

Spectral Action for Robertson–Walker metrics Part II: Multifractal Cosmologies

Matilde Marcolli

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Reference

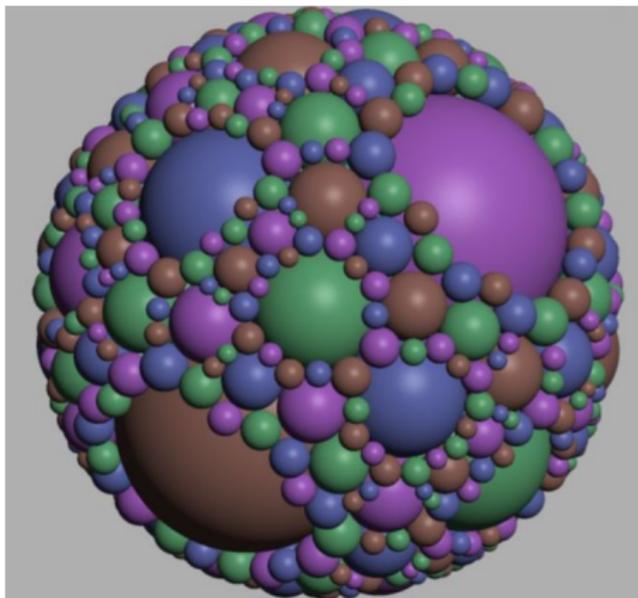
- Farzad Fathizadeh, Yeorgia Kafkoulis, Matilde Marcolli, *Bell polynomials and Brownian bridge in Spectral Gravity models on multifractal Robertson-Walker cosmologies*, arXiv:1811.02972

Other references

- A. Ball, M. Marcolli, *Spectral Action Models of Gravity on Packed Swiss Cheese Cosmology*, *Classical and Quantum Gravity*, 33 (2016), no. 11, 115018, 39 pp.

Packed Swiss Cheese Cosmology

- \mathcal{P} Apollonian packing of 3-spheres radii $\{a_{n,k} : n \in \mathbb{N}, k = 1, \dots, 6 \cdot 5^{n-1}\}$
- iterative construction of packing: at n -th step $6 \cdot 5^{n-1}$ spheres $S_{a_{n,k}}^3$ are added
- spacetime that are isotropic but not homogeneous



- two possible choices of associated Robertson–Walker metrics
 - 1 round scaling (of full 4-dim spacetime)

$$ds_{n,k}^2 = a_{n,k}^2 (dt^2 + a(t)^2 d\sigma^2), \quad n \in \mathbb{N}, k = 1, \dots, 6 \cdot 5^{n-1}$$

- 2 non-round scaling (of spatial sections only)

$$ds_{n,k}^2 = dt^2 + a(t)^2 a_{n,k}^2 d\sigma^2, \quad n \in \mathbb{N}, k = 1, \dots, 6 \cdot 5^{n-1}$$

- $D_{n,k}$ resulting Dirac operators on $\mathbb{R} \times \mathcal{S}_{a_{n,k}}^3$
- entire (multifractal) spacetime $\mathbb{R} \times \mathcal{P}$
- spectral triple for $\mathbb{R} \times \mathcal{P}$: \mathcal{A} subalgebra of $C_0(\mathbb{R} \times \mathcal{P})$, Hilbert space $\mathcal{H} = \bigoplus_{n,k} \mathcal{H}_{n,k}$ with $\mathcal{H}_{n,k} = L^2(\mathcal{S}_{a_{n,k}}, \mathbb{S})$ and Dirac

$$D = D_{\mathbb{R} \times \mathcal{P}} := \bigoplus_{n \in \mathbb{N}} \bigoplus_{k=1}^{6 \cdot 5^{n-1}} D_{n,k}$$

Mellin Transform and Zeta Functions

- meromorphic function $\phi(z)$ with poles at $\mathcal{S} \subset \mathbb{C}$, Laurent series expansion at a pole $z_0 \in \mathcal{S}$

$$\phi(z) = \sum_{-N \leq k} c_k (z - z_0)^k$$

- singular element** at $z_0 \in \mathcal{S}$

$$S(\phi, z_0) := \sum_{-N \leq k \leq 0} c_k (z - z_0)^k$$

- singular expansion** of ϕ

$$S_\phi(z) := \sum_{z \in \mathcal{S}} S(\phi, z)$$

- Example: for the Gamma function

$$\Gamma(z) \asymp \sum_{k \geq 0} \frac{(-1)^k}{k!} \frac{1}{z + k}$$

- Mellin transform

$$\phi(z) = \mathcal{M}(f)(z) = \int_0^\infty f(\tau)\tau^{z-1}d\tau$$

- relation between asymptotic expansion at $u \rightarrow 0$ of a function $f(u)$ and singular expansion of its Mellin transform

$$\phi(z) = \mathcal{M}(f)(z)$$

- small time asymptotic expansion

$$f(u) \sim_{u \rightarrow 0^+} \sum_{\alpha \in \mathcal{S}, k_\alpha} c_{\alpha, k_\alpha} u^\alpha \log(u)^{k_\alpha}$$

- coefficients c_{α, k_α} determined by singular expansion of Mellin transform

$$\mathcal{M}(f)(z) \asymp S_{\mathcal{M}(f)}(z) = \sum_{\alpha \in \mathcal{S}, k_\alpha} c_{\alpha, k_\alpha} \frac{(-1)^{k_\alpha} k_\alpha!}{(s + \alpha)^{k_\alpha + 1}}$$

- index k_α ranges over terms in singular element of $\phi(z) = \mathcal{M}(f)(z)$ at $z = \alpha$, up to order of pole at α

Example: Packing of 4-Spheres

- round S^4 is a Robertson–Walker metric $dt^2 + a(t)^2 d\sigma^2$ with $a(t) = \sin t$ ($0 \leq t \leq \pi$) and $d\sigma^2$ round metric on S^3
- spectrum of Dirac operator on S_r^{D-1} radius $r > 0$

$$\text{Spec}(D_{S_r^{D-1}}) = \left\{ \lambda_{\ell, \pm} = \pm r^{-1} \left(\frac{D-1}{2} + \ell \right) \mid \ell \in \mathbb{Z}_+ \right\}$$

multiplicities

$$m_{\ell, \pm} = 2^{\lfloor \frac{D-1}{2} \rfloor} \binom{\ell + D}{\ell}.$$

- zeta function of Dirac operator

$$\zeta_D(s) = \text{Tr}(|D_{S_r^4}|^{-s}) = \sum_{\ell, \pm} m_{\ell, \pm} |\lambda_{\ell, \pm}|^{-s} = \frac{4}{3} r^s (\zeta(s-3) - \zeta(s-1))$$

$\zeta(s)$ Riemann zeta function

- **fractal string zeta function** $\zeta_{\mathcal{L}}(s) = \sum_{n,k} a_{n,k}^s$ of Apollonian packing \mathcal{P} of $S_{a_{n,k}}^3$ with radii sequence $\mathcal{L} = \{a_{n,k}\}$
- resulting Dirac operator $\mathcal{D}_{\mathcal{P}}$ on associated packing of 4-spheres (each 3-sphere equator in a fixed hyperplane of a corresponding 4-sphere)
- zeta function of Dirac $\mathcal{D}_{\mathcal{P}}$ factors as product of zetas

$$\begin{aligned} \zeta_{\mathcal{D}_{\mathcal{P}}}(s) &= \text{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n,k} \frac{4}{3} a_{n,k}^s (\zeta(s-3) - \zeta(s-1)) \\ &= \zeta_{\mathcal{L}}(s) \zeta_{D_{S^4}}(s) \end{aligned}$$

- Mellin transform relation between the zeta function of the Dirac operator and the heat-kernel of the Dirac Laplacian

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \frac{1}{\Gamma(s/2)} \int_0^\infty \mathrm{Tr}(e^{-t\mathcal{D}_{\mathcal{P}}^2}) t^{s/2-1} dt$$

- use to compute spectral action leading terms from zeta function: $\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim$

$$f(0)\zeta_{\mathcal{D}_{\mathcal{P}}}(0) + f_2\Lambda^2 \frac{\zeta_{\mathcal{L}}(2)}{2} + f_4\Lambda^4 \frac{\zeta_{\mathcal{L}}(4)}{2} + \sum_{\sigma \in \mathcal{S}(\mathcal{L})} f_\sigma \Lambda^\sigma \frac{\zeta_{D_{S^4}}(\sigma)}{2} \mathcal{R}_\sigma$$

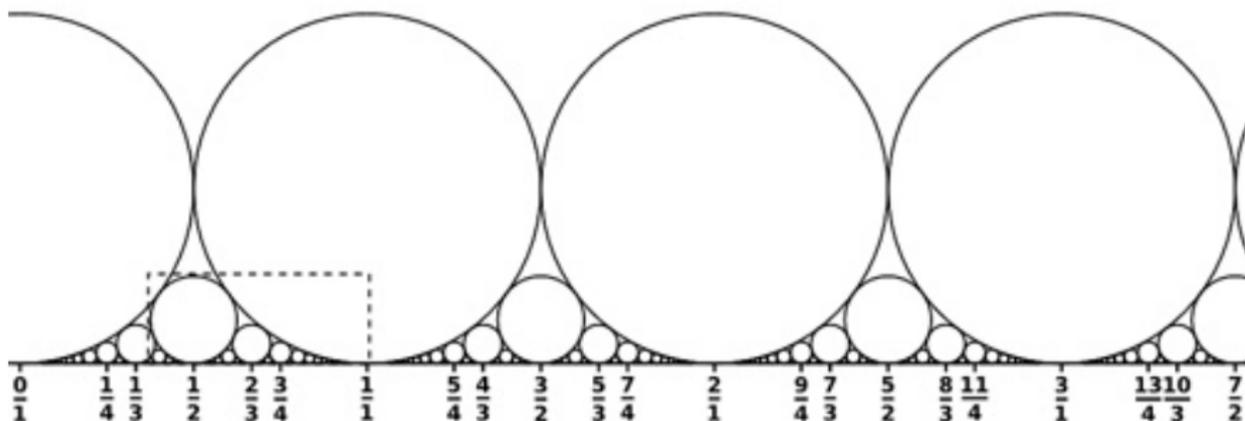
- $\mathcal{S}(\mathcal{L})$ set of poles of fractal string zeta $\zeta_{\mathcal{L}}(s)$ residues

$$\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma} \zeta_{\mathcal{L}}(s)$$

- $\zeta_{\mathcal{L}}(2)$ and $\zeta_{\mathcal{L}}(4)$ replace radii r^2 and r^4 for a single sphere S_r^4 : zeta regularization of $\sum_{n,k} a_{n,k}^2$ and $\sum_{n,k} a_{n,k}^4$

Example: Lower Dimensional Apollonian Ford Circles

- **Ford circles**: tangent to the real line at points $(k/n, 0)$ with centers at points $(k/n, 1/(2n^2))$



- number of circles of radius $r_n = (2n^2)^{-1}$ is number of integers $1 \leq k \leq n$ coprime to n : multiplicity $m(r_n)$ given by Euler totient function

$$m(r_n) = \varphi(n),$$

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

product over the distinct prime numbers dividing n

- Dirichlet series generating function of the Euler totient function

$$\mathcal{D}_\varphi(s) = \sum_{n \geq 1} \frac{\varphi(n)}{n^s}$$

- fractal string zeta function

$$\zeta_{\mathcal{L}}(s) = \sum_{n \geq 1} \varphi(n) (2n^2)^{-s} = 2^{-s} \sum_{n \geq 1} \varphi(n) n^{-2s} = 2^{-s} \mathcal{D}_{\varphi}(2s)$$

- using $\varphi(p^k) = p^k - p^{k-1}$

$$1 + \sum_k \varphi(p^k) p^{-sk} = \frac{1 - p^{-s}}{1 - p^{1-s}}$$

- using Euler product formula

$$\mathcal{D}_{\varphi}(s) = \frac{\zeta(s-1)}{\zeta(s)}$$

- so fractal string zeta function of Ford circles

$$\zeta_{\mathcal{L}}(s) = 2^{-s} \frac{\zeta(2s-1)}{\zeta(2s)}$$

- build a packing of 4-spheres over the Ford circles: collection of 2-spheres with Ford circles as equators in a given hyperplane, then same as equators of a collection of 3-spheres and then of 4-spheres
- for the resulting packing of 4-spheres spectral action

$$\begin{aligned} \mathrm{Tr}(f(\mathcal{D}_P/\Lambda)) &\sim \frac{11}{140} f(0) + \frac{f_1}{\pi^2} \Lambda + \frac{45 \zeta(3)}{4\pi^4} f_2 \Lambda^2 + \frac{4725 \zeta(7)}{16\pi^8} f_4 \Lambda^4 \\ &+ \sum_{k \in \mathbb{N}} \frac{2^{k+1} f_{-k}}{3} \frac{\zeta(-k-3) - \zeta(-k-1)}{\zeta(-2k-1)} \Lambda^{-k} \\ &+ \sum_{\sigma = a+ib} \frac{2^{-a} \cos(b \log 2)}{3} \Re(Z_\sigma) r(f)_\sigma \cos(b \log \Lambda) \Lambda^a \end{aligned}$$

σ over nontrivial zeros of $\zeta(2s)$ with
 $r(f)_\sigma = \int_0^\infty f(u) u^{a-1} \cos(bu) du$ and
 $Z_\sigma = (\zeta(\sigma-3) - \zeta(\sigma-1)) \zeta(2\sigma-1)$

General Case $\mathbb{R} \times \mathcal{P}$ with round scaling $a_{n,k}^2(dt^2 + a(t)^2 d\sigma^2)$

- $R = \{r_n\}$ sequence of $r_n \in \mathbb{R}_+^*$ so that $\zeta_R(z) = \sum_n r_n^{-z}$ converges for $\Re(z) > C$ for some $C > 0$
- function $f(\tau)$ with small time asymptotics

$$f(\tau) \sim \sum_N c_N \tau^N$$

- associated series

$$g_R(\tau) = \sum_n f(r_n \tau)$$

- then small time asymptotic expansion of $g_R(\tau)$

$$g_R(\tau) \sim_{\tau \rightarrow 0^+} \sum_N c_N \zeta_R(-N) \tau^N + \sum_{\sigma \in \mathcal{S}(\zeta_R)} \mathcal{R}_{R,\sigma} \mathcal{M}(f)(\sigma) \tau^{-\sigma}$$

with $\mathcal{S}(\zeta_R)$ poles of $\zeta_R(z)$

$$\mathcal{R}_{R,\sigma} := \text{Res}_{z=\sigma} \zeta_R(z)$$

Sketch of proof:

- write associated series as

$$g_R(\tau) \sim \sum_{N,n} c_N r_n^N \tau^N = \sum_N \zeta_R(-N) \tau^N$$

- Mellin transform $\mathcal{M}(g)(z) = \int_0^\infty g(\tau) \tau^{z-1} d\tau$ gives

$$\mathcal{M}(g)(z) = \left(\sum_n r_n^{-z} \right) \int_0^\infty \sum_N c_N u^{N+z-1} du = \zeta_R(z) \cdot \mathcal{M}(f)(z)$$

- asymptotic expansion of $g_R(\tau)$ from Mellin transform $\mathcal{M}(g_R)(z)$ singular expansion

$$S_{\mathcal{M}(g_R)}(z) = \sum_{\sigma \in \mathcal{S}(\zeta_R)} \frac{\mathcal{R}_{R,\sigma} \mathcal{M}(f)(\sigma)}{z - \sigma} + \sum_{\sigma \in \mathcal{S}(\mathcal{M}(f))} \frac{\zeta_R(\sigma) c_\sigma}{z - \sigma}$$

- and from small time asymptotics of $f(\tau)$ know

$$S_{\mathcal{M}(f)}(z) = \sum_N \frac{c_N}{z + N}$$

Feynman–Kac formula on $\mathbb{R} \times \mathcal{P}$

- on each $\mathbb{R} \times S_{a_{n,k}}^3$ decompose Dirac $D_{a_{n,k}}$ using operators

$$H_{m,n,k} = -a_{n,k}^{-2} \frac{d^2}{dt^2} + V_{m,n,k}(t)$$

$$V_{m,n,k} = \frac{(m + \frac{3}{2})}{a_{n,k}^2 \cdot a(t)^2} \left((m + \frac{3}{2}) - a_{n,k} \cdot a'(t) \right)$$

as in Chamseddine–Connes

- Feynman–Kac formula

$$\begin{aligned} e^{-\tau^2 H_{m,n,k}}(t, t) &= e^{-\frac{\tau^2}{a_{n,k}^2} \left(\frac{d^2}{dt^2} + a_{n,k}^2 V_{m,n,k} \right)}(t, t) \\ &= \frac{a_{n,k}}{2\sqrt{\pi\tau}} \int \exp\left(-\tau^2 \int_0^1 V_{m,n,k}\left(t + \sqrt{2} \frac{\tau}{a_{n,k}} \alpha(u)\right) du\right) D[\alpha] \end{aligned}$$

- Poisson summation to replace sum

$$\sum_m \mu(m) e^{-\tau^2 H_{m,n,k}(t, t)}$$

with multiplicities $\mu(m)$ with the integral

$$\int_{-\infty}^{\infty} f_{\tau, n, k}(x) dx$$

$$f_{\tau, n, k}(x) = \left(x^2 - \frac{1}{4} \right) e^{-x^2 a_{n,k}^{-2} U - x a_{n,k}^{-1} V}$$

with U and V as in single sphere case

$$\begin{aligned} \sum_m \mu(m) e^{-\tau^2 H_{m,n,k}(t, t)} &= \\ \int \frac{a_{n,k}}{\tau} \left(\frac{e^{\frac{V^2}{4U}}}{4} (-a_{n,k} U^{-1/2} + 2a_{n,k}^3 U^{-3/2} + a_{n,k}^3 V^2 U^{-5/2}) \right) D[\alpha] \\ &= \int \frac{1}{\tau} \left(\frac{e^{\frac{V^2}{4U}}}{4} (-a_{n,k}^2 U^{-1/2} + a_{n,k}^4 (2U^{-3/2} + V^2 U^{-5/2})) \right) D[\alpha] \end{aligned}$$

- same Taylor expansion method

$$e^{\frac{V^2}{4U}} U^r V^\ell = \tau^{2(r+\ell)} \sum_{M=0}^{\infty} a_{n,k}^{-M-2(r+\ell)} C_M^{(r,\ell)} \tau^M$$

with $C_M^{(r,\ell)}$ as in single sphere case $dt^2 + a(t)^2 d\sigma^2$

- resulting expansion

$$\begin{aligned} & \frac{1}{\tau} \left(\frac{e^{\frac{V^2}{4U}}}{4} (-a_{n,k}^2 U^{-1/2} + a_{n,k}^4 (2U^{-3/2} + V^2 U^{-5/2})) \right) = \\ & \frac{1}{4} \sum_{M=0}^{\infty} \left(C_M^{(-5/2,2)} - C_M^{(-1/2,0)} \right) \zeta_{\mathcal{L}}(-M+2) \tau^{M-2} \\ & \quad + \frac{1}{2} \sum_{M=0}^{\infty} C_M^{(-3/2,0)} \zeta_{\mathcal{L}}(-M+4) \tau^{M-4}. \end{aligned}$$

- Feynman–Kac formula for the whole $\mathbb{R} \times \mathcal{P}$

$$\sum_{n,k} \sum_m \mu(m) e^{-\tau^2 H_{m,n,k}}(t, t) =$$

$$\sum_{M=0}^{\infty} \tau^{2M-4} \zeta_{\mathcal{L}}(-2M+4) \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} (C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}) \right) D[\alpha]$$

with only the term $\frac{1}{2} C_0^{(-3/2,0)}$ when $M = 0$

- obtained as a series

$$g_{\mathcal{L}}(\tau) = \sum_{n,k} f(a_{n,k}^{-1} \tau)$$

$$f(\tau) \sim \sum_M \tau^{2M-4} \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} (C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}) \right) D[\alpha]$$

Result: Spectral Action on Multifractal Robertson–Walker $\mathbb{R} \times \mathcal{P}$

$$\mathrm{Tr}(f(\mathcal{D}/\Lambda)) \sim$$

$$\sum_{M=0}^{\infty} \Lambda^{n_M} f_{n_M} \zeta_{\mathcal{L}}(n_M) \int \left(\frac{1}{2} C_{4-n_M}^{(-3/2,0)} + \frac{1}{4} (C_{2-n_M}^{(-5/2,2)} - C_{2-n_M}^{(-1/2,0)}) \right) D[\alpha] \\ + \sum_{\sigma \in \mathcal{S}_{\mathcal{L}}} \tilde{f}(\sigma) \cdot f_{\sigma} \cdot \mathrm{Res}_{z=\sigma} \zeta_{\mathcal{L}} \cdot \Lambda^{\sigma}$$

$n_M = 4 - 2M$, set of poles $\mathcal{S}_{\mathcal{L}}$ of $\zeta_{\mathcal{L}}$ Mellin transform $\tilde{f}(z) = \mathcal{M}(f)(z)$ of $f(\tau) = \mathrm{Tr}(\exp(-\tau^2 D^2))$ with Dirac on $\mathbb{R} \times S^3$ with $dt^2 + a(t)^2 d\sigma^2$

Conclusion: presence of fractality detected by two types of effects

- 1 zeta regularization of coefficients $\zeta_{\mathcal{L}}(4 - 2M)$ in terms Λ^{4-2M} (including effective gravitational and cosmological constant in top terms)
- 2 additional terms from non-real poles of order $\Lambda^{\Re\sigma}$ (and log periodic) with $3 < \Re\sigma = \dim_H \mathcal{P} < 4$ between cosmological and Einstein–Hilbert term

Multifractal Robertson–Walker with non-round scaling

$$dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$$

- rescaling $ds_a^2 = dt^2 + a^2 \cdot a(t)^2 d\sigma^2$ with $a > 0$ gives $U \mapsto a^{-2} U$ and $V \mapsto a^{-1} V$
- this gives rescaling

$$\frac{1}{4} \sum_{M=0}^{\infty} (a^3 C_M^{(-5/2,2)} - a C_M^{(-1/2,0)}) \tau^{M-2} + \frac{1}{2} \sum_{M=0}^{\infty} a^3 C_M^{(-3/2,0)} \tau^{M-4}$$

- expect presence of zeta regularized coefficients $\zeta_{\mathcal{L}}(3)$, $\zeta_{\mathcal{L}}(1)$
- to see this use a Mellin transform with respect to the “multiplicity variable” x in $f_s(x)$

Kummer confluent hypergeometric function

- notation: $a^{(n)} := a(a+1)\cdots(a+n-1)$ and $a^{(0)} := 1$
- Kummer confluent hypergeometric function defined by series

$${}_1F_1(a, b, t) = \sum_{n=0}^{\infty} \frac{a^{(n)} t^n}{b^{(n)} n!}$$

- solution of the Kummer equation

$$t \frac{d^2 f}{dt^2} + (b - t) \frac{df}{dt} - af = 0.$$

Mellin transform and hypergeometric function

- Mellin transform in the x -variable of the function

$$f_{s,-}(x) := f_s(x) = \left(x^2 - \frac{1}{4}\right)e^{-x^2U-xV}$$

given by

$$\mathcal{M}\left(\left(x^2 - \frac{1}{4}\right)e^{-x^2U-xV}\right)(z) = \frac{1}{8}U^{-(z+3)/2} \times$$

$$\left(U^{1/2} \Gamma\left(\frac{z}{2}\right) \left(-U {}_1F_1\left(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) + 2z {}_1F_1\left(\frac{z+2}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) \right) \right. \\ \left. + V \Gamma\left(\frac{z+1}{2}\right) \left(U {}_1F_1\left(\frac{z+1}{2}, \frac{3}{2}, \frac{V^2}{4U}\right) - 2(z+1) {}_1F_1\left(\frac{z+3}{2}, \frac{3}{2}, \frac{V^2}{4U}\right) \right) \right)$$

- similar expression for transform of $f_{s,+}(x) := \left(x^2 - \frac{1}{4}\right)e^{-x^2U+xV}$

- multiplicity integral

$$\int_{-\infty}^{\infty} f_s(x) dx = \int_0^{\infty} f_{s,-}(x) dx + \int_0^{\infty} f_{s,+}(x) dx$$

$$f_{s,\pm}(x) = \left(x^2 - \frac{1}{4}\right) e^{-x^2 U \pm x V}$$

- multiplicity integral as special value at $z = 1$ of Mellin

$$\int_{-\infty}^{\infty} f_s(x) dx = \mathcal{M}(f_{s,-})(z)|_{z=1} + \mathcal{M}(f_{s,+})(z)|_{z=1}$$

- Mellin transform $\mathcal{M}(f_{s,-})(z) + \mathcal{M}(f_{s,+})(z)$

$$= -\frac{1}{4} U^{-1-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \left(U {}_1F_1\left(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) - 2z {}_1F_1\left(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) \right)$$

- value at $z = 1$

$$\left(-\frac{1}{4} U^{-(1+\frac{z}{2})} \Gamma\left(\frac{z}{2}\right) \left(U {}_1F_1\left(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) - 2z {}_1F_1\left(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) \right) \right) \Big|_{z=1} =$$

$$e^{\frac{V^2}{4U}} \frac{\sqrt{\pi}}{4} (-U^{-1/2} + 2U^{-3/2} + V^2 U^{-5/2})$$

Effect of scaling

- notation:

$$H_\lambda(\tau, z) := U^{-z/2} \Gamma(z/2) {}_1F_1\left(\frac{z}{2}, \lambda, \frac{V^2}{4U}\right)$$

$$H(\tau, z) := H_{1/2}(\tau, z) = U^{-z/2} \Gamma(z/2) {}_1F_1\left(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right)$$

$$H_{\mathcal{L}}(\tau, z) := U^{-z/2} \zeta_{\mathcal{L}}(z) \Gamma(z/2) {}_1F_1\left(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) = \zeta_{\mathcal{L}}(z) H(\tau, z)$$

- multiplicity integral with scaling $dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

$$\begin{aligned} \mathcal{M}(f_{s,n,k,-})(z) + \mathcal{M}(f_{s,n,k,+})(z) = \\ -\frac{1}{4} a_{n,k}^z U^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) {}_1F_1\left(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) \\ + a_{n,k}^{z+2} U^{1-\frac{z}{2}} \Gamma\left(1 + \frac{z}{2}\right) {}_1F_1\left(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}\right) \\ - \frac{1}{4} a_{n,k}^z H(\tau, z) + a_{n,k}^{z+2} H(\tau, z + 2). \end{aligned}$$

- multiplicity integral over the full sphere packing $\mathbb{R} \times \mathcal{P}$

$$f_{\mathcal{P},s}(x) = \left(x^2 - \frac{1}{4}\right) \sum_{n,k} e^{-x^2 a_{n,k}^{-2}} U - x a_{n,k}^{-1} V$$

- as value of Mellin transform

$$\int_{-\infty}^{\infty} f_{\mathcal{P},s}(x) dx = \mathcal{M}(f_{\mathcal{P},s,-})(z)|_{z=1} + \mathcal{M}(f_{\mathcal{P},s,+})(z)|_{z=1}$$

$$f_{\mathcal{P},s,\pm} = (x^2 - 1/4) \sum_{n,k} \exp(-x^2 a_{n,k}^{-2}) U \pm x a_{n,k}^{-1} V$$

- Mellin transforms

$$\mathcal{M}(f_{\mathcal{P},s,-})(z) + \mathcal{M}(f_{\mathcal{P},s,+})(z) = -\frac{1}{4} H_{\mathcal{L}}(\tau, z) + H_{\mathcal{L}}(\tau, z + 2)$$

- this shows one gets zeta regularized $\zeta_{\mathcal{L}}(3)$ and $\zeta_{\mathcal{L}}(1)$

Sketch of how to see the log periodic terms for non-round scaling
 $dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

- τ expansion

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!} \tau^n, \quad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!} \tau^n$$

- gives expansion of confluent hypergeometric function

$$H_{\mathcal{L}}(\tau, z) = \zeta_{\mathcal{L}}(z) \Gamma(z/2) U^{-z/2} \sum_{n=0}^{\infty} \frac{(z/2)_n}{4^n n! (1/2)_n} V^{2n} U^{-n}$$

with $(a)_n = a(a+1) \cdots (a+n-1)$ the rising factorial

- the term $U^{-z/2}$ contributes a term with τ^z times a power series in τ , while confluent hypergeometric function contributes a power series in τ

- to see why τ^z gives rise to log periodic terms in the spectral action consider simplified case
- product of the Mellin transforms $\mathcal{M}(f_1)(z) \cdot \mathcal{M}(f_2)(z)$ is Mellin transform of convolution

$$\mathcal{M}(f_1)(z) \cdot \mathcal{M}(f_2)(z) = \mathcal{M}(f_1 \star f_2)(z),$$

$$(f_1 \star f_2)(x) = \int_0^\infty f_1\left(\frac{x}{u}\right) f_2(u) \frac{du}{u}$$

- Mellin transform of a delta distribution

$$\tau^{z-1} = \mathcal{M}(\delta(x - \tau))$$

- Mellin transform of distribution

$$\Lambda_{\mathcal{P}, \tau} := \sum_{n,k} \tau a_{n,k} \delta(x - \tau \cdot a_{n,k})$$

$$\left\langle \sum_{n,k} \tau a_{n,k} \delta(x - \tau \cdot a_{n,k}), \phi(x) \right\rangle = \sum_{n,k} \tau a_{n,k} \phi(\tau a_{n,k})$$

given by

$$\tau^z \zeta_{\mathcal{L}}(z) = \mathcal{M}\left(\sum_{n,k} \tau a_{n,k} \delta(x - \tau \cdot a_{n,k})\right)$$

- given function $g(x)$
will want $g_\gamma(x) := \mathcal{M}^{-1}(\Gamma(z/2) {}_1F_1(z/2, 1/2, \gamma))$

$$\begin{aligned} \mathcal{M}(\Lambda_{\mathcal{P},\tau})(z) \cdot \mathcal{M}(g)(z) &= \mathcal{M}(\Lambda_{\mathcal{P},\tau} \star g)(z) \\ &= \mathcal{M}\left(\sum_{n,k} \tau a_{n,k} \int_0^\infty \delta(u - \tau a_{n,k}) g\left(\frac{x}{u}\right) \frac{du}{u}\right) = \sum_{n,k} \mathcal{M}\left(g\left(\frac{x}{\tau \cdot a_{n,k}}\right)\right) \end{aligned}$$

- take $h_z(\tau) := \mathcal{M}\left(g\left(\frac{x}{\tau}\right)\right)$

$$L_z(\tau) := \sum_{n,k} h_z(\tau \cdot a_{n,k})$$

- asymptotic expansion for this function through singular expansion of Mellin transform in τ

$$\mathcal{M}_\tau(L_z(\tau))(\beta) = \zeta_{\mathcal{L}}(\beta) \cdot \mathcal{M}(h_z(\tau))(\beta)$$

- contributions from poles of $\zeta_{\mathcal{L}}(\beta)$ and of $\mathcal{M}(h_z(\tau))(\beta)$:
log-periodic and zeta regularized terms as expected