NONCOMMUTATIVE GEOMETRY AND ARITHMETIC

MATILDE MARCOLLI

Abstract. This is an overview of recent results aimed at developing a geometry of noncommutative tori with real multiplication, with the purpose of providing a parallel, for real quadratic fields, of the classical theory of elliptic curves with complex multiplication for imaginary quadratic fields. This talk concentrates on two main aspects: the relation of Stark numbers to the geometry of noncommutative tori with real multiplication, and the shadows of modular forms on the noncommutative boundary of modular curves, that is, the moduli space of noncommutative tori.

1. Introduction

The last few years have seen the development of a new line of investigation, aimed at applying methods of noncommutative geometry and theoretical physics to address questions in number theory. A broad overview of some of the main directions in which this area has progressed can be found in the recent monographs [41] and [14]. In this talk I am going to focus mostly on a particular, but in my opinion especially promising, aspect of this new and rapidly growing field, which did not get sufficient attention in [14], [41]: the question of developing an appropriate geometry underlying the abelian extensions of real quadratic fields. This line of investigation was initially proposed by Manin in [27], [28], as the "real multiplication program" and it aims at developing within noncommutative geometry a parallel to the classical theory of elliptic curves with complex multiplication, and their role in the explicit construction of abelian extensions of imaginary quadratic fields, which would work for real quadratic fields. I am going to give an overview of the current state of the art in addressing this problem, by focusing on those aspects I have been more closely involved with.

There are two complementary approaches to developing a noncommutative geometry of real quadratic fields. One is based on working with noncommutative tori as substitutes for elliptic curves, focussing on those whose real parameter is a quadratic irrationality, which have non trivial self Morita equivalences, analogous to the complex multiplication phenomenon for elliptic curves. This approach requires constructing suitable functions on these spaces, which replace the coordinates of the torsion points of elliptic curves, hence the problem of finding suitable algebraic models for noncommutative tori. I will concentrate here especially on the question of how to express certain numbers, the Stark numbers, which conjecturally generate abelian extensions of real quadratic fields, in terms of the geometry of noncommutative tori.

The other complementary approach deals with a noncommutative space that parameterizes noncommutative tori up to Morita equivalence. This is sometimes referred to as the "invisible boundary" of the modular curves, since it parameterizes those degenerations of elliptic curves with level structure that are no longer expressible in algebro-geometric terms but that continue to exist as noncommutative tori. A related adelic version includes degenerations of the level structure and gives rise to a quantum statistical mechanical system based on the commensurability relation of lattices with possibly degenerate level structures, whose zero temperature equilibrium states, evaluated on an algebra of arithmetic elements should conjecturally provide generators of abelian extensions. The main problem in this approach is to obtain the right algebra of functions on this invisible boundary, which should consist of holographic images, or "shadows", that modular forms on the bulk space cast upon the invisible boundary.

2. Elliptic curves and noncommutative tori

Elliptic curves are among the most widely studied objects in mathematics, whose pervasive presence in geometry, arithmetic and physics has made them a topic of nearly universal interest across mathematical disciplines. In number theory, one of the most famous manifestations of elliptic curves is through the theory of complex multiplication and the abelian class field theory problem (Hilbert 12th problem) in the case of imaginary quadratic fields.

The analytic model of an elliptic curve is the complex manifold realized as a quotient $E_\tau(\mathbb{C}) = \mathbb{C}^2/\Lambda$ with $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ or with the Jacobi uniformization $E_q(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$ with $|q| < 1$. The endomorphism ring of an elliptic curves is a copy of $\mathbb{Z}$, except in the special case of elliptic curves with complex multiplication where $\mathrm{End}(E_\tau) = \mathbb{Z} + fO_{\mathbb{K}}$, with $O_{\mathbb{K}}$ the ring of integers of an imaginary quadratic field and $f \geq 1$ an integer (the conductor).

A beautiful result in number theory relates the geometry of elliptic curves with complex multiplication to the explicit class field theory problem for imaginary quadratic fields: the explicit construction of generators of abelian extensions with the Galois action.

There are two formulations of this construction, one that works directly with the CM elliptic curves, and the coordinates of their torsion points, and one that works with the values of modular forms on the CM points in the moduli space of elliptic curves. (We refer the reader to [25], [49] for more information on this topic.)

As I will explain in the rest of the paper, both approaches have a noncommutative geometry analog in the case of real quadratic fields, which is in the process of being developed into a tool suitable for the investigation of the corresponding class field theory problem.

In the elliptic curve point of view, one knows that the maximal abelian extension $\mathbb{K}^{ab}$ of an imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ has explicit generators

$$\mathbb{K}^{ab} = \mathbb{K}(t(E_{\mathbb{K},\mathrm{tors}}), j(E_{\mathbb{K}})),$$

where t is a coordinate on the quotient $E_{\mathbb{K}}/\mathrm{Aut}(E_{\mathbb{K}}) \simeq \mathbb{P}^1$ and $j(E_{\mathbb{K}})$ is the j-invariant.

I will explain below, based on a result of [37], how one can obtain an analog of the quotient $E_{\mathbb{K}}/\mathrm{Aut}(E_{\mathbb{K}})$ in the noncommutative geometry context for real quadratic fields. I will also mention some current approaches aimed at identifying the correct analog of the j-invariant in that setting.

Currently, the main problem in extending this approach to real quadratic fields via noncommutative geometry lies in the fact that, while elliptic curves have, besides the analytic model as quotients, an algebraic model as algebraic curves defined by polynomial equations, their noncommutative geometry analogs, the noncommutative tori, have a good analytic model, but not yet a fully satisfactory algebraic model. I will comment more on the current state of the art on this question in §3.2 below.

The other point of view, based on the moduli space, considers all elliptic curves, parameterized by the modular curve $X_\Gamma(\mathbb{C}) = \mathbb{H}/\Gamma$, with $\mathbb{H}$ the complex upper half plane and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acting on it by fractional linear transformations. One considers then the field

F of modular functions. In this setting, the explicit class field theory result for imaginary quadratic field is stated in terms of the generators

$$\mathbb{K}^{ab} = \mathbb{K}(f(\tau),\ f \in F,\ \tau \in \text{ CM points of } X_\Gamma),$$

the values of modular functions at CM points. The Galois action of $\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$ is induced by the action of the automorphism group $\mathrm{Aut}(F)$ of the modular field.

The case of the explicit class field theory of $\mathbb{Q}$, the Kronecker–Weber theorem, can be formulated in terms of a special degenerate case of elliptic curves. When the parameter q in the elliptic curve $E_q(\mathbb{C})$ tends to a root of unity, or equivalently when the parameter $\tau \in \mathbb{H}$ tends to a rational points in the real line, the elliptic curve degenerates to a cylinder, the multiplicative group $\mathbb{C}^* = \mathbb{G}_m(\mathbb{C})$. The maximal abelian extension of $\mathbb{Q}$ is then generated by the torsion points of this degenerate elliptic curve,

$$\mathbb{Q}^{ab} = \mathbb{Q}(\mathbb{G}_{m,\mathrm{tors}}),$$

that is, by the roots of unity, the cyclotomic extensions.

The first case of number fields for which a solution to the explicit class field theory problem is not known is that of the real quadratic fields $\mathbb{K} = \mathbb{Q}(\sqrt{d})$. The approach currently being developed via noncommutative geometry is based on the idea of relating this case also to a special degenerate case of elliptic curves, the *noncommutative tori*. Manin's "Real multiplication program" [27], [28], to which I will return in the following, aims at building for noncommutative tori a parallel to the theory of complex multiplication for elliptic curves.

When the modulus q of the elliptic curve $E_q(\mathbb{C})$ tends to a point $\exp(2\pi i\theta)$ on the unit circle $S^1 \subset \mathbb{C}^*$ which is not a root of unity, or equivalently when $\tau \in \mathbb{H}$ tends to an irrational point on the real line, the elliptic curve degenerates in a much more drastic way. The action of $\mathbb{Z}$ by irrational rotations on the unit circle has dense orbits, so that the quotient, in the usual sense, does not deliver any interesting space that can be used to the purpose of doing geometry. This prevents one from considering such degenerations of elliptic curves in the usual algebro-geometric or complex-analytic world.

Noncommutative geometry, however, is explicitly designed in such a way as to treat "bad quotients" so that one can continue to make sense of ordinary geometry on them as if they were smooth objects. The main idea of how one does that is, instead of collapsing points via the equivalence relation of the quotient operation, one keeps all the identifications explicit in the groupoid describing the equivalence. More precisely, the algebra of functions on the quotient is replaced by a noncommutative algebra of functions on the graph of the equivalence relation with the associative convolution product dictated by the transitivity property of the equivalence relation,

$$(f_1 \star f_2)(x,y) = \sum_{x\sim z\sim y} f_1(x,z) f_2(z,y).$$

More precisely, in the case of the action of a discrete group G on a (compact) topological space X, the resulting algebra of (continuous) functions on the quotient is the *crossed product algebra* $C(X) \rtimes_\alpha G$, where the associative, noncommutative product is given by $(fU_g)(hU_{g'}) = f\alpha_g(h)U_{gg'}$, with $\alpha_g(h)(x) = h(g^{-1}(x))$.

In the case of the quotient of S^1 by the action of $\mathbb{Z}$ generated by $\exp(2\pi i\theta)$, an irrational rotation $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$, the quotient is therefore described by the algebra $C(S^1) \rtimes_\theta \mathbb{Z}$. This is by definition the algebra $\mathcal{A}_\theta$ of continuous functions on the noncommutative torus $\mathbb{T}_\theta$ of modulus θ.

An equivalent description of the irrational rotation algebra $\mathcal{A}_\theta$ is as the universal C^*-algebra generated by two unitaries U, V with the commutation relation $VU = e^{2\pi i\theta}UV$. It has a smooth structure given by the smooth subalgebra of series $\sum_{n,m} a_{n,m} U^n V^m$ with rapidly decaying coefficients (*cf.* [8]).

Morita equivalence is the correct notion of isomorphism for noncommutative spaces, and it can be formulated in terms of the existence of a bimodule that implements an equivalence between the categories of modules for the two algebras. The algebras $\mathcal{A}_{\theta_1}$ and $\mathcal{A}_{\theta_2}$ are Morita equivalent if and only if there exists a $g \in \mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{R}$ by fractional linear transformations, such that $\theta_1 = g\theta_2$, see [8], [48]. The bimodules realizing the Morita equivalences between noncommutative tori are obtained explicitly in [8] in terms of spaces of Schwartz functions on the line, and in [48] via a construction of projectors.

One can also describe the irrational rotation algebra of the noncommutative torus as a twisted group algebra $C^*(\mathbb{Z}^2, \sigma_\theta)$, with the cocyle

$$\sigma_\theta((n,m),(n',m')) = \exp(-2\pi i(\xi_1 nm' + \xi_2 mn')), \tag{2.1}$$

with $\theta = \xi_2 - \xi_1$. This is the norm closure of the action of the twisted group ring on $\ell^2(\mathbb{Z}^2)$ with the generators U and V are given by

$$Uf(n,m) = e^{-2\pi i\xi_2 n} f(n, m+1), \qquad Vf(n,m) = e^{-2\pi i\xi_1 m} f(n+1, m).$$

This description of the noncommutative torus is especially useful in the noncommutative geometry models of the integer quantum Hall effect, where this noncommutative space replaces the Brillouin zone in the presence of a magnetic field, see [3], [43].

3. L-functions, solvmanifolds, and noncommutative tori

I give an overview here of recent progress in understanding the geometry of a special class of noncommutative tori, which have real multiplication, realized by nontrivial self Morita equivalences. These are the quantum tori $\mathbb{T}_\theta$ with $\theta \in \mathbb{R}$ a quadratic irrationality. In particular, I will focus on a results from [37] that realizes certain L-functions associated to real quadratic fields in terms of Riemannian and Loretzian geometry on the noncommutative tori with real multiplication.

3.1. **Noncommutative tori with real multiplication.** The starting observation of Manin's "Real multiplication program" is the following. The elliptic curves with complex multiplication are the only ones that have additional nontrivial endomorphisms, by the ring of integers $O_\mathbb{K}$ of an imaginary quadratic field, and they correspond to lattices $\Lambda \subset \mathbb{C}$ that are $O_\mathbb{K}$-submodules with $\Lambda \otimes_{O_\mathbb{K}} \mathbb{K} \cong \mathbb{K}$, which corresponds to the parameter τ being a CM point of $\mathbb{H}$ for the imaginary quadratic field $\mathbb{K}$. In the same way, the noncommutative tori $\mathcal{A}_\theta$ for which the modulus θ is a real multiplication point in $\mathbb{R}$, in a real quadratic field $\mathbb{K} \subset \mathbb{R}$, have non-trivial self Morita equivalences, which play the role of the additional automorphisms of the CM elliptic curve.

3.2. **Analytic versus algebraic model.** A good part of the recent work on noncommutative tori with real multiplication was aimed at developing an algebraic model for these objects, in addition to the analytic model as quotients and crossed product algebras.

The most interesting approach to algebraic models for noncommutative tori is the one developed in [47], which is based on enriching the bimodules that give the self Morita equivalences with a "complex structure", in the sense of [16]. These are parameterized by the choice of an auxiliary elliptic curve E, or equivalently by a modulus $\tau \in \mathbb{H}$ up to $\mathrm{SL}_2(\mathbb{Z})$. By a suitable construction of morphisms, one obtains in this way a category of holomorphic vector bundles and a fully faithful functor to the derived category $D^b(E)$ of coherent sheaves on the auxiliary elliptic curve. The image is given by stable objects in the heart of a nonstandard t-structure, which depends on the parameter θ of the irrational rotation algebra $\mathcal{A}_\theta$ of the noncommutative torus. The real multiplication gives rise to autoequivalences of $D^b(E)$ preserving the t-structure.

This then makes it possible to associate to a noncommutative torus $\mathbb{T}_\theta$ with real multiplication a noncommutative algebraic variety, in the sense of [1]. These are described by graded algebras of the form

$$A_{F,O} = \bigoplus_{n\geq 0} \mathrm{Hom}(O, F^n(O)) \tag{3.1}$$

where O is an object of an additive category and F is an additive functor. In the case of the noncommutative tori of [47], the additive category is the heart of the t-structure in $D^b(E)$, the object O is $\mathcal{A}_\theta$, and F is induced by real multiplication, tensoring with the bimodule that generates the nontrivial self Morita equivalences.

The resulting ring was then related in [53] to the ring of quantum theta functions. These provide a good theory of theta functions for noncommutative tori developed in [29], [30]. As in the case of the classical theta functions, these can be constructed in terms of Heisenberg groups as a deformation of the classical case, see [29] (further elaborated upon in [46].) The relation between the quantum theta functions and the explicit construction of bimodules over noncommutative tori via projections was established in [4].

The arithmetic properties of the algebras of [47] were studied in [45], in terms of an explicit presentation of the twisted homogeneous coordinate rings (3.1) or real multiplication noncommutative tori, which involves modular forms of cusp type with level specified by an explicitly determined congruence subgroup. A field of definition for these arithmetic structures on noncommutative tori can then be specified in terms of the field of definition of the auxiliary elliptic curve. It is not yet clear whether this approach to defining algebraic models for noncommutative tori with real multiplication can be successfully employed to provide a substitute for the coordinates of torsion points of elliptic curves in the CM case.

There is, however, another approach which works directly with the analytic model of noncommutative tori and with the candidate generators for abelian extensions of real quadratic fields given by Stark numbers.

3.3. **Stark numbers and L-functions.** There is in number theory a conjectural candidate for explicit generators of abelian extensions of real quadratic fields, in the form of Stark numbers, [51]. These are obtained by considering a family of L-functions associated to lattices $L \subset \mathbb{K}$ in a real quadratic field. In the notation of [27], one considers an $\ell_0 \in O_{\mathbb{K}}$, with the property that the ideals $\mathfrak{b} = (L, \ell_0)$ and $\mathfrak{a} = (\ell_0)\mathfrak{b}^{-1}$ are coprime with $\mathfrak{f} = L\mathfrak{b}^{-1}$. Let U_L denote the set of units of $\mathbb{K}$ such that $u(\ell_0 + L) = \ell_0 + L$, and let $'$ denote the Galois conjugate, with $N(\ell) = \ell\ell'$. One then considers the function

$$\zeta(L, \ell_0, s) = \mathrm{sign}(\ell_0')\ N(\mathfrak{b})^s \sum_{\ell \in (\ell_0 + L)/U_L} \frac{\mathrm{sign}(\ell')}{|N(\ell)|^s}. \tag{3.2}$$

The associated Stark number is then

$$S_0(L, \ell_0) = \exp(\frac{d}{ds}\zeta(L, \ell_0, s)|_{s=0}). \tag{3.3}$$

Part of the "real multiplication program" of [27], [28] is the question of providing an interpretation of these numbers directly in terms of the geometry of noncommutative tori with real multiplication.

To understand how one can relate these numbers to RM noncommutative tori and to a suitable noncommutative space that plays the role of the quotient $E/\mathrm{Aut}(E)$ of a CM elliptic curve, we concentrate here on the case of a closely related L-function, the Shimizu L-function of a lattice in a real quadratic field.

The lattice $L \subset \mathbb{K}$ define a lattice $\Lambda = \iota(L) \subset \mathbb{R}^2$ via the two embeddings $L \ni \ell \mapsto (\ell, \ell')$. The group V of units of $\mathbb{K}$ satisfying

$$V = \{u \in O_{\mathbb{K}}^* \,|\, uL \subset L,\ \iota(u) \in (\mathbb{R}_+^*)^2\}$$

has generator a unit ϵ and it acts on Λ by $(x, y) \mapsto (\epsilon x, \epsilon' y)$. The Shimizu L-function is then given by

$$L(\Lambda, s) = \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} \frac{\mathrm{sign}(N(\mu))}{|N(\mu)|^s}. \tag{3.4}$$

This corresponds to the case $\ell_0 = 0$ of (3.2), with the sum avoiding the point $0 \in \Lambda$.

3.4. **Solvmanifolds and noncommutative spaces.** The hint on how the L-function (3.4) is related to RM noncommutative tori comes from a well known result of Atiyah–Donnelly–Singer [2], which proved a conjecture of Hirzebruch relating the Shimizu L-function to the signature of the Hilbert modular surfaces, through the computation of the eta invariant of a 3-dimensional solvmanifold which is the link of an isolated singularity of the Hilbert modular surface. The result of [2] is in fact more generally formulated for Hilbert modular varieties and L-functions of totally real fields, but for our purposes we concentrate on the real quadratic case only.

Although it does not look like it at first sight, and it was certainly not formulated in those terms, the result of [2] is in fact saying something very useful about the geometry of noncommutative tori with real multiplication, as I explained in [37].

A first observation is the fact that, in noncommutative geometry, one often has a way to construct a commutative model, up to homotopy, of a noncommutative space describing a bad quotient. The idea is similar to the use of homotopy quotients in topology, and is closely related to the Baum–Connes conjecture. In fact, the latter can be seen as the statement that invariants of noncommutative spaces, such as K-theory, can be computed geometrically using a commutative model as homotopy quotient.

As we recalled above, a “bad quotient” can be described by a noncommutative space with algebra of functions an associative convolution algebra, the crossed product algebra in the case of a group action. In particular, we consider the noncommutative space describing the quotient $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$ of a noncommutative torus with real multiplication by the automorphisms coming from the group V of units in the real quadratic field $\mathbb{K}$ preserving the lattice $L \subset \mathbb{K}$. The quotient of the action of the group of units V on the noncommutative torus with real multiplication is described by the crossed product algebra $\mathcal{A}_\theta \rtimes V$. This can also be described by a twisted group algebra of the form

$$\mathcal{A}_\theta \rtimes V \cong C^*(\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}, \tilde{\sigma}_\theta), \tag{3.5}$$

where, after identifying the lattice Λ with $\mathbb{Z}^2$ on a given basis, the action of the generator ϵ of V on Λ is implemented by a matrix $\varphi_\epsilon \in \mathrm{SL}_2(\mathbb{Z})$, and one correspondingly identifies the semidirect product $S(\Lambda, V) = \Lambda \rtimes_\epsilon V$ with $\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}$. The cocycle $\tilde{\sigma}$ is given by

$$\tilde{\sigma}_\theta((n, m, k), (n', m', k')) = \sigma_\theta((n, m), (n', m')\varphi_\epsilon^k). \tag{3.6}$$

This is indeed a cocycle for $S(\Lambda, V)$, for $\xi_2 = -\xi_1 = \theta/2$, since in this case (2.1) satisfies $\sigma((n, m)\gamma, (n', m')\gamma) = \sigma((n, m), (n', m'))$, for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Groups of the form $S(\Lambda, V)$ satisfy the Baum–Connes conjecture. This implies that the quotient noncommutative space $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$, with algebra of coordinates $C^*(S(\Lambda, V), \tilde{\sigma}_\theta)$, admits a good homotopy quotient model. In this case, as shown in [37], this homotopy quotient can be identified explicitly as the 3-dimensional smooth solvmanifold X_ϵ obtained as the quotient of $\mathbb{R}^2 \rtimes_\epsilon \mathbb{R}$ by the group $S(\Lambda, V)$. This is the same 3-manifold that gives the link of the singularity in the Hilbert modular surface in [2], whose eta invariant computes the signature defect.

Another way to describe this 3-dimensional solvmanifold, with its natural metric, is in terms of Hecke lifts of geodesics to the space of lattices (see [27] and [37]). For $t \in \mathbb{R}$ one considers the lattice in $\mathbb{R}^2$ of the form

$$\iota_t(L) = \{(xe^t, ye^{-t}) \,|\, (x, y) \in \Lambda\},$$

with $\iota_1(L) = \Lambda$, as above. Then one has a fibration $T^2 \to S(\Lambda, V) \to S^1$, where the base S^1 is a circle of length $\log \epsilon$, identified with the closed geodesic in $X_\Gamma(\mathbb{C})$ corresponding to the geodesic in $\mathbb{H}$ with endpoints $\theta, \theta' \in \mathbb{R}$, for $\{1, \theta\}$ a basis of the real quadratic field $\mathbb{K}$. The fiber over $t \in S^1$ is the 2-torus $T^2_t = \mathbb{R}^2/\iota_t(L)$.

The result of [2] can then be reintepreted as saying that the spectral theory of the Dirac operator on the 3-dimensional solvmanifold X_ϵ can be decomposed into a contribution coming from the underlying noncommutative space $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$, and an additional spurious part, which depends on the choice of a homotopy model for this quotient. The part coming from the underlying noncommutative torus is the one that recovers the Shimizu L-function and that is responsible for the signature defect computed in [2].

3.5. **Spectral triples.** To understand how the Dirac operator on the manifold X_ϵ can be related to a Dirac operator on the noncommutative space, one can resort to the general formalism of *spectral triples* in noncommutative geometry [9]. One encodes metric geometry on a noncommutative space by means of the data $(\mathcal{A}, \mathcal{H}, D)$ of a representation on a Hilbert space $\mathcal{H}$ of a dense subalgebra $\mathcal{A}$ of the algebra of coordinates, together with a self-adjoint (unbounded) operator D on $\mathcal{H}$ with compact resolvent, satisfying the condition that commutators $[D, a]$ with elements of the algebra are bounded operators. This plays the role of an abstract Dirac operator which provides the metric structure.

3.6. **The Shimizu L-function and noncommutative tori.** One can then relate the Dirac operator on X_ϵ to a spectral triple on the noncommutative torus with real multiplication, which recovers the Shimizu L-function, in two steps, [37]. The first makes use of the isospectral deformations of manifolds introduced in [12]. Given a smooth spin Riemannian manifold X, which admits an action of a torus T^2 by isometries, one can construct a deformation of X to a family of noncommutative spaces X_η, parameterized by a real parameter $\eta \in \mathbb{R}$, with algebras of coordinates $\mathcal{A}_{X_\eta}$, in such a way that, if $(C^\infty(X), L^2(X, S), D)$ is the original spectral triple describing the ordinary spin geometry on X, then the data $(\mathcal{A}_{X_\eta}, L^2(X, S), D)$ still give a spectral triple on X_η. In this way, one can isospectrally deform the fibration $T^2 \to X_\epsilon \to S^1$ to a noncommutative space $X_{\epsilon,\theta}$, which is a fibration $\mathbb{T}_\theta \to X_{\epsilon,\theta} \to S^1$, where $\mathbb{T}_\theta$ is the noncommutative torus with real multiplication. One then checks that, up to a unitary equivalence, the restriction of the Dirac operator to the fiber $\mathbb{T}_\theta$ gives a spectral triple on this noncommutative torus with Dirac operator of the form

$$D_{\theta,\theta'} = \begin{pmatrix} 0 & \delta_{\theta'} - i\delta_\theta \\ \delta_{\theta'} + i\delta_\theta & 0 \end{pmatrix}, \tag{3.7}$$

with $\{1, \theta\}$ the basis for the real quadratic field $\mathbb{K}$ and θ' the Galois conjugate of θ. The derivations δ_θ and $\delta_{\theta'}$ act as

$$\delta_\theta \psi_{n,m} = (n + m\theta)\psi_{n,m}, \quad \text{and} \quad \delta_\theta \psi_{n,m} = (n + m\theta')\psi_{n,m},$$

and they correspond to leafwise derivations $e^t\partial_x$ and $e^{-t}\partial_y$ on the tori T^2_t. The Dirac operator $D_{\theta,\theta'}$ decomposes into a product of an operator with spectrum $\mathrm{sign}(N(\mu))|N(\mu)|^{1/2}$, which recovers the Shimizu L-function, and a term whose spectrum only depends on the powers ϵ^k on the unit ϵ, see §7 of [37].

3.7. **Lorentzian geometry.** An important problem in the context of noncommutative geometry is extending the formalism of spectral triples from Riemannian to Lorentzian geometries. This is especially important in the particle physics and cosmology models based on spectral triples and the spectral action functionals, see [7], [44]. A proposal for Lorentzian noncommutative geometries, based on Krein spaces replacing Hilbert spaces in the indefinite signature context, was developed in [52].

Another interesting aspect of the geometry of noncommutative tori with real multiplication is the fact that the spectral triples described above admit a continuation to a Lorentzian geometry, based on considering the norm of the real quadratic field $N(\lambda) = \lambda_1\lambda_2 = (n + m\theta)(n + m\theta')$ as the analog of the wave operator in momentum space, with modes $\Box_\lambda = N(\lambda)$. The Krein involution is constructed using the Galois conjugation of the real quadratic field, and the Wick rotation to Euclidean signature of the resulting Lorentzian Dirac operator $\mathcal{D}_{\mathbb{K}}$ on $\mathbb{T}_\theta$, with $\mathcal{D}^2_{\mathbb{K},\lambda} = \Box_\lambda$, recovers the Dirac operator $D_{\theta,\theta'}$. The eta function of the Lorentzian spectral triple is a product

$$\eta_{\mathcal{D}_{\mathbb{K}}}(s) = L(\Lambda, V, s) Z(\epsilon, s),$$

of the Shimizu L-function and a function that only depends on the unit ϵ.

3.8. **Quantum field theory and noncommutative tori.** This result of [37] recalled above explains how certain number theoretic L-functions associated to real quadratic fields, such as the Shimizu L-function or, more generally, the zeta functions of (3.2) arise from the noncommutative geometry of noncommutative tori with real multiplication $\mathbb{T}_\theta$ and their quotients $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$.

One would then like to explain the meaning in terms of noncommutative geometry of numbers of the form $\exp(L'(0))$, where $L(s)$ is one of these L-functions, since this is the class of numbers that the Stark conjectures propose as conjectural generators of abelian extensions. While there is at present no completely satisfactory answer to this second question, I describe here some work in progress in which I am trying to provide such interpretation in terms of quantum field theory.

It should not come as a surprise that one would aim at realizing numbers of arithmetic significance in terms of quantum field theory. In fact, there is a broad range of results (see [40] for an overview) relating the evaluation of Feynman integrals in quantum field theory to the arithmetic geometry of motives.

Here the point of connection is the zeta function regularization method in quantum field theory. This expresses the functional integral that gives the partition function as

$$\int e^{-\langle \phi, D\phi \rangle} \mathcal{D}[\phi] \sim (\det(D))^{-1/2},$$

where the quantity $\det(D)$ here is obtained through the zeta function regularization, using the zeta function $\zeta_D(s) = \mathrm{Tr}(|D|^{-s})$ of the operator D and setting $\det(D) = \exp(-\zeta'_D(0))$.

To adapt this to the setting described above of spectral triples on a noncommutative torus with real multiplication, one can use the fact that there is a well developed method [21] for doing quantum field theory on finite projective modules, that is, for fields that are sections of "bundles over noncommutative spaces". This formalism was developed completely explicitly in [21] for the case of finite projective modules over noncommutative tori. In the case with real multiplication, one has a preferred choice of a QFT, namely the one associated to the bimodule that generates the non-trivial self Morita equivalences that give RM structure. A description of the numbers (3.3) in terms of this quantum field theory is work in progress [42].

4. The noncommutative boundary of modular curves

In the case of the imaginary quadratic fields, as we mentioned above, the other approach to constructing abelian extensions is by considering, instead of individual CM elliptic curves, the CM points on the moduli space of elliptic curves.

In terms of noncommutative tori, one can similarly consider a moduli space that parameterizes the equivalence classes under Morita equivalence. This itself is described by a noncommutative space, which corresponds to the quotient of $\mathbb{P}^1(\mathbb{R})$ by the action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ by fractional linear transformations. As a noncommutative space, this is described by the crossed product algebra $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$. This space parameterizes degenerate lattices where $\tau \in \mathbb{H}$ becomes a point $\theta \in \mathbb{R}$. One thinks of this space as the "invisible boundary of the modular curve $X_\Gamma(\mathbb{C})$. It complements the usual boundary $\mathbb{P}^1(\mathbb{Q})/\Gamma$ (the cusp point corresponding to the degeneration of the elliptic curve $E_\tau(\mathbb{C})$ to the multiplicative group $\mathbb{G}_m(\mathbb{C})$) with the irrational points $(\mathbb{R} \smallsetminus \mathbb{Q})/\Gamma$, treated as a noncommutative space. These irrational points account for the degenerations to noncommutative tori, "invisible" to the usual world of algebraic geometry but nonetheless existing as noncommutative spaces.

In this approach, the main question becomes identifying what remnants of modularity one can have on this "invisible boundary" and what replaces evaluating a modular form at a CM point in this setting.

4.1. **Modular shadow play.**

A phenomenon similar to the "holography principle" (also known as AdS/CFT correspondence) of string theory relates the noncommutative geometry of the invisible boundary of the modular curves to the algebraic geometry of the classical "bulk space" $X_\Gamma(\mathbb{C})$ (see [36]). For example, it was shown in [34] that the K-theory of the crossed product algebra $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$ recovers Manin's modular complex [31], which gives an explicit presentation of the homology of the modular curves $X_\Gamma(\mathbb{C})$.

A way of inducing on the noncommutative boundary $\mathbb{P}^1(\mathbb{R})/\Gamma$ a class of functions corresponding to modular forms on the bulk space $X_\Gamma(\mathbb{C})$ was given in [34], [35] in terms of a Lévy–Mellin transform, which can be thought of as creating a "holographic image" of a modular form on the boundary.

Consider a complex valued function f which is defined on pairs (q, q') of coprime integers $q \geq q' \geq 1$, satisfying $f(q, q') = O(q^{-\epsilon})$ for some $\epsilon > 0$. For $x \in (0, 1]$ set

$$\ell(f)(x) = \sum_{n=1}^{\infty} f(q_n(x), q_{n-1}(x)),$$

where the $q_n(x)$ are successive denominators of the continued fraction expansion of x. Lévy's lemma (see [34]) shows that one has

$$\int_0^1 \ell(f)(x)dx = \sum_{q \geq q' \geq 1; (q,q')=1} \frac{f(q, q')}{q(q + q')}.$$

This identity can be used to recast identities of modular forms in terms of integrals on the boundary $\mathbb{P}^1(\mathbb{R})$. For example, it is shown in [34] that one can use the function

$$f(q, q') = \frac{q + q'}{q^{1+t}} \{0, q'/q\}$$

with $\Re(t) > 0$ and $\{0, q'/q\}$ the classical modular symbol, together with the identity of [31],

$$\sum_{d|m}\sum_{b=1}^{d}\int_{\{0,b/d\}}\omega = (\sigma(m) - c_m)\int_0^{i\infty}\pi_\Gamma^*(\omega),$$

where $\pi_\Gamma^*(\omega)/dz$ is a cusp form for $\Gamma = \Gamma_0(p)$, with p a prime, which is an eigenvector of the Hecke operator T_m with eigenvalue c_m, with $p \not| m$, and with $\sigma(m)$ the sum of the divisors of m. One then obtains an identity of the form

$$\int_0^1 dx \sum_{n=0}^{\infty}\frac{q_{n+1}(x)+q_n(x)}{q_{n+1}(x)^{1+t}}\int_{\{0,q_n(x)/q_{n+1}(x)\}}\omega = \left(\frac{\zeta(1+t)}{\zeta(2+t)} - \frac{L_\omega^{(p)}(2+t)}{\zeta^{(p)}(2+t)^2}\right)\int_0^{i\infty}\pi^*(\omega),$$

where $L_\omega^{(p)}$ and $\zeta^{(p)}$ are the Mellin transform and zeta function with omitted p-th Euler factor. Other such examples were given in [34], [38].

This type of identities, recasting integrals of cusp forms on modular symbols in terms of integrals along the invisible boundary of a transform of the modular form producing a function on the boundary, can be formulated more generally and more abstractly as a way of obtaining "shadows" of modular forms on the boundary. In [35] one considers pseudomeasures associated to pair of rational points on the boundary with values in an abelian group, $\mu : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to W$, satisfying $\mu(x,x) = 0$, $\mu(x,y) + \mu(y,x) = 0$, and $\mu(x,y)+\mu(y,z)+\mu(z,x) = 0$. In particular the modular pseudomeasures satisfy $\mu\gamma(x,y) = \gamma\mu(x,y)$, or an analogous identity twisted by a character, where $\gamma(x,y) = (\gamma(x),\gamma(y))$ is the action of a finite index $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ by fractional linear transformations. The classical Hecke operators act on modular pseudomeasures. Pseudomeasures can be equivalently formulated in terms of currents on the tree $\mathcal{T}$ of $\mathrm{PSL}_2(\mathbb{Z})$ embedded in the hyperbolic plane $\mathbb{H}$. In terms of noncommutative spaces, they can also be described as group homomorphisms $\mu : K_0(C(\partial\mathcal{T}) \rtimes \Gamma) \to W$.

Integration along geodesics in $\mathbb{H}$ of holomorphic functions vanishing at cusps define pseudomeasures on the boundary. It is shown in [35] that one can obtain "shadows" of modular symbols on the boundary by the following procedure. Given a finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and a weight $w \in \mathbb{N}$, let $\mathcal{S}_{w+2}(\Gamma)$ be the $\mathbb{C}$-vector space of cusp forms $f(z)$ of weight $w+2$ for Γ, holomorphic on $\mathbb{H}$ and vanishing at cusps. Let $\mathcal{P}_w$ be the space of homogeneous polynomials of degree w in two variables and let W be the space of linear functionals on $\mathcal{S}_{w+2} \otimes \mathcal{P}_w$. Then

$$\mu(x,y) : f \otimes P \mapsto \int_x^y f(z)P(z,1)dz$$

defines a W-valued modular pseudomeasure, which is the shadow of the higher weight modular symbol of [50].

A general formulation is the given in [35], which encompasses the averaging techniques over successive convergents of the continued fraction expansion, used in [34] to relate Mellin transforms of weight-two cusp forms to quantities defined entirely on the noncommutative boundary of the modular curves. One considers a class of functions $\ell(f)(x) = \sum_I f(I)\chi_I(x)$ that are formal infinite linear combinations of characteristic functions of "primitive intervals" in $[0,1]$, with coefficients $f(I)$ in an abelian group. More generally, this may depend on an additional regularization parameter, $\ell(f)(x,s)$. The primitive intervals are those of the form $I = (g(\infty), g(0))$ with $g \in \mathrm{GL}_2(\mathbb{Z})$. Pseudomeasures are completely determined by their values on these intervals. The Lévy–Mellin transform is then defined in [35] as $\mathcal{LM}(s) = \int_0^{1/2}\ell(f)(x,s)dx$. The integration over $[0,1/2]$ instead of $[0,1]$ keeps symmetry into account. When applied to a pseudomeasure obtained as the shadow of a modular symbol, for an $\mathrm{SL}_2(\mathbb{Z})$-cusp form this gives back the usual Mellin transform.

The formalism of pseudomeasures was also used in [33] to describe modular symbols for Maass wave forms, based on the work of Lewis–Zagier [26]. In particular, Manin gives in [33] an interpretation of the Lévy–Mellin transform of [35] as an analog at arithmetic infinity (at the archimedan prime) of the p-adic Mellin–Mazur transform.

4.2. **Modular shadows and the Kronecker limit formula.** Modular pseudomeasures with values in a Γ-module W, with $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, give rise to 1-cocycles, by setting $\phi^\mu_x(\gamma) = \mu(\gamma x, x)$. The cocycle condition $\phi(\gamma_1\gamma_2) = \phi(\gamma_1) + \gamma_1\phi(\gamma_2)$ follows from the modularity of μ together with the relations $\mu(x, x) = 0$, $\mu(x, y)+\mu(y, x) = 0$, and $\mu(x, y)+\mu(y, z) + \mu(z, x) = 0$, see [35]. Conversely, any cocycle with $\phi(\sigma) = \phi(\tau)$, where σ and τ are the generators of order two and three of $\Gamma = \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$. In fact, a pseudomeasure is determined by the relations $(1 + \sigma)\mu(0,\infty) = 0$ and $(1 + \tau + \tau^2)\mu(0,\infty) = 0$, while a 1-cocycle is determined by the relations $(1 + \sigma)\phi(\sigma) = 0$ and $(1 + \tau + \tau^2)\phi(\tau) = 0$.

An interesting recent result [54] gives a construction of a modular pseudomeasure involved in a higher Kronecker limit formula for real quadratic fields. The pseudomeasure takes values in $C(\mathbb{P}^1(\mathbb{R}))$ with the action of Γ of weight $2k$. One considers a function

$$\psi_{2k}(x) = \mathrm{sign}(x) \sum^{*}_{p,q\geq 0} (p|x| + q)^k,$$

where the $*$ on the sum means that the sum is for $(p, q) \neq (0, 0)$ and that the terms with $p = 0$ or $q = 0$ are counted with a coefficient $1/2$. The modular pseudomeasure is given by setting $\mu(0,\infty) = \psi_{2k}$, since $\psi_{2k} = \phi(\sigma) = \phi(\tau)$ determines a 1-cocycle. For $x > 0$ the function $\psi_{2k}(x)$ is also expressed in terms of the derivatives of the functions $\mathcal{F}_{2k}$, constructed in terms of the digamma function $\Gamma'(x)/\Gamma(x)$, which give the higher Kronecker limit formula proved in [54] as

$$\zeta(\mathfrak{b}, k) = \sum_{Q\in \mathrm{Red}(\mathfrak{b})} (\mathcal{D}_{k-1}\mathcal{F}_{2k})(Q),$$

where $\zeta(\mathfrak{b}, s) = \sum_{\mathfrak{n}\in\mathfrak{b}} N(\mathfrak{n})^{-s}$ and $\mathrm{Red}(\mathfrak{b})$ is the set of reduced quadratic forms in the class $\mathfrak{b}$, by seeing narrow ideal classes as Γ-orbits on the set of integer quadratic forms. The $\mathcal{D}_{k-1}$ are differential operators of order k mapping differentiable functions of one variable to functions of two variables, given explicitly in [54].

In particular, as shown in [54], one can use this higher Kronecker limit formula to evaluate Stark numbers, as values at $k = 1$ of the zeta-functions rather than as derivatives at zero. This provides an alternative way of connecting Stark numbers to the geometry of noncommutative tori, not by working with a single noncommutative torus with real multiplication, but with their noncommutative moduli space and the modular shadows.

4.3. **Quantum modular forms.** There is at present another approach to extending modularity to the boundary, in a form that arises frequently in very different contexts, such quantum invariants of 3-manifolds. Zagier recently developed [55] a notion of *quantum modular forms*, which encompasses all these phenomena. The idea is that, instead of the usual properties of modular forms, namely a holomorphic function on $\mathbb{H}$ satisfying the modularity property

$$(f|_k\gamma)(z) := f(\frac{az + b}{cz + d})(cz + d)^{-k} = f(z),$$

one has a function f defined on $\mathbb{P}^1(\mathbb{Q})$, for which the function

$$h_\gamma(x) = f(x) - (f|_k\gamma)(x), \tag{4.1}$$

which measures the failure of modularity, extends to a continuous or even (piecewise) analytic function on $\mathbb{P}^1(\mathbb{R})$.

A more refined notion of "strong" quantum modular form prescribes that, besides having evaluations at all rational points, the function f also has a formal Taylor series expansion at all $x \in \mathbb{Q}$, and (4.1) is an identity of formal power series. Typical examples of strong quantum modular forms described in [55] have the additional property that the function $f : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{C}$ extends to a function $f : (\mathbb{C} \smallsetminus \mathbb{R}) \cup \mathbb{Q} \to \mathbb{C}$, which is analytic on $\mathbb{C} \smallsetminus \mathbb{R}$, and whose asymptotic expansion approaching a point $x \in \mathbb{Q}$ along vertical lines agrees with the formal Taylor series of f at x. Such quantum modular forms can be thought of as two analytic functions, on the upper and lower half plane, respectively, that communicate across the rational points on the boundary.

There are two observations one can make to relate this setting to noncommutative geometry. One is that, in the case of quantum modular forms, one is dealing with functions f defined on the rational points of the boundary, while the "invisible boundary" consisting of the irrational points is seen only through the associated function h_γ which measures the failure of modularity of f. Thus, the object that should be interpreted in terms of the noncommutative space $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$ is the h_γ rather than the quantum modular form f itself.

Another observation is that a similar setting, with functions that have evaluations and Taylor expansions at all rational points, is provided by the Habiro ring of "analytic functions of roots of unity" [23]. This was, in fact, also developed to deal with the same phenomenology of quantum invariants of 3-manifolds, such as the Witten–Reshetikhin–Turaev invariants, which typically have a value at each root of unity as well as a formal Taylor expansion, the Ohtsuki series. Those strong quantum modular forms that satisfy an additional integrality condition needed in the construction of the Habiro ring may be thought of as objects satisfying a partial modularity property (through the associated h_γ) among these analytic functions of roots of unity. Several significant examples of quantum modular forms given in [55] indeed define elements in the Habiro ring.

The functions in the Habiro ring were recently interpreted in [32] as providing the right class of functions to do analytic geometry over the "field with one element" $\mathbb{F}_1$. This was then reformulated in the setting of noncommutative geometry in [39] using the notion of endomotives developed in [10] (see also §4 of [14]) which is a category of noncommutative spaces combining Artin motives with semigroup actions, together with the relation between the endomotive associated to abelian extensions of $\mathbb{Q}$ and Soulé's notion of geometry over $\mathbb{F}_1$, established in [11]. The same noncommutative space and some natural multivariable generlizations are related in [39] to another notion of geometry over $\mathbb{F}_1$ developed by Borger [5] in terms of consistent lifts of Frobenius encoded in the structure of a Λ-ring.

5. Quantum statistical mechanics and number fields

The description of the boundary of modular curves in terms of the noncommutative space $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$, for Γ a finite index subgroup of the modular group, accounts for degenerations of lattices with level structures, to degenerate lattices (pseudolattices in the terminology of [27]). In the adelic description, this would correspond to degenerating lattices with level structures at the archimedean component. In fact, one can also consider degenerating the level structures at the non-archimedean components. This leads to another noncommutative space, which contains the usual modular curves, and which also contains in its compactification the invisible boundary described above.

In [13] such a noncommutative space of adelic degenerations of lattices with level structures was described as the moduli space of 2-dimensional $\mathbb{Q}$-lattices up to commensurability and up to a scaling relation. A $\mathbb{Q}$-lattice is a pair of a lattice Λ together with a group homomorphism $\phi : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}\Lambda/\Lambda$ which is a possibly degenerate level structure (it is not required to be an isomorphism). Commensurability means that $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2$ modulo $\Lambda_1 + \Lambda_2$. The scaling is by an action of $\mathbb{C}^*$. The corresponding noncommutative space is the convolution algebra of functions $f((\Lambda, \phi), (\Lambda', \phi'))$ of pairs of commensurable lattices that are of degree zero for the $\mathbb{C}^*$-action, with the convolution product

$$(f_1 \star f_2)((\Lambda, \phi), (\Lambda', \phi')) = \sum_{(\Lambda'', \phi'') \sim (\Lambda, \phi)} f_1((\Lambda, \phi), (\Lambda'', \phi''))f_2((\Lambda'', \phi''), (\Lambda', \phi')).$$

This admits a convenient parameterization in terms of coordinates (g, ρ, z) with $g \in \mathrm{GL}_2^+(\mathbb{Q})$, $\rho \in M_2(\hat{\mathbb{Z}})$, and $z \in \mathbb{H}$.

The advantage of adopting this point of view is that the resulting noncommutative space, whose algebra of coordinates I denote here by $\mathcal{A}_{GL(2),\mathbb{Q}}$, has a natural time evolution, by the covolume of lattices

$$\sigma_t(f)((\Lambda, \phi), (\Lambda', \phi')) = \left(\frac{\mathrm{covol}(\Lambda')}{\mathrm{covol}(\Lambda)}\right)^{it} f((\Lambda, \phi), (\Lambda', \phi')).$$

5.1. **Zero temperature states and modular forms.** The extremal low temperature KMS equilibrium states for the dynamical system $(\mathcal{A}_{GL(2),\mathbb{Q}}, \sigma)$ are parameterized by those $\mathbb{Q}$-lattices for which ϕ is an isomorphism (the invertible ones). Thus the set of extremal low temperature KMS states can be identified ([13], [14] §3) with the usual Shimura variety $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/\mathbb{C}^*$. This can be thought of as the set of the classical points of the noncommutative space $\mathcal{A}_{GL(2),\mathbb{Q}}$.

The adelic group $\mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ acts as symmetries of this quantum statistical mechanical system, with the subgroup $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) = \mathrm{GL}_2^+(\mathbb{Q}) \cdot \mathrm{GL}_2(\hat{\mathbb{Z}})$ acting by automorphisms, and $\mathrm{GL}_2^+(\mathbb{Q})$ by endomorphisms, and the quotient by $\mathbb{Q}^*$ eliminating the inner symmetries that act trivially on the KMS states.

The zero temperature extremal KMS states, defined in [13] as weak limits of the positive temperature ones, have the property that, when evaluated at elements of a $\mathbb{Q}$-algebra $\mathcal{M}_{GL(2),\mathbb{Q}}$ of unbounded multipliers of $\mathcal{A}_{GL(2),\mathbb{Q}}$, they give values that are evaluations of modular forms $f \in F$ at points in $\mathbb{H}$. Under the identification $\mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \cong \mathrm{Aut}(F)$, for a generic set of points $\tau \in \mathbb{H}$ the action of symmetries of the dynamical system is intertwined with the action of automorphisms of the modular field. This is very much like the $GL(1)$-case of [6], which corresponds, in the same setting, to the case of 1-dimensional $\mathbb{Q}$-lattices.

5.2. **Imaginary quadratic fields.** One can recast in this setting of quantum statistical mechanical systems the case of imaginary quadratic fields, [15]. One considers a similar convolution algebra for 1-dimensional $\mathbb{K}$-lattices, for $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ and realizes it as a subalgebra $\mathcal{A}_\mathbb{K}$ of the algebra of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices recalled above. In this case, the extremal low temperature KMS states are parameterized by the invertible $\mathbb{K}$-lattices, which are labelled by a CM point in $\mathbb{H}$ and an element in $\hat{O}_\mathbb{K}$. The evaluation of extremal zero temperature KMS states on the restriction of the algebra $\mathcal{M}_{GL(2),\mathbb{Q}}$ to $\mathcal{A}_\mathbb{K}$ then give evaluations of modular forms at CM points and the action of symmetries induces the correct action of $\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$.

5.3. **Quantum statistical mechanical systems for number fields.** The construction of [15] of quantum statistical mechanical systems $(\mathcal{A}_\mathbb{K}, \sigma)$ associated to imaginary quadratic fields, using the system for 2-dimensional $\mathbb{Q}$-lattices of [13], was generalized in [22] to a construction of a similar system for an arbitrary number field, using a generalization of the $GL(2)$-system to quantum statistical mechanical systems associated to arbitrary Shimura varieties. Rewritten in the notation of [24] these quantum statistical mechanical systems $(\mathcal{A}_{\mathbb{K}}, \sigma)$ for number fields are given by semigroup crossed product algebras of the form

$$\mathcal{A}_{\mathbb{K}} = C(G_{\mathbb{K}}^{ab} \times_{\hat{O}_{\mathbb{K}}^*} \hat{O}_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^+,$$

where $J_{\mathbb{K}}^+$ is the semigroup of integral ideals and $G_{\mathbb{K}}^{ab} = \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$. These also admit an interpretation as convolution algebras of commensurability classes of 1-dimensional $\mathbb{K}$-lattices, see [24]. The time evolution is by the norm of ideals

$$\sigma_t(f) = f, \quad \forall f \in C(G_{\mathbb{K}}^{ab} \times_{\hat{O}_{\mathbb{K}}^*} \hat{O}_{\mathbb{K}}), \qquad \text{and} \qquad \sigma_t(\mu_{\mathfrak{n}}) = N(\mathfrak{n})^{it}\, \mu_{\mathfrak{n}}, \quad \forall \mathfrak{n} \in J_{\mathbb{K}}^+.$$

An explicit presentation for the algebras $\mathcal{A}_{\mathbb{K}}$ was obtained in [20], by embedding them into larger crossed product algebras. What is still missing in this general construction is the "algebra of arithmetic elements" replacing $\mathcal{M}_{GL(2),\mathbb{Q}}$, on which to evaluate the zero temperature extremal KMS states to get candidate generators of abelian extensions. In the particular case of the real quadratic fields, such an algebra would contain the correct replacement for the modular functions on the invisible boundary of the modular curves.

5.4. **Noncommutative geometry and anabelian geometry.** The quantum statistical mechanical systems for number fields described above are explicitly designed to carry information on the abelian extensions of the field, hence they involve the abelianization of the absolute Galois group. However, it appears that these noncommutative spaces may in fact contain also the full "anabelian" geometry of number fields. This is presently being investigated in my joint work with Cornelissen [19]. The question is to what extent one can reconstruct the number field from the system $(\mathcal{A}_{\mathbb{K}}, \sigma)$. The fact that the partition function of this quantum statistical mechanical system is the Dedekind zeta function and that the evaluation of low temperature KMS states on elements in the algebra can be written in terms of Dirichlet series, shows that at least the system recovers the arithmetic equivalence class of the field. A similar results should in fact hold for function fields, where a version of these quantum statistical mechanical systems in the positive characteristic setting with partition function the Goss zeta function was developed in [17] (see [18] for the role of the Goss zeta function for arithmetic equivalence.) It is more subtle to see whether the system $(\mathcal{A}_{\mathbb{K}}, \sigma)$ recovers not only the field up to arithmetic equivalence but also up to isomorphism, [19].

References

[1] M. Artin, M. van der Bergh, *Twisted homogeneous coordinate rings*, J. Algebra 133 (1990) 249–271.

[2] M.F. Atiyah, H. Donnelly, I.M. Singer, *Eta invariants, signature defects of cusps, and values of L-functions*, Annals of Mathematics 118 (1983) 131–177.

[3] J. Bellissard, *Noncommutative geometry and quantum Hall effect.* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 1238–1246, Birkhäuser, 1995.

[4] F.P. Boca, *Projections in rotation algebras and theta functions*, Commun. Math. Phys. 202 (1999) 325–357.

[5] J. Borger, *Lambda-rings and the field with one element*, arXiv:0906.3146.

[6] J.B. Bost, A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory.* Selecta Math. (N.S.) Vol.1 (1995) N.3, 411–457.

[7] A. Chamseddine, A. Connes, M. Marcolli, *Gravity and the standard model with neutrino mixing*, Advances in Theoretical and Mathematical Physics, 11 (2007) 991–1090.

[8] A. Connes, *C^* algèbres et géométrie differentielle.* C.R. Acad. Sci. Paris, Ser. A-B , 290 (1980) 599–604.

[9] A. Connes, *Geometry from the spectral point of view.* Lett. Math. Phys. 34 (1995), no. 3, 203–238.

[10] A. Connes, C. Consani, M. Marcolli, *Noncommutative geometry and motives: the thermodynamics of endomotives*, Advances in Math. 214 (2) (2007), 761–831.

[11] A. Connes, C. Consani, M. Marcolli, *Fun with $\mathbb{F}_1$*, J. Number Theory, Vol.129 (2009), N.6, 1532–1561.

[12] A. Connes, G. Landi, *Noncommutative manifolds, the instanton algebra and isospectral deformations.* Comm. Math. Phys. 221 (2001), no. 1, 141–159.

[13] A. Connes, M. Marcolli, *From physics to number theory via noncommutative geometry, Part I: Quantum statistical mechanics of $\mathbb{Q}$-lattices*, in Frontiers in Number Theory, Physics and Geometry, I pp.269–350, Springer Verlag, 2006.

[14] A. Connes, M. Marcolli, *Noncommutative geometry, quantum fields and motives*, Colloquium Publications, Vol.55, AMS, 2008.

[15] A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication*, Selecta Math. (N.S.) Vol.11 (2005) N.3–4, 325–347.

[16] A. Connes, M. Rieffel, *Yang–Mills for noncommutative two-tori*. in "Operator algebras and mathematical physics", pp.237–266, Contemp. Math., 62, AMS, 1987.

[17] C. Consani, M. Marcolli, *Quantum statistical mechanics over function fields*, Journal of Number Theory **123** (2) (2007) 487–528.

[18] G. Cornelissen, A. Kontogeorgis, L. van der Zalm, *Arithmetic equivalence, the Goss zeta function, and a generalisation*, preprint arxiv:0906.4424, to appear in J. Number Theory, 2009.

[19] G. Cornelissen, M. Marcolli, *Quantum statistical mechanics, L-series and anabelian geometry*, in preparation.

[20] J. Cuntz, X. Li, *C^*-algebras associated with integral domains and crossed products by actions on adele spaces*, arXiv:0906.4903.

[21] V. Gayral, J.H. Jureit, T. Krajewski, R. Wulkenhaar, *Quantum field theory on projective modules.* J. Noncommut. Geom. 1 (2007), no. 4, 431–496.

[22] E. Ha, F. Paugam, *Bost-Connes-Marcolli systems for Shimura varieties. I. Denitions and formal analytic properties*, IMRP Int. Math. Res. Pap. (2005), no. 5, 237–286.

[23] K. Habiro, *Cyclotomic completions of polynomial rings*, Publ. RIMS Kyoto Univ. (2004) Vol.40, 1127–1146.

[24] M. Laca, N.S. Larsen, S. Neshveyev, *On Bost-Connes types systems for number fields*, J. Number Theory 129 (2009), no. 2, 325–338.

[25] S. Lang, *Elliptic Functions*, (Second Edition), Graduate Texts in Mathematics, Vol.112, Springer Verlag, 1987.

[26] J. Lewis, D. Zagier, *Period functions for Maass wave forms, I*, Ann. of Math. (2) 153 (2001) N.1, 191–258.

[27] Yu.I. Manin, *Real multiplication and noncommutative geometry (ein Alterstraum)*. The legacy of Niels Henrik Abel, 685–727, Springer, Berlin, 2004.

[28] Yu.I. Manin, *Von Zahlen und Figuren*, in "Géométrie au XXe siècle. Histoire et horizons", pp.24–44, Hermann, 2005.

[29] Yu.I. Manin, *Theta functions, quantum tori and Heisenberg groups*. Lett. Math. Phys. 56 (2001), no. 3, 295–320.

[30] Yu.I. Manin, *Quantized theta-functions*. in "Common trends in mathematics and quantum field theories" (Kyoto, 1990). Progr. Theoret. Phys. Suppl. No. 102 (1990), 219–228 (1991).

[31] Yu.I. Manin, *Parabolic points and zeta functions of modular curves*, in "Selected Papers", pp.268–290, World Scientific, 1996.

[32] Yu.I. Manin, *Cyclotomy and analytic geometry over F_1*, arXiv:0809.1564.

[33] Yu.I. Manin, *Remarks on modular symbols for Maass wave forms*, arXiv:0803.3270.

[34] Yu.I. Manin, M. Marcolli, *Continued fractions, modular symbols, and noncommutative geometry*, Selecta Mathematica (New Series) Vol.8 N.3 (2002) 475-520.

[35] Yu.I. Manin, M. Marcolli, *Modular shadows and the Levy-Mellin infinity-adic transform*, in "Modular forms on Schiermonnikoog", Cambridge University Press, 2008, pp. 189–238.

[36] Yu.I. Manin, M. Marcolli, *Holography principle and arithmetic of algebraic curves*, Advances in Theoretical and Mathematical Physics, Vol.5, N.3 (2001) 617–650.

[37] M. Marcolli, *Solvmanifolds and noncommutative tori with real multiplication*, Communications in Number Theory and Physics, Volume 2, No. 2 (2008) 423–479.

[38] M. Marcolli, *Limiting modular symbols and the Lyapunov spectrum*, Journal of Number Theory, Vol.98 N.2 (2003) 348-376.

[39] M. Marcolli, *Cyclotomy and endomotives*, P-Adic Numbers, Ultrametric Analysis and Applications, Vol.1 (2009) N.3, 217–263.

[40] M. Marcolli, *Feynman motives*, World Scientific, 2009.

[41] M. Marcolli, *Arithmetic noncommutative geometry*, University Lectures Series, Vol.35, AMS, 2005.

[42] M. Marcolli, *Quantum field theory on noncommutative tori with real multiplication*, in preparation.

[43] M. Marcolli, V. Mathai, *Towards the fractional quantum Hall effect, a noncommutative geometry perspective*, in "Noncommutative Geometry and Number Theory", pp.235–262. Vieweg Verlag, 2006.

[44] M. Marcolli, E. Pierpaoli, *Early Universe models from noncommutative geometry*, arXiv:0908.3683.

[45] J. Plazas, *Arithmetic structures on noncommutative tori with real multiplication*. Int. Math. Res. Not. IMRN 2008, no. 2, Art. ID rnm147, 41 pp.

[46] J. Plazas, *Heisenberg modules over real multiplication noncommutative tori and related algebraic structures*, arXiv:0712.0279.
[47] A. Polishchuk, *Noncommutative two-tori with real multiplication as noncommutative projective varieties*, J. Geom. Phys. 50 (2004), no. 1-4, 162–187.
[48] M.A. Rieffel, C^**-algebras associated to irrational rotations*, Pacific J. Math. 93 (1981) 415-429.
[49] G. Shimura, *Arithmetic theory of automorphic functions*, Princeton, 1971.
[50] V. Shokurov, *Shimura integrals of cusp forms*, Math. USSR Izvestiya, Vol.16 (1981) N.3, 603–646.
[51] H.M. Stark, *L-functions at* $s=1$*. IV. First derivatives at* $s=0$, Adv. Math. 35 (1980) 197–235.
[52] A. Strohmaier, *On noncommutative and pseudo-Riemannian geometry*. J. Geom. Phys. 56 (2006), no. 2, 175–195.
[53] M. Vlasenko, *The graded ring of quantum theta functions for noncommutative torus with real multiplication*. Int. Math. Res. Not. 2006, Art. ID 15825, 19 pp.
[54] M. Vlasenko, D. Zagier, *Higher Kronecker "limit" formulas for real quadratic fields*, preprint 2010.
[55] D. Zagier, *Quantum modular forms*, preprint 2009.

Division of Physics, Mathematics, and Astronomy, California Institute of Technology, Mail Code 253-37, 1200 E.California Blvd, Pasadena, CA 91125, USA

E-mail address: matilde@caltech.edu