

# Noncommutative Geometry and the Quantum Hall Effect

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Ma148b: topics in Mathematical Physics, Caltech Winter 2021

## Based on

- M. Marcolli, V. Mathai, *Towards the fractional quantum Hall effect: a noncommutative geometry perspective*, in “Noncommutative geometry and number theory”, 235–261, Aspects Math., E37, Vieweg, 2006
- M. Marcolli, V. Mathai, *Twisted index theory on good orbifolds, II: fractional quantum numbers*, Communications in Mathematical Physics, 217 (2001) no.1, 55–87
- M. Marcolli, V. Mathai, *Twisted index theory on good orbifolds, I: noncommutative Bloch theory*, Communications in Contemporary Mathematics, 1 (1999) 553–587

## Integer Quantum Hall Effect and NCG

- earliest application of Noncommutative Geometry to Physics
  - J. Bellissard, A. van Elst, H. Schulz-Baldes, *The noncommutative geometry of the quantum Hall effect. Topology and physics*. J. Math. Phys. 35 (1994), no. 10, 5373–5451
  - quantization of Hall conductance in the presence of magnetic field sees as values of an index theorem
  - magnetic field renders the Brillouin zone (torus for an electron moving in a periodic structure) into a noncommutative torus
  - index theorem on a noncommutative torus

## Electrons in solids – Bloch theory

Crystals Bravais lattice  $\Gamma \subset \mathbb{R}^d$  ( $d = 2, 3$ )

Periodic potential (electron–ions interaction)

$$U(x) = \sum_{\gamma \in \Gamma} u(x - \gamma) \quad T_\gamma U = U, \forall \gamma \in \Gamma$$

Mutual interaction of electrons

$$\sum_{i=1}^N (-\Delta_{x_i} + U(x_i)) + \frac{1}{2} \sum_{i \neq j} W(x_i - x_j)$$

## Independent electron approximation

$$\sum_{i=1}^N (-\Delta_{x_i} + V(x_i))$$

modification  $V$  of single electron potential

$\psi(x_1, \dots, x_N) = \det(\psi_i(x_j))$  for  $(-\Delta + V)\psi_i = E_i\psi_i$  so  
 $\sum(-\Delta_{x_i} + V(x_i))\psi = E\psi$  with  $E = \sum E_i$  (reduction to  
single electron)

Inverse problem: determine  $V$

## Bloch electrons

$T_\gamma$  = unitary operator on  $\mathcal{H} = L^2(\mathbb{R}^d)$  translation by  $\gamma \in \Gamma$

$$T_\gamma H T_{\gamma^{-1}} = H \quad \forall \gamma \in \Gamma$$

for  $H = -\Delta + V$

Simultaneously diagonalize: characters of  $\Gamma \Rightarrow$   
Pontrjagin dual  $\hat{\Gamma}$

$T_\gamma \psi = c(\gamma)\psi$ , with  $c : \Gamma \rightarrow U(1)$  since  $T_{\gamma_1 \gamma_2} = T_{\gamma_1} T_{\gamma_2}$

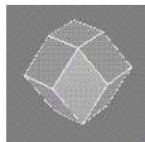
$$c(\gamma) = e^{i\langle k, \gamma \rangle}, \quad k \in \hat{\Gamma}$$

$\hat{\Gamma}$  compact group isom to  $T^d$  (dual of  $\mathbb{R}^d$  mod  
reciprocal lattice  $\Gamma^\#$ )

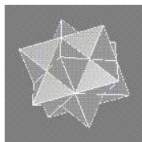
$$\Gamma^\# = \{k \in \mathbb{R}^d : \langle k, \gamma \rangle \in 2\pi\mathbb{Z}, \forall \gamma \in \Gamma\}$$

## Brillouin zones

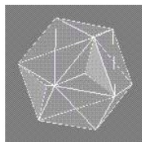
Fundamental domains of  $\Gamma^\sharp$ :  $n$ -th zone, points such that line to the origin crosses exactly  $(n-1)$  Bragg hyperplanes of the crystal



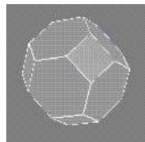
BCC Zone 1



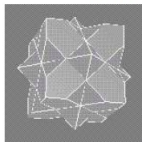
Zone 2



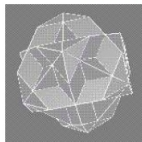
Zone 3



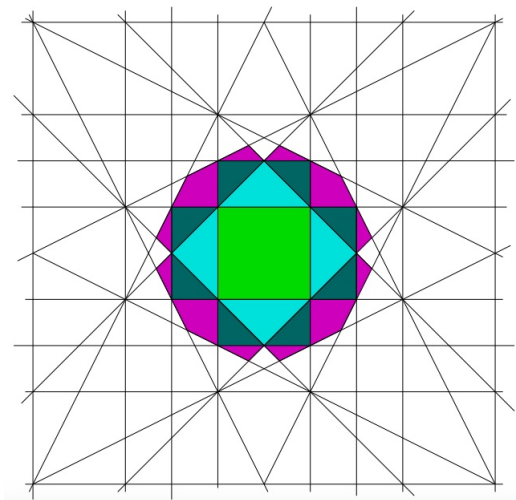
FCC Zone 1



Zone 2



Zone 3



## Band structure

Self-adjoint elliptic boundary value problem

$$D_k = \begin{cases} (-\Delta + V)\psi = E\psi \\ \psi(x + \gamma) = e^{i\langle k, \gamma \rangle} \psi(x) \end{cases}$$

Eigenvalues  $\{E_1(k), E_2(k), E_3(k), \dots\}$

$$E(k) = E(k + u) \quad \forall u \in \Gamma^\sharp$$

Plot  $E_n(k)$  over the n-the Brillouin zone

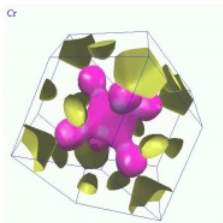
$$k \mapsto E(k) \quad k \in \mathbb{R}^d$$

energy–crystal momentum dispersion relation

## Fermi surface

Surface  $F$  in space of crystal momenta  $k$  determines electric and optical properties of the solid

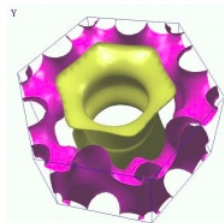
$$F_\lambda(\mathbb{R}) = \{k \in \mathbb{R}^d : E(k) = \lambda\}$$



Chromium



Vanadium



Yttrium

## Complex Bloch variety

$$B(V) = \left\{ (k, \lambda) \in \mathbb{C}^{d+1} : \begin{array}{l} \exists \psi \text{ nontriv sol of} \\ (-\Delta + V)\psi = \lambda\psi \\ \psi(x + \gamma) = e^{i\langle k, \gamma \rangle} \psi(x) \end{array} \right\}$$

$$F_\lambda(\mathbb{C}) = \pi^{-1}(\lambda) \subset B(V)$$

complex hypersurface in  $\mathbb{C}^d \Rightarrow$  singular projective

$$F_\lambda \cap \mathbb{R}^d = F_\lambda(\mathbb{R}) \text{ cycle in } H_{d-1}(F_\lambda(\mathbb{C}), \mathbb{Z})$$

## Integrated density of states

$$\rho(\lambda) = \lim_{\ell \rightarrow \infty} \frac{1}{\ell} \#\{\text{eigenvalues of } H \leq \lambda\}$$

$$H = -\Delta + V \text{ on } L^2(\mathbb{R}^d/\ell\Gamma)$$

## Period

$$\frac{d\rho}{d\lambda} = \int_{F_\lambda(\mathbb{R})} \omega_\lambda$$

$\omega_\lambda =$  holom differential on  $F_\lambda(\mathbb{C})$

## Discretization on $\ell^2(\mathbb{Z}^d)$ Random Walk

$$\mathcal{R}\psi(n_1, \dots, n_d) = + \sum_{i=1}^d \psi(n_1, \dots, n_i + 1, \dots, n_d) \\ + \sum_{i=1}^d \psi(n_1, \dots, n_i - 1, \dots, n_d)$$

## Discretized Laplacian

$$\Delta\psi(n_1, \dots, n_d) = (2d - \mathcal{R}) \psi(n_1, \dots, n_d)$$

## Bloch variety (polynomial equation in $z_i, z_i^{-1}$ )

$$B(V) = \left\{ (z_1, \dots, z_d, \lambda) : \begin{array}{l} \exists \psi \in \ell^2(\Gamma) \text{ nontriv sol of} \\ (\mathcal{R} + V)\psi = (\lambda + 2d)\psi \\ R_{\gamma_i}\psi = z_i \psi \end{array} \right\}$$

$$R_{\gamma_i}\psi(n_1, \dots, n_d) = \psi(n_1, \dots, n_i + a_i, \dots, n_d)$$

## Random walk for discrete groups on $\mathcal{H} = \ell^2(\Gamma)$

Symmetric set of generators  $\gamma_i$  of  $\Gamma$

$$\mathcal{R}\psi(\gamma) = \sum_{i=1}^r R_{\gamma_i}\psi(\gamma) = \sum_{i=1}^r \psi(\gamma\gamma_i)$$

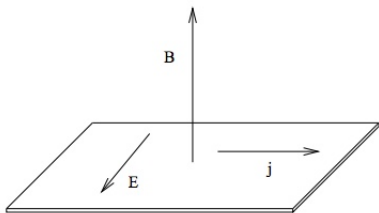
Discretized Laplacian  $\Delta = r - \mathcal{R}$

Breakdown of classical Bloch theory:

- magnetic field (quantum Hall)
- aperiodicity (quasicrystals)

## Hall Effect (Classical 1879)

thin metal, orthogonal strong magnetic field  $\mathbf{B}$ , current  $\mathbf{j} \Rightarrow$  electric field  $\mathbf{E}$ , transverse Hall current



equilibrium of forces  $\Rightarrow$  linear relation

$$Ne\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = 0$$

## Hall conductance

$$\sigma_H = \frac{Ne\delta}{B}$$

intensity of Hall current/intensity of magnetic field

$$\sigma_H = \frac{\nu}{R_H}$$

filling factor = density of charge carriers  $\cdot \frac{h}{eB}$

Hall resistance  $R_H = h/e^2$  universal constant  
(fine structure constant  $e^2/\hbar c$ )

## Integer Quantum Hall Effect (von Klitzing 1980)

- $\sigma_H$  as function of  $\nu$  has plateaux at integer multiples of  $e^2/h$
- at values of  $\nu$  corresponding to plateaux, conductivity along current density axis vanishes

## Magnetic field

2-form  $\omega = d\eta$  (field and potential  $\mathbf{B} = \text{curl}\mathbf{A}$ )

## Magnetic Schrödinger operator

$$-(\nabla - i\eta)^2 + V$$

magnetic Laplacian  $\Delta^\eta := (d - i\eta)^* (d - i\eta)$

Periodicity:  $\gamma^*\omega = \omega$  (e.g. constant field)  $0 = \omega - \gamma^*\omega = d(\eta - \gamma^*\eta)$

$$\eta - \gamma^*\eta = d\phi_\gamma$$

Magnetic Laplacian no longer commutes with  $\Gamma$  action

## Magnetic Translations

$$\phi_\gamma(x) = \int_{x_0}^x (\eta - \gamma^* \eta)$$

$$T_\gamma^\phi \psi := \exp(i\phi_\gamma) T_\gamma \psi$$

$$\text{then } (d - i\eta) T_\gamma^\phi = T_\gamma^\phi (d - i\eta)$$

Magnetic translations no longer commute

$$T_\gamma^\phi T_{\gamma'}^\phi = \sigma(\gamma, \gamma') T_{\gamma\gamma'}^\phi$$

projective representation of  $\Gamma$

$$\sigma(\gamma, \gamma') = \exp(-i\phi_\gamma(\gamma' x_0))$$

$$\phi_\gamma(x) + \phi_{\gamma'}(\gamma x) - \phi_{\gamma\gamma'}(x) \text{ indep of } x$$

## Pontrjagin dual vs. group $C^*$ -algebra

$\Gamma$  discrete abelian  $\Leftrightarrow \hat{\Gamma}$  compact abelian

Pontrjagin dual  $e^{i\langle k, \gamma \rangle}$  characters

Algebra of functions

$$C(\hat{\Gamma}) \cong C_r^*(\Gamma)$$

$C^*$ -algebra generated by  $\Gamma$  regular representation on

$$\ell^2(\Gamma) = \{\psi : \Gamma \rightarrow \mathbb{C} : \sum_{\gamma} |\psi(\gamma)|^2 < \infty\}$$

Fourier transform

When  $\Gamma$  non-abelian,  $C_r^*(\Gamma)$  still makes sense  
(non-abelian)

$\hat{\Gamma}$  exists as a noncommutative space

## Harper operator (discretized magnetic Laplacian)

$$\begin{aligned}H_{\alpha_1, \alpha_2} \psi(m, n) &= e^{-i\alpha_1 n} \psi(m+1, n) \\ &+ e^{i\alpha_1 n} \psi(m-1, n) \\ &+ e^{-i\alpha_2 m} \psi(m, n+1) \\ &+ e^{i\alpha_2 m} \psi(m, n-1)\end{aligned}$$

Let  $\sigma((m', n'), (m, n)) = \exp(-i(\alpha_1 m' n + \alpha_2 m n'))$   
(2-cocycle  $\sigma : \Gamma \times \Gamma \rightarrow U(1)$ )

Magnetic translations  $U = R_{(0,1)}^\sigma$ ,  $V = R_{(1,0)}^\sigma$

$$U\psi(m, n) = \psi(m, n+1)e^{-i\alpha_2 m}$$

$$V\psi(m, n) = \psi(m+1, n)e^{-i\alpha_1 n}$$

$\Rightarrow$  Noncommutative torus ( $\theta = \alpha_2 - \alpha_1$ )

$$UV = e^{i\theta} VU$$

## Harper operators for discrete groups

**multiplier**  $\sigma : \Gamma \times \Gamma \rightarrow U(1)$  (2-cocycle)

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)$$

$$\sigma(\gamma, 1) = \sigma(1, \gamma) = 1$$

Hilbert space:  $\ell^2(\Gamma)$

### Left/right $\sigma$ -regular representations

$$L_\gamma^\sigma \psi(\gamma') = \psi(\gamma^{-1}\gamma')\sigma(\gamma, \gamma^{-1}\gamma')$$

$$R_\gamma^\sigma \psi(\gamma') = \psi(\gamma'\gamma)\sigma(\gamma', \gamma)$$

satisfy

$$L_\gamma^\sigma L_{\gamma'}^\sigma = \sigma(\gamma, \gamma')L_{\gamma\gamma'}^\sigma \quad R_\gamma^\sigma R_{\gamma'}^\sigma = \sigma(\gamma, \gamma')R_{\gamma\gamma'}^\sigma$$

**Harper Operator** symmetric set of generators  $\{\gamma_i\}$  of  $\Gamma$

$$\mathcal{R}_\sigma = \sum_{i=1}^r R_{\gamma_i}^\sigma$$

## Twisted group algebras

$\mathbb{C}(\Gamma, \sigma)$  algebra of observables (magnetic translations), rep in  $\mathcal{B}(\ell^2(\Gamma))$  by right  $\sigma$ -regular rep  $R_\gamma^\sigma$

- weak closure  $\mathcal{U}(\Gamma, \sigma)$  twisted group von Neumann algebra
- norm closure  $C_r^*(\Gamma, \sigma)$  twisted (reduced) group  $C^*$ -algebra

## NC Torus as Twisted Group $C^*$ -algebra

- multiplier

$$\sigma((m', n'), (m, n)) = \exp(-i(\alpha_1 m' n + \alpha_2 n' m))$$

for fixed  $\alpha_1, \alpha_2 \in \mathbb{R}$  and for all  $(n, m), (n', m') \in \mathbb{Z}^2$

- $\theta = \alpha_2 - \alpha_1$
- $(Uf)(m', n') = f(m', n' + 1)e^{-i\alpha_2 m'}$  and  
 $(Vf)(m', n') = f(m' + 1, n')e^{-i\alpha_1 n'}$

$$UV = e^{i\theta} VU$$

## Spectral theory

Magnetic Schrödinger  $\mathcal{R}^\sigma + V$

potential  $V \in \mathbb{C}(\Gamma, \sigma)$

Self-adjoint bounded  $\Rightarrow \text{Spec}(\mathcal{R}^\sigma + V) \subset \mathbb{R}$

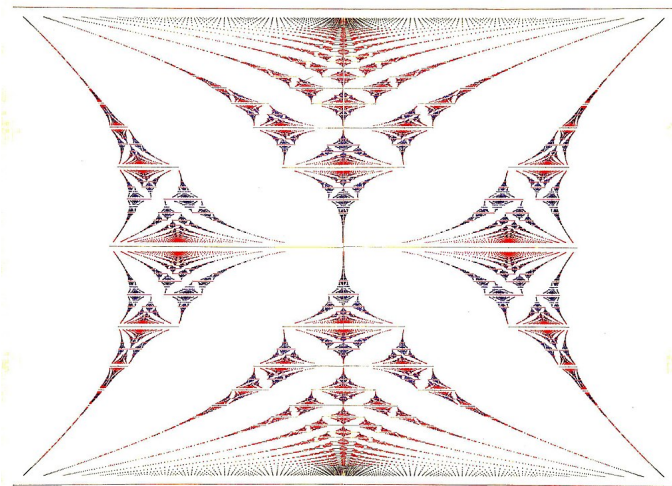
complement = collection of intervals

- Finitely many (Band structure, gaps)
- Infinitely many (Cantor-like spectrum)

For  $\Gamma = \mathbb{Z}^2$ , depends on rationality/irrationality of flux

$$\Phi = \langle [\sigma], [\Gamma] \rangle$$

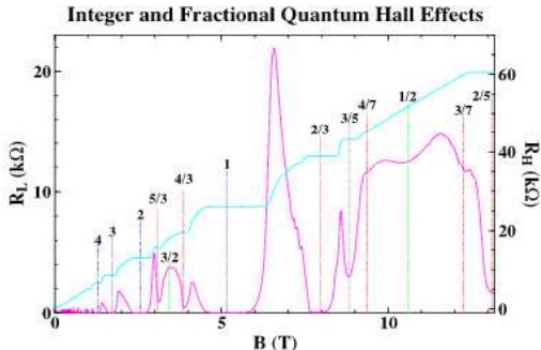
## Hofstadter butterfly



spectrum (energy) as function of magnetic flux

# Fractional Quantum Hall Effect

(Stormer and Tsui, 1983)



is there a NCG model for FQHE?

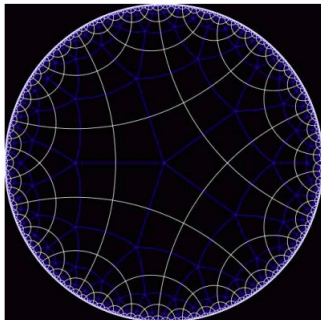
- plateaux at certain rational multiples of  $e^2/h$
- strongly interacting electrons (many particles not single particle model)

## Toy model using a single particle model but with interaction *simulated by curvature*

- need to:
  - account for strong electron interactions
  - recover observed fractions (+predictions)
  - account for varying width of plateaux
- simulation of interaction by curvature: single electron in curved geometry behaves “as if” subject to an average strong multi-electron interaction potential

## Hall effect in the hyperbolic plane

Cocompact Fuchsian group  $\Gamma = \Gamma(g, \nu_1, \dots, \nu_n)$   
 $\Gamma \subset \text{PSL}(2, \mathbb{R})$  discrete cocompact, genus  $g$ , elliptic elements order  $\nu_1, \dots, \nu_n$



**Presentation:**  $\Gamma = \{a_i, b_i, c_j \mid \prod_{i=1}^g [a_i, b_i] c_1 \cdots c_n = 1, c_j^{\nu_j} = 1\}$

**Good orbifolds**  $\Sigma = \underline{B}\Gamma = \Gamma \backslash \underline{E}\Gamma$

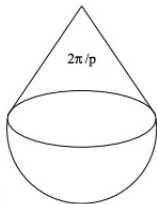
(orbifold-covered by a smooth manifold)

$$\Sigma_{g'} \xrightarrow{G} \Sigma(g; \nu_1, \dots, \nu_n) = \Gamma \backslash \mathbb{H}^2$$

$\Sigma_{g'}$  = smooth compact Riemann surface genus

$$g' = 1 + \frac{\#G}{2}(2(g-1) + (n-\nu))$$

$$\nu = \sum_{j=1}^n \nu_j^{-1}$$



Exception: teardrop orbifold

## Orbifold Euler characteristic $\chi_{orb}(\Sigma) \in \mathbb{Q}$

Multiplicative over orbifold covers, usual  $\chi$  for smooth, inclusion–exclusion

$$\begin{aligned}\chi_{orb}(\Sigma_1 \cup \dots \cup \Sigma_r) = & \sum_i \chi_{orb}(\Sigma_i) - \sum_{i,j} \chi_{orb}(\Sigma_i \cap \Sigma_j) \\ & \dots + (-1)^{r+1} \chi_{orb}(\Sigma_1 \cap \dots \cap \Sigma_r)\end{aligned}$$

$$\chi_{orb}(\Sigma(g; \nu_1, \dots, \nu_n)) = 2 - 2g + \nu - n$$

## Spectral theory

$$H_\sigma \in \mathbb{C}(\Gamma, \sigma) \subset C_r^*(\Gamma, \sigma) \subset \mathcal{U}(\Gamma, \sigma)$$

Spectral projections  $P_E = \chi_{(-\infty, E]}(H_\sigma) \in \mathcal{U}(\Gamma, \sigma)$

$$E \notin \text{Spec}(H_\sigma) \Rightarrow P_E \in C_r^*(\Gamma, \sigma)$$

$\exists F$  holom function on neighb of  $\text{Spec}(H_\sigma)$

$$P_E = \chi_{(-\infty, E]}(H_\sigma) = F(H_\sigma) = \int_C \frac{d\lambda}{\lambda - H_\sigma}$$

$C$  = contour around  $\text{Spec}(H_\sigma)$  left of  $E$

Counting gaps in the spectrum  $\Leftrightarrow$  counting projections in  $C_r^*(\Gamma, \sigma)$

Counting gaps in the spectrum  $\Leftrightarrow$  counting projections in  $C_r^*(\Gamma, \sigma)$

Trace  $\tau(T) = \langle Te_1, e_1 \rangle_{\ell^2(\Gamma)}$

$$\text{tr} = \tau \otimes \text{Tr} : \text{Proj}(C_r^*(\Gamma, \sigma) \otimes \mathcal{K}) \rightarrow \mathbb{R}$$

$[\text{tr}]$  on  $K_0(C_r^*(\Gamma, \sigma))$

## Range of the Trace

Morita equiv  $(A \otimes C_0(G)) \rtimes \Gamma \simeq C_0(\Gamma \backslash G, \mathcal{E})$

$\mathcal{E} = A \rtimes_{\Gamma} G \rightarrow \Gamma \backslash G$

$$K_{\bullet}(C_r^*(\Gamma)) \cong K_{SO(2)}^{\bullet}(P(g; \nu_1, \dots, \nu_n))$$

$$\cong K_{orb}^{\bullet}(\Sigma(g; \nu_1, \dots, \nu_n)) \cong \begin{cases} \mathbb{Z}^{2-n+\sum \nu_j} & \bullet = \text{even} \\ \mathbb{Z}^{2g} & \bullet = \text{odd} \end{cases}$$

Twisted case  $C_0(\Gamma \backslash G, \mathcal{E}) \simeq C_0(\Gamma \backslash G, \mathcal{E}_{\sigma})$  provided

$$\delta(\sigma) = 0$$

$\delta : H^2(\Gamma, U(1)) \rightarrow H^3(\Gamma, \mathbb{Z})$  surjection from

long exact sequence of  $1 \rightarrow \mathbb{Z} \hookrightarrow \mathbb{R} \xrightarrow{\exp(2\pi i \cdot)} U(1) \rightarrow 1$

$$[\text{tr}](K_0(C_r^*(\Gamma, \sigma))) = \mathbb{Z} + \theta\mathbb{Z} + \sum_j \nu_j^{-1}\mathbb{Z}$$

$\theta = \langle [\sigma], [\Gamma] \rangle$  rational  $\Rightarrow$  finitely many gaps

Fundamental class  $[\Gamma] = \frac{[\Sigma_d]}{\#G}$

### Dense \*-subalgebra

For  $\gamma \in \Gamma$  set  $De_\gamma = \ell(\gamma)e_\gamma$  with  $\ell(\gamma) =$  word length

Unbounded closed derivation  $\delta = [D, \cdot]$  on  $C_r^*(\Gamma, \sigma)$

$$\mathcal{R} := \bigcap_{k \in \mathbb{N}} \text{Dom}(\delta^k)$$

$\mathbb{C}(\Gamma, \sigma) \subset \mathcal{R}$ , closed under holo functional calculus

Cyclic cocycles  $t : \mathcal{R} \times \cdots \times \mathcal{R} \rightarrow \mathbb{C}$

$$\begin{aligned} t(a_0, a_1, \dots, a_n) &= t(a_n, a_0, a_1, \dots, a_{n-1}) \\ \dots &= t(a_1, \dots, a_n, a_0) \end{aligned}$$

and

$$\begin{aligned} t(aa_0, a_1, \dots, a_n) &- t(a, a_0a_1, \dots, a_n) \\ \dots &(-1)^{n+1} t(a_na, a_0, \dots, a_{n-1}) = 0 \end{aligned}$$

Cyclic cocycles pair with  $K$ -theory

## Conductance

In lattice  $\Gamma = \mathbb{Z}^2$ , current density in  $e_1$  direction functional derivative  $\delta_1$  of  $H_\sigma$  by  $A_1$  (component of magnetic potential)

Expected value of current  $\text{tr}(P\delta_1 H)$  for state  $P$

Using  $\partial_t P = i[P, H]$  and  $\partial_t = \frac{\partial A_2}{\partial t} \times \delta_2$  (for  $e_2 \perp e_1$ ) get

$$i \text{tr}(P[\partial_t P, \delta_1 P]) = -i E_2 \text{tr}(P[\delta_2 P, \delta_1 P])$$

(electrostatic potential gauged away:  $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ )

- **zero temperature limit:** charge carriers occupy all levels below Fermi level, so  $P = P_F$

$$\sigma_H = \text{tr}(P_F[\delta_1 P_F, \delta_2 P_F])$$

## In curved geometry

On Riemann surfaces changes of potential by real and imaginary parts of holomorphic 1-forms

$$H^1(\Gamma, \mathbb{Z}) = \mathbb{Z}^{2g} \quad \{a_i, b_i\}_{i=1, \dots, g} \text{ symplectic basis}$$

By effect of electron-electron interaction, to a moving electron the directions  $\{e_1, e_2\}$  appear split into  $\{e_i, e_{i+g}\}_{i=1, \dots, g}$  corresponding to  $a_i, b_i$

## Kubo formula

1-cocycle  $a$  on  $\Gamma \Rightarrow$  derivation

$$\delta_a(f)(\gamma) = a(\gamma)f(\gamma) \quad \text{on } f \in \mathbb{C}(\Gamma, \sigma)$$

Conductance cocycle:

$$\sum_{i=1}^g \text{tr} \left( f^0 \left( \delta_{a_i}(f^1) \delta_{b_i}(f^2) - \delta_{b_i}(f^1) \delta_{a_i}(f^2) \right) \right)$$

defines cyclic 2-cocycle  $\text{tr}^K(f^0, f^1, f^2)$  on  $\mathcal{R}$

$$C(\gamma_1, \gamma_2) = \text{Area}(\Delta(x_0, \gamma_1^{-1}x_0, \gamma_2x_0))$$

hyperbolic area of geodesic triangle with given vertices

Restriction of  $\Gamma \subset \text{PSL}(2, \mathbb{R})$  area cocycle on  $\Gamma$

Area cocycle:

$$\sum_{\gamma_0\gamma_1\gamma_2=1} f^0(\gamma_0)f^1(\gamma_1)f^2(\gamma_2)C(\gamma_1, \gamma_2)\sigma(\gamma_1, \gamma_2)$$

defines cyclic 2-cocycle  $\text{tr}^C(f^0, f^1, f^2)$  on  $\mathcal{R}$

Coboundaries  $t_1, t_2$  cyclic 2-cocycles

$$t_1(a_0, a_1, a_2) - t_2(a_0, a_1, a_2) = \lambda(a_0a_1, a_2) - \lambda(a_0, a_1a_2) + \lambda(a_2a_0, a_1),$$

where  $\lambda$  is a cyclic 1-cocycle

*Conductance and area differ by coboundary*

## Values of the conductance

(Connes–Moscovici higher index theorem, twisted)

$$\text{Ind}_{c,\Gamma,\sigma}(\not{D}_{\mathcal{E}}^+ \otimes \nabla) = \frac{1}{2\pi\#G} \int_{\Sigma_{g'}} \hat{A} \text{tr}(e^{R_{\mathcal{E}}}) e^{\omega} u_c$$

$\omega = d\eta$  2-form of magnetic field  $\nabla^2 = i\omega$

cocycle  $c$  and lift  $u_c$  to 2-form on  $\Sigma_{g'}$

By dimension

$$\text{Ind}_{c,\Gamma,\sigma}(\not{D}_{\mathcal{E}}^+ \otimes \nabla) = \frac{\text{rank}\mathcal{E}}{2\pi\#G} \int_{\Sigma_{g'}} u_c$$

Dependence on magnetic field only through  $\mathcal{E} =$  orbifold bundle representing class of  $P_E$  spectral projection in  $K_0(C_r^*(\Gamma, \sigma))$  (using Baum–Connes)

For area cocycle  $c$  2-form  $u_c$  is hyperbolic volume form

$$\int_{\Sigma_{g'}} u_c = 2\pi(2g' - 2)$$

Orbifold Euler characteristic

$$\frac{(2g' - 2)}{\#G} = -\chi_{orb}(\Sigma) \in \mathbb{Q}$$

$$\sigma_H = \text{tr}^K(P_F, P_F, P_F) = \text{tr}^C(P_F, P_F, P_F) \in \mathbb{Z}\chi_{orb}(\Sigma)$$

experimental	$g = 0 \ n = 3$
1/3	$\Sigma(0; 3, 6, 6)$
2/5	$\Sigma(0; 5, 5, 5)$
2/3	$\Sigma(0; 9, 9, 9)$
3/5	$\Sigma(0; 5, 10, 10)$
4/9	$\Sigma(0; 3, 9, 9)$
5/9	$\Sigma(0; 6, 6, 9)$
4/5	$\Sigma(0; 15, 15, 15)$
3/7	$\Sigma(0; 4, 4, 14)$
4/7	$\Sigma(0; 7, 7, 7)$
5/7	$\Sigma(0; 7, 14, 14)$

predicted	$g = 0 \ n = 3$
8/15	$\Sigma(0; 5, 6, 10)$
11/15	$\Sigma(0; 10, 10, 15)$
7/9	$\Sigma(0; 12, 12, 18)$
11/21	$\Sigma(0; 6, 6, 7)$
16/21	$\Sigma(0; 12, 12, 14)$

## Problems

- does not discriminate against even denominators (and too many fractions)
- Relation to Chern–Simons approach? Anyons?
- Effective field theories that interpolate between this one-particle toy model and QFT models?
- What happens to Bloch variety and periods in NCG Quantum Hall case?

More recent work addressing some of these problems

- M. Marcolli, K. Seipp, *Twisted index theory on orbifold symmetric products and the fractional quantum Hall effect*, Adv. Theor. Math. Phys. 21 (2017), no. 2, 451–501.
- D. Li, *The algebraic geometry of Harper operators*, J. Phys. A 44 (2011), no. 40, 405204, 27 pp.