

# Finite Spectral Triple

Matilde Marcolli

Ma148b Topics in Mathematical Physics  
Winter 2021 Caltech

## References

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- **Finite space**  $X = \{p_1, \dots, p_N\}$  finite set
- **Algebra** of functions  $f : X \rightarrow \mathbb{C}$  is  $C(X) = \mathbb{C}^N$ ,  
 $x = (x_i) = (f(p_i))$
- **Noncommutative algebras** (space of  $N$  pts with inner structure)

$$\mathcal{A} = \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$$

- finite dimensional **Hilbert space**: vector space  $\mathcal{H}$  with hermitian inner product  $\langle v, w \rangle$
- $\star$ -algebra of linear operators  $\mathcal{L}(\mathcal{H})$ , product = composition, involution = adjoint, norm

$$\|T\| = \sup_{v: \|v\|=1} \|Tv\|$$

$\sqrt{\lambda}$  largest eigenvalue of  $T^*T$

- **Representations:**  $\star$ -algebra homomorphism

$$\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$$

irreducible: only  $\mathcal{A}$ -invariant subspaces are  $\{0\}$  and  $\mathcal{H}$

- **Commutant**

$$\pi(\mathcal{A})' = \{T \in \mathcal{L}(\mathcal{H}) : \pi(a)T = T\pi(a), \forall a \in \mathcal{A}\}$$

$\pi(\mathcal{A})'$  also a  $\star$ -algebra

- Representation  $\pi$  irreducible iff  $\pi(\mathcal{A})'$  scalar multiples of identity
- **Unitary equivalence:**  $(\mathcal{H}_1, \pi_1) \sim (\mathcal{H}_2, \pi_2)$

$$\pi_1(a) = U^* \pi_2(a) U, \quad \forall a \in \mathcal{A}$$

with unitary  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$

- **Structure space:**  $\hat{\mathcal{A}}$  set of unitary equivalence classes of representations of  $\mathcal{A}$

- **Modules:** algebra  $\mathcal{A}$ , left  $\mathcal{A}$ -module: vector space  $\mathcal{E}$  with left action  $(a_1 a_2)v = a_1(a_2 v)$ ; right  $\mathcal{A}$  module with right action  $v(a_1 a_2) = (v a_1) a_2$
- (left)  $\mathcal{A}$ -module homomorphism  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  with  $\phi(av) = a\phi(v)$
- Representation  $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  makes  $\mathcal{H}$  into left  $\mathcal{A}$ -module
- **Bimodules:** algebras  $\mathcal{A}$  and  $\mathcal{B}$  is left  $\mathcal{A}$ -module and right  $\mathcal{B}$ -module, commuting left  $\mathcal{A}$ -action and right  $\mathcal{B}$ -action
- balanced tensor product

$$\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{F} = \mathcal{E} \otimes \mathcal{F} / \left\{ \sum_i v_i a_i \otimes w_i - v_i \otimes a_i w_i \right\}$$

$$v_i \in \mathcal{E}, w_i \in \mathcal{F}, a_i \in \mathcal{A}$$

- **Hilbert bimodule**  ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$  with  $\mathcal{B}$ -valued inner product

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{B}$$

$$\langle v_1, av_2 \rangle = \langle a^* v_1, v_2 \rangle$$

$$\langle v_1, v_2 b \rangle = \langle v_1, v_2 \rangle b$$

$$\langle v_1, v_2 \rangle^* = \langle v_2, v_1 \rangle$$

$$\langle v, v \rangle \geq 0; \quad \langle v, v \rangle = 0 \text{ iff } v = 0$$

where  $b \geq 0$  in  $\mathcal{B}$  means  $b = h^* h$

- set of Hilbert bimodules  $KK_f(\mathcal{A}, \mathcal{B})$
- $\star$ -algebra homomorphism  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  makes  $\mathcal{B}$  into a Hilbert bimodule with  $\langle b, b' \rangle = b^* b'$  and  $\phi(a)$  as  $\mathcal{A}$ -action

- **Kasparov product**  $KK_f(\mathcal{A}, \mathcal{B}) \times KK_f(\mathcal{B}, \mathcal{C}) \rightarrow KK_f(\mathcal{A}, \mathcal{C})$

$$({}_\mathcal{A}\mathcal{E}_\mathcal{B}, {}_\mathcal{B}\mathcal{F}_\mathcal{C}) \mapsto {}_\mathcal{A}\mathcal{E}_\mathcal{B} \otimes_\mathcal{B} {}_\mathcal{B}\mathcal{F}_\mathcal{C}$$

with  $\mathcal{C}$ -valued inner product

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_\mathcal{C} = \langle w_1, \langle v_1, v_2 \rangle_\mathcal{B} w_2 \rangle_\mathcal{C}$$

- **Morita equivalence**  $\mathcal{A} \overset{M}{\simeq} \mathcal{B}$  iff there are  $\mathcal{E} \in KK_f(\mathcal{A}, \mathcal{B})$  and  $\mathcal{F} \in KK_f(\mathcal{B}, \mathcal{A})$  such that

$$\mathcal{E} \otimes_\mathcal{B} \mathcal{F} = \mathcal{A}, \quad \mathcal{F} \otimes_\mathcal{A} \mathcal{E} = \mathcal{B}$$

these give equivalences of the categories of modules by tensoring

–  $\otimes_\mathcal{A} \mathcal{E}$  and –  $\otimes_\mathcal{B} \mathcal{F}$

- **Fact:** Algebras  $\mathcal{A} = \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$  and  $\mathcal{B} = \bigoplus_{j=1}^M M_{m_j}(\mathbb{C})$  Morita equivalent iff  $N = M$
- $M_n(\mathbb{C})$  has a unique irreducible representation  $\mathbb{C}^n$
- bimodules

$$\mathcal{E} = \bigoplus_{i=1}^N \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}, \quad \mathcal{F} = \bigoplus_{i=1}^N \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}$$

with  $\mathcal{A}$  acting on the  $\mathbb{C}^{n_i}$  factors and  $\mathcal{B}$  on the  $\mathbb{C}^{m_i}$

$$\begin{aligned} \mathcal{E} \otimes_{\mathcal{B}} \mathcal{F} &= \bigoplus_i \mathbb{C}^{n_i} \otimes (\mathbb{C}^{m_i} \otimes_{M_{m_i}(\mathbb{C})} \mathbb{C}^{m_i}) \otimes \mathbb{C}^{n_i} \\ &= \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i} = \bigoplus_i M_{n_i}(\mathbb{C}) \end{aligned}$$

same form  $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{E}$

- $N = \#\hat{\mathcal{A}}$  and  $M = \#\hat{\mathcal{B}}$  structure spaces, Morita invariant

## Finite Spectral triples as Metric Spaces

- finite space  $X = \{p_1, \dots, p_N\}$ , metric  $d_{ij} = \text{dist}(p_i, p_j)$  with  $d_{ij} = d_{ji}$ ,  $d_{ij} \geq 0$  and  $d_{ij} = 0$  iff  $i = j$ ;  $d_{ij} \leq d_{ik} + d_{jk}$
- algebra  $\mathcal{A} = C(X) = \mathbb{C}^N$ : there is rep  $(\mathcal{H}, \pi)$  and  $D = D^*$  on  $\mathcal{H}$

$$d_{ij} = \sup_{f \in \mathcal{A}} \{|f(p_i) - f(p_j)| : \|[D, \pi(f)]\| \leq 1\}$$

- construct inductively:  $N = 2$ ,  $\mathcal{H} = \mathbb{C}^2$

$$\pi(f) = \begin{pmatrix} f(p_1) & 0 \\ 0 & f(p_2) \end{pmatrix}, \quad D = \begin{pmatrix} 0 & d_{12}^{-1} \\ d_{12}^{-1} & 0 \end{pmatrix}$$

$$\|[D, \pi(f)]\| = d_{12}^{-1} |f(p_1) - f(p_2)|$$

- given  $(\mathcal{H}_N, \pi_N, D_N)$  take

$$\mathcal{H}_{N+1} = \mathcal{H}_N \oplus \bigoplus_{i=1}^N \mathbb{C}^2$$

$$\pi_{N+1}(f(p_1), \dots, f(p_{N+1})) = \pi_N(f(p_1), \dots, f(p_N)) \oplus \bigoplus_{i=1}^N \begin{pmatrix} f(p_i) & 0 \\ 0 & f(p_{N+1}) \end{pmatrix}$$

$$D_{N+1} = D_N \oplus \bigoplus_{i=1}^N \begin{pmatrix} 0 & d_{1,N+1}^{-1} \\ d_{1,N+1}^{-1} & 0 \end{pmatrix}$$

- **Finite Spectral Triple:**  $(\mathcal{A}, \mathcal{H}, D)$  finite dimensional involutive  $\mathbb{C}$ -algebra, representation  $(\mathcal{H}, \pi)$  on a finite dimensional Hilbert space,  $D = D^*$  on  $\mathcal{H}$

Wedderburn theorem:  $\mathcal{A} = \bigoplus_i M_{n_i}(\mathbb{C})$

- **1-forms:** given  $(\mathcal{A}, \mathcal{H}, D)$  finite

$$\Omega_D^1(\mathcal{A}) = \left\{ \sum_k a_k [D, b_k] : a_k, b_k \in \mathcal{A} \right\}$$

$d : \mathcal{A} \rightarrow \Omega_D^1(\mathcal{A})$  with  $da = [D, a]$

$$d(ab) = d(a)b + ad(b), \quad d(a^*) = -d(a)^*$$

$\Omega_D^1(\mathcal{A})$  is an  $\mathcal{A}$ -bimodule

- Example:  $(M_n(\mathbb{C}), \mathbb{C}^n, D)$  with  $D \neq \lambda I$ , then  $\Omega_D^1(\mathcal{A}) = M_n(\mathbb{C})$

- **Unitary equivalence**  $(\mathcal{A}, \mathcal{H}_1, D_1) \sim (\mathcal{A}, \mathcal{H}_2, D_2)$ : unitary operator  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$

$$U\pi_1(a)U^* = \pi_2(a), \quad UD_1U^* = D_2$$

case of inner  $u \in U(\mathcal{A})$ : gives  $uD_1u^* = D + u[D, u^*]$  shifted by a 1-form

- **Morita equivalences** of finite spectral triples

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$$

$\mathcal{E} \in KK_f(\mathcal{B}, \mathcal{A})$  and Dirac operator

$$D'(v \otimes \xi) = v \otimes D\xi + \nabla(v)\xi$$

$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A})$  connection

$$\nabla(va) = \nabla(v)a + v \otimes [D, a]$$

- **Role of the connection:** if take only  $D'(v \otimes \xi) = v \otimes D\xi$  does not preserve ideal defining balanced product  $va \otimes \xi - v \otimes a\xi$ ; this problem corrected precisely by connection

$$D'(va \otimes \xi - v \otimes a\xi) = va \otimes D\xi + \nabla(va)\xi - v \otimes D(a\xi) - \nabla(v)a\xi = 0$$

so  $D'$  well defined on quotient by the ideal

- **Conclusion:** given  $(\mathcal{A}, \mathcal{H}, D)$  and  $\mathcal{E} \in KK_f(\mathcal{B}, \mathcal{A})$

$$(\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, D')$$

also a finite spectral triple if the connection  $\nabla$  satisfies

$$\langle v_1, \nabla(v_2) \rangle_{\mathcal{A}} - \langle \nabla(v_1), v_2 \rangle_{\mathcal{A}} = d\langle v_1, v_2 \rangle_{\mathcal{A}}$$

*compatibility condition:* ensures that  $D'^* = D'$

## Classifying finite spectral triples

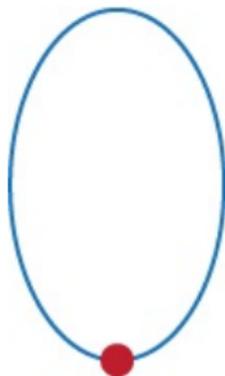
- Algebra  $\mathcal{A} = \sum_{i=1}^N M_{n_i}(\mathbb{C})$ , representation (faithful)  
 $\mathcal{H} = \bigoplus_{i=1}^N \mathbb{C}^{n_i} \otimes V_i$ , multiplicity  $V_i$  of dim  $r_i$

$$D_{ij} : \mathbb{C}^{n_i} \otimes V_i \rightarrow \mathbb{C}^{n_j} \otimes V_j$$

with  $D_{ij} = D_{ji}^*$  components of Dirac operator

- **Decorated graphs**:  $(V, E)$  vertices, edges (possible looping edges and multiple edges);  $\#V = N$ : decorate each vertex  $v$  by non-negative integers  $n_i$  (rank) and  $r_i$  (multiplicity); edge between vertices  $v(n_i, r_i)$  and  $v(n_j, r_j)$  if  $D_{ij} \neq 0$

- Example:  $(M_n(\mathbb{C}), \mathbb{C}^n, D = D_e + D_e^*)$ , loop with one vertex decorated by  $n$  and one edge decorated by  $D_e$



## Finite Spectral Triples with Real Structure

- $(\mathcal{A}, \mathcal{H}, D)$  finite spectral triple as before
- **even**:  $\mathbb{Z}/2\mathbb{Z}$ -grading  $\gamma$  on  $\mathcal{H}$  with  $\gamma^* = \gamma$ ,  $\gamma^2 = 1$  and

$$\gamma D + D\gamma = 0, \quad \gamma\pi(a) = \pi(a)\gamma$$

for all  $a \in \mathcal{A}$

- **anti-unitary**:  $J : \mathcal{H} \rightarrow \mathcal{H}$

$$\langle J\xi_1, J\xi_2 \rangle = \langle \xi_2, \xi_1 \rangle, \quad \forall \xi_1, \xi_2 \in \mathcal{H}$$

- **bimodule**:  $\pi^0(a) = J\pi(a)^*J^{-1}$  right action of  $\mathcal{A}$  on  $\mathcal{H}$ , with  $\pi^0(ab) = \pi^0(b)\pi^0(a)$  and

$$[\pi(a), \pi^0(b)] = 0, \quad \forall a, b \in \mathcal{A}$$

- **order one condition**: for all  $a, b \in \mathcal{A}$

$$[[D, \pi(a)]\pi^0(b)] = 0$$

Notation: usually write  $[D, a]$  for  $[D, \pi(a)]$  and  $a^0$  for  $\pi^0(a)$ .

- KO-dimension:** antilinear isometry  $J : \mathcal{H} \rightarrow \mathcal{H}$

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon''\gamma J$$

<b>n</b>	0	1	2	3	4	5	6	7
$\varepsilon$	1	1	-1	-1	-1	-1	1	1
$\varepsilon'$	1	-1	1	1	1	-1	1	1
$\varepsilon''$	1		-1		1		-1	

- Extended KO-dimension:** additional cases that do not correspond to the behavior of classical manifolds but are needed for a good theory of products and locally product-like spectral triples:

<b>n</b>	$0_+$	$0_-$	1	$2_+$	$2_-$	3	$4_+$	$4_-$	5	$6_+$	$6_-$	7
$\varepsilon$	1	1	1	-1	1	-1	-1	-1	-1	1	-1	1
$\varepsilon'$	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1
$\varepsilon''$	1	1		-1	-1		1	1		-1	-1	

## Products of spectral triples (modelled on manifolds) $J = J_1 \otimes J_2$

- ① even/odd  $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$  and  $(\mathcal{A}_2, \mathcal{H}_2, D_2)$

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D = D_1 \otimes 1 + \gamma_1 \otimes D_2)$$

- ② odd/even  $(\mathcal{A}_1, \mathcal{H}_1, D_1)$  and  $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2)$

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D = D_1 \otimes \gamma_2 + 1 \otimes D_1)$$

- ③ even/even  $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$  and  $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2)$  use either of the above: the two operators are unitarily equivalent

$$D_1 \otimes 1 + \gamma_1 \otimes D_2 \sim D_1 \otimes \gamma_2 + 1 \otimes D_1$$

- ④ odd/odd  $(\mathcal{A}_1, \mathcal{H}_1, D_1)$  and  $(\mathcal{A}_2, \mathcal{H}_2, D_2)$

$$(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}^2, D = D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_2, \gamma = 1 \otimes 1 \otimes \sigma_3)$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- **Unitary equivalences:**  $(\mathcal{A}, \mathcal{H}_1, D_1, J_1, \gamma_1) \simeq (\mathcal{A}, \mathcal{H}_2, J_2, \gamma_2)$   
unitary  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$

$$U\pi_1(a)U^* = \pi_2(a), \quad UD_1U^* = D_2, \quad U\gamma_1U^* = \gamma_2, \quad UJ_1U^* = J_2$$

- **Morita equivalences:**  $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$  and  $\mathcal{E} \in KK_f(\mathcal{B}, \mathcal{A})$   
conjugate module  $\mathcal{E}^\circ = \{\bar{v} : v \in \mathcal{E}\}$  with  $a\bar{v}b = \bar{b}^*va^*$  Hilbert  
bimodule for  $(\mathcal{B}^\circ, \mathcal{A}^\circ)$  opposite algebras

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}^\circ} \mathcal{E}^\circ, \quad J'(v \otimes \xi \otimes \bar{w}) = v \otimes J\xi \otimes \bar{w}$$

$$D'(v \otimes \xi \otimes \bar{w}) = \nabla(v)\xi \otimes \bar{w} + v \otimes D\xi \otimes \bar{w} + v \otimes \xi \bar{\nabla}(\bar{w}),$$

with connection  $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A})$  and  $\bar{\nabla} = \tau \circ \nabla$  for  
 $\tau : \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A}) \rightarrow \Omega_D^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}^\circ$  by  $\tau(v \otimes \omega) = -\omega^* \otimes \bar{v}$

$$\bar{\nabla}(a\bar{v}) = [D, a] \otimes \bar{v} + a\bar{\nabla}(\bar{v})$$

right action of 1-forms:  $\xi \mapsto \epsilon' J\omega^* J^{-1}\xi$  and  $\gamma' = 1 \otimes \gamma \otimes 1$

## Classifying Finite Real Spectral Triples: Krajewski diagrams

- again algebra  $\mathcal{A} = \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$
- the Hilbert space  $\mathcal{H}$  is now a bimodule: rep of  $\mathcal{A} \otimes \mathcal{A}^\circ$

$$\mathcal{H} = \bigoplus_{i,j=1}^N \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij}$$

multiplicities  $V_{ij}$ ; irreducible rep  $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ}$

- anti-unitary  $J : \mathcal{H} \rightarrow \mathcal{H}$  with  $J^2 = \pm 1$
- $J^2 = 1$ : o.n. basis  $\{e_k^{ij}\}$  of  $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij}$  with  $Je_k^{ij} = e_k^{ij}$
- $J^2 = -1$  o.n. basis  $\{e_k^{ij}, f_k^{ji}\}$  with  $e_k^{ij} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij}$  and  $f_k^{ji} \in \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^\circ} \otimes V_{ij}$  with  $Je_k^{ij} = f_k^{ji}$  and  $Jf_k^{ji} = -e_k^{ij}$
- Dirac:  $D_{ij,kl}^* = D_{kl,ij}$

$$D_{ij,kl} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij} \rightarrow \mathbb{C}^{n_k} \otimes \mathbb{C}^{n_l^\circ} \otimes V_{kl}$$

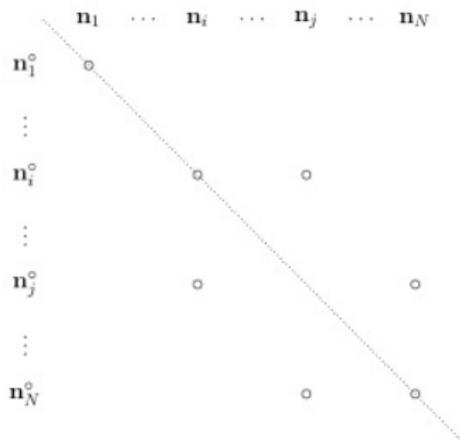
need compatibility with  $J$  and order-one condition

- $JD = \pm DJ$  and  $[[D, a]b^0] = 0$ : edges in the diagram only horizontal or vertical or looping edges at a vertex: order one condition for  $a, b$  diagonal  $a = \lambda_1 1_{n_1} \oplus \cdots \oplus \lambda_N 1_{n_N}$  and  $b = \mu_1 1_{n_1} \oplus \cdots \oplus \mu_N 1_{n_N}$

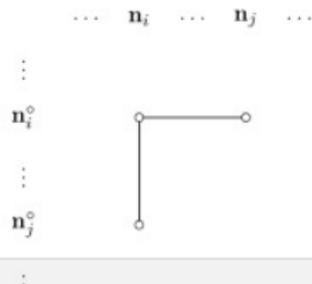
$$[[D, a]b^0]_{ij,kl} = D_{ij,kl}(\lambda_i - \lambda_j)(\bar{\mu}_j - \bar{\mu}_i) = 0$$

gives  $D_{ij,kl} = 0$  when  $i \neq j$  and  $k \neq l$  so only vertical and horizontal arrows or looping; compatibility with  $J$  relates  $D_{ij,kl}$  with  $D_{ji,lk}$  (preserves diagonal symmetry of diagram)

**Fig. 3.2** The presence of the real structure  $J$  implies a symmetry in the diagram along the diagonal



**Fig. 3.3** The lines between two nodes represent a non-zero  $D_{i,j} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^{\circ}} \rightarrow \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^{\circ}}$ , as well as its adjoint  $D_{j,i} : \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^{\circ}} \rightarrow \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^{\circ}}$ . The non-zero components  $D_{i,i,j}$  and  $D_{j,i,i}$  are related to  $\pm D_{i,j}$  and  $\pm D_{j,i}$ , respectively, according to  $J D = \pm D J$



from Walter van Suijlekom, *Noncommutative Geometry and Particle Physics*, Springer 2014

## The case of real algebras

- for  $\mathbb{R}$ -algebras Wedderburn theorem

$$\mathcal{A} = \bigoplus_{i=1}^N M_{n_i}(\mathbb{F}_i), \quad \mathbb{F}_i = \mathbb{C}, \mathbb{R}, \text{ or } \mathbb{H}$$

quaternions

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = M_2(\mathbb{C}) \text{ and } M_k(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} = M_{2k}(\mathbb{C})$$

- representation  $\mathbb{R}$ -linear  $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  complex  $\mathcal{H}$ ; one-to-one correspondence with complex representations of  $\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$
- Krajewski diagrams with vertex labels by  $(n_i, n_j, v_{ij}, \mathbb{F}_i, \mathbb{F}_j)$  and edges labelled by  $D_{ij,kl}$
- irreducible finite real spectral triples classified in
  - A. Chamseddine, A. Connes, *Why the Standard Model?* J. Geom. Phys. 58 (2008) 38–47

## Moduli spaces of Dirac operators

- in earlier literature on (finite) spectral triples two additional properties are assumed, by analogy with the geometry of manifolds
  - Orientability:  $\mathcal{A}$ -bimodule  $(\mathcal{H}, \gamma)$  orientable if  $\exists a_1, \dots, a_k, b_1, \dots, b_k \in \mathcal{A}$

$$\gamma = \sum_{i=1}^k \pi(a_i) \pi^0(b_i)$$

- Poincaré duality: for projections  $e, f \in \mathcal{A}$  bilinear form  $\text{Tr}(\gamma \pi(e) \pi^0(f))$  is non-degenerate

**Note:** physically interesting finite spectral triples do not necessarily satisfy these properties (also do not have KO-dim zero)

- structure  $P$  on  $\mathcal{A}$ -bimodule  $\mathcal{H}$  (even, real, ...) then  $\mathcal{D}(\mathcal{A}, \mathcal{H}, P)$  is  $\mathbb{R}$ -vector space of  $D$  with  $D^* = D$ , order one  $[[D, \pi(a)], \pi^0(b)] = 0$ , and  $D\gamma + \gamma D = 0$ ,  $DJ = \epsilon'JD$ , ...
- **Moduli space** of Dirac operators (up to unitary equivalence)

$$\mathcal{M}(\mathcal{A}, \mathcal{H}, P) = \mathcal{D}(\mathcal{A}, \mathcal{H}, P) / U_{\mathcal{A}}^{LR}(\mathcal{H}, P)$$

where  $U_{\mathcal{A}}^{LR}(\mathcal{H}, P)$  is  $U : \mathcal{H} \rightarrow \mathcal{H}$  unitaries compatible with  $P$

- **Complete classification** of these moduli spaces (depending on KO-dim and other data  $P$ ) given in  
Branimir Ćaćić, *Moduli spaces of Dirac operators for finite spectral triples*, arXiv:0902.2068
- We will discuss in detail moduli space for Standard Model case: physically renormalization group flow is a flow on this moduli space