Noncommutative Geometry and Arithmetic

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ICM 2010 – Mathematical Physics Section

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The geometry of imaginary quadratic fields

Elliptic curves
$E_q(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}} = \mathbb{C}^2/(\mathbb{Z} + \tau\mathbb{Z})$

Complex multiplication $\mathrm{End}(E_{\tau,\mathbb{K}}) = \mathbb{Z} + fO_{\mathbb{K}}$
$\mathbb{K} = \mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{-d})$, ring of integers $O_{\mathbb{K}}$, $f \geq 1$ integer (conductor)

Abelian extensions of imaginary quadratic fields (torsion points)

$$\mathbb{K}^{ab} = \mathbb{K}(t(E_{\tau,\mathbb{K},\mathrm{tors}}), j(E_{\tau,\mathbb{K}}))$$

$t =$ coordinate on quotient $E_\tau/\mathrm{Aut}(E_\tau) \simeq \mathbb{P}^1$
$j(E_{\tau,\mathbb{K}})$ j-invariant

The moduli space viewpoint

Elliptic curves E_τ up to isomorphism

modular curve $X_\Gamma(\mathbb{C}) = \mathbb{H}/\Gamma$, upper half plane mod $\mathrm{PSL}_2(\mathbb{Z})$

$+$ level structure: $X_G(\mathbb{C}) = \mathbb{H}/G$, finite index $G \subset \Gamma$

complex multiplication case $\tau \in \mathbb{H}$ CM points, in some $\mathbb{K} = \mathbb{Q}(\tau) = \mathbb{Q}(\sqrt{-d})$

F field of modular functions on the tower

$$Sh(\mathrm{GL}_2, \mathbb{H}^\pm) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \times \mathbb{H}^\pm$$

abelian extensions of imaginary quadratic fields:

$$\mathbb{K}^{ab} = \mathbb{K}(f(\tau),\ f \in F,\ \tau \in \text{ CM points of } X_\Gamma)$$

values of modular functions at CM points

Galois action $\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$ induced by $\mathrm{Aut}(F) = \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$

Case of $\mathbb{Q}$: Kronecker–Weber

$$\mathbb{Q}^{ab} = \mathbb{Q}(\mathbb{G}_{m,\mathrm{tors}}),$$

torsion points of multiplicative group $\mathbb{G}_m$, roots of unity, cyclotomic extensions tower

$$Sh(\mathrm{GL}_1, \pm 1) = \mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q},f}) \times \{\pm 1\}$$

Observation the multiplicative group $\mathbb{C}^* = \mathbb{G}_m(\mathbb{C})$ is a degenerate elliptic curve

$$q \to e^{2\pi i \theta}, \quad \theta \in \mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^1(\mathbb{R}) = \partial \mathbb{H}$$

Other possible degenerations of $E_q(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$ when $q \to e^{2\pi i \theta}$ with $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$???
No longer within algebraic geometry but noncommutative geometry

Quotients in NCG are replaced by crossed product algebras!

Other number fields? Real quadratic fields?
Hilbert's 12th problem (explicit class field theory)

Manin's program: Noncommutative tori and real multiplication

Goal: find a geometric analog of CM elliptic curves for real quadratic fields $\mathbb{Q}(\sqrt{d})$

Noncommutative tori $\mathcal{A}_\theta = C(S^1) \rtimes_\theta \mathbb{Z}$ irrational rotation
Two unitaries with $VU = e^{2\pi i\theta} UV$
Twisted group C^*-algebra $C^*(\mathbb{Z}^2, \sigma)$

$$\sigma_\theta((n,m),(n',m')) = \exp(-2\pi i(\xi_1 nm' + \xi_2 mn')), \qquad \theta = \xi_2 - \xi_1$$

Real multiplication when $\theta \in \mathbb{Q}(\sqrt{d})$ non-trivial self Morita equivalences of the NC torus

Geometric idea: noncommutative geometry describes bad quotients
X = nice geometric object (smooth manifold, variety, etc)
$\sim$= equivalence relation
In general quotient $Y = X/\sim$ no longer nice
Functions $C(Y) = \{f \in C(X) \mid \sim \text{invariant}\}$ too small
(for instance $C(Y) = \mathbb{C}$)
Better algebra of functions $C(\mathcal{R})$ functions on $\mathcal{R} \subset X \times X$ graph of the equivalence relation

$$f_1 \star f_2(x,y) = \sum_{x \sim z \sim y} f_1(x,z) f_2(z,y)$$

convolution product: associative, non-commutative
Algebra of function on the "noncommutative space" $Y = X/\sim$
Leaves identification explicit: groupoid (cf stacks in alg geom)

Real quadratic fields candidate generators for abelian extensions

Stark numbers: lattices $L \subset \mathbb{K} = \mathbb{Q}(\sqrt{d})$, family of L-functions

$$S_0(L, \ell_0) = \exp(\frac{d}{ds}\zeta(L, \ell_0, s)|_{s=0})$$

Prototype example: Shimizu L-function

$$L(\Lambda, s) = \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} \frac{\mathrm{sign}(N(\mu))}{|N(\mu)|^s}$$

$\Lambda = \iota(L) \subset \mathbb{R}^2$ lattice from two embeddings of $L \subset \mathbb{K}$ in $\mathbb{R}$,

$$V = \{u \in O_{\mathbb{K}}^* \,|\, uL \subset L,\ \iota(u) \in (\mathbb{R}_+^*)^2\} = \epsilon^{\mathbb{Z}}$$

units, and $N(\mu) = \mu\mu'$ norm

$\Rightarrow$ in terms of geometry of NC tori with real multiplication?

$\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$ analog of $E_{\mathbb{K}}/\mathrm{Aut}(E_{\mathbb{K}})$ for NC tori?
(hint from Atiyah–Donnelly–Singer proof of Hirzebruch conjecture)
Solvmanifold $X_\epsilon = \mathbb{R}^2 \rtimes_\epsilon \mathbb{R}/S(\Lambda, V)$

$$\pi_1(X_\epsilon) = S(\Lambda, V) = \mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z} = \Lambda \rtimes_\epsilon V$$

$T^2 \to X_\epsilon \to S^1$ fibration (mapping torus)
Commutative homotopy quotient model (Baum–Connes) of NC space $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$ given by

$$\mathcal{A}_\theta \rtimes V \cong C^*(\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}, \tilde{\sigma}_\theta)$$

$$\tilde{\sigma}_\theta((n,m,k),(n',m',k')) = \sigma_\theta((n,m),(n',m')\varphi_\epsilon^k)$$

Isospectral deformation of X_ϵ to NC space: all fiber T^2 become NC tori $\mathbb{T}_\theta$, spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (NC Riemannian manifold) fiberwise Dirac operator on RM noncommutative torus

$$D_{\theta,\theta'} = \begin{pmatrix} 0 & \delta_{\theta'} - i\delta_\theta \\ \delta_{\theta'} + i\delta_\theta & 0 \end{pmatrix}$$

$$\delta_\theta \psi_{n,m} = (n + m\theta)\psi_{n,m}, \quad \text{and} \quad \delta_{\theta'} \psi_{n,m} = (n + m\theta')\psi_{n,m}$$

Eta function $\Rightarrow$ Shimizu L-function
Wick rotation of a Lorentzian geometry (Lorentzian spectral triple)
$N(\lambda) = \lambda_1\lambda_2 = (n + m\theta)(n + m\theta')$ modes of wave operator
$\Box_\lambda = N(\lambda)$, Lorentzian Dirac operator $\mathcal{D}^2_{\mathbb{K},\lambda} = \Box_\lambda$

The noncommutative boundary of modular curves

NC tori are degenerations of elliptic curves at the irrational points $\tau \to \theta$ of the boundary $\mathbb{P}^1(\mathbb{R})$ of $\mathbb{H}$

Moduli space viewpoint:

NC space $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$ as moduli space of NC tori

(with level structure, if $G \subset \Gamma$ finite index)

holography principle: NCG on the boundary recovers AG in the bulk space, holographic image of modular forms? "modular shadows"

Bulk/boundary correspondence for modular curves

• K-theory of NC boundary $\Leftrightarrow$ Manin's modular complex $H_1(X_G)$

• modular symbols $\{x, y\}$ between cusps $\mathbb{P}^1(\mathbb{Q})/G$ extend to "limiting modular symbols" at irrational points (limiting cycles)

• Selberg zeta function of X_G as Fredholm determinant of Ruelle transfer operator on NC boundary

• Manin's identities for periods of modular forms become integral averages of "Lévy–Mellin transforms" on the NC boundary

Key: orbits of Γ on $\mathbb{P}^1(\mathbb{R}) \smallsetminus \mathbb{P}^1(\mathbb{Q}) \Leftrightarrow$ orbits of the shift of the continued fraction expansion

NC spaces of $\mathbb{Q}$-lattices

Degenerations of elliptic curves to NC tori $\Leftrightarrow$ degenerations of lattices $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ to pseudolattices $L = \mathbb{Z} + \theta\mathbb{Z}$

Adelic description of lattices $\Rightarrow$ can also degenerate at the non-archimedean components $\Leftrightarrow$ degenerations of level structures

$\mathbb{Q}$-lattices (Λ, ϕ) with $\Lambda \subset \mathbb{R}^n$ lattice and $\phi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda$ group homom

Commensurability $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$

Generalized for number fields or function fields $\mathbb{K}$ instead of $\mathbb{Q}$

Quotient by commensurability $=$ NC space

• 1-dimensional $\mathbb{Q}$-lattices

The cyclotomic tower $Sh(\mathrm{GL}_1, \pm 1) = \mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q},f}) \times \{\pm 1\}$ replaced by noncommutative

$$Sh^{nc}(\mathrm{GL}_1, \pm 1) = \mathrm{GL}_1(\mathbb{Q})\backslash \mathbb{A}_{\mathbb{Q},f} \times \{\pm 1\}$$

C^*-algebra $C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}_+^*$ Morita equivalent to $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$ (Bost–Connes algebra)

• 2-dimensional $\mathbb{Q}$-lattices

The Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm}) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \times \mathbb{H}^{\pm}$ of the modular tower replaced by noncommutative

$$Sh^{nc}(\mathrm{GL}_2, \mathbb{H}^{\pm}) = \mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A}_{\mathbb{Q},f}) \times \mathbb{P}^1(\mathbb{C})$$

a groupoid C^*-algebra (more delicate: Γ-isomorphisms)

Quantum statistical mechanics

Algebra of observables: (unital) C^*-algebra $\mathcal{A}$

Time evolution: $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$

States: $\varphi : \mathcal{A} \to \mathbb{C}$, $\varphi(a^* a) \geq 0$, $\varphi(1) = 1$, probability measures (extremal = points)

KMS Equilibrium states (Kubo-Martin-Schwinger) at inverse temperature β: $\forall a, b \in \mathcal{A}, \exists F_{a,b}(z)$

$$\varphi(a\sigma_t(b)) = F_{a,b}(t), \quad \varphi(\sigma_t(b)a) = F_{a,b}(t + i\beta)$$

$F_{a,b}$ holomorphic on horizontal strip $I_\beta = \{0 < \Im(z) < \beta\}$, bounded continuous on ∂I_β

φ_β fails to be a trace by amount controlled by interpolation by a holomorphic function

QSM systems of $\mathbb{Q}$-lattices

1-dimensional $\mathbb{Q}$-lattices up to commensurability and scaling: algebra $\mathcal{A} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$, time evolution

$$\sigma_t(f)(L, L') = (\frac{covol(L')}{covol(L)})^{it} f(L, L')$$

$\sigma_t(e(r)) = e(r)$, $\sigma_t(\mu_n) = n^{it}\mu_n$

Bost–Connes quantum statistical mechanical system

Analog for 2-dimensional $\mathbb{Q}$-lattices

Idea: Equilibrium states of a QSM at inverse temperature β are like "points" for a NC space (extremal KMS states)

Idea Low temperature equilibrium states recover classical (algebro-geometric) spaces

• 1-dimensional $\mathbb{Q}$-lattices
Low temperature extremal KMS states
$Sh(\mathrm{GL}_1, \pm 1) = \mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q},f}) \times \{\pm 1\}$, with symmetries $\hat{\mathbb{Z}}^*$; values of KMS states on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ torsion points of $\mathbb{G}_m$ (roots of unity) generators of $\mathbb{Q}^{ab}$

• 2-dimensional $\mathbb{Q}$-lattices
Low temperature extremal KMS states
$Sh(\mathrm{GL}_2, \mathbb{H}^{\pm}) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \times \mathbb{H}^{\pm}$, with symmetries $\mathrm{Aut}(F) = \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$

QSM of imaginary quadratic fields $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$

Commensurability classes of 1-dimensional $\mathbb{K}$-lattices, convolution algebra

$$(f_1 \star f_2)((\Lambda,\phi),(\Lambda',\phi')) = \sum_{(\Lambda'',\phi'')\sim(\Lambda,\phi)} f_1((\Lambda,\phi),(\Lambda'',\phi''))f_2((\Lambda'',\phi''),(\Lambda',\phi'))$$

Restriction of algebra of 2-dim $\mathbb{Q}$-lattices to 1-dim $\mathbb{K}$-lattices

Same time evolution: norms of ideals

Symmetries: $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^* \simeq \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$

(automorphisms $\hat{\mathcal{O}}^*/\mathcal{O}^*$, endomorphisms $Cl(\mathcal{O})$, class number)

Zero temperature extremal KMS states $\Rightarrow$ values of modular functions at CM points (explicit class field theory)

QSM of number fields

Ha–Paugam: generalization of 2-dim $\mathbb{Q}$-lattices to Shimura varieties, from these QSM systems of number fields by specialization, reformulation gives

$$\mathcal{A}_{\mathbb{K}} = C(G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K}}) \rtimes J^+_{\mathbb{K}},$$

$J^+_{\mathbb{K}}$ semigroup of integral ideals, $G^{ab}_{\mathbb{K}} = \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$

Time evolution by norms of nonzero ideals $\sigma_t(\mu_a) = n(a)^{it}\mu_a$

Partition function Dedekind zeta function $\zeta_{\mathbb{K}}(\beta) = \sum_a n(a)^{-\beta}$

No solution of Hilbert's 12th problem (arithmetic subalgebra to evaluate zero temperature KMS states?)

From noncommutative to anabelian geometry

How much does $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}})$ know about $\mathbb{K}$?

Neukirch–Uchida: $\mathbb{K} \simeq \mathbb{L}$ isomorphic as fields iff absolute Galois groups isomorphic as topological groups

The QSM system $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}})$ seems to involve only the abelianization $G_{\mathbb{K}}^{ab}$, but ...

Thm (Cornelissen-M.) $\mathbb{K} \simeq \mathbb{L}$ isomorphic as fields iff $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}})$ and $(\mathcal{A}_{\mathbb{L}}, \sigma_{\mathbb{L}})$ isomorphic QSM

Also equivalent to identity of all L-series with Hecke characters (induced by a homeom of idele class groups)

Where is the anabelian geometry hidden in the QSM $(\mathcal{A}_{\mathbb{K}}, \sigma_{\mathbb{K}})$?

Outline of proof start with isomorphism of QSM:

$\varphi : \mathcal{A}_{\mathbb{K}} \to \mathcal{A}_{\mathbb{L}}$ isom of C^*-algebras with $\sigma_{\mathbb{L}}\varphi = \varphi\sigma_{\mathbb{K}}$

This gives:

- Homeomorphism of space of extremal KMS$_\beta$ states
- $\zeta_{\mathbb{K}}(\beta) = \zeta_{\mathbb{L}}(\beta)$ arithmetic equivalence of fields
- Homeomorphism of $X_{\mathbb{K}}$ and $X_{\mathbb{L}}$ with $X_{\mathbb{K}} = G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K}}$
- Locally constant (in $X_{\mathbb{K}}$) isomorphism of semigroups $J^+_{\mathbb{K}}$ and $J^+_{\mathbb{L}}$
- Isomorphism of $G^{ab}_{\mathbb{K}}$ and $G^{ab}_{\mathbb{L}}$ as endomorphisms of the QSM
- Locally constant $J^+_{\mathbb{K}} \simeq J^+_{\mathbb{L}}$ is constant
- Induced isoms $\hat{\mathcal{O}}^*_{\mathbb{K}} \simeq \hat{\mathcal{O}}^*_{\mathbb{L}}$, $\mathbb{A}^*_{\mathbb{K},f} \simeq \mathbb{A}^*_{\mathbb{L},f}$, and $\mathcal{O}^\times_{\mathbb{K}} \simeq \mathcal{O}^\times_{\mathbb{L}}$

Outline of proof next step

• Isom $J_{\mathbb{K}}^{+} \simeq J_{\mathbb{L}}^{+}$ induces isom of additive groups of residue fields $(\bar{\mathbb{K}}_{\wp}, +) \simeq (\bar{\mathbb{L}}_{\varphi(\wp)}, +)$ at prime ideals (using Galois cohomology)

• Same map induces isom of multiplicative groups of integers and of additive groups of residue fields $\Rightarrow \mathbb{K}$ and $\mathbb{L}$ isomorphic as fields

Matching of L-series low temperature KMS states

$$\omega_{\beta}(f) = \frac{\chi(\rho\gamma)}{\zeta_{\mathbb{K}}(\beta)} \sum_{a \in J_{\mathbb{K},B}^{+}} \frac{\tilde{\chi}(a)}{N_{\mathbb{K}}(a)^{\beta}}$$

$f(\gamma, \rho) = \chi(\gamma\rho)$, Hecke character whose restriction to $\hat{\mathcal{O}}^{*}$ depends on set of places B, Dirichlet character $\tilde{\chi}$