The Spectral Action and Cosmic Topology

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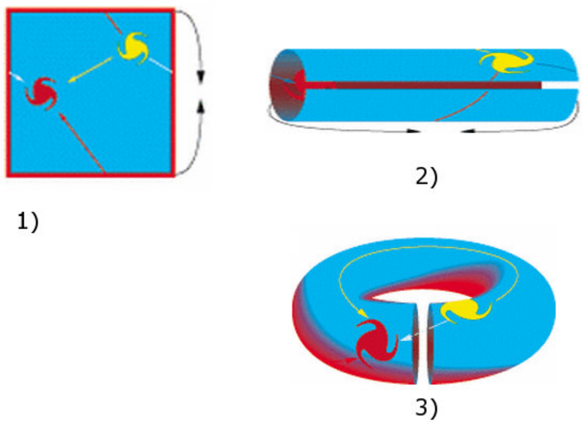
Ma148b: Topics in Mathematical Physics, Caltech Winter 2021

This lecture based on

- Matilde Marcolli, Elena Pierpaoli, Kevin Teh, *The spectral action and cosmic topology*, Commun.Math.Phys.304 (2011) 125–174
- Matilde Marcolli, Elena Pierpaoli, Kevin Teh, *The coupling of topology and inflation in noncommutative cosmology*, Comm. Math. Phys. 309 (2012), no. 2, 341–369
- Branimir Ćaćić, Matilde Marcolli, Kevin Teh, *Coupling of gravity to matter, spectral action and cosmic topology*, J. Noncommut. Geom. 8 (2014), no. 2, 473?504
- Kevin Teh, *Nonperturbative spectral action of round coset spaces of $SU(2)$*, J. Noncommut. Geom. 7 (2013), no. 3, 677–708.

The question of Cosmic Topology:

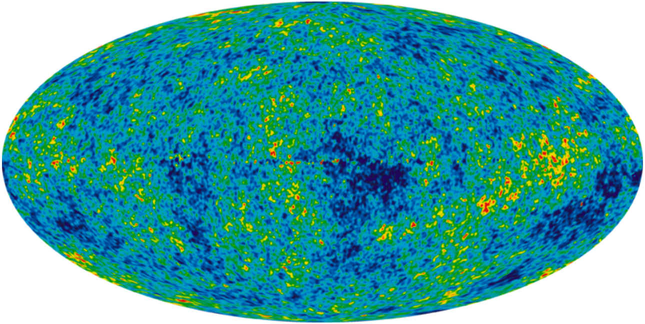


Nontrivial (non-simply-connected) spatial sections of spacetime, homogeneous spherical or flat spaces: how can this be detected from cosmological observations?

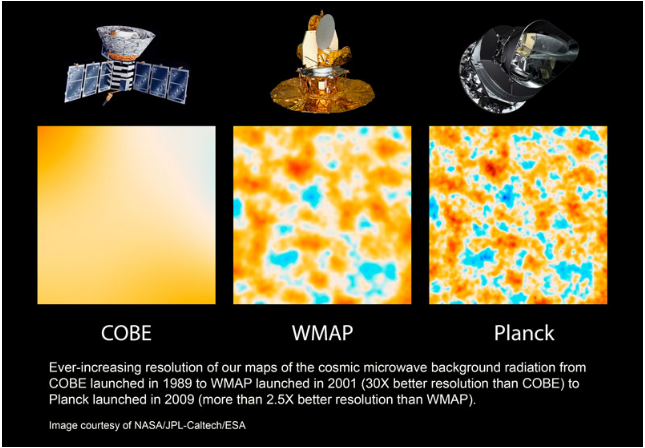
Our approach:

- NCG provides a modified gravity model through the spectral action
- The nonperturbative form of the spectral action determines a slow-roll inflation potential
- The underlying geometry (spherical/flat) affects the shape of the potential
- Different inflation scenarios depending on geometry and topology of the cosmos
- Shape of the inflation potential readable from cosmological data (CMB)

Cosmic Microwave Background best source of cosmological data on which to test theoretical models (modified gravity models, cosmic topology hypothesis, particle physics models)



- COBE satellite (1989)
- WMAP satellite (2001)
- Planck satellite (2009)

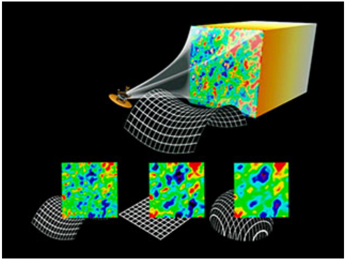


Ever-increasing resolution of our maps of the cosmic microwave background radiation from COBE launched in 1989 to WMAP launched in 2001 (30X better resolution than COBE) to Planck launched in 2009 (more than 2.5X better resolution than WMAP).

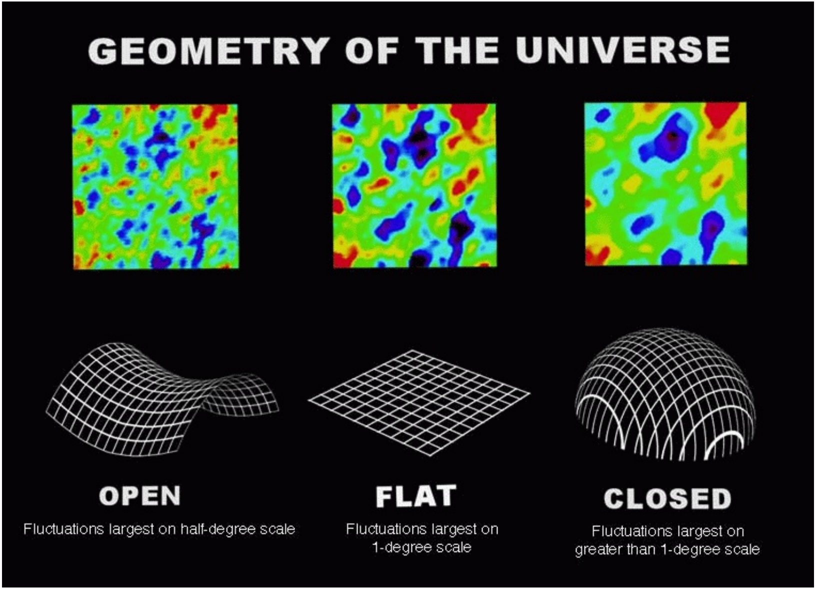
Image courtesy of NASA/JPL-Caltech/ESA

Cosmic topology and the CMB

- Einstein equations determine geometry not topology (don't distinguish S^3 from S^3/Γ with round metric)
- Cosmological data (BOOMERanG experiment 1998, WMAP data 2003): spatial geometry of the universe is flat or slightly positively curved
- Homogeneous and isotropic compact case: spherical space forms S^3/Γ or Bieberbach manifolds T^3/Γ



Is cosmic topology detected by the Cosmic Microwave Background (CMB)? Search for signatures of multiconnected topologies


GEOMETRY OF THE UNIVERSE
OPEN
Fluctuations largest on half-degree scale
FLAT
Fluctuations largest on 1-degree scale
CLOSED
Fluctuations largest on greater than 1-degree scale

CMB sky and spherical harmonics temperature fluctuations

$$\frac{\Delta T}{T} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}$$

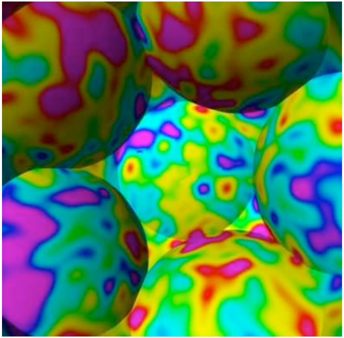
$Y_{\ell m}$ spherical harmonics

Methods to address cosmic topology problem

- Statistical search for matching circles in the CMB sky: identify a nontrivial fundamental domain
- Anomalies of the CMB: quadrupole suppression, the small value of the two- point temperature correlation function at angles above 60 degrees, and the anomalous alignment of the quadrupole and octupole
- Residual gravity acceleration: gravitational effects from other fundamental domains
- Bayesian analysis of different models of CMB sky for different candidate topologies

Results: no conclusive evidence of a non-simply connected topology

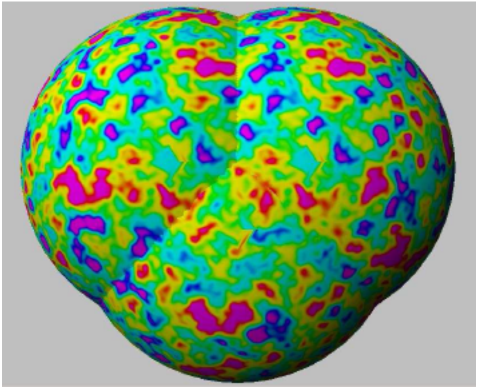
Simulated CMB sky: Laplace spectrum on spherical space forms



(Luminet, Lehoucq, Riazuelo, Weeks, et al.)

Best spherical candidate: Poincaré homology 3-sphere (dodecahedral cosmology)

Simulated CMB sky for a flat Bieberbach G6-cosmology



(from Riazuelo, Weeks, Uzan, Lehoucq, Luminet, 2003)

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- J. Weeks, J. Gundermann, *Dodecahedral topology fails to explain quadrupole-octupole alignment*, Class. Quantum Grav. 24 (2007) 1863–1866.
- J. Weeks, R. Lehoucq, J.P. Uzan, *Detecting topology in a nearly flat spherical universe*, Class. Quant. Grav. 20 (2003) 1529–1542.

Slow-roll models of inflation in the early universe
Minkowskian Friedmann metric on $Y \times \mathbb{R}$

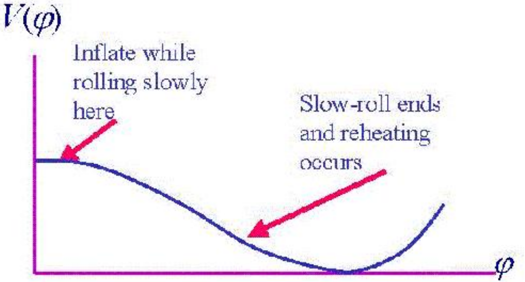
$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

accelerated expansion $\frac{\ddot{a}}{a} = H^2(1-\epsilon)$ Hubble parameter

$$H^2(\phi)\left(1 - \frac{1}{3}\epsilon(\phi)\right) = \frac{8\pi}{3m_{Pl}^2} V(\phi)$$

m_{Pl} Planck mass, inflation phase $\epsilon(\phi) < 1$

A potential $V(\phi)$ for a scalar field ϕ that runs the inflation



Slow roll parameters

$$\epsilon(\phi) = \frac{m_{Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

$$\eta(\phi) = \frac{m_{Pl}^2}{8\pi}\frac{V''(\phi)}{V(\phi)}$$

$$\xi(\phi) = \frac{m_{Pl}^4}{64\pi^2}\frac{V'(\phi)V'''(\phi)}{V^2(\phi)}$$

$\Rightarrow$ measurable quantities

$$n_s \simeq 1 - 6\epsilon + 2\eta, \quad n_t \simeq -2\epsilon, \quad r = 16\epsilon,$$

$$\alpha_s \simeq 16\epsilon\eta - 24\epsilon^2 - 2\xi, \quad \alpha_t \simeq 4\epsilon\eta - 8\epsilon^2$$

spectral index n_s, tensor-to-scalar ratio r, etc.

Slow roll parameters and the CMB

Friedmann metric (expanding universe)

$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

Separate tensor and scalar perturbation h_{ij} of metric (traceless and trace part) ⇒ Fourier modes: power spectra for scalar and tensor fluctuations, $\mathcal{P}_s(k)$ and $\mathcal{P}_t(k)$ satisfy power law

$$\mathcal{P}_s(k) \sim \mathcal{P}_s(k_0) \left(\frac{k}{k_0}\right)^{1-n_s+\frac{\alpha_s}{2}\log(k/k_0)}$$

$$\mathcal{P}_t(k) \sim \mathcal{P}_t(k_0) \left(\frac{k}{k_0}\right)^{n_t+\frac{\alpha_t}{2}\log(k/k_0)}$$

Amplitudes and exponents: constrained by observational parameters and predicted by models of *slow roll inflation* (slow roll potential)

Poisson summation formula: $h \in \mathcal{S}(\mathbb{R})$ rapidly decaying function

$$\sum_{k\in\mathbb{Z}} h(x+2\pi k) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \hat{h}(n) e^{inx}$$

function $f(x) = \sum_{k\in\mathbb{Z}} h(x+2\pi k)$ is 2π-periodic with Fourier coefficients

$$\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_0^{2\pi} h(x+2\pi k) e^{-inx} dx$$

$$= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{2\pi k}^{2\pi(k+1)} g(x) e^{-inx} dx = \frac{1}{2\pi}\int_{\mathbb{R}} h(x) e^{-inx} dx = \frac{1}{2\pi}\hat{h}(n)$$

Spectral action and Poisson summation formula

$$\sum_{n\in\mathbb{Z}} h(x+\lambda n) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} \exp\left(\frac{2\pi i n x}{\lambda}\right)\ \widehat{h}(\frac{n}{\lambda})$$

$\lambda \in \mathbb{R}_+^*$ and $x \in \mathbb{R}$ with

$$\widehat{h}(x) = \int_{\mathbb{R}} h(u)\, e^{-2\pi i u x}\, du$$

Idea: write $\mathrm{Tr}(f(D/\Lambda))$ as sums over lattices

- Need explicit spectrum of D with multiplicities
- Need to write as a union of arithmetic progressions $\lambda_{n,i}$, $n \in \mathbb{Z}$
- Multiplicities polynomial functions $m_{\lambda_{n,i}} = P_i(\lambda_{n,i})$

$$\mathrm{Tr}(f(D/\Lambda)) = \sum_i \sum_{n\in\mathbb{Z}} P_i(\lambda_{n,i}) f(\lambda_{n,i}/\Lambda)$$

The standard topology S^3 Dirac spectrum $\pm a^{-1}(\frac{1}{2}+n)$ for $n \in \mathbb{Z}$, with multiplicity $n(n+1)$

$$\mathrm{Tr}(f(D/\Lambda)) = (\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{4}(\Lambda a)\widehat{f}(0) + O((\Lambda a)^{-k})$$

with $\widehat{f}^{(2)}$ Fourier transform of $v^2 f(v)$ 4-dimensional Euclidean $S^3 \times S^1$

$$\mathrm{Tr}(h(D^2/\Lambda^2)) = \pi \Lambda^4 a^3 \beta \int_0^\infty u\, h(u)\, du - \frac{1}{2}\pi \Lambda a \beta \int_0^\infty h(u)\, du + O(\Lambda^{-k})$$

$$g(u,v) = 2P(u)\, h(u^2(\Lambda a)^{-2} + v^2(\Lambda\beta)^{-2})$$

$$\widehat{g}(n,m) = \int_{\mathbb{R}^2} g(u,v) e^{-2\pi i(xu+yv)}\, du\, dv$$

Spectral action in this case computed in

- Ali Chamseddine, Alain Connes, *The uncanny precision of the spectral action*, arXiv:0812.0165

A slow roll potential: perturbation $D^2 \mapsto D^2 + \phi^2$ gives potential $V(\phi)$ scalar field coupled to gravity

$$\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2))) = \pi\Lambda^4\beta a^3 \int_0^\infty u h(u) du - \frac{\pi}{2}\Lambda^2\beta a \int_0^\infty h(u) du$$

$$+\pi\Lambda^4\beta a^3\, \mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a\, \mathcal{W}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u)) du, \qquad \mathcal{W}(x) = \int_0^x h(u) du$$

Parameters: $a =$ radius of 3-sphere, $\beta =$ auxiliary inverse temperature parameter (choice of Euclidean S^1-compactification), $\Lambda =$ energy scale

Slow-roll parameters from spectral action: case $S = S^3$

$$\epsilon(x) = \frac{m_{Pl}^2}{16\pi}\left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}\right)^2$$

$$\eta(x) = \frac{m_{Pl}^2}{8\pi}\frac{h'(x) + 2\pi(\Lambda a)^2 h(x)}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}$$

- In Minkowskian Friedmann metric $\Lambda(t) \sim 1/a(t)$
- Also independent of β (artificial Euclidean compactification)

Slow-roll potential, cases of spherical and flat topologies:

- Matilde Marcolli, Elena Pierpaoli, Kevin Teh, *The spectral action and cosmic topology*, arXiv:1005.2256
- Matilde Marcolli, Elena Pierpaoli, Kevin Teh, *The coupling of topology and inflation in noncommutative cosmology*, arXiv:1012.0780

The quaternionic space $SU(2)/Q8$ (quaternion units $\pm 1, \pm\sigma_k$)

Dirac spectrum (Ginoux)

$$\frac{3}{2} + 4k \quad \text{with multiplicity} \quad 2(k+1)(2k+1)$$

$$\frac{3}{2} + 4k + 2 \quad \text{with multiplicity} \quad 4k(k+1)$$

Polynomial interpolation of multiplicities

$$P_1(u) = \frac{1}{4}u^2 + \frac{3}{4}u + \frac{5}{16}$$

$$P_2(u) = \frac{1}{4}u^2 - \frac{3}{4}u - \frac{7}{16}$$

Spectral action

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{8}(\Lambda a)^3 \widehat{f^{(2)}}(0) - \frac{1}{32}(\Lambda a)\widehat{f}(0) + O(\Lambda^{-k})$$

(1/8 of action for S^3) with $g_i(u) = P_i(u) f(u/\Lambda)$:

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{4}\left(\widehat{g}_1(0) + \widehat{g}_2(0)\right) + O(\Lambda^{-k})$$

from Poisson summation ⇒ Same slow-roll parameters

Other spherical space forms: method of generating functions to compute multiplicities (C. Bär)

- Spin structures on S^3/Γ: homomorphisms $\epsilon : \Gamma \to \mathrm{Spin}(4) \cong SU(2) \times SU(2)$ lifting inclusion $\Gamma \hookrightarrow SO(4)$ under double cover $\mathrm{Spin}(4) \to SO(4)$, $(A, B) \mapsto AB$
- Dirac spectrum for S^3/Γ subset of spectrum of S^3
- Multiplicities given by a generating function: ρ^+ and ρ^- two half-spin irreducible reps, $\chi^\pm$ their characters

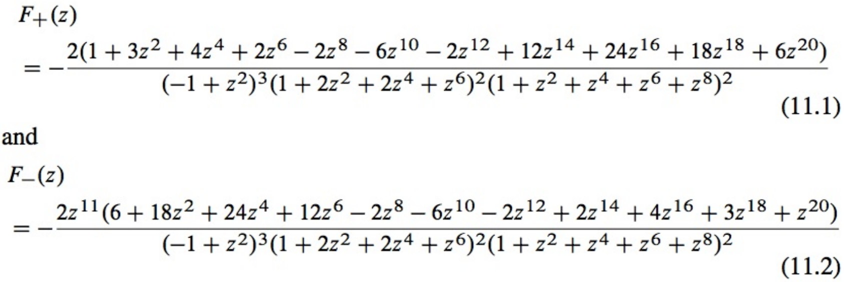
$$F_+(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^-(\epsilon(\gamma)) - z\chi^+(\epsilon(\gamma))}{\det(1 - z\gamma)}$$

$$F_-(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^+(\epsilon(\gamma)) - z\chi^-(\epsilon(\gamma))}{\det(1 - z\gamma)}$$

Then $F_+(z)$ and $F_-(z)$ generating functions of spectral multiplicities

$$F_+(z) = \sum_{k=0}^{\infty} m(\frac{3}{2} + k, D) z^k \quad F_-(z) = \sum_{k=0}^{\infty} m(-(\frac{3}{2} + k), D) z^k$$

The dodecahedral space Poincaré homology sphere S^3/Γ
binary icosahedral group 120 elements
using generating function method (Bär):



$$F_+(z) = -\frac{2(1+3z^2+4z^4+2z^6-2z^8-6z^{10}-2z^{12}+12z^{14}+24z^{16}+18z^{18}+6z^{20})}{(-1+z^2)^3(1+2z^2+2z^4+z^6)^2(1+z^2+z^4+z^6+z^8)^2} \tag{11.1}$$

and

$$F_-(z) = -\frac{2z^{11}(6+18z^2+24z^4+12z^6-2z^8-6z^{10}-2z^{12}+2z^{14}+4z^{16}+3z^{18}+z^{20})}{(-1+z^2)^3(1+2z^2+2z^4+z^6)^2(1+z^2+z^4+z^6+z^8)^2} \tag{11.2}$$

from K.Teh, *Nonperturbative spectral action of round coset spaces of $SU(2)$*, arXiv:1010.1827

Polynomial interpolation of multiplicities: 60 polynomials $P_i(u)$

$$\sum_{j=0}^{59} P_j(u) = \frac{1}{2}u^2 - \frac{1}{8}$$

Spectral action: functions $g_j(u) = P_j(u) f(u/\Lambda)$

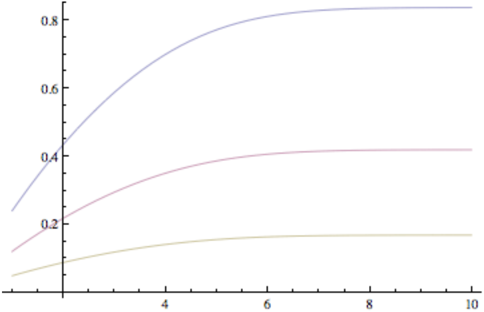
$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{60} \sum_{j=0}^{59} \widehat{g}_j(0) + O(\Lambda^{-k})$$

$$= \frac{1}{60} \int_{\mathbb{R}} \sum_j P_j(u) f(u/\Lambda) du + O(\Lambda^{-k})$$

by Poisson summation ⇒ 1/120 of action for S^3

Same slow-roll parameters

But ... different amplitudes of power spectra:
multiplicative factor of potential $V(\phi)$



$$\mathcal{P}_s(k) \sim \frac{V^3}{(V')^2}, \quad \mathcal{P}_t(k) \sim V$$

$$V \mapsto \lambda V \quad \Rightarrow \mathcal{P}_s(k_0) \mapsto \lambda \mathcal{P}_s(k_0), \quad \mathcal{P}_t(k_0) \mapsto \lambda \mathcal{P}_t(k_0)$$

$\Rightarrow$ distinguish different spherical topologies

Topological factors (spherical cases):

- Spherical forms $Y = S^3/\Gamma$, up to $O(\Lambda^{-\infty})$:

$$\mathrm{Tr}(f(D_Y/\Lambda)) = \frac{1}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) = \frac{1}{\#\Gamma}\mathrm{Tr}(f(D_{S^3}/\Lambda))$$

Y spherical	λ_Y
sphere	1
lens N	$1/N$
binary dihedral $4N$	$1/(4N)$
binary tetrahedral	$1/24$
binary octahedral	$1/48$
binary icosahedral	$1/120$

Note: λ_Y does not distinguish all of them

- Kevin Teh, *Nonperturbative Spectral Action of Round Coset Spaces of SU(2)*, arXiv:1010.1827.

The flat tori: Dirac spectrum (Bär)

$$\pm 2\pi \parallel (m,n,p) + (m_0,n_0,p_0) \parallel, \tag{1}$$

$(m,n,p) \in \mathbb{Z}^3$ multiplicity 1 and constant vector (m_0,n_0,p_0) depending on spin structure

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \sum_{(m,n,p)\in\mathbb{Z}^3} 2f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Poisson summation

$$\sum_{\mathbb{Z}^3} g(m,n,p) = \sum_{\mathbb{Z}^3} \widehat{g}(m,n,p)$$

$$\widehat{g}(m,n,p) = \int_{\mathbb{R}^3} g(u,v,w) e^{-2\pi i(mu+nv+pw)} dudvdw$$

$$g(m,n,p) = f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Spectral action for the flat tori

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2)\,du\,dv\,dw + O(\Lambda^{-k})$$

$X = T^3 \times S^1_\beta$:

$$\mathrm{Tr}(h(D_X^2/\Lambda^2)) = \frac{\Lambda^4 \beta \ell^3}{4\pi}\int_0^\infty u h(u)\,du + O(\Lambda^{-k})$$

using

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} 2\ h\left(\frac{4\pi^2}{(\Lambda\ell)^2}((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)+\frac{1}{(\Lambda\beta)^2}(r+\frac{1}{2})^2\right)$$

$$g(u,v,w,y) = 2\ h\left(\frac{4\pi^2}{\Lambda^2}(u^2+v^2+w^2)+\frac{y^2}{(\Lambda\beta)^2}\right)$$

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} g(m+m_0,n+n_0,p+p_0,r+\frac{1}{2}) = \sum_{(m,n,p,r)\in\mathbb{Z}^4} (-1)^r\,\widehat{g}(m,n,p,r)$$

Different slow-roll potential and parameters Introducing the perturbation $D^2 \mapsto D^2 + \phi^2$:

$$\mathrm{Tr}(h((D_X^2 + \phi^2)/\Lambda^2)) = \mathrm{Tr}(h(D_X^2/\Lambda^2)) + \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

slow-roll potential

$$V(\phi) = \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u\,(h(u+x) - h(u))\,du$$

Slow-roll parameters (different from spherical cases)

$$\epsilon = \frac{m_{Pl}^2}{16\pi} \left(\frac{\int_x^\infty h(u)du}{\int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta = \frac{m_{Pl}^2}{8\pi} \left(\frac{h(x)}{\int_0^\infty u(h(u+x) - h(u))du} \right)$$

Bieberbach manifolds

Quotients of T^3 by group actions: $G2$, $G3$, $G4$, $G5$, $G6$
spin structures

	δ_1	δ_2	δ_3
(a)	± 1	1	1
(b)	± 1	-1	1
(c)	± 1	1	-1
(d)	± 1	-1	-1

$G2(a)$, $G2(b)$, $G2(c)$, $G2(d)$, etc.
Dirac spectra known (Pfäffle)
Note: spectra often different for different spin structures
... but spectral action same!

Bieberbach cosmic topologies (t_i = translations by a_i)

- $G2$ = half turn space

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, and $a_3 = (T, S, 0)$, with $H, L, S \in \mathbb{R}_+^*$ and $T \in \mathbb{R}$

$$\alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}$$

- $G3$ = third turn space

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$ and $a_3 = (-\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, for H and L in $\mathbb{R}_+^*$

$$\alpha^3 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1}$$

- $G4$ = quarter turn space

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, and $a_3 = (0, L, 0)$, with $H, L > 0$

$$\alpha^4 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1}$$

- $G5$ = sixth turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$ and $a_3 = (\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, $H, L > 0$

$$\alpha^6 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3$$

- $G6$ = Hantzsche–Wendt space (π-twist along each coordinate axis)

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (0,S,0)$, with $H, L, S > 0$

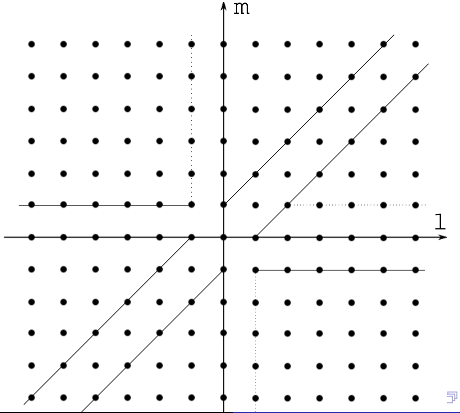
$$\begin{array}{ccc} \alpha^2 = t_1, & \alpha t_2 \alpha^{-1} = t_2^{-1}, & \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ \beta^2 = t_2, & \beta t_1 \beta^{-1} = t_1^{-1}, & \beta t_3 \beta^{-1} = t_3^{-1}, \\ \gamma^2 = t_3, & \gamma t_1 \gamma^{-1} = t_1^{-1}, & \gamma t_2 \gamma^{-1} = t_2^{-1}, \\ & \gamma\beta\alpha = t_1 t_3. & \end{array}$$

Lattice summation technique for Bieberbach manifolds:

Example G3 case: $\lambda^{\pm}_{klm}$ symmetries $R : l \mapsto -l, m \mapsto -m$, $S : l \mapsto m, m \mapsto l$, $T : l \mapsto l - m, m \mapsto -m$

$$\mathbb{Z}^3 = I \cup R(I) \cup S(I) \cup RS(I) \cup T(\tilde{I}) \cup RT(\tilde{I}) \cup \{l = m\}$$

$I = \{(k, l, m) \in \mathbb{Z}^3 : l \geq 1, m = 0, \ldots, l - 1\}$ and
$\tilde{I} = \{(k, l, m) \in \mathbb{Z}^3 : l \geq 2, m = 1, \ldots, l - 1\}$



Topological factors (flat cases):

- Bieberbach manifolds spectral action

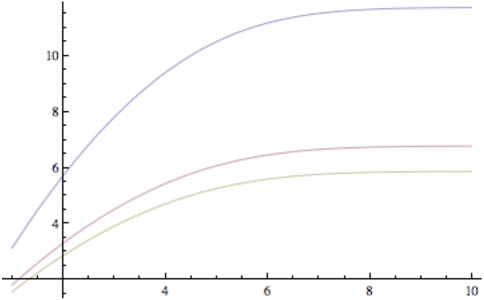
$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{\lambda_Y \Lambda^3}{4\pi^3} \int_{\mathbb{R}^3} f(u^2 + v^2 + w^2) du dv dw$$

up to oder $O(\Lambda^{-\infty})$ with factors

$$\lambda_Y = \begin{cases} \frac{HSL}{2} & G2 \\ \frac{HL^2}{2\sqrt{3}} & G3 \\ \frac{HL^2}{4} & G4 \\ \frac{HLS}{4} & G6 \end{cases}$$

Note lattice summation technique not immediately suitable for $G5$, but expect like $G3$ up to factor of 2

Topological factors and inflation slow-roll potential



⇒ Multiplicative factor in amplitude of power spectra

Adding the coupling to matter $Y \times F$
Not only product but nontrivial fibration
Vector bundle V over 3-manifold Y, fiber $\mathcal{H}_F$ (fermion content)
Dirac operator D_Y twisted with connection on V (bosons)

Spectra of twisted Dirac operators on spherical manifolds (Cisneros–Molina)

Similar computation with Poisson summation formula

$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{N}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right)$$

up to order $O(\Lambda^{-\infty})$
representation V dimension N; spherical form $Y = S^3/\Gamma$
$\Rightarrow$ topological factor $\lambda_Y \mapsto N\lambda_Y$

Variant: almost commutative geometries

$$(C^\infty(M,\mathcal{E}), L^2(M,\mathcal{E}\otimes S), \mathcal{D}_\mathcal{E})$$

- M smooth manifold, $\mathcal{E}$ algebra bundle: fiber $\mathcal{E}_x$ finite dimensional algebra $\mathcal{A}_F$
- $C^\infty(M,\mathcal{E})$ smooth sections of a algebra bundle $\mathcal{E}$
- Dirac operator $\mathcal{D}_\mathcal{E} = c\circ(\nabla^\mathcal{E}\otimes 1 + 1\otimes\nabla^S)$ with spin connection ∇^S and hermitian connection on bundle
- Compatible grading and real structure

An equivalent intrinsic (abstract) characterization in:

- Branimir Ćaćić, *A reconstruction theorem for almost-commutative spectral triples*, arXiv:1101.5908

Basic Setup

- $\Gamma \subset SU(2)$ finite group isometries of S^3

- spinor bundle on spherical form S^3/Γ given by $S^3 \times_\sigma \mathbb{C}^2 \to S^3/\Gamma$, with σ representation of Γ (standard representation of $SU(2)$ on $\mathbb{C}^2$)

- unitary representation $\alpha : \Gamma \to U(N)$ defines a flat bundle $\mathcal{V}_\alpha = S^3 \times_\alpha \mathbb{C}^N$ with a canonical flat connection

- twisting Dirac operator with flat bundle, D_α^Γ on S^3/Γ acting on twisted spinors: Γ-equivariant sections $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^\Gamma$

- Γ acts by isometries on S^3 and by $\sigma \otimes \alpha$ on $\mathbb{C}^2 \otimes \mathbb{C}^N$

- D_α^Γ restriction of the Dirac operator $D \otimes \mathrm{id}_{\mathbb{C}^N}$ to subspace

$$C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^\Gamma \subset C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)$$

Dirac spectrum (Cisneros-Molina)

- $\alpha : \Gamma \to GL_N(\mathbb{C})$ representation of Γ and Dirac operator D_α^Γ
- $\dim_{\mathbb{C}} \mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N)$, in terms of pairing $\langle \chi_{E_k}, \chi_{\sigma\otimes\alpha}\rangle_\Gamma$ of characters of corresponding Γ-representations
- eigenvalues of D_α^Γ on S^3/Γ

$$
\begin{array}{ll}
-\frac{1}{2} - (k+1) \text{ with multiplicity } \langle \chi_{E_{k+1}}, \chi_\alpha\rangle_\Gamma (k+1), & \text{if } k \geq 0, \\
-\frac{1}{2} + (k+1) \text{ with multiplicity } \langle \chi_{E_{k-1}}, \chi_\alpha\rangle_\Gamma (k+1), & \text{if } k \geq 1.
\end{array}
$$

- c_Γ least common multiple of orders of elements in Γ
- $k = c_\Gamma l + m$ with $0 \leq m < c_\Gamma$

① If $-1 \in \Gamma$, then

$$\langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma = \begin{cases} \frac{c_\Gamma l}{\#\Gamma}(\chi_\alpha(1) + \chi_\alpha(-1)) + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma & \text{if } k \text{ is even,} \\ \frac{c_\Gamma l}{\#\Gamma}(\chi_\alpha(1) - \chi_\alpha(-1)) + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma & \text{if } k \text{ is odd.} \end{cases}$$

② If $-1 \notin \Gamma$, then

$$\langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma = \frac{N c_\Gamma l}{\#\Gamma} + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma .$$

Poisson Summation Formula again to compute spectral action

Character tables

- Example: binary icosahedral group order 120

Class	1_+	1_-	30	20_+	20_-	12_{a+}	12_{b+}	12_{a-}	12_{b-}
Order	1	2	4	6	3	10	5	5	10
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	-2	0	1	-1	μ	ν	$-\mu$	$-\nu$
χ_3	2	-2	0	1	-1	$-\nu$	$-\mu$	ν	μ
χ_4	3	3	-1	0	0	$-\nu$	μ	$-\nu$	μ
χ_5	3	3	-1	0	0	μ	$-\nu$	μ	$-\nu$
χ_6	4	4	0	1	1	-1	-1	-1	-1
χ_7	4	-4	0	-1	1	1	-1	-1	1
χ_8	5	5	1	-1	-1	0	0	0	0
χ_9	6	-6	0	0	0	-1	1	1	-1

with $\mu = \frac{\sqrt{5}+1}{2}$, and $\nu = \frac{\sqrt{5}-1}{2}$

Polynomials P_m^+ and P_m^- interpolating multiplicities of positive and negative spectrum

$$\sum_{m=1}^{c_\Gamma} P_m^+(u) = \sum_{m=0}^{c_\Gamma - 1} P_m^-(u) = \frac{N c_\Gamma}{\#\Gamma}\left(u^2 - \frac{1}{4}\right).$$

Spectral Action after Poisson summation

$$\mathrm{Tr} f(D/\Lambda) = \frac{N}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty})$$

with α an N-dimensional representation and with $\widehat{f}^{(2)}$ the Fourier transform of $u^2 f(u)$

Heat Kernel argument

- $f(x) = \mathcal{L}[\phi](x^2)$ for some measurable $\phi : \mathbb{R}_+ \to \mathbb{C}$ then

$$\mathrm{Tr}\left(f(D/\Lambda)\right) = \int_0^\infty \mathrm{Tr}\left(e^{-sD^2/\Lambda^2}\right)\phi(s)ds$$

- $\mathcal{V}$ self-adjoint Clifford module bundle on a manifold M and D Dirac-type operator on $\mathcal{V}$

$$\mathrm{Tr}\left(f(D/\Lambda)\right) = \int_0^\infty \left[\int_M \mathrm{tr}\left(K(s/\Lambda^2, x, x)\right) dvol(x)\right]$$

with $K(t, x, y)$ heat kernel of D^2

- Asymptotic expansion

$$\mathrm{Tr}\left(f(D/\Lambda)\right) \sim \sum_{k=-\dim M}^{\infty} \Lambda^{-k}\phi_k \int_M a_{k+\dim M}(x, D^2) dvol(x)$$

$a_n(x, D^2)$ Seeley-DeWitt coefficients and $\phi_n = \int_0^\infty \phi(s)s^{n/2}ds$

- $\widetilde{M} \to M$ covering, $\widetilde{\mathcal{V}} \to \widetilde{M}$ be a Γ-equivariant self-adjoint Clifford module bundle with $\widetilde{D}$ a Γ-equivariant symmetric Dirac-type operator on $\widetilde{\mathcal{V}}$
- quotient $\mathcal{V} := \widetilde{\mathcal{V}}/\Gamma \to M = \widetilde{M}/\Gamma$ with $\widetilde{D}$ descending to symmetric Dirac-type operator D on $\mathcal{V}$
- then Spectral Action:

$$\mathrm{Tr}\,(f(D/\Lambda)) = \frac{1}{\#\Gamma}\mathrm{Tr}\left(f(\widetilde{D}/\Lambda)\right) + O(\Lambda^{-\infty})$$

- from heat kernel and relation between spectral action and heat kernel

$$\begin{aligned}\mathrm{Tr}\left(e^{-tD^2}\right) = & \frac{1}{\#\Gamma}\mathrm{Tr}\left(e^{-t\widetilde{D}^2}\right) \\ + & \frac{1}{\#\Gamma}\sum_{\gamma\in\Gamma\setminus\{e\}}\int_{\widetilde{M}} \mathrm{tr}\left(\rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\right) dvol(\widetilde{x}),\end{aligned}$$

- also version with $D^2 + \phi^2$ and inflation potential $V(\phi)$

Conclusion (for now)

A modified gravity model based on the spectral action can distinguish between the different cosmic topology in terms of the slow-roll parameters (distinguish spherical and flat cases) and the amplitudes of the power spectral (distinguish different spherical space forms and different Bieberbach manifolds).

Inflation potential also gets an overall multiplicative factor from the number of fermion generations in the model.

Different inflation scenarios in different topologies