Models based on Finite Spectral Triple

Matilde Marcolli

Ma148b: Topics in Mathematical Physics, Caltech Winter 2021

References

- J.W. Barrett, *Matrix geometries and fuzzy spaces as finite spectral triples*, arXiv:1502.05383
- J.W. Barrett, L. Glaser, *Monte Carlo simulations of random non-commutative geometries*, arXiv:1510.01377
- M.Marcolli, W.van Suijlekom, *Gauge Networks in Noncommutative Geometry*, J. Geom. Phys., Vol.75 (2014) 71–91
- S.Azarfar, M.Khalkhali, *Random Finite Noncommutative Geometries and Topological Recursion*, arXiv:1906.09362
- M.Khalkhali, N.Pagliaroli, *Phase transition in random noncommutative geometries*, arXiv:2006.02891
- E.Gesteau, *Renormalizing Yukawa interactions in the standard model with matrices and noncommutative geometry*, J. Phys. A: Math. Theor. 54 (2021) 035203 (18pp)

John Barret's Random noncommutative geometries

• a geometry: $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ finite spectral triple with real structure

• random geometry: fixed fermion space $(\mathcal{A}, \mathcal{H}, J, \gamma)$ and varying Dirac operator D up to unitary equivalences

• a random geometry is a "random" (in a suitable probability distribution) point in the moduli space of Dirac operators

• want measure to reflect some action functional, as in path integral:

$$e^{-S(D)}\, dD$$

• view this as a random matrix model where the matrices D are constrained by the properties of Dirac operators of finite spectral triples

• take action functional as a spectral action

$$S(D) = \mathrm{Tr}(f(D)) = \sum_{\lambda \in Spec(D)} f(\lambda)$$

• here want some function $f(x)$ with $f(x) \to \infty$ for $|x| \to \infty$ for convergence of

$$Z = \int_{\mathcal{M}} e^{-S(D)} \, dD$$

• simplest choice quartic polynomial: $g_4 > 0$ (or $g_4 = 0$, $g_2 > 0$)

$$f(D) = g_2 D^2 + g_4 D^4$$

• observables $\mathcal{O}(D)$ functions of D

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\mathcal{M}} \mathcal{O}(D) e^{-S(D)} \, dD$$

behavior in limit $N \to \infty$ of large matrices

• use only Dirac operators that resemble those on manifolds

• different possibilities for Dirac operators: action on $\mathcal{H} = V \otimes M_n(\mathbb{C})$ with $V = \mathbb{C}^k$ a Clifford module signature (p, q) (with $k = 2^{d/2}$ or $k = 2^{(d-1)/2}$)

• express all the possibilities for (p, q) writing Dirac operators in terms of gamma matrices and commutators $[L, \cdot]$ or anticommutators $\{H, \cdot\}$ with given hermitian matrices H and anti-hermitian L

• Example: $(1, 0)$ has $D = \{H, \cdot\}$ and $(0, 1)$ has $D = -i[L, \cdot]$

• Example: $(1, 1)$ has $(\gamma^1)^2 = 1$ and $(\gamma^2)^2 = -1$ and

$$D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot]$$

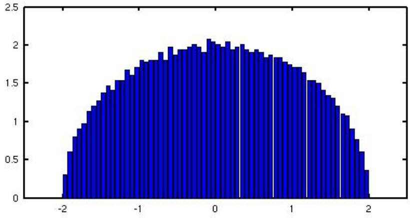
etc.

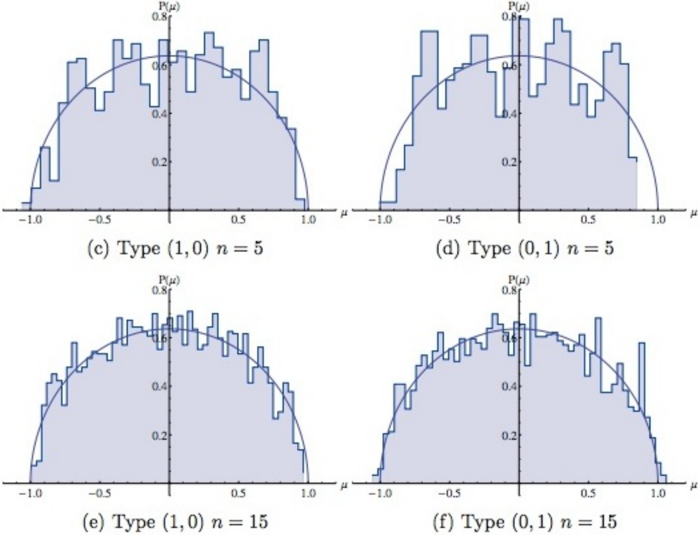
Monte Carlo simulation

• start with random D and construct $D + \delta D$ by δH_i and δL_i

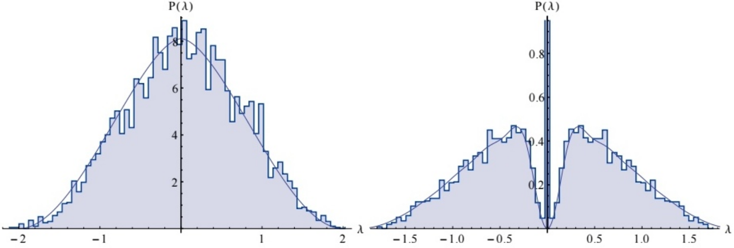
• accept if $\Delta S(D) = S(D_{new}) - S(D_{old}) < 0$ or (to escape local minima) if $\exp(S(D_{old}) - S(D_{new})) > p$ uniformly distributed random number on $[0, 1]$ otherwise keep D_{old}

• compare results with Wigner's semicircle law for random matrix model with real symmetric matrices large order N



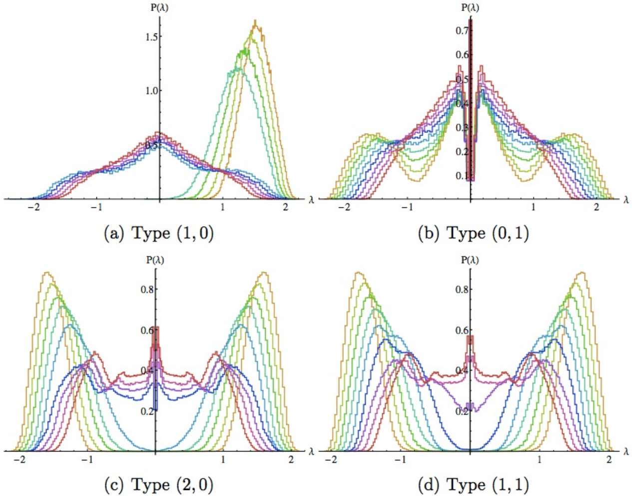


Density of states for H and L from Barrett and Glaser arXiv:1510.01377, Gaussian case, with $\mathrm{Tr}(D^2)$ action



(a) $(1,0)$ $n=5$ (b) $(0,1)$ $n=5$

Eigenvalue density distribution for the Dirac operator as a combination of H and L, Gaussian case, from Barrett and Glaser arXiv:1510.01377



quartic action $\mathrm{Tr}(g_2 D^2 + g_4 D^4)$, with g_2 ranging from -5 to -1 from Barrett and Glaser arXiv:1510.01377

- in three of four cases in last figure the graphs show a phase transition
- the eigenvalue distribution at the critical value of g_2 resembles the eigenvalue distribution on a manifold, power law $|\lambda|^{d-1}$ for dimension d
- finite spectral triples as an approximation to an "emergent" manifold-like spacetime?
- what is a good rigorous random matrix model for the phenomena observed in Barrett and Glaser? (recent work of S. Azarfar, N. Pagliaroli and M. Khalkhali)

Some Background on Random Matrix Theory

- H an $N \times N$ real matrix whose entries are independently sampled from a Gaussian probability distribution
- $H_s = (H + H^t)/2$ symmetrization
- GOE Gaussian Orthogonal Ensemble
- similarly with complex or quaternionic entries (and hermitianization)
- GUE Gaussian Unitary Ensemble and GSE Gaussian Symplectic Ensemble
- generate n such matrices and plot histogram of the N eigenvalues of these matrices
- what is the shape in the limit $N \to \infty$?
- there is a limiting shape (Wigner semicircle law)

- for randomly sampled matrix H independent Gaussian variables

$$\rho[H] = \prod_{i,j=1}^{N} \exp\left(-\frac{H_{ij}^2}{2}\right) / \sqrt{2\pi}$$

- for the symmetrization $H_{s,ij} = (H_{ij} + H_{ji})/2$

$$\rho[H_s] = \prod_{i=1}^{N} \left(\exp(-\frac{H_{ii}^2}{2})/\sqrt{2\pi}\right) \cdot \prod_{i<j} (\exp(-H_{s,ij}^2)/\sqrt{\pi})$$

  variance of off-diagonal entries is half of variance of diagonal

- distribution of eigenvalues?

Coulomb Gas (Dyson, Wigner)

- 2D fluid of charges particles (electrostatic potential is logarithmic) confined on a 1D line
- probability distribution

$$\rho(x_1, \ldots, x_N) = \frac{1}{\mathcal{Z}_{N,\beta}} e^{-\frac{1}{2}\sum_{i=1}^N x_i^2} \prod_{j<k} |x_j - x_k|^\beta$$

- tension between exponential confinement and electrostatic repulsion
- rescaling $x_i \mapsto x_i \sqrt{\beta N}$ normalization factor

$$C_{N,\beta} = (\sqrt{\beta N})^{N+\beta N(N-1)/2}$$

- partition function

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} e^{-\frac{\beta N}{2}\sum_{i=1}^N x_i^2} \prod_{j<k} |x_j - x_k|^\beta \prod_{j=1}^N dx_j$$

- rewrite partition function in terms of an *energy* functional $\mathcal{E}[x]$

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} e^{-\beta N^2 \mathcal{E}[x]} \prod_{j=1}^{N} dx_j$$

$$\mathcal{E}[x] = \frac{1}{2N} \sum_i x_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \log |x_i - x_j|$$

- this describes a fluid of particles with positions $x_1, \ldots, x_N$ on a line in equilibrium with Boltzmann–Gibbs distribution $e^{-\beta N^2 \mathcal{E}[x]}$ at inverse temperature β (no kinetic term in $\mathcal{E}[x]$: static fluid)
- Note: limit $N \to \infty$ thermodynamic limit; because of factor βN^2 can also take zero-temperature limit
- zero-temperature equilibrium from minimization of the *free energy*

$$F = -\beta^{-1} \log \mathcal{Z}_{N,\beta}$$

- behavior of free energy $F = -\beta^{-1} \log \mathcal{Z}_{N,\beta}$ for large N
- normalized counting measure

$$n(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)$$

- a functional integral way of writing this

$$1 = \int \delta \left(n(x) - \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \right) \mathcal{D}(n(x))$$

functional integral over all normalized non-negative $n(x)$

- use to rewrite partition function $\mathcal{Z}_{N,\beta}$ as functional integral

- partition function as functional integral

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int \mathcal{D}(n(x)) \int_{\mathbb{R}^N} \prod_j dx_j \, e^{-\beta N^2 \mathcal{E}[x]} \, \delta \left(n(x) - \frac{1}{N} \sum_{i=1}^N \delta(x - x_i) \right)$$

- replace in energy functional sums by integrals over counting distribution

$$\sum_i f(x_i) = N \int_{\mathbb{R}} n(x) f(x) dx, \quad \sum_{ij} g(x_i, x_j) = N^2 \int_{\mathbb{R}^2} dx dy n(x) n(y) g(x, y)$$

- partition function

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int \mathcal{D}(n(x)) e^{-\beta N^2 \mathcal{V}(n(x))} \mathcal{I}_N(n(x))$$

$$\mathcal{V}(n(x)) = \frac{1}{2} \int_{\mathbb{R}} dx \, x^2 n(x) - \frac{1}{2} \int_{\mathbb{R}^2} dx dy n(x) n(y) \log|x - y|$$

(with a cutoff that regularizes the short-distance divergence of the log integral)

$$\mathcal{I}_N(n(x)) = \int_{\mathbb{R}^N} \prod_j dx_j \, \delta \left(n(x) - \frac{1}{N} \sum_{i=1}^N \delta(x - x_i) \right)$$

For details of computations see

- G.Livan, M.Novaes, P.Vivo, *Introduction to Random Matrices. Theory and Practice*, Springer, 2018.
- estimates of the terms $\mathcal{I}_N(n(x))$ and $\mathcal{V}(n(x))$ give $\mathcal{Z}_{N,\beta}$

$$C_{N,\beta} \int \mathcal{D}(n(x)) e^{-\beta N^2 \mathcal{F}_0(n(x)) + \frac{\beta}{2} N \log N + (\frac{\beta}{2}-1) N \mathcal{F}_1(n(x)) - \frac{\beta}{2} N \log C + o(N)}$$

$$\mathcal{F}_0(n(x)) = \frac{1}{2}\int_{\mathbb{R}} dx\, x^2\, n(x) - \frac{1}{2}\int_{\mathbb{R}^2} dxdy\, n(x) n(y) \log|x-y|$$

$$\mathcal{F}_1(n(x)) = \int_{\mathbb{R}} dx n(x) \log n(x)$$

- constraint on normalization of $n(x)$ as exponential (Fourier transform)

$$\delta(1 - \int n(x) dx) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(1-\int n(x)dx)}$$

and rescale $ik \mapsto \beta N^2 \kappa$

- get estimate of partition function (leading terms)

$$\mathcal{Z}_{N,\beta} \sim C_{N,\beta} \int \mathcal{D}(n(x)) \int d\kappa e^{-\beta N^2 \mathcal{S}(n(x),\kappa)}$$

$$\mathcal{S}(n(x),\kappa) = \mathcal{F}_0(n(x)) - \kappa(1 - \int n(x)dx)$$

- saddle point evaluation

$$\mathcal{Z}_{N,\beta} \sim \exp(-\beta N^2 \mathcal{S}(n^*(x),\kappa^*))$$

with $n^*(x)$ and κ^* solutions of variational problem

$$0 = \frac{\delta}{\delta n(x)}\mathcal{S}(n(x),\kappa) = \frac{x^2}{2} - \int_{\mathbb{R}} dy\, n(y) \log|x-y| - \kappa$$

$$0 = \frac{\partial}{\partial \kappa}\mathcal{S}(n(x),\kappa)$$

the latter imposing $\int n(x)dx = 1$

- so want solutions $n^*(x)$ of integral problem

$$\frac{x^2}{2} - \int_{\mathbb{R}} dy\, n(y) \log|x-y| - \kappa = 0$$

with $n^*(x) \geq 0$ and $\int n^*(x) dx = 1$

- search for solutions support in some interval $(a,b) \subset \mathbb{R}$
- by differentiation: $\log|x-y|$ not differentiable but it is in the distributional sense
- distributional derivative of $u(x) = \int_{\mathbb{R}} dy\, n(y) \log|x-y|$ is principal value

$$\mathrm{Pr} \int dy \frac{n(y)}{x-y}$$

- solve for

$$\mathrm{Pr} \int_a^b dy \frac{n(y)}{x-y} = x$$

- known from theory of integral equations

$$\mathrm{Pr}\int_a^b dy \frac{f(y)}{x-y} = g(x) \Rightarrow f(x) = \frac{C - \mathrm{Pr}\int_a^b \frac{dt}{\pi}\frac{\sqrt{(t-a)(b-t)}}{x-t} g(t)}{\pi\sqrt{(x-a)(b-x)}}$$

- so get after normalization $\int n(x)dx = 1$

$$n^*(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}(1 - x^2 + \frac{1}{2}(a+b)x + \frac{1}{8}(b-a)^2)$$

- now deal with dependence on parameters a, b
- dependence in the term $\mathcal{F}_0(n^*(x))$ get:

$$\mathcal{F}_0(n^*(x)) = \frac{1}{4}\int_a^b dx\, n^*(x)x^2 + \frac{a^2}{2} - \frac{1}{2}\int_a^b dx\, n^*(x)\log(x-a)$$

- inserting $n^*(x)$ and integrating

$$\frac{1}{512}(-9a^4 + 4a^3 b + 2a^2(5b^2+48) + 4ab(b^2+16) - 256\log(b-a) - 9b^4 + 96b^2 + 512\log 2)$$

- minimize over a, b gives $a = -\sqrt{2}$ and $b = \sqrt{2}$

$$n^*(x) = \frac{1}{\pi}\sqrt{2 - x^2}$$

Wigner semicircle law

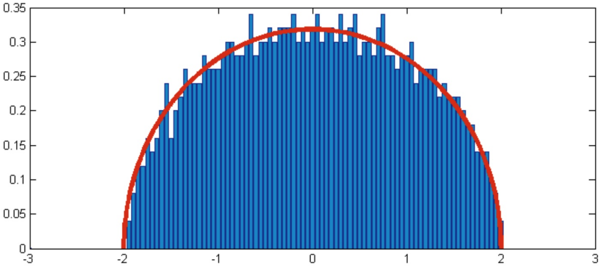


Figure 1: Simulation of the semicircle law using 1000 samples of the eigenvalues of 1000 by 1000 matrices. Bin size is 0.05.

Coulomb Gas and Eigenvalues of Random Matrices

- Dyson index $\beta = 1, 2, 4$ for GOE, GUE, GSE
- GOE case want to relate Coulomb gas distribution

$$\rho[x] = \frac{1}{\mathcal{Z}_{N,1}} e^{-\frac{1}{2}\sum_i x_i^2} \prod_{j<k} |x_j - x_k|$$

with the distribution

$$\rho[H] = \prod_i \frac{e^{-H_{ii}^2/2}}{\sqrt{2\pi}} \prod_{i<j} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}}$$

- Stiefel manifold $\mathbb{V}_N \subset \mathbb{R}^{N^2}$ of orthogonal matrices $O^t O = 1$

$$\mathrm{Vol}(\mathbb{V}_N) = \frac{2^N \pi^{N^2/2}}{\Gamma_N(N/2)}$$

$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma(a - (i-1)/2)$

- change of coordinates from matrix entries $H = (H_{ij})$ to eigenvalues via diagonalization $H = O^t \mathrm{diag}(x) O$
- Jacobian of the change of coordinates $H \mapsto (x, O)$ given by Vandermonde determinant

$$V(x) = \prod_{j>k} (x_j - x_k)$$

- distribution for the eigenvalues

$$\rho_{\rm eigenv}(x) = \int_{\mathbb{V}_N} \rho_{\rm entries}(x, O)\, V(x)\, dO$$

- write entries distribution in an invariant way

$$\rho[H] = \prod_i \frac{e^{-H_{ii}^2/2}}{\sqrt{2\pi}} \prod_{i<j} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}} = (2\pi)^{-N/2} \pi^{-(N^2-N)/4} \exp(-\frac{1}{2} \mathrm{Tr}(H^2))$$

- trace term invariant under OHO^t, gives $\exp(-\frac{1}{2}\sum_i x_i^2)$
- factor 2^{-N} normalizing for ambiguity $v \mapsto -v$ in choice of eigenvectors in O get numerical factor $\pi^{N^2/2}/\Gamma_N(N/2)$

Work of Shahab Azarfar and Masoud Khalkhali on Finite Spectral Triples and Random Matrices

• case of type $(1,0)$ in Barrett's classification $D = \{H, \cdot\}$ anticommutation with Hermitian matrix

- The Dirac operator

$$D = \{H, \cdot\}\ , \quad H \in \mathcal{H}_N$$

- Initial form of the action functional

$$\tilde{\mathcal{S}}(D) = \mathrm{Tr}\left(\tilde{\mathcal{V}}(D)\right)\ , \quad \text{where} \quad \tilde{\mathcal{V}}(x) = \frac{1}{2}\left(\frac{x^2}{2} - \sum_{l=3}^{d} t_l \frac{x^l}{l}\right)$$

- We decompose $\tilde{\mathcal{S}}(D)$ as $\tilde{\mathcal{S}}(D) = \tilde{\mathcal{S}}_1(D) + \tilde{\mathcal{S}}_2(D)$, where

$$\tilde{\mathcal{S}}_1(D) = 2N\,\mathrm{Tr}\left(\tilde{\mathcal{V}}(H)\right)$$

$$\tilde{\mathcal{S}}_2(D) = \frac{1}{2}\left[(\mathrm{Tr}(H))^2 - \sum_{l=3}^{d} \frac{t_l}{l} \sum_{k=1}^{l-1} \binom{l}{k} \mathrm{Tr}\left(H^{l-k}\right) \mathrm{Tr}\left(H^k\right)\right]$$

• more general form of action functional (formal multi-trace Hermitian models)

$$\mathcal{S}(D) = t^{-1}\widetilde{\mathcal{S}}_1(D) + r\widetilde{\mathcal{S}}_2(D)$$

• distribution for this matrix model

$$e^{-\mathcal{S}(D)}\,\mathrm{d}D = \exp\left(-N\,\mathrm{Tr}\left(\mathcal{V}(H)\right) + \sum_{(l_1,l_2)\in\mathfrak{L}} \frac{t_{l_1,l_2}}{2\,l_1 l_2}\,\mathrm{Tr}(H^{l_1})\,\mathrm{Tr}(H^{l_2})\right)\,\mathrm{d}H$$

$$\mathcal{V}(x) = \frac{1}{t}\left(\frac{x^2}{2} - \sum_{l=3}^{d} t_l\,\frac{x^l}{l}\right)$$

$$\mathfrak{L} = (\mathbb{Z}_+)^2 \cap \left\{(x,y)\in\mathbb{R}^2 \mid 2 \le x+y \le d\right\}$$

Schwinder–Dyson equation for correlators recursive equation

For a matrix model with

$$\mathrm{dP}_N(H) = \frac{1}{Z_N} \exp\left(-N\,\mathrm{Tr}(\mathcal{V}(H))\right) \mathrm{d}H \,,$$

the n-point correlators of the form $\mathbb{E}_{\mathrm{P}_N}\left[\prod_{i=1}^{n} \mathrm{Tr}(H^{l_i})\right]$ satisfy the following SDE:

$$\sum_{k=0}^{l_1-1} \mathbb{E}_{\mathrm{P}_N}\left[\mathrm{Tr}(H^k)\,\mathrm{Tr}(H^{l_1-1-k}) \prod_{j=2}^{n} \mathrm{Tr}(H^{l_j})\right]$$
$$-\,N\,\mathbb{E}_{\mathrm{P}_N}\left[\mathrm{Tr}\left(H^{l_1}\mathcal{V}'(H)\right) \prod_{j=2}^{n} \mathrm{Tr}(H^{l_j})\right]$$
$$+\sum_{j=2}^{n} l_j\,\mathbb{E}_{\mathrm{P}_N}\left[\mathrm{Tr}(H^{l_j+l_1-1}) \prod_{i=2,i\neq j}^{n} \mathrm{Tr}(H^{l_i})\right] = 0\,.$$

Surface counting: matrix model with potential

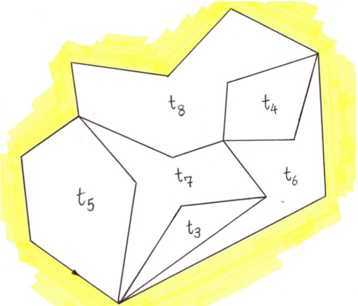
$$\mathcal{V}(x) = \frac{1}{t}\left(\frac{x^2}{2} - \sum_{\ell=3}^{d} t_\ell \frac{x^\ell}{t_\ell}\right)$$

with t, t_ℓ formal parameters

• computation expressible as an enumeration of polygonal maps (discretized surfaces): each term

$$\tau_{\ell_k} = t_{\ell_k} \frac{N}{t} \frac{\mathrm{Tr}(H^{\ell_k})}{\ell_k}$$

corresponds to an ℓ_k-gon counted with weight t_{ℓ_k}



Multi-trace matrix models

$$d\rho_N(H) = \exp\left(\sum_{\substack{n\geq 1\\ h\geq 0}} \frac{1}{n!} (N/t)^{2-2h-n} \sum_{l_1,\cdots,l_n} t^h_{l_1,\cdots,l_n} \prod_{i=1}^n \frac{\mathrm{Tr}(H^{l_i})}{l_i} \right) dH$$

- An elementary 2-cell of topology (h,n) and perimeters $(l_1,\cdots,l_n)$ is a surface of genus h whose boundary consists of the 1-skeleton of l_i-gons $i=1,\cdots,n$

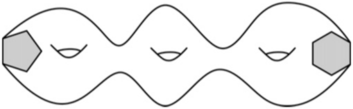
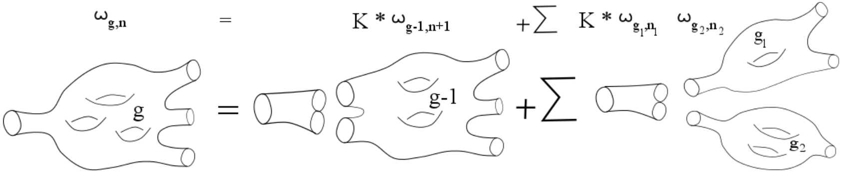


Figure: An elementary 2-cell of topology $(h,n)=(3,2)$ and perimeters $(l_1,l_2)=(5,6)$

- enumeration of "stuffed maps"

Topological Recursion Borot, Eynard, Orantin

- Schwinder–Dyson equation for correlators
  - expansion of correlators $W_n(x, x_I) = \sum_{g \geq 0} N^{2-2g-n} W_n^g(x, x_I)$
  - terms $W_n^g(x, x_I) \in \mathcal{O}(\mathbb{C} \setminus \Gamma)$ for Γ union of intervals in $\mathbb{R}$ (where particles of Coulomb gas are distributed)
  - analytic continuation of $W_n^g(x, x_I)$: Riemann surface Σ and differentials $\omega_{n,g}$ of degree n (sections of $K^{\boxtimes n} \to \Sigma^n$ external tensor of canonical line bundle)



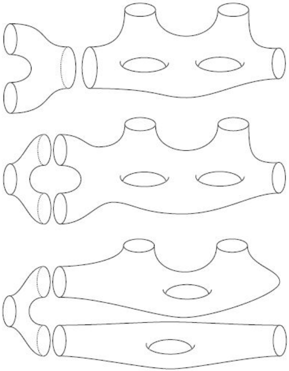
• recursion: a Riemann surface (spectral curve) with a family $\omega_{n,g}$ of differential forms; initial terms $\omega_{0,1}$ and $\omega_{0,2}$ given; remaining terms obtained via a universal recursive formula by removing pairs of pants

$(g, n) \Longrightarrow (g, n-1)$

$(g, n) \Longrightarrow (g-1, n+1)$

$(g, n) \Longrightarrow (g_1, n_1) + (g_2, n_2)$

$$\begin{cases} g = g_1 + g_2 \\ n = n_1 + n_2 - 1 \end{cases}$$



This approach to matrix model for spectral action on finite spectral triples via Borot–Eynard–Orantin topological recursion presented from

- S.Azarfar, M.Khalkhali, *Random Finite Noncommutative Geometries and Topological Recursion*, arXiv:1906.09362
- M.Khalkhali, N.Pagliaroli, *Phase transition in random noncommutative geometries*, arXiv:2006.02891

Also recent work on matrix model based on finite spectral triple computing the beta function and renormalization of the Yukawa coupling terms in the Standard Model of elementary particle physics

- E.Gesteau, *Renormalizing Yukawa interactions in the standard model with matrices and noncommutative geometry*, J. Phys. A: Math. Theor. 54 (2021) 035203 (18pp)

Gauge networks

(NCG generalization of gauge theory on a lattice/graph)

• using finite spectral triple for a model combining gauge theory on a lattice (or graph) and spin networks approach to gravity

• an action functional (in terms of Dirac operator) that recovers the Wilson action (which in continuum limit gives Yang–Mills) will additional terms for a Higgs field in adjoint representation

• build a category of finite spectral triples with morphisms built from algebra morphisms and unitary operators

• representations of quivers (oriented graphs) in this category of finite spectral triples

• configuration space (of such representation) modulo gauge action

• morphisms between gauge networks by correspondences (bimodules); Hamiltonian and time evolution

• discretized Dirac operator and continuum limit

$\mathcal{C}_0$ Category of finite spectral triples with trivial Dirac $D = 0$

• objects $(\mathcal{A}, \pi, \mathcal{H})$, fin. dim. algebra $\mathcal{A}$ and fin. Hilbert space rep. $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$

• morphisms $\Phi : (\mathcal{A}_1, \pi_1, \mathcal{H}_1) \to (\mathcal{A}_2, \pi_2, \mathcal{H}_2)$ pair $\Phi = (\phi, L)$ $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ morphism of unital $\star$-algebras, $L : \mathcal{H}_1 \to \mathcal{H}_2$ unitary

$$L\pi_1(a)L^* = \pi_2(\phi(a))$$

$\mathcal{C}$ Category of finite spectral triples

• objects $(\mathcal{A}, \pi, \mathcal{H}, D)$ fin spectral triples

• morphisms $\Phi : (\mathcal{A}_1, \pi_1, \mathcal{H}_1, D_1) \to (\mathcal{A}_2, \pi_2, \mathcal{H}_2, D_2)$ as above with also $LD_1L^* = D_2$

Bratteli diagrams

• Wedderburn theorem:

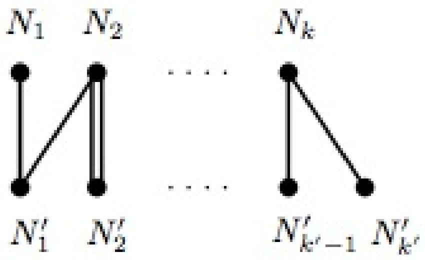
$$\mathcal{A}_1 = \bigoplus_{i=1}^{k} M_{N_i}(\mathbb{C}), \quad \mathcal{A}_2 = \bigoplus_{j=1}^{k'} M_{N'_j}(\mathbb{C})$$

• unital $*$-algebra morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ direct sum

$$\phi_j : \bigoplus_{i=1}^{k} M_{N_i}(\mathbb{C}) \to M_{N'_j}(\mathbb{C})$$

ϕ_j splits as a direct sum of representation $\phi_{ij} : M_{N_i}(\mathbb{C}) \to M_{N'_j}(\mathbb{C})$ with multiplicity $d_{ij} \geq 0$, with $N'_j = \sum_i d_{ij} N_i$

• Bratteli diagrams: two rows of vertices: top k vertices labeled $N_1, \dots, N_k$, bottom k' vertices labeled by $N'_1, \dots, N'_{k'}$; d_{ij} edges between vertex i (top row) and j (bottom row)



$\phi : \mathcal{A}_1 \to \mathcal{A}_2$ unital, so all vertices in bottom row reached by an edge, but top row can have vacant vertices

Example

- $\mathcal{A}_1 = \mathbb{C} \oplus M_2(\mathbb{C})$, $\mathcal{H}_1 = \mathbb{C} \oplus \mathbb{C}^2$, $\mathcal{A}_2 = M_3(\mathbb{C})$, $\mathcal{H}_2 = \mathbb{C}^3$
- unital $*$-algebra map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ two possibilities

$$(z, a) \in \mathbb{C} \oplus M_2(\mathbb{C}) \quad \mapsto u \begin{pmatrix} z & \\ & a \end{pmatrix} u^* \in M_3(\mathbb{C})$$

with $u \in U(3)$ or

$$(z, a) \in \mathbb{C} \oplus M_2(\mathbb{C}) \quad \mapsto z 1_3 \in M_3(\mathbb{C})$$

with kernel $M_2(\mathbb{C})$

- unitary map of $\mathcal{H}_1$ to $\mathcal{H}_2$

$$(x, y) \in \mathbb{C} \oplus \mathbb{C}^2 \mapsto U \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^3$$

with $U \in U(3)$

- compatibility of ϕ and L: first case OK with $u = U$

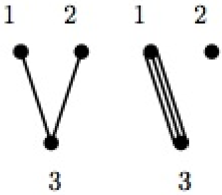
$$u \begin{pmatrix} z & \\ & a \end{pmatrix} u^* = U \begin{pmatrix} z & \\ & a \end{pmatrix} U^*.$$

but in second case

$$z 1_3 = U \begin{pmatrix} z & \\ & a \end{pmatrix} U^*.$$

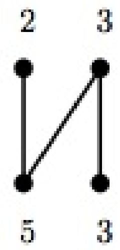
cannot be satisfied for arbitrary $(z, a) \in \mathcal{A}_1$

- so get $\mathrm{Hom}((A_1, H_1), (A_2, H_2)) \simeq U(3)$ and Bratteli diagram



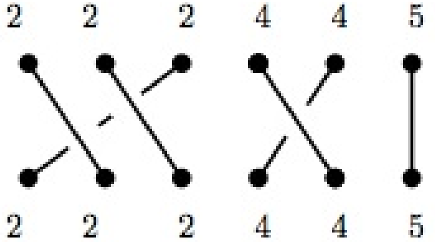
(a) (b)

Example



Bratteli diagram for the only unital $*$-algebra map $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ given $(a, b) \mapsto (a \oplus b, b)$

• to better take care also of permutations of matrix blocks of the same dimension: braid Bratteli diagrams



braid Bratteli diagram with permutations of matrix blocks of same dim in $M_2(\mathbb{C})^{\oplus 3} \oplus M_4(\mathbb{C})^{\oplus 2} \oplus M_5(\mathbb{C})$

• any Bratteli diagram $\mathbb{B}$ for a pair (A_1, A_2) gives homomorphism $\phi_{\mathbb{B}} : A_1 \to A_2$ embedding matrix blocks of A_1 into those of A_2 following lines in $\mathbb{B}$

• any other unital $*$-algebra morphisms $\phi : A_1 \to A_2$ can be obtained from $\phi_{\mathbb{B}}$ by unitary change of basis
$\phi(\cdot) = U\phi_{\mathbb{B}}(\cdot)U^* =: \mathrm{Ad}\, U\phi_{\mathbb{B}}(\cdot)$ some unitary U in A_2

• representation λ of A on finite dim Hilbert space H, two-sided ideal $\mathrm{Ker}(\lambda)$ with $A = \widetilde{A} \oplus \mathrm{Ker}(\lambda)$ and $\widetilde{A} \simeq \lambda(A)$

• morphisms (ϕ, L) with $\star$-homomorphisms $\phi : A_1 \to A_2$ and unitary $L : H_1 \to H_2$ with $L\lambda_1(a)L^* = \lambda_2(\pi(a))$

• decompose as $\phi = \widetilde{\phi} + \phi_0$ with $\widetilde{\phi} : \widetilde{A}_1 \to \widetilde{A}_2$ with $\widetilde{\phi}(\widetilde{a}) = L\widetilde{a}L^*$ and $\phi_0 : A_1 \to \mathrm{Ker}(\lambda_2)$

• identify $\mathrm{Aut}((A, \lambda, H)) \simeq \mathcal{U}(\widetilde{A}) \rtimes S(\widetilde{A}; H) \times P\mathcal{U}(\ker\lambda) \rtimes S(\ker\lambda)$ with $S(\widetilde{A}; H)$ and $S(\ker\lambda)$ groups of permutations of matrix blocks of equal dimension in $\widetilde{A}$ and H and $\ker\lambda$ (proj unitary group because adjoint action of center of $\mathcal{U}(\ker\lambda)$ on $\ker\lambda$ trivial)

- for algebras and Hilbert spaces

$$A_1 = \bigoplus_{i=1}^{k+l} M_{N_i}(\mathbb{C}) \qquad A_2 = \bigoplus_{j=1}^{k'+l'} M_{N'_j}(\mathbb{C})$$

$$H_1 = \bigoplus_{i=1}^{k} n_i \mathbb{C}^{N_i} \qquad H_2 = \bigoplus_{j=1}^{k'} n'_j \mathbb{C}^{N'_j}$$

- any morphism (ϕ, L) given by

$$\phi = \mathrm{Ad}\, U \phi_{\widetilde{\mathbb{B}}} + \mathrm{Ad}\, V \phi_{\mathbb{B}_0} \qquad L = U L_{\widetilde{\mathbb{B}}}$$

  - unitary $U \in \mathrm{Aut}_{\widetilde{A}_2}(H_2) \simeq \prod_{j=1}^{k'} U(n_j N_j)$
  - unitary $V \in \mathcal{U}(\ker \lambda_2) \simeq \prod_{j=k'+1}^{k'+l'} U(N_j)$
  - Bratteli diagrams $\widetilde{\mathbb{B}}, \mathbb{B}_0$ of $*$-algebra maps $\widetilde{A}_1 \hookrightarrow \widetilde{A}_2$ and $A_1 \to \ker \lambda_2$
  - unitary map $L_{\widetilde{\mathbb{B}}} : H_1 \to H_2$ implements $*$-algebra map $\phi_{\widetilde{\mathbb{B}}} : \widetilde{A}_1 \to \widetilde{A}_2$

$$L_{\widetilde{\mathbb{B}}} \widetilde{a} L^*_{\widetilde{\mathbb{B}}} = \phi_{\widetilde{\mathbb{B}}}(\widetilde{a}) \qquad \forall \widetilde{a} \in \widetilde{A}_1$$

Quiver representations in categories

• Quiver Γ directed graph

• representation π of a quiver Γ in a category $\mathcal{C}$:
- object π_v for each vertex v
- morphism π_e in $\mathrm{Hom}(\pi_{s(e)}, \pi_{t(e)})$ for each directed edge e.

• two representations π, π' of Γ in same category equivalent if $\pi_v = \pi'_v$, for all $v \in V(\Gamma)$ and $\exists$ family of invertible morphisms $\phi_v \in \mathrm{Hom}(\pi(v), \pi(v))$ for $v \in V(\Gamma)$ such that

$$\pi_e = \phi_{t(e)} \circ \pi'_e \circ \phi^{-1}_{s(e)}$$

• For categories $\mathcal{C}$ (or $\mathcal{C}_0$) of finite spectral triples, representation π of a quiver Γ assigns
- spectral triples $(\mathcal{A}_v, \mathcal{H}_v, D_v)$ ($D_v = 0$ for $\mathcal{C}_0$) to vertices $v \in V(\Gamma)$
- pairs $(\phi, L) \in \mathrm{Hom}((\mathcal{A}_{s(e)}, \mathcal{H}_{s(e)}, D_{s(e)}), (\mathcal{A}_{t(e)}, \mathcal{H}_{t(e)}, D_{t(e)}))$ to edges $e \in E(\Gamma)$

Equivalence of quiver representations

• two representations π, π' of Γ in the same category are equivalent if

- $\pi_v = \pi'_v$ for all $v \in \Gamma^{(0)}$
- there exists a family of invertible morphisms $\phi_v \in \mathrm{Hom}(\pi(v), \pi(v))$ indexed by the vertices v such that

$$\pi_e = \phi_{t(e)} \circ \pi'_e \circ \phi_{s(e)}^{-1}$$

• if we view a quiver Γ itself as a category, a representation is a functor π from Γ to a category; equivalent representations coincide on objects and are related via invertible natural transformations

Example $U(N)$ spin networks (John Baez)

• If $(\mathcal{A}_v, \mathcal{H}_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ and $D = 0$, unitary $u_e \in U(N)$ along each edge and gauge action $g_v \in U(N)$ at each vertex with

$$u_e \mapsto g_{t(e)} u_e g^*_{s(e)}$$

• only possible Bratteli diagram in this case for $\phi : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ is single edge between one upper row vertex and one lower row vertex

• J.C. Baez, *Spin network states in gauge theory*, Adv. Math. 117 (1996) 253–272

General case: gauge networks

$$\{\Gamma, (A_v, \lambda_v, H_v; \iota_v)_v, (\rho_e, \mathbb{B}_e)_e\}$$

- Γ directed graph
- (A_v, λ_v, H_v) is an object in the category $\mathcal{C}_0$ for each vertex $v \in V(\Gamma)$
- Edge $e \in E(\Gamma)$: representation ρ_e of unitary group $G_e = \mathrm{Aut}_{\tilde{A}_{t(e)}}(H_{t(e)}) \times \mathcal{U}(\ker \lambda_{t(e)})$
- Edge $e \in E(\Gamma)$: Bratteli diagram $\mathbb{B}_e$ for $*$-algebra maps $A_{s(e)} \to A_{t(e)}$
- subdiagrams $\tilde{\mathbb{B}}$ for $\tilde{A}_{s(e)} \to \tilde{A}_{t(e)}$ and $\mathbb{B}_0$ for $A_{s(e)} \to \ker \lambda_{t(e)}$
- "intertwiners at vertices" between representations ρ_e associated to edges (more on this later)

• space of representations of Γ in $\mathcal{C}_0$

$$\mathcal{X} = \coprod_{\{A_v, H_v\}_v} \prod_{e \in E(\Gamma)} \mathcal{X}_e$$

$\mathcal{X}_e = \mathrm{Hom}((A_{s(e)}, \lambda_{s(e)}, H_{s(e)}), (A_{t(e)}, \lambda_{t(e)}, H_{t(e)}))$

• elements $(\phi_e, L_e) \in \mathcal{X}_e$

$$\phi_e = \mathrm{Ad}\, U \phi_{\widetilde{\mathbb{B}}_e} + \mathrm{Ad}\, V \phi_{\mathbb{B}_{e0}}; \qquad L_e = U L_{\widetilde{\mathbb{B}}_e}$$

unitaries $U \in \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)})$, $V \in \mathcal{U}(\ker \lambda_{t(e)})$ and a Bratteli diagram $\mathbb{B}_e$ (with subdiagrams $\widetilde{\mathbb{B}}_e, \mathbb{B}_{e0}$) for each edge e

• this means unitary group $\mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)})$ together with all $\mathcal{U}(\ker \lambda_{t(e)})$-orbits of $\phi_{\mathbb{B}_{e0}}$ for all such $\mathbb{B}_{e0}$ gives all of $\mathcal{X}_e$

• Orbit-stabilizer: isotropy subgroup $\mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$ of $\phi_{\mathbb{B}_{e0}}$

$$\mathcal{X}_e = \coprod_{\mathbb{B}_e} \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)}) \times \mathcal{U}(\ker \lambda_{t(e)}) / \mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$$

• elements in $\mathcal{X}$ by $(U_e, [V_e], \mathbb{B}_e)_e$ with $U_e \in \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)})$ and $V_e \in \mathcal{U}(\ker \lambda_{t(e)})$

• equivalence of quiver representations: collection of unitaries

$$(g_v, \sigma_v) := (\widetilde{g}_v, \widetilde{\sigma}_v; g_{v0}, \sigma_{v0}) \in \mathcal{G}_v$$

$$\mathcal{G}_v := \mathrm{Aut}_{\widetilde{A}_v}(H_v) \rtimes S(\widetilde{A}_v; H_v) \times P\mathcal{U}(\ker \lambda_v) \rtimes S(\ker \lambda_v)$$

mapping $(U_e, [V_e], \mathbb{B}_e) \in \mathcal{X}_e$ to

$$(\widetilde{g}_{t(e)} U_e \phi_{\widetilde{\mathbb{B}}_e}(\widetilde{g}^*_{s(e)}), [g_{t(e)0} V_e \phi_{\mathbb{B}_{e0}}(g^*_{s(e)})], \sigma_{t(e)} \circ \mathbb{B}_e \circ \sigma_{s(e)})$$

• Peter-Weyl theorem for compact Lie groups G

$$L^2(G) \simeq \bigoplus_{\rho \in \widehat{G}} \rho \otimes \rho^*$$

with $\widehat{G}$ irreducible unitary reps, isomorphism of $G \times G$-representations with

$$((g_1, g_2)f)(x) = f(g_1^{-1} x g_2) \qquad \forall f \in L^2(G)$$

$$(g_1, g_2)(y_1 \otimes y_2) = \rho(g_1)y_1 \otimes \rho^*(g_2)(y_2) \qquad \forall y_1 \in \rho, y_2 \in \rho^*$$

• this means orthonormal basis for $L^2(G)$ (Haar measure) constructed using matrix coefficients $\langle \pi(g)e_i, e_j \rangle$ for $g \in G$, over representatives π of isomorphism classes of irreducible unitary representations

• G compact Lie group, K and H mutually commuting closed subgroups

$$L^2(G/K) \simeq L^2(G)^K \simeq \bigoplus_{\rho \in \widehat{G}} \rho \otimes (\rho^*)^K$$

isomorphism of $G \times H$-representations, with ρ^K the K-invariant subspace of the G-representation ρ

- then get for the space of representations of Γ in $\mathcal{C}_0$

$$L^2(\mathcal{X}) \simeq \bigoplus_{\{A_v,H_v\}} \bigotimes_{e} \bigoplus_{\mathbb{B}_e} L^2(G_e/K_{\mathbb{B}_e})$$

$$G_e := \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)}) \times \mathcal{U}(\ker \lambda_{t(e)}) \qquad K_{\mathbb{B}_e} := \{e\} \times \mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$$

by Peter-Weyl theorem

$$L^2(\mathcal{X}) \simeq \bigoplus_{\{A_v,H_v\}} \bigotimes_{e} \bigoplus_{\mathbb{B}_e} \bigoplus_{\rho_e \in \widehat{G_e}} \rho_e \otimes (\rho_e^*)^{K_{\mathbb{B}_e}}$$

- action of $\mathcal{G}$

$$\bigoplus_{\{A_v,H_v\}} \bigotimes_{e} \bigoplus_{\mathbb{B}_e} \bigoplus_{\rho_e \in \widehat{G_e}} \rho_e(g_{t(e)}) \otimes \rho_e^* \circ \phi_{\mathbb{B}}(g_{s(e)}),$$

• rewrite $L^2(\mathcal{X})$ in the form

$$L^2(\mathcal{X}) \simeq \bigoplus_{\substack{\{A_v,H_v\} \\ \{\rho_e,\mathbb{B}_e\}}} \bigotimes_v \left(\bigotimes_{e\in T(v)} \rho_e \otimes \bigotimes_{e\in S(v)} (\rho_e^*)^{K_{\mathbb{B}_e}} \right)$$

with $S(v)$, $T(v)$) sets of edges with v as a source, target

• group $\mathcal{G}$ acts by

$$\bigoplus_{\substack{\{A_v,H_v\} \\ \{\rho_e,\mathbb{B}_e\}}} \bigotimes_v \left(\bigotimes_{e\in T(v)} \rho_e(g_v) \otimes \bigotimes_{e\in S(v)} \rho_e^* \circ \phi_{\mathbb{B}}(g_v) \right)$$

• orthonormal basis decomposition of $L^2(\mathcal{X}/\mathcal{G}) \equiv L^2(\mathcal{X})^{\mathcal{G}}$

$$L^2(\mathcal{X}/\mathcal{G}) \simeq \bigoplus_{\substack{\{A_v,H_v\}\\ \{\rho_e,\mathbb{B}_e\}}} \bigotimes_v \mathrm{Inv}(v,\rho),$$

where $\mathrm{Inv}(v,\rho)$ are intertwining operators ι_v on each vertex v, *i.e.*

$$\iota_v : \bigotimes_{e\in T(v)} \rho_e \to \bigotimes_{e\in S(v)} (\rho_e)^{K_{\mathbb{B}_e}} \circ \phi_{\mathbb{B}}$$

as representations of the group $U(A_v)$
(with ρ_e a representation of $U(A_{t(e)})$)

Intertwiners at vertices of gauge networks

additional data for gauge networks

- Vertex v with $e'_1, \dots, e'_k$ incoming edges and $e_1, \dots, e_l$ outgoing edges at v
- the intertwiners ι_v for the group $\mathcal{G}_v = U(\mathcal{A}_v) \rtimes S(\mathcal{A}_v)$:

$$\iota_v : \rho_{e'_1} \otimes \cdots \otimes \rho_{e'_k} \to \rho_{e_1}^{K_{\mathbb{B}_{e_1}}} \circ \phi_{\mathbb{B}} \otimes \cdots \otimes \rho_{e_l}^{K_{\mathbb{B}_{e_l}}} \circ \phi_{\mathbb{B}}$$

isotropy group $K_{\mathbb{B}_e} = \mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$

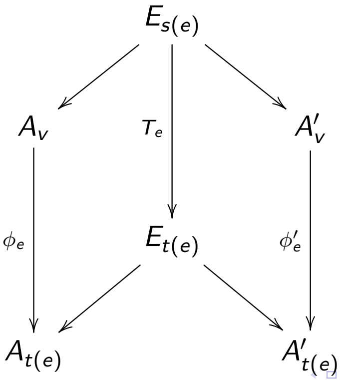
Correspondences between gauge networks

- two π, π' quiver reps of Γ
- $\mathcal{A}_v - \mathcal{A}'_v$ Bimodules $\mathcal{E}_v$

$$\mathcal{H}_v = \mathcal{E} \otimes_{\mathcal{A}'_v} \mathcal{H}'_v$$

- morphisms $T_e : \mathcal{E}_{s(e)} \to \mathcal{E}_{t(e)}$ compatible with alg maps ϕ_e, ϕ'_e

$$T_e(a\eta b) = \phi_e(a) T_e(\eta) \phi'_e(b), \quad a \in A_{s(e)}, \eta \in E_{s(e)}, b \in A'_{s(e)}$$



Algebra of gauge networks and correspondences

- given gauge networks

$$\psi = (\Gamma, (A_v, H_v, \iota_v)_v, (\rho_e, \mathbb{B}_e)_e), \qquad \psi' = (\Gamma, (A'_v, H'_v, \iota_v)_v, (\rho'_e, \mathbb{B}'_e)_e)$$

and correspondences ${}_{\psi}\Psi_{\psi'}$

$$\Psi = \{\Gamma, ({}_{A_v}E_{A'_v}, \iota_v \otimes \iota'_v)_v, (\rho_e \otimes \rho'_e, \mathbb{B}_e \times \mathbb{B}'_e)_e\}$$

- composition of correspondences (tensor product of bimodules)

$$\Psi_1 = \{\Gamma, ({}_{A_v}E_{A'_v}, \iota_v \otimes \iota'_v)_v, (\rho_e \otimes \rho'_e, \mathbb{B}_e \times \mathbb{B}'_e)_e\}$$

$$\Psi_2 = \{\Gamma, ({}_{A'_v}F_{A''_v}, \iota'_v \otimes \iota''_v)_v, (\rho'_e \otimes \rho''_e, \mathbb{B}'_e \times \mathbb{B}''_e)_e\}$$

$$\Psi_1 \circ \Psi_2 = \{\Gamma, ({}_{A_v}E \otimes_{A'_v} F_{A''_v}, \iota_v \otimes \iota''_v)_v, (\rho_e \otimes \rho''_e, \mathbb{B}_e \times \mathbb{B}''_e)_e\}$$

• $\mathcal{S}$ = category of gauge networks with correspondences as morphisms

• algebra $\mathbb{C}[\mathcal{S}]$ elements $a = \sum_\Psi a_\Psi \Psi$ convolution product

$$(a * b)_\Psi = \sum_{\Psi=\Psi_1\circ\Psi_2} a_{\Psi_1} b_{\Psi_2}.$$

• can be completed to a C^*-algebra represented on a Hilbert space

• dynamical: Hamiltonian and time evolution, built using quadratic Casimir (kind of Lie group Laplacian) on $\mathcal{U}(A_{t(e)})$

Spectral action and lattice field theory

• Γ embedded in a Riemannian spin manifold M: pullback spin geometry of M to Γ

• $\mathcal{S}$ fiber of spinor bundle on M; take $\mathcal{S}^{V(\Gamma)}$ space of spinors on Γ

• holonomy $\mathrm{Hol}(e, \nabla^S)$ of spin connection along edges e of Γ

$$\mathrm{Hol}(e, \nabla^S) = \mathcal{P} e^{\int_e \omega \cdot dx} \sim 1 + l_e \omega_e(s(e)) + \mathcal{O}(l_e^2)$$

$\omega_e(v)$ pairing of 1-form ω and vector $\dot{e}$ at vertex v

• Dirac operator on Γ:

$$(D_\Gamma \psi)_v = \sum_{t(e)=v} \frac{1}{2 l_e} \gamma_e \, \mathrm{Hol}(e, \nabla^S) \psi_{s(e)} + \sum_{s(e)=v} \frac{1}{2 l_{\bar{e}}} \gamma_{\bar{e}} \, \mathrm{Hol}(\bar{e}, \nabla^S) \psi_{t(e)};$$

l_e = geodesic length of embedded edge e; $\bar{e}$ = opposite orientation

• gamma matrices γ_e defined so that (discretization/continuum)

$$\sum_{e \in S(v)} \gamma_e \omega_e = \gamma^\mu \omega_\mu$$

Continuum limit of Dirac operator

• lattice spacing l_e goes to zero; assume $l_e = l$ for all edges and square lattice

$$(D_\Gamma \psi)_v = \sum_{v_1, v_2} \frac{1}{2l} \gamma_e (\psi_{v_1} - \psi_{v_2}) + \frac{1}{2} \gamma_e \omega_e(v) (\psi_{v_1} + \psi_{v_2}) + \mathcal{O}(l).$$

sum over all collinear $v_1 \xrightarrow[e']{} v \xrightarrow[e]{} v_2$

• formally, when $l \to 0$

$$(D_\Gamma \psi)_v \longrightarrow \gamma^\mu (\partial_\mu + \omega_\mu) \psi(v)$$

Dirac twisted with finite spectral triples

• if also quiver representation of Γ in the category of finite spectral triples

$$(D_{\Gamma,L}\psi)_v = \sum_{t(e)=v} \frac{1}{2l_e}\gamma_e \left(\mathrm{Hol}(e,\nabla^S)\otimes L_e\right)\psi_{s(e)}$$

$$+\sum_{s(e)=v} \frac{1}{2l_{\bar{e}}}\gamma_{\bar{e}} \left(\mathrm{Hol}(\bar{e},\nabla^S)\otimes L_{\bar{e}}\right)\psi_{t(e)} + \gamma D_v\psi_v$$

where $L_{\bar{e}} = L_e^*$ and γ grading on spinor bundle of M if even dimensional

• if $(A_v, H_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ at all vertices v, then morphism (ϕ, L) unitary in $U(N)$ holonomy of some gauge connection 1-form A_μ, then Dirac on Γ reduces to Dirac on M twisted by gauge field

Spectral action: finite spectral triples

$$S[\{L_e\},\{D_v\}] = \mathrm{Tr} f(D_{\Gamma,L})$$

some function f on the real line

- lattice gauge fields on $M = \mathbb{R}^4$, cutoff $\Lambda \propto l^{-1}$

$$S_\Lambda[\{L_e\},\{D_v\}] := \mathrm{Tr} f(D_{\Gamma,L}/\Lambda) \equiv l^4 \mathrm{Tr}((D_{\Gamma,L})^4)$$

- on square lattice $\mathbb{Z}^4$ find

$$S_\Lambda[\{L_e\},\{D_v\}] = -\frac{1}{4} \sum_{\partial p = e_4 \cdots e_1} \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1}\right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4}\right)\right) + \text{const}$$

$$+ \sum_v l^4 \mathrm{Tr} D_v^4 + 4 l^2 \sum_e \left(\mathrm{Tr} D_{s(e)}^2 + \mathrm{Tr} D_{t(e)}^2 - \mathrm{Tr} L_e^* D_{t(e)} L_e D_v \right)$$

from counting contributions of different cycles in the lattice

• flat case: holonomy of spin connection trivial: $S_\Lambda[\{L_e\}]$ is

$$= 4l^4 \sum_{\partial p = \bar{e}_4 \bar{e}_3 e_2 e_1} \frac{1}{(2l)^4} \mathrm{Tr}(\gamma_\nu \gamma_\mu)^2 \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1} \right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4} \right) \right)$$

plus constant terms

$$= -\frac{1}{4} \sum_{\partial p = \bar{e}_4 \bar{e}_3 e_2 e_1} \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1} \right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4} \right) \right) + \mathrm{const}$$

Similar argument for the other terms

Continuum limit and Wilson action

• μ direction of e and A_μ continuous gauge field at $s(e)$

$$L_e = \mathcal{P} e^{i \int_e A \cdot dx} \sim e^{i A_\mu l} \quad \text{for } l \to 0$$

• with $(A_v, H_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ at all vertices v, limit $l \to 0$ and $\Lambda \propto l^{-1}$ spectral action S_Λ becomes

$$\frac{1}{4} \int_M \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + 2 \int_M \mathrm{Tr}(\partial_\mu \Phi - [iA_\mu, \Phi])(\partial^\mu \Phi - [iA^\mu, \Phi])$$

$$+ 8\Lambda^2 \int_M \mathrm{Tr}\Phi^2 + \int_M \mathrm{Tr}\Phi^4.$$

Yang–Mills coupled to a Higgs field with quartic potential

- For a plaquette

$$\mathrm{Tr}\,(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1}) = \mathrm{Tr} e^{-ilA_\nu(x)} e^{-ilA_\mu(x+l\hat{\nu})} e^{ilA_\nu(x+l\hat{\mu})} e^{ilA_\mu(x)}$$

$$\sim \mathrm{Tr} e^{il^2 F_{\mu\nu}} \quad \text{for } l \to 0$$

and similarly for $\mathrm{Tr}\,(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4})$

- so for $l \to 0$ (and $\Lambda \to \infty$)

$$S_\Lambda \sim \frac{1}{4} \int_M \mathrm{tr} F_{\mu\nu} F^{\mu\nu}$$

- Higgs terms: vertex v at position x

$$\mathrm{Tr} e^{-iA_\mu l} \Phi(x + l\hat{\mu}) e^{iA_\mu l} \Phi(x) \sim$$

$$\mathrm{Tr}\Big(\Phi(x)\Phi(x + l\hat{\mu}) + l\Phi(x + l\hat{\mu})[iA_\mu, \Phi(x)]$$

$$-\frac{1}{2} l^2 [iA_\mu, \Phi(x + l\hat{\mu})][iA_\mu, \Phi(x)] \Big) + \mathcal{O}(l^3)$$

$\Phi(x)$ continuous (hermitian) Higgs field corresponding to D_x and L_e is expanded in A_μ

- modulo $\mathcal{O}(l^3)$ find in S_Λ

$$S_\Lambda = -\frac{1}{4}\sum_{\partial p=\bar{e}_4\bar{e}_3 e_2 e_1} \left(\mathrm{Tr}\left(L_{\bar{e}_4}L_{\bar{e}_3}L_{e_2}L_{e_1}\right) + \mathrm{Tr}\left(L_{\bar{e}_1}L_{\bar{e}_2}L_{e_3}L_{e_4}\right)\right)$$

$$+ \sum_v l^4 \mathrm{Tr} D_v^4 + 4l^2 \sum_e \left(\mathrm{Tr} D_{s(e)}^2 + \mathrm{Tr} D_{t(e)}^2 - \mathrm{Tr} L_e^* D_{t(e)} L_e D_{s(e)}\right)$$

$$\sim \frac{1}{2}\mathrm{Tr} e^{il^2 F_{\mu\nu}} + l^4 \mathrm{Tr}\Phi^4(x) + 2l^2 \sum_\mu \mathrm{tr}\Phi^2(x) + \mathrm{tr}\Phi^2(x + l\hat{\mu})$$

$$+ 2l^4 \sum_\mu \frac{1}{l^2}\mathrm{Tr}(\Phi(x + l\hat{\mu}) - \Phi(x))^2$$

$$-\frac{2}{l}\mathrm{Tr}\Phi(x + l\hat{\mu})[iA_\mu(x), \Phi(x)] + \mathrm{Tr}([iA_\mu(x), \Phi(x)])^2$$

- M.Marcolli, W.van Suijlekom, *Gauge Networks in Noncommutative Geometry*, J. Geom. Phys., Vol.75 (2014) 71–91