (no kinetic term in $\mathcal{E}[x]$: static fluid)
- Note: limit $N \to \infty$ thermodynamic limit; because of factor βN^2 can also take zero-temperature limit
- zero-temperature equilibrium from minimization of the *free energy*

$$F = -\beta^{-1} \log \mathcal{Z}_{N,\beta}$$

- behavior of free energy $F = -\beta^{-1} \log \mathcal{Z}_{N,\beta}$ for large N
- normalized counting measure

$$n(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)$$

- a functional integral way of writing this

$$1 = \int \delta \left(n(x) - \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \right) \mathcal{D}(n(x))$$

functional integral over all normalized non-negative $n(x)$

- use to rewrite partition function $\mathcal{Z}_{N,\beta}$ as functional integral

- partition function as functional integral

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int \mathcal{D}(n(x)) \int_{\mathbb{R}^N} \prod_j dx_j \, e^{-\beta N^2 \mathcal{E}[x]} \, \delta \left(n(x) - \frac{1}{N} \sum_{i=1}^N \delta(x - x_i) \right)$$

- replace in energy functional sums by integrals over counting distribution

$$\sum_i f(x_i) = N \int_{\mathbb{R}} n(x) f(x) dx, \quad \sum_{ij} g(x_i, x_j) = N^2 \int_{\mathbb{R}^2} dxdy n(x) n(y) g(x,y)$$

- partition function

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int \mathcal{D}(n(x)) e^{-\beta N^2 \mathcal{V}(n(x))} \mathcal{I}_N(n(x))$$

$$\mathcal{V}(n(x)) = \frac{1}{2} \int_{\mathbb{R}} dx \, x^2 n(x) - \frac{1}{2} \int_{\mathbb{R}^2} dxdy n(x) n(y) \log|x-y|$$

(with a cutoff that regularizes the short-distance divergence of the log integral)

$$\mathcal{I}_N(n(x)) = \int_{\mathbb{R}^N} \prod_j dx_j \, \delta \left(n(x) - \frac{1}{N} \sum_{i=1}^N \delta(x - x_i) \right)$$

For details of computations see

- G.Livan, M.Novaes, P.Vivo, *Introduction to Random Matrices. Theory and Practice*, Springer, 2018.
- estimates of the terms $\mathcal{I}_N(n(x))$ and $\mathcal{V}(n(x))$ give $\mathcal{Z}_{N,\beta}$

$$C_{N,\beta} \int \mathcal{D}(n(x)) e^{-\beta N^2 \mathcal{F}_0(n(x)) + \frac{\beta}{2} N \log N + (\frac{\beta}{2} - 1) N \mathcal{F}_1(n(x)) - \frac{\beta}{2} N \log C + o(N)}$$

$$\mathcal{F}_0(n(x)) = \frac{1}{2} \int_{\mathbb{R}} dx\, x^2\, n(x) - \frac{1}{2} \int_{\mathbb{R}^2} dxdy\, n(x) n(y) \log |x - y|$$

$$\mathcal{F}_1(n(x)) = \int_{\mathbb{R}} dx n(x) \log n(x)$$

- constraint on normalization of $n(x)$ as exponential (Fourier transform)

$$\delta(1 - \int n(x) dx) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(1 - \int n(x) dx)}$$

and rescale $ik \mapsto \beta N^2 \kappa$

- get estimate of partition function (leading terms)

$$\mathcal{Z}_{N,\beta} \sim C_{N,\beta} \int \mathcal{D}(n(x)) \int d\kappa e^{-\beta N^2 \mathcal{S}(n(x),\kappa)}$$

$$\mathcal{S}(n(x),\kappa) = \mathcal{F}_0(n(x)) - \kappa(1 - \int n(x)dx)$$

- saddle point evaluation

$$\mathcal{Z}_{N,\beta} \sim \exp(-\beta N^2 \mathcal{S}(n^*(x),\kappa^*))$$

with $n^*(x)$ and κ^* solutions of variational problem

$$0 = \frac{\delta}{\delta n(x)} \mathcal{S}(n(x),\kappa) = \frac{x^2}{2} - \int_{\mathbb{R}} dy\, n(y) \log|x-y| - \kappa$$

$$0 = \frac{\partial}{\partial \kappa} \mathcal{S}(n(x),\kappa)$$

the latter imposing $\int n(x)dx = 1$

- so want solutions $n^*(x)$ of integral problem

$$\frac{x^2}{2} - \int_{\mathbb{R}} dy\, n(y) \log|x-y| - \kappa = 0$$

with $n^*(x) \geq 0$ and $\int n^*(x) dx = 1$

- search for solutions support in some interval $(a,b) \subset \mathbb{R}$
- by differentiation: $\log|x-y|$ not differentiable but it is in the distributional sense
- distributional derivative of $u(x) = \int_{\mathbb{R}} dy\, n(y) \log|x-y|$ is principal value

$$\mathrm{Pr} \int dy \frac{n(y)}{x-y}$$

- solve for

$$\mathrm{Pr} \int_a^b dy \frac{n(y)}{x-y} = x$$

- known from theory of integral equations

$$\mathrm{Pr}\int_a^b dy \frac{f(y)}{x-y} = g(x) \Rightarrow f(x) = \frac{C - \mathrm{Pr}\int_a^b \frac{dt}{\pi}\frac{\sqrt{(t-a)(b-t)}}{x-t}g(t)}{\pi\sqrt{(x-a)(b-x)}}$$

- so get after normalization $\int n(x)dx = 1$

$$n^*(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}(1 - x^2 + \frac{1}{2}(a+b)x + \frac{1}{8}(b-a)^2)$$

- now deal with dependence on parameters a, b
- dependence in the term $\mathcal{F}_0(n^*(x))$ get:

$$\mathcal{F}_0(n^*(x)) = \frac{1}{4}\int_a^b dx\, n^*(x)x^2 + \frac{a^2}{2} - \frac{1}{2}\int_a^b dx\, n^*(x)\log(x-a)$$

- inserting $n^*(x)$ and integrating

$$\frac{1}{512}(-9a^4 + 4a^3 b + 2a^2(5b^2 + 48) + 4ab(b^2 + 16) - 256\log(b-a) - 9b^4 + 96b^2 + 512\log 2)$$

- minimize over a, b gives $a = -\sqrt{2}$ and $b = \sqrt{2}$

$$n^*(x) = \frac{1}{\pi}\sqrt{2 - x^2}$$

Wigner semicircle law

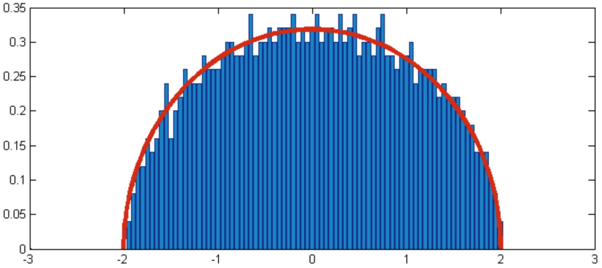


Figure 1: Simulation of the semicircle law using 1000 samples of the eigenvalues of 1000 by 1000 matrices. Bin size is 0.05.

Coulomb Gas and Eigenvalues of Random Matrices

- Dyson index $\beta = 1, 2, 4$ for GOE, GUE, GSE
- GOE case want to relate Coulomb gas distribution

$$\rho[x] = \frac{1}{\mathcal{Z}_{N,1}} e^{-\frac{1}{2}\sum_i x_i^2} \prod_{j<k} |x_j - x_k|$$

with the distribution

$$\rho[H] = \prod_i \frac{e^{-H_{ii}^2/2}}{\sqrt{2\pi}} \prod_{i<j} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}}$$

- Stiefel manifold $\mathbb{V}_N \subset \mathbb{R}^{N^2}$ of orthogonal matrices $O^t O = 1$

$$\mathrm{Vol}(\mathbb{V}_N) = \frac{2^N \pi^{N^2/2}}{\Gamma_N(N/2)}$$

$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma(a - (i-1)/2)$

- change of coordinates from matrix entries $H = (H_{ij})$ to eigenvalues via diagonalization $H = O^t \mathrm{diag}(x) O$
- Jacobian of the change of coordinates $H \mapsto (x, O)$ given by Vandermonde determinant

$$V(x) = \prod_{j>k} (x_j - x_k)$$

- distribution for the eigenvalues

$$\rho_{\mathrm{eigenv}}(x) = \int_{\mathbb{V}_N} \rho_{\mathrm{entries}}(x, O)\, V(x)\, dO$$

- write entries distribution in an invariant way

$$\rho[H] = \prod_i \frac{e^{-H_{ii}^2/2}}{\sqrt{2\pi}} \prod_{i<j} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}} = (2\pi)^{-N/2} \pi^{-(N^2-N)/4} \exp(-\frac{1}{2}\mathrm{Tr}(H^2))$$

- trace term invariant under OHO^t, gives $\exp(-\frac{1}{2}\sum_i x_i^2)$
- factor 2^{-N} normalizing for ambiguity $v \mapsto -v$ in choice of eigenvectors in O get numerical factor $\pi^{N^2/2}/\Gamma_N(N/2)$

Work of Shahab Azarfar and Masoud Khalkhali on Finite Spectral Triples and Random Matrices

• case of type $(1,0)$ in Barrett's classification $D = \{H, \cdot\}$ anticommutation with Hermitian matrix

- The Dirac operator

$$D = \{H, \cdot\}\ , \quad H \in \mathcal{H}_N$$

- Initial form of the action functional

$$\tilde{\mathcal{S}}(D) = \mathrm{Tr}\left(\tilde{\mathcal{V}}(D)\right)\ , \quad \text{where} \quad \tilde{\mathcal{V}}(x) = \frac{1}{2}\left(\frac{x^2}{2} - \sum_{l=3}^{d} t_l \frac{x^l}{l}\right)$$

- We decompose $\tilde{\mathcal{S}}(D)$ as $\tilde{\mathcal{S}}(D) = \tilde{\mathcal{S}}_1(D) + \tilde{\mathcal{S}}_2(D)$, where

$$\tilde{\mathcal{S}}_1(D) = 2N\,\mathrm{Tr}\left(\tilde{\mathcal{V}}(H)\right)$$

$$\tilde{\mathcal{S}}_2(D) = \frac{1}{2}\left[(\mathrm{Tr}(H))^2 - \sum_{l=3}^{d} \frac{t_l}{l} \sum_{k=1}^{l-1} \binom{l}{k} \mathrm{Tr}\left(H^{l-k}\right) \mathrm{Tr}\left(H^k\right)\right]$$

• more general form of action functional (formal multi-trace Hermitian models)

$$\mathcal{S}(D) = t^{-1}\widetilde{\mathcal{S}}_1(D) + r\widetilde{\mathcal{S}}_2(D)$$

• distribution for this matrix model

$$e^{-\mathcal{S}(D)}\,\mathrm{d}D = \exp\left(-N\,\mathrm{Tr}\,(\mathcal{V}(H)) + \sum_{(l_1,l_2)\in\mathfrak{L}} \frac{t_{l_1,l_2}}{2\,l_1 l_2}\,\mathrm{Tr}(H^{l_1})\,\mathrm{Tr}(H^{l_2})\right)\,\mathrm{d}H$$

$$\mathcal{V}(x) = \frac{1}{t}\left(\frac{x^2}{2} - \sum_{l=3}^{d} t_l\,\frac{x^l}{l}\right)$$

$$\mathfrak{L} = (\mathbb{Z}_+)^2 \cap \left\{(x,y)\in\mathbb{R}^2 \mid 2 \le x+y \le d\right\}$$

Schwinder–Dyson equation for correlators recursive equation

For a matrix model with

$$\mathrm{dP}_N(H) = \frac{1}{Z_N} \exp\left(-N \,\mathrm{Tr}(\mathcal{V}(H))\right) \mathrm{d}H \,,$$

the n-point correlators of the form $\mathbb{E}_{\mathrm{P}_N}\left[\prod_{i=1}^n \mathrm{Tr}(H^{l_i})\right]$ satisfy the following SDE:

$$\sum_{k=0}^{l_1-1} \mathbb{E}_{\mathrm{P}_N}\left[\mathrm{Tr}(H^k)\,\mathrm{Tr}(H^{l_1-1-k}) \prod_{j=2}^n \mathrm{Tr}(H^{l_j})\right]$$

$$-\,N\,\mathbb{E}_{\mathrm{P}_N}\left[\mathrm{Tr}\left(H^{l_1}\mathcal{V}'(H)\right) \prod_{j=2}^n \mathrm{Tr}(H^{l_j})\right]$$

$$+\sum_{j=2}^n l_j\,\mathbb{E}_{\mathrm{P}_N}\left[\mathrm{Tr}(H^{l_j+l_1-1}) \prod_{i=2, i\neq j}^n \mathrm{Tr}(H^{l_i})\right] = 0\,.$$

Surface counting: matrix model with potential

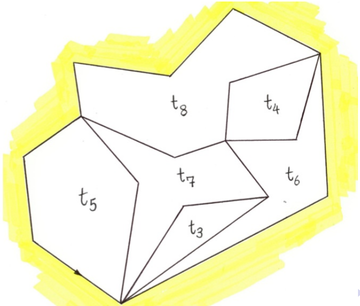
$$\mathcal{V}(x) = \frac{1}{t}\left(\frac{x^2}{2} - \sum_{\ell=3}^{d} t_\ell \frac{x^\ell}{t_\ell}\right)$$

with t, t_ℓ formal parameters

• computation expressible as an enumeration of polygonal maps (discretized surfaces): each term

$$\tau_{\ell_k} = t_{\ell_k} \frac{N}{t} \frac{\mathrm{Tr}(H^{\ell_k})}{\ell_k}$$

corresponds to an ℓ_k-gon counted with weight t_{ℓ_k}



Multi-trace matrix models

$$d\rho_N(H) = \exp\left(\sum_{\substack{n\geq 1\\ h\geq 0}} \frac{1}{n!} (N/t)^{2-2h-n} \sum_{l_1,\cdots,l_n} t^h_{l_1,\cdots,l_n} \prod_{i=1}^{n} \frac{\mathrm{Tr}(H^{l_i})}{l_i} \right) dH$$

- An elementary 2-cell of topology (h, n) and perimeters $(l_1, \cdots, l_n)$ is a surface of genus h whose boundary consists of the 1-skeleton of l_i-gons $i = 1, \cdots, n$

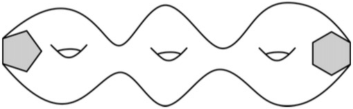
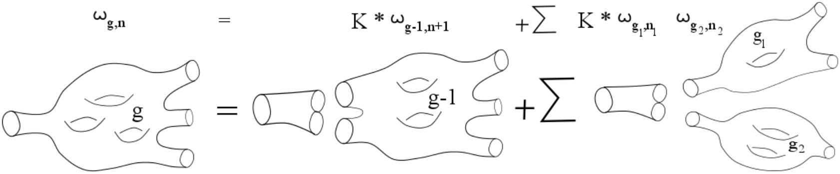


Figure: An elementary 2-cell of topology $(h, n) = (3, 2)$ and perimeters $(l_1, l_2) = (5, 6)$

- enumeration of "stuffed maps"

Topological Recursion Borot, Eynard, Orantin

- Schwinder–Dyson equation for correlators
  - expansion of correlators $W_n(x, x_I) = \sum_{g \geq 0} N^{2-2g-n} W_n^g(x, x_I)$
  - terms $W_n^g(x, x_I) \in \mathcal{O}(\mathbb{C} \smallsetminus \Gamma)$ for Γ union of intervals in $\mathbb{R}$ (where particles of Coulomb gas are distributed)
  - analytic continuation of $W_n^g(x, x_I)$: Riemann surface Σ and differentials $\omega_{n,g}$ of degree n (sections of $K^{\boxtimes n} \to \Sigma^n$ external tensor of canonical line bundle)



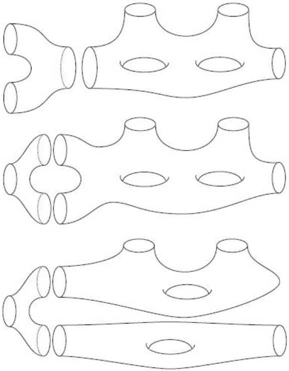
• recursion: a Riemann surface (spectral curve) with a family $\omega_{n,g}$ of differential forms; initial terms $\omega_{0,1}$ and $\omega_{0,2}$ given; remaining terms obtained via a universal recursive formula by removing pairs of pants

$$(g,n) \Longrightarrow (g,n-1)$$

$$(g,n) \Longrightarrow (g-1,n+1)$$

$$(g,n) \Longrightarrow (g_1,n_1)+(g_2,n_2)$$

$$\begin{cases} g = g_1 + g_2 \\ n = n_1 + n_2 - 1 \end{cases}$$



This approach to matrix model for spectral action on finite spectral triples via Borot–Eynard–Orantin topological recursion presented from

- Shahab Azarfar, *Topological Recursion and Random Finite Noncommutative Geometries* , PhD Thesis, University of Western Ontario, August 2018.