Quantum statistical mechanics, L-series, Anabelian Geometry

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General philosophy:

- Zeta functions are counting devices: spectra of operators with spectral multiplicities, counting ideals with given norm, number of periodic orbits, rational points, etc.
- Zeta function does not determine object: isospectral manifolds, arithmetically equivalent number fields, isogeny
- but ... sometimes a family of zeta functions does
- Zeta functions occur as partition functions of physical systems

Number fields: finite extensions $\mathbb{K}$ of the field of rational numbers $\mathbb{Q}$.

- zeta functions: Dedekind $\zeta_{\mathbb{K}}(s)$ (for $\mathbb{Q}$ Riemann zeta)
- symmetries: $G_{\mathbb{K}} = Gal(\bar{\mathbb{K}}/\mathbb{K})$ absolute Galois group; abelianized $G_{\mathbb{K}}^{ab}$
- adeles $\mathbb{A}_{\mathbb{K}}$ and ideles $\mathbb{A}_{\mathbb{K}}^*$, Artin map $\vartheta_{\mathbb{K}} : \mathbb{A}_{\mathbb{K}}^* \to G_{\mathbb{K}}^{ab}$
- topology: analogies with 3-manifolds (arithmetic topology)

How well do we understand them?
Analogy with manifolds: are there complete invariants?

Recovering a Number Field from invariants

- Dedekind zeta function $\zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s)$ arithmetic equivalence Gaßmann examples:

$$\mathbb{K} = \mathbb{Q}(\sqrt[8]{3}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt[8]{3 \cdot 2^4})$$

not isomorphism $\mathbb{K} \neq \mathbb{L}$

- Adeles rings $\mathbb{A}_{\mathbb{K}} \cong \mathbb{A}_{\mathbb{L}}$ adelic equivalence $\Rightarrow$ arithmetic equivalence; Komatsu examples:

$$\mathbb{K} = \mathbb{Q}(\sqrt[8]{2 \cdot 9}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt[8]{2^5 \cdot 9})$$

not isomorphism $\mathbb{K} \neq \mathbb{L}$

- Abelianized Galois groups: $G_{\mathbb{K}}^{ab} \cong G_{\mathbb{L}}^{ab}$ also not isomorphism; Onabe examples:

$$\mathbb{K} = \mathbb{Q}(\sqrt{-2}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt{-3})$$

  not isomorphism $\mathbb{K} \neq \mathbb{L}$

- But ... absolute Galois groups $G_{\mathbb{K}} \cong G_{\mathbb{L}} \Rightarrow$ isomorphism $\mathbb{K} \cong \mathbb{L}$: Neukirch–Uchida theorem (Grothendieck's anabelian geometry)

Question: Can combine $\zeta_{\mathbb{K}}(s)$, $\mathbb{A}_{\mathbb{K}}$ and $G_{\mathbb{K}}^{ab}$ to something as strong as $G_{\mathbb{K}}$ that determines isomorphism class of $\mathbb{K}$?

Answer: Yes! Combine as a Quantum Statistical Mechanical system

Main Idea:

- Construct a QSM system associated to a number field
- Time evolution and equilibrium states at various temperatures
- Low temperature states are related to L-series
- Extremal equilibrium states determine the system
- System recovers the number field up to isomorphism

Purely number theoretic consequence:

An identity of all L-functions with Größencharakter gives an isomorphism of number fields

Quantum Statistical Mechanics (minimalist sketch)

- $\mathscr{A}$ unital C^*-algebra of observables
- σ_t time evolution, $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathscr{A})$
- states $\omega : \mathscr{A} \to \mathbb{C}$ continuous, normalized $\omega(1) = 1$, positive

$$\omega(a^* a) \geq 0$$

- equilibrium states $\omega(\sigma_t(a)) = \omega(a)$ all $t \in \mathbb{R}$
- representation $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, Hamiltonian H

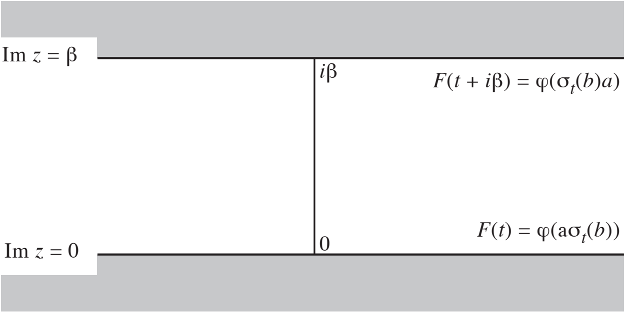
$$\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH}$$

- partition function $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$
- Gibbs states (equilibrium, inverse temperature β):

$$\omega_\beta(a) = \frac{\mathrm{Tr}(\pi(a) e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$$

- Generalization of Gibbs states: KMS states (Kubo–Martin–Schwinger) $\forall a, b \in A$, $\exists$ holomorphic $F_{a,b}$ on strip $I_\beta = \{0 < \mathrm{Im}\, z < \beta\}$, bounded continuous on ∂I_β,

$$F_{a,b}(t) = \omega(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \omega(\sigma_t(b)a)$$



- Fixed $\beta > 0$: KMS$_\beta$ state convex simplex: extremal states (like points in NCG)

Isomorphism of QSM systems: $\varphi : (\mathscr{A}, \sigma) \to (\mathscr{B}, \tau)$

$$\varphi : \mathscr{A} \stackrel{\simeq}{\to} \mathscr{B}, \quad \varphi \circ \sigma = \tau \circ \varphi$$

C^*-algebra isomorphism intertwining time evolution

- Algebraic subalgebras $\mathscr{A}^\dagger \subset \mathscr{A}$ and $\mathscr{B}^\dagger \subset \mathscr{B}$: stronger condition: QSM isomorphism also preserves "algebraic structure"
$$\varphi : \mathscr{A}^\dagger \stackrel{\simeq}{\to} \mathscr{B}^\dagger$$
- Pullback of a state: $\varphi^* \omega(a) = \omega(\varphi(a))$

Why QSM and Number theory? (a historical note)

1995: Bost–Connes QSM system $\mathscr{A}_{BC} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$

- generators $e(r)$, $r \in \mathbb{Q}/\mathbb{Z}$ and μ_n, $n \in \mathbb{N}$ and relations

$$\mu_n \mu_m = \mu_m \mu_n, \quad \mu_m^* \mu_m = 1$$

$$\mu_n \mu_m^* = \mu_m^* \mu_n \quad \text{if} \quad (n,m) = 1$$

$$e(r+s) = e(r)e(s), \quad e(0) = 1$$

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

- time evolution $\sigma_t(f) = f$ and $\sigma_t(\mu_n) = n^{it} \mu_n$

- representations $\pi_\rho : \mathscr{A}_{BC} \to \ell^2(\mathbb{N})$, $\rho \in \hat{\mathbb{Z}}^*$

$$\pi_\rho(\mu_n)\epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(r))\epsilon_m = \zeta_r^m \epsilon_m$$

  $\zeta_r = \rho(e(r))$ root of unity
- Hamiltonian $H\epsilon_m = \log(m)\,\epsilon_m$, partition function

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \zeta_{\mathbb{Q}}(\beta)$$

  Riemann zeta function
- Low temperature KMS states: L-series normalized by zeta
- Galois action on zero temperature states (class field theory)

Further generalizations: other QSM's with similar properties

- Bost-Connes as GL_1-case of QSM for moduli spaces of $\mathbb{Q}$-lattices up to commensurability (Connes-M.M. 2006) $\Rightarrow \mathrm{GL}_2$-case, modular curves and modular functions
- QSM systems for imaginary quadratic fields (class field theory): Connes-M.M.-Ramachandran
- B.Jacob and Consani-M.M.: QSM systems for function fields (Weil and Goss L-functions as partition functions)
- Ha-Paugam: QSM systems for Shimura varieties $\Rightarrow$ QSM systems for arbitrary number fields (Dedekind zeta function) further studied by Laca-Larsen-Neshveyev

We use these QSM systems for number fields

The Noncommutative Geometry viewpoint:

• Equivalence relation $\mathscr{R}$ on X: quotient $Y = X/\mathscr{R}$. Even for very good $X \Rightarrow X/\mathscr{R}$ pathological!

• Functions on the quotient $\mathscr{A}(Y) := \{f \in \mathscr{A}(X) \,|\, f\mathscr{R} - \text{invariant}\}$
$\Rightarrow$ often too few functions: $\mathscr{A}(Y) = \mathbb{C}$ only constants

• NCG: $\mathscr{A}(Y)$ noncommutative algebra $\mathscr{A}(Y) := \mathscr{A}(\Gamma_{\mathscr{R}})$ functions on the graph $\Gamma_{\mathscr{R}} \subset X \times X$ of the equivalence relation with involution $f^*(x,y) = \overline{f(y,x)}$ and convolution product

$$(f_1 * f_2)(x,y) = \sum_{x \sim u \sim y} f_1(x,u) f_2(u,y)$$

• $\mathscr{A}(\Gamma_{\mathscr{R}})$ associative noncommutative $\Rightarrow Y = X/\mathscr{R}$ *noncommutative space* (as good as X to do geometry, but new phenomena: time evolutions, thermodynamics, quantum phenomena)

In the various cases QSM system semigroup action on a space:

Bost–Connes revisited (Connes–M.M. 2006)

• $\mathbb{Q}$-lattices: (Λ, ϕ) $\mathbb{Q}$-lattice in $\mathbb{R}^n$: lattice $\Lambda \subset \mathbb{R}^n$ + group homomorphism

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

• Commensurability: $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$

• Quotient $\mathbb{Q}$-lattices/Commensurability $\Rightarrow$ NC space

• 1-dimensional $\mathbb{Q}$-lattices up to scaling $C(\hat{\mathbb{Z}})$

$$(\Lambda, \phi) = (\lambda\,\mathbb{Z}, \lambda\,\rho) \quad \lambda > 0$$

$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$

• with action of semigroup $\mathbb{N}$ commensurability

$$\alpha_n(f)(\rho) = f(n^{-1}\rho) \quad \text{or zero}$$

$C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$ Bost–Connes algebra: moduli space

QSM systems for number fields: algebra and time evolution (A, σ)

$$A_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^{+}, \quad \text{with} \quad X_{\mathbb{K}} := G_{\mathbb{K}}^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^{*}} \hat{\mathcal{O}}_{\mathbb{K}},$$

$\hat{\mathcal{O}}_{\mathbb{K}}$ = ring of finite integral adeles, $J_{\mathbb{K}}^{+}$ = is the semigroup of ideals, acting on $X_{\mathbb{K}}$ by Artin reciprocity

- Crossed product algebra $A_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^{+}$, generators and relations: $f \in C(X_{\mathbb{K}})$ and $\mu_{\mathfrak{n}}$, $\mathfrak{n} \in J_{\mathbb{K}}^{+}$

$$\mu_{\mathfrak{n}}\mu_{\mathfrak{n}}^{*} = e_{\mathfrak{n}}; \;\; \mu_{\mathfrak{n}}^{*}\mu_{\mathfrak{n}} = 1; \;\; \rho_{\mathfrak{n}}(f) = \mu_{\mathfrak{n}} f \mu_{\mathfrak{n}}^{*};$$

$$\sigma_{\mathfrak{n}}(f)e_{\mathfrak{n}} = \mu_{\mathfrak{n}}^{*} f \mu_{\mathfrak{n}}; \;\; \sigma_{\mathfrak{n}}(\rho_{\mathfrak{n}}(f)) = f; \;\; \rho_{\mathfrak{n}}(\sigma_{\mathfrak{n}}(f)) = f e_{\mathfrak{n}}$$

- Artin reciprocity map $\vartheta_{\mathbb{K}} : \mathbb{A}^*_{\mathbb{K}} \to G^{ab}_{\mathbb{K}}$, write $\vartheta_{\mathbb{K}}(\mathfrak{n})$ for ideal $\mathfrak{n}$ seen as idele by non-canonical section s of

$$\mathbb{A}^*_{\mathbb{K},f} \twoheadrightarrow J_{\mathbb{K}} \quad (\text{section } s) \qquad : \qquad (x_{\mathfrak{p}})_{\mathfrak{p}} \mapsto \prod_{\mathfrak{p} \text{ finite}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}$$

- semigroup action: $\mathfrak{n} \in J^+_{\mathbb{K}}$ acting on $f \in C(X_{\mathbb{K}})$ as

$$\rho_{\mathfrak{n}}(f)(\gamma, \rho) = f(\vartheta_{\mathbb{K}}(\mathfrak{n})\gamma, s(\mathfrak{n})^{-1}\rho) e_{\mathfrak{n}},$$

$e_{\mathfrak{n}} = \mu_{\mathfrak{n}}\mu^*_{\mathfrak{n}}$ projector onto $[(\gamma, \rho)]$ with $s(\mathfrak{n})^{-1}\rho \in \hat{\mathcal{O}}_{\mathbb{K}}$

- partial inverse of semigroup action:

$$\sigma_{\mathfrak{n}}(f)(x) = f(\mathfrak{n} * x) \quad \text{with} \quad \mathfrak{n} * [(\gamma, \rho)] = [(\vartheta_{\mathbb{K}}(\mathfrak{n})^{-1}\gamma, \mathfrak{n}\,\rho)]$$

- Time evolution $\sigma_{\mathbb{K}}$ acts on $J^+_{\mathbb{K}}$ as a phase factor $N(\mathfrak{n})^{it}$

$$\sigma_{\mathbb{K},t}(f) = f \quad \text{and} \quad \sigma_{\mathbb{K},t}(\mu_{\mathfrak{n}}) = N(\mathfrak{n})^{it}\, \mu_{\mathfrak{n}}$$

for $f \in C(G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K}})$ and for $\mathfrak{n} \in J^+_{\mathbb{K}}$

Algebraic structure: covariance algebra

Algebraic subalgebra $A^{\dagger}_{\mathbb{K}}$ of C^*-algebra $A_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J^+_{\mathbb{K}}$:

$A^{\dagger}_{\mathbb{K}}$ unital, non-involutive algebra generated by $C(X_{\mathbb{K}})$ and the $\mu_{\mathfrak{n}}$, $\mathfrak{n} \in J^+_{\mathbb{K}}$ (but not $\mu^*_{\mathfrak{n}}$), with relations

$$(\text{using } \mu^*_{\mathfrak{n}}\mu_{\mathfrak{n}} = 1) \quad f\mu_{\mathfrak{n}} = \mu_{\mathfrak{n}}\sigma_{\mathfrak{n}}(f), \quad \mu_{\mathfrak{n}} f = \rho_{\mathfrak{n}}(f)\mu_{\mathfrak{n}}$$

Comment: presence of an algebraic subalgebra also in previous examples of arithmetic QSM

Comment: similar NCG interpretation as moduli spaces of $\mathbb{K}$-lattices up to commensurability

QSM isomorphism: two number fields $\mathbb{K}$ and $\mathbb{L}$

$$\varphi : A_{\mathbb{K}} \xrightarrow{\sim} A_{\mathbb{L}}$$

C^*-algebra isomorphism

$$\varphi \circ \sigma_{\mathbb{K}} = \sigma_{\mathbb{L}} \circ \varphi$$

intertwines the time evolutions

$$\varphi : A^{\dagger}_{\mathbb{K}} \xrightarrow{\sim} A^{\dagger}_{\mathbb{L}}$$

preserves the covariance algebras

Theorem The following are equivalent:

1. $\mathbb{K} \cong \mathbb{L}$ are isomorphic number fields
2. Quantum Statistical Mechanical systems are isomorphic

$$(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \simeq (A_{\mathbb{L}}, \sigma_{\mathbb{L}})$$

   C^*-algebra isomorphism $\varphi : A_{\mathbb{K}} \to A_{\mathbb{L}}$ compatible with time evolution, $\sigma_{\mathbb{L}} \circ \varphi = \varphi \circ \sigma_{\mathbb{K}}$ and covariance $\varphi : A^{\dagger}_{\mathbb{K}} \xrightarrow{\sim} A^{\dagger}_{\mathbb{L}}$
3. There is a group isomorphism $\psi : \hat{G}^{ab}_{\mathbb{K}} \to \hat{G}^{ab}_{\mathbb{L}}$ of Pontrjagin duals of abelianized Galois groups with

$$L_{\mathbb{K}}(\chi, s) = L_{\mathbb{L}}(\psi(\chi), s)$$

   identity of all L-functions with Großencharakter

Comments:

- Generalization of arithmetic equivalence:
  $\chi = 1$ gives $\zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s)$
- Now also a purely number theoretic proof of (3) $\Rightarrow$ (1) available by Hendrik Lenstra and Bart de Smit
- L-functions $L(\chi, s)$, for $s = \beta > 1$ is product of $\zeta_{\mathbb{K}}(\beta)$ and evaluation of an extremal KMS$_\beta$ state of the QSM system $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ at a test function $f_\chi \in C(X_{\mathbb{K}})$

Scheme of proof: (2) $\Rightarrow$ (1)

- QSM isomorphism $\Rightarrow$ arithmetic equivalence $\zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s)$
- $A^{\dagger}_{\mathbb{K}} \simeq A^{\dagger}_{\mathbb{L}}$ gives homeomorphism $X_{\mathbb{K}} \simeq X_{\mathbb{L}}$ and compatible semigroup isomorphism $J^{+}_{\mathbb{K}} \simeq J^{+}_{\mathbb{L}}$
- Group isomorphism $G^{ab}_{\mathbb{K}} \simeq G^{ab}_{\mathbb{L}}$
- This preserves ramification $\Rightarrow$ isomorphism of local units $\hat{\mathcal{O}}^{*}_{\wp} \stackrel{\sim}{\to} \hat{\mathcal{O}}^{*}_{\varphi(\wp)}$ and products $\varphi : \hat{\mathcal{O}}^{*}_{\mathbb{K}} \stackrel{\sim}{\to} \hat{\mathcal{O}}^{*}_{\mathbb{L}}$
- Semigroup isomorphism $\mathbb{A}^{*}_{\mathbb{K},f} \cap \hat{\mathcal{O}}_{\mathbb{K}} \stackrel{\sim}{\to} \mathbb{A}^{*}_{\mathbb{L},f} \cap \hat{\mathcal{O}}_{\mathbb{L}}$
- Endomorphism action of these $\Rightarrow$ inner: $\mathcal{O}^{\times}_{\mathbb{K},+} \stackrel{\sim}{\to} \mathcal{O}^{\times}_{\mathbb{L},+}$ (tot pos non-zero integers)
- Recover additive structure (mod any totally split prime) $\varphi(x+y) = \varphi(x) + \varphi(y) \bmod p$

$\Rightarrow \mathcal{O}_{\mathbb{K}} \simeq \mathcal{O}_{\mathbb{L}} \Rightarrow \mathbb{K} \simeq \mathbb{L}$

Scheme of proof: (2) $\Rightarrow$ (3)

- QSM isomorphism $\Rightarrow G^{ab}_{\mathbb{K}} \simeq G^{ab}_{\mathbb{L}}$ preserving ramification (as above)
- character groups $\psi : \hat{G}^{ab}_{\mathbb{K}} \stackrel{\sim}{\to} \hat{G}^{ab}_{\mathbb{L}}$
- character χ to function $f_\chi \in C(X_{\mathbb{K}})$, matching $\varphi(f_\chi) = f_{\psi(\chi)}$
- $\chi(\vartheta_{\mathbb{K}}(\mathfrak{n})) = \psi(\chi)(\vartheta_{\mathbb{L}}(\varphi(\mathfrak{n})))$
- Matching KMS$_\beta$ states: $\omega^{\mathbb{L}}_{\gamma,\beta}(\varphi(f)) = \omega^{\mathbb{K}}_{\tilde{\gamma},\beta}(f)$
- using arithmetic equivalence: $L_{\mathbb{K}}(\chi, s) = L_{\mathbb{L}}(\psi(\chi), s)$

QSM isomorphism $\Rightarrow$ matching of L-series

Scheme of proof: (3) $\Rightarrow$ (1)

- need compatible isomorphisms $J_{\mathbb{K}}^{+} \stackrel{\sim}{\to} J_{\mathbb{L}}^{+}$ and $C(X_{\mathbb{K}}) \stackrel{\sim}{\to} C(X_{\mathbb{L}})$
- know same number of primes $\wp$ above same p with inertia degree f want to match compatibly with Artin map
- use combinations of L-series as counting functions: on finite quotients $\pi_G : G_{\mathbb{K}}^{ab} \to G$

$$\sum_{\substack{\mathfrak{n}\in J_{\mathbb{K}}^{+} \\ N_{\mathbb{K}}(\mathfrak{n})}} \left(\sum_{\widehat{G}} \chi(\pi_G(\gamma)^{-1})\chi(\vartheta_{\mathbb{K}}(\mathfrak{n})) \right) = b_{\mathbb{K},G,n}(\gamma)$$

$$b_{\mathbb{K},G,n}(\gamma) = \#\{\mathfrak{n} \in J_{\mathbb{K}}^{+} \ : \ N_{\mathbb{K}}(\mathfrak{n}) = n \text{ and } \pi_G(\vartheta_{\mathbb{K}}(\mathfrak{n})) = \pi_G(\gamma)\}$$

- For $G_{\mathbb{L},n}^{\rm ab}$ = Gal of max ab ext unram over n, get unique $\mathfrak{m} \in J_{\mathbb{L}}^{+}$ with $N_{\mathbb{L}}(\mathfrak{m}) = N_{\mathbb{K}}(\mathfrak{n})$ and

$$\pi_{G_{\mathbb{K},n}^{\rm ab}}(\vartheta_{\mathbb{L}}(\mathfrak{m})) = \pi_{G_{\mathbb{L},n}^{\rm ab}}((\psi^{-1})^{*}(\vartheta_{\mathbb{K}}(\mathfrak{n})))$$

- Use stratification of $X_{\mathbb{K}}$ to extend $\psi \ : \ C(G_{\mathbb{K}}^{ab}) \stackrel{\sim}{\to} C(G_{\mathbb{L}}^{ab})$ to $\varphi : C(X_{\mathbb{K}}) \stackrel{\sim}{\to} C(X_{\mathbb{L}})$ compatibly with semigroup actions

One more equivalent formulation: $\mathbb{K}$ and $\mathbb{L}$ isomorphic iff $\exists$

- topological group isomorphism $\hat{\psi} : G^{ab}_{\mathbb{K}} \xrightarrow{\sim} G^{ab}_{\mathbb{L}}$
- semigroup isomorphism $\Psi : J^+_{\mathbb{K}} \xrightarrow{\sim} J^+_{\mathbb{L}}$

with compatibility conditions

- Norm compatibility: $N_{\mathbb{L}}(\Psi(\mathfrak{n})) = N_{\mathbb{K}}(\mathfrak{n})$ for all $\mathfrak{n} \in J^+_{\mathbb{K}}$
- Artin map compatibility: for every finite abelian extension $\mathbb{K}' = (\mathbb{K}^{ab})^N/\mathbb{K}$, with $N \subset G^{ab}_{\mathbb{K}}$: prime $\mathfrak{p}$ of $\mathbb{K}$ unramified in $\mathbb{K}'$ $\Rightarrow$ prime $\Psi(\mathfrak{p})$ unramified in $\mathbb{L}' = (\mathbb{L}^{ab})^{\hat{\psi}(N)}/\mathbb{L}$ and

$$\hat{\psi}(\mathrm{Frob}_{\mathfrak{p}}) = \mathrm{Frob}_{\Psi(\mathfrak{p})}$$

Conclusions

- Is Quantum Statistical Mechanics a “noncommutative version" of anabelian geometry?
- What about function fields? QSM systems exist, purely NT proof seems not to work, but this QSM proof may work

General philosophy L-functions as coordinates determining underlying geometry

Examples:

- Cornelissen-M.M.: zeta functions of a spectral triple on limit set of Schottky uniformized Riemann surface determine conformal structure
- Cornelissen–J.W.de Jong: family of zeta functions of spectral triple of Riemannian manifold determine manifold up to isometry

Anabelian versus Noncommutative

- Anabelian geometry describes a number field $\mathbb{K}$ in terms of the absolute Galois group $G_{\mathbb{K}}$
- But... no description of $G_{\mathbb{K}}$ in terms of internal data of $\mathbb{K}$ only (Kronecker's hope)
- Langlands: relate to internal data via automorphic forms
- For abelian extensions yes: $G^{ab}_{\mathbb{K}}$ in terms of internal data: adeles, ideles (class field theory)
- But... $G^{ab}_{\mathbb{K}}$ does not recover $\mathbb{K}$
- Noncommutative geometry replaces $G_{\mathbb{K}}$ with the QSM system $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ to reconstruct $\mathbb{K}$
- $A_{\mathbb{K}} = C(X_{\mathbb{K}}) \rtimes J^{+}_{\mathbb{K}}$ is built only from internal data of $\mathbb{K}$ (primes, adeles, $G^{ab}_{\mathbb{K}}$)

More details on the proof of (2) $\Rightarrow$ (1): Stratification of $X_{\mathbb{K}}$

- $\hat{\mathcal{O}}_{\mathbb{K},n} := \prod_{\mathfrak{p}|n} \hat{\mathcal{O}}_{\mathbb{K},\mathfrak{p}}$ and

$$X_{\mathbb{K},n} := G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K},n} \quad \text{with} \quad X_{\mathbb{K}} = \varinjlim_n X_{\mathbb{K},n}$$

- Topological groups

$$G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}^*_{\mathbb{K},n} \simeq G^{ab}_{\mathbb{K}} / \vartheta_{\mathbb{K}}(\hat{\mathcal{O}}^*_{\mathbb{K},n}) = G^{ab}_{\mathbb{K},n}$$

  Gal of max ab ext *unramified* at primes dividing n
- $J^+_{\mathbb{K},n} \subset J^+_{\mathbb{K}}$ subsemigroup gen by prime ideals dividing n
- Decompose $X_{\mathbb{K},n} = X^1_{\mathbb{K},n} \coprod X^2_{\mathbb{K},n}$

$$X^1_{\mathbb{K},n} := \bigcup_{\mathfrak{n} \in J^+_{\mathbb{K},n}} \vartheta_{\mathbb{K}}(\mathfrak{n}) G^{ab}_{\mathbb{K},n} \quad \text{and} \quad X^2_{\mathbb{K},n} := \bigcup_{\mathfrak{p}|n} Y_{\mathbb{K},\mathfrak{p}}$$

  where $Y_{\mathbb{K},\mathfrak{p}} = \{(\gamma,\rho) \in X_{\mathbb{K},n} \,:\, \rho_{\mathfrak{p}} = 0\}$
- $X^1_{\mathbb{K},n}$ dense in $X_{\mathbb{K},n}$ and $X^2_{\mathbb{K},n}$ has $\mu_{\mathbb{K}}$-measure zero
- Algebra $C(X_{\mathbb{K},n})$ is generated by functions

$$f_{\chi,\mathfrak{n}} \,:\, \gamma \mapsto \chi(\vartheta_{\mathbb{K}}(\mathfrak{n}))\chi(\gamma), \quad \chi \in \widehat{G}^{ab}_{\mathbb{K},n}, \quad \mathfrak{n} \in J^+_{\mathbb{K},n}$$

First Step of (2) $\Rightarrow$ (1): $(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \simeq (A_{\mathbb{L}}, \sigma_{\mathbb{L}}) \Rightarrow \zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s)$

- QSM (A, σ) and representation $\pi : A \to B(\mathscr{H})$ gives Hamiltonian

$$\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH}$$

$$H_{\sigma_{\mathbb{K}}} \varepsilon_{\mathfrak{n}} = \log N(\mathfrak{n}) \ \varepsilon_{\mathfrak{n}}$$

  Partition function $\mathscr{H} = \ell^2(J_{\mathbb{K}}^+)$

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \zeta_{\mathbb{K}}(\beta)$$

- Isomorphism $\varphi : (A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \simeq (A_{\mathbb{L}}, \sigma_{\mathbb{L}}) \Rightarrow$ homeomorphism of sets of extremal KMS$_\beta$ states by pullback $\omega \mapsto \varphi^*(\omega)$

- KMS$_\beta$ states for $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ classified [LLN]: $\beta > 1$

$$\omega_{\gamma,\beta}(f) = \frac{1}{\zeta_{\mathbb{K}}(\beta)} \sum_{\mathfrak{m} \in J_{\mathbb{K}}^+} \frac{f(\vartheta_{\mathbb{K}}(\mathfrak{m})\gamma)}{N_{\mathbb{K}}(\mathfrak{m})^\beta}$$

  parameterized by $\gamma \in G_{\mathbb{K}}^{ab} / \vartheta_{\mathbb{K}}(\hat{\mathscr{O}}_{\mathbb{K}}^*)$

- Comparing GNS representations of $\omega \in \mathrm{KMS}_\beta(A_{\mathbb{L}}, \sigma_{\mathbb{L}})$ and $\varphi^*(\omega) \in \mathrm{KMS}_\beta(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ find Hamiltonians

$$H_{\mathbb{K}} = U\, H_{\mathbb{L}}\, U^* + \log \lambda$$

for some U unitary and $\lambda \in \mathbb{R}_+^*$

- Then partition functions give

$$\zeta_{\mathbb{L}}(\beta) = \lambda^{-\beta} \zeta_{\mathbb{K}}(\beta)$$

identity of Dirichlet series

$$\sum_{n\geq 1} \frac{a_n}{n^\beta} \quad \text{and} \quad \sum_{n\geq 1} \frac{b_n}{(\lambda n)^\beta}$$

with $a_1 = b_1 = 1$, taking limit as $\beta \to \infty$

$$a_1 = \lim_{\beta\to\infty} b_1 \lambda^{-\beta} \quad \Rightarrow \lambda = 1$$

Conclusion of first step: arithmetic equivalence $\zeta_{\mathbb{L}}(\beta) = \zeta_{\mathbb{K}}(\beta)$

Consequences:
From arithmetic equivalence already know $\mathbb{K}$ and $\mathbb{L}$ have same degree over $\mathbb{Q}$, discriminant, normal closure, unit groups, archimedean places.

But... not class group (or class number)

Second Step of (2) $\Rightarrow$ (1): unraveling the crossed product

$$\varphi : C(X_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^{+} \stackrel{\simeq}{\to} C(X_{\mathbb{L}}) \rtimes J_{\mathbb{L}}^{+} \quad \text{with} \quad \sigma_{\mathbb{L}} \circ \varphi = \varphi \circ \sigma_{\mathbb{K}}$$

and preserving the covariance algebra $\varphi : A_{\mathbb{K}}^{\dagger} \stackrel{\simeq}{\to} A_{\mathbb{L}}^{\dagger}$

- Restrict to finitely many isometries $\mu_{\wp}$, $N_{\mathbb{K}}(\wp) = p$
- $A_{\mathbb{K}}$ generated by $\mu_{\mathfrak{n}} f \mu_{\mathfrak{m}}^{*}$; in $A_{\mathbb{K}}^{\dagger}$ only $\mu_{\mathfrak{n}} f$
- Eigenspaces of time evolution in $A_{\mathbb{K}}^{\dagger}$ preserved: so $C(X_{\mathbb{K}}) \stackrel{\simeq}{\to} C(X_{\mathbb{L}})$ and $\varphi(\mu_{\mathfrak{n}}) = \sum \mu_{\mathfrak{m}} f_{\mathfrak{n},\mathfrak{m}}$
- Commutators $[f, \mu_{\mathfrak{n}}] = (f - \rho_{\mathfrak{n}}(f))\mu_{\mathfrak{n}}$: match maximal ideals (mod commutators) so that homeomorphism $\Phi : X_{\mathbb{K}} \stackrel{\simeq}{\to} X_{\mathbb{L}}$ compatible with semigroup actions $\gamma_{\alpha_x(\mathfrak{n})}(\Phi(x)) = \Phi(\gamma_{\mathfrak{n}}(x))$ with locally constant $\alpha_x : J_{\mathbb{K}}^{+} \to J_{\mathbb{L}}^{+}$ (that is, $\varphi(\mu_{\mathfrak{n}}) = \sum \delta_{\mathfrak{m},\alpha_x(\mathfrak{n})} \mu_{\mathfrak{n}}$)
- $\alpha_x = \alpha$ constant: know [LLN] ergodic action of $J_{\mathbb{K}}^{+}$ on $X_{\mathbb{K}}$, level sets would be clopen invariant subsets

Third Step of (2) ⇒ (1): isomorphism $G^{ab}_{\mathbb{K}} \xrightarrow{\sim} G^{ab}_{\mathbb{L}}$

- Projectors $e_{\mathbb{K},\mathfrak{n}} = \mu_{\mathfrak{n}}\mu_{\mathfrak{n}}^*$ mapped to projector $e_{\mathbb{L},\varphi(\mathfrak{n})}$
- Fix $\mathfrak{m} \in J^+_{\mathbb{K}}$ and $\hat{\mathcal{O}}_{\mathbb{K},\mathfrak{m}} = \prod_{\mathfrak{p}|\mathfrak{m}} \hat{\mathcal{O}}_{\mathbb{K},\mathfrak{p}}$, then

$$V_{\mathbb{K},\mathfrak{m}} := \bigcap_{(\mathfrak{m},\mathfrak{n})=1} \mathrm{Range}(e_{\mathbb{K},\mathfrak{n}}) = G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \{(0,\ldots,0,\hat{\mathcal{O}}_{\mathbb{K},\mathfrak{m}},0,\ldots,0)\}$$

$$\Phi(V_{\mathbb{K},\mathfrak{m}}) = \bigcap_{(\mathfrak{m},\mathfrak{n})=1} \Phi(\mathrm{Range}(e_{\mathbb{K},\mathfrak{n}})) = \bigcap_{(\varphi(\mathfrak{m}),\varphi(\mathfrak{n}))=1} \mathrm{Range}(e_{\mathbb{L},\varphi(\mathfrak{n})})$$

$$= G^{ab}_{\mathbb{L}} \times_{\hat{\mathcal{O}}^*_{\mathbb{L}}} \{(0,\ldots,0,\hat{\mathcal{O}}_{\mathbb{L},\varphi(\mathfrak{m})},0,\ldots,0)\} = V_{\mathbb{L},\varphi(\mathfrak{m})}$$

- $1_{\mathfrak{m}}$ integral adele $= 1$ at the prime divisors of $\mathfrak{m}$, zero elsewhere

$$H_{\mathbb{K},\mathfrak{m}} := G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \{1_{\mathfrak{m}}\} \subseteq X_{\mathbb{K}} \xrightarrow{\varphi} G^{ab}_{\mathbb{L}} \times_{\hat{\mathcal{O}}^*_{\mathbb{L}}} \{y_{\varphi(\mathfrak{m})}\} \subseteq X_{\mathbb{L}}$$

- check that $y \in \hat{\mathcal{O}}^*_{\mathbb{L},\mathfrak{m}}$ is a unit
- then $H_{\mathbb{K},\mathfrak{m}}$ classes $[(\gamma, 1_{\mathfrak{m}})] \sim [(\gamma', 1_{\mathfrak{m}})] \iff \exists u \in \hat{\mathcal{O}}^*_{\mathbb{K}}$ with $\gamma' = \vartheta_{\mathbb{K}}(u)^{-1}\gamma$ and $1_{\mathfrak{m}} = u 1_{\mathfrak{m}}$

- then for $\mathring{G}^{ab}_{\mathbb{K},\mathfrak{m}}$ Gal of max ab ext unram *outside* prime div of $\mathfrak{m}$

$$H_{\mathbb{K},\mathfrak{m}} \cong G^{ab}_{\mathbb{K}} / \vartheta_{\mathbb{K}} \left(\prod_{\mathfrak{q} \nmid \mathfrak{m}} \hat{\mathcal{O}}^*_{\mathfrak{q}} \right) \cong \mathring{G}^{ab}_{\mathbb{K},\mathfrak{m}}$$

- $\mathring{G}^{ab}_{\mathbb{K},\mathfrak{m}}$ has dense subgroup gen by $\vartheta_{\mathbb{K}}(\mathfrak{n})$, ideals coprime to $\mathfrak{m}$ $\Rightarrow H_{\mathbb{K},\mathfrak{m}}$ gen by these $\gamma_{\mathfrak{n}} := [(\vartheta_{\mathbb{K}}(\mathfrak{n})^{-1}, 1_{\mathfrak{m}})]$
- with $\mathbf{1}_{\mathfrak{m}} = [(1, 1_{\mathfrak{m}})]$ and $\Phi(\mathbf{1}_{\mathfrak{m}}) = [(x_{\mathfrak{m}}, y_{\mathfrak{m}})]$ get

$$\Phi(\gamma_{\mathfrak{n}_1} \cdot \gamma_{\mathfrak{n}_2}) = \Phi([(\vartheta_{\mathbb{K}}(\mathfrak{n}_1\, \mathfrak{n}_2)^{-1}, 1_{\mathfrak{m}})])$$

$$= \Phi([(\vartheta_{\mathbb{K}}(\mathfrak{n}_1\, \mathfrak{n}_2)^{-1}, \mathfrak{n}_1\, \mathfrak{n}_2\, 1_{\mathfrak{m}})]) \text{ (since } \mathfrak{n}_1, \mathfrak{n}_2 \text{ coprime to } \mathfrak{m})$$

$$= \Phi(\mathfrak{n}_1\, \mathfrak{n}_2 * \mathbf{1}_{\mathfrak{m}}) = \varphi(\mathfrak{n}_1\, \mathfrak{n}_2) * \Phi(\mathbf{1}_{\mathfrak{m}}) = [(\vartheta_{\mathbb{L}}(\varphi(\mathfrak{n}_1\, \mathfrak{n}_2))^{-1} x_{\mathfrak{m}}, \varphi(\mathfrak{n}_1\, \mathfrak{n}_2) y_{\mathfrak{m}})]$$

- $\lim_{m \to +\infty} \mathbf{1}_m = 1 \Rightarrow \lim_{m \to +\infty} \Phi(\mathbf{1}_m) = \Phi(1)$ and get

$$\widetilde{\Phi}(\gamma_1 \gamma_2) = \Phi(\gamma_1 \cdot \gamma_2) \Phi(1)^{-1} = \Phi(\gamma_1)\Phi(\gamma_2)\Phi(1)^{-2} = \widetilde{\Phi}(\gamma_1) \cdot \widetilde{\Phi}(\gamma_2)$$

Fourth step of (2): Preserving ramification

$N \subset G^{ab}_{\mathbb{K}}$ subgroup, $G^{ab}_{\mathbb{K}}/N \stackrel{\sim}{\to} G^{ab}_{\mathbb{L}}/\Phi(N)$

$$\mathfrak{p} \text{ ramifies in } \mathbb{K}'/\mathbb{K} \iff \varphi(\mathfrak{p}) \text{ ramifies in } \mathbb{L}'/\mathbb{L}$$

where $\mathbb{K}' = (\mathbb{K}^{ab})^N$ finite extension and $\mathbb{L}' := (\mathbb{L}^{ab})^{\Phi(N)}$

- seen have isomorphism $\Phi : \mathring{G}^{ab}_{\mathbb{K},\mathfrak{m}} \stackrel{\sim}{\to} \mathring{G}^{ab}_{\mathbb{L},\varphi(\mathfrak{m})}$ (Gal of max ab ext $\mathbb{K}_{\mathfrak{m}}$ unram outside prime div of $\mathfrak{m}$)
- $\mathbb{K}' = (\mathbb{K}^{ab})^N$ fin ext ramified precisely above $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in J^+_{\mathbb{K}}$
- By previous $\mathbb{L}' := (\mathbb{L})^{\Phi(N)}$ contained in $\mathbb{L}_{\varphi(\mathfrak{p}_1)\cdots\varphi(\mathfrak{p}_r)}$ but not in any $\mathbb{L}_{\varphi(\mathfrak{p}_1)\cdots\widehat{\varphi(\mathfrak{p}_i)}\cdots\varphi(\mathfrak{p}_r)} \Rightarrow \mathbb{L}'/\mathbb{L}$ ramified precisely above $\varphi(\mathfrak{p}_1), \ldots, \varphi(\mathfrak{p}_r)$

Fifth Step of (2) $\Rightarrow$ (1): from QSM isomorphism get also

- Isomorphism of local units

$$\varphi : \hat{\mathcal{O}}_{\mathfrak{p}}^* \stackrel{\sim}{\to} \hat{\mathcal{O}}_{\varphi(\mathfrak{p})}^*$$

max ab ext where $\mathfrak{p}$ unramified = fixed field of inertia group $I_{\mathfrak{p}}^{ab}$, by ramification preserving

$$\Phi(I_{\mathfrak{p}}^{ab}) = I_{\varphi(\mathfrak{p})}^{ab}$$

and by local class field theory $I_{\mathfrak{p}}^{ab} \simeq \hat{\mathcal{O}}_{\mathfrak{p}}^*$

- by product of the local units: isomorphism

$$\varphi : \hat{\mathcal{O}}_{\mathbb{K}}^* \stackrel{\sim}{\to} \hat{\mathcal{O}}_{\mathbb{L}}^*$$

- Semigroup isomorphism

$$\varphi : (\mathbb{A}_{\mathbb{K},f}^* \cap \hat{\mathcal{O}}_{\mathbb{K}}, \times) \stackrel{\sim}{\to} (\mathbb{A}_{\mathbb{L},f}^* \cap \hat{\mathcal{O}}_{\mathbb{L}}, \times)$$

by exact sequence

$$0 \to \hat{\mathcal{O}}_{\mathbb{K}}^* \to \mathbb{A}_{\mathbb{K},f}^* \cap \hat{\mathcal{O}}_{\mathbb{K}} \to J_{\mathbb{K}}^+ \to 0$$

(non-canonically) split by choice of uniformizer $\pi_{\mathfrak{p}}$ at every place

Recover multiplicative structure of the field

- Endomorphism action of $\mathbb{A}^*_{\mathbb{K},f} \cap \hat{\mathcal{O}}_{\mathbb{K}}$

$$\epsilon_s(f)(\gamma,\rho) = f(\gamma, s^{-1}\rho) e_\tau, \quad \epsilon_s(\mu_{\mathfrak{n}}) = \mu_{\mathfrak{n}}\, e_\tau$$

  e_τ char function of set $s^{-1}\rho \in \hat{\mathcal{O}}_{\mathbb{K}}$
- $\hat{\mathcal{O}}^*_{\mathbb{K}}$ = part acting by automorphisms
- $\overline{\mathcal{O}^*_{\mathbb{K},+}}$ (closure of tot pos units): trivial endomorphisms
- $\mathcal{O}^\times_{\mathbb{K},+} = \mathcal{O}_{\mathbb{K},+} - \{0\}$ (non-zero tot pos elements of ring of integers): *inner endomorphisms* (isometries in $A^\dagger_{\mathbb{K}}$ eigenv of time evolution)
- $\varphi(\varepsilon_s) = \varepsilon_{\varphi(s)}$ for all $s \in \mathbb{A}^*_{\mathbb{K},f} \cap \hat{\mathcal{O}}_{\mathbb{K}}$

Conclusion: isom of multiplicative semigroups of tot pos non-zero elements of rings of integers

$$\varphi : (\mathcal{O}^\times_{\mathbb{K},+}, \times) \xrightarrow{\sim} (\mathcal{O}^\times_{\mathbb{L},+}, \times)$$

Last Step of (2) ⇒ (1): Recover additive structure of the field

Extend by $\varphi(0) = 0$ the map $\varphi : (\mathscr{O}^{\times}_{\mathbb{K},+}, \times) \xrightarrow{\sim} (\mathscr{O}^{\times}_{\mathbb{L},+}, \times)$, Claim: it is additive

- Start with induced multipl map of local units $\varphi : \hat{\mathscr{O}}^{*}_{\mathbb{K},\mathfrak{p}} \xrightarrow{\sim} \hat{\mathscr{O}}^{*}_{\mathbb{L},\varphi(\mathfrak{p})}$ (from ramification preserving)
- set $1_{\mathfrak{p}} = (0, \dots, 0, 1, 0, \dots, 0)$ and $\mathbf{1}_{\mathfrak{p}} := [(1, 1_{\mathfrak{p}})] \in X_{\mathbb{K}}$; for $u \in \hat{\mathscr{O}}_{\mathbb{K},\mathfrak{p}}$, integral idele $u_{\mathfrak{p}} := (1, \dots, 1, u, 1, \dots, 1)$:

$$[(1, u_{\mathfrak{p}})] = [(\vartheta_{\mathbb{K}}(u_{\mathfrak{p}})^{-1}, 1)] \mapsto \Phi([(\vartheta_{\mathbb{K}}(u_{\mathfrak{p}})^{-1}), 1)]) =: [(1, \varphi(u)_{\varphi(\mathfrak{p})})]$$

- Group isom to image $\lambda_{\mathbb{K},\mathfrak{p}} : \hat{\mathscr{O}}^{*}_{\mathbb{K},\mathfrak{p}} \to X_{\mathbb{K}} \xrightarrow{[\cdot \mathbf{1}_{\mathfrak{p}}]} Z_{\mathbb{K},\mathfrak{p}} \subset X_{\mathbb{K}}$

$$u \mapsto [(1, u_{\mathfrak{p}})] \mapsto [(1, u_{\mathfrak{p}} \cdot 1_{\mathfrak{p}})] = [(1, (0, \dots, 0, u, 0, \dots, 0)]$$

- Commutative diagram

$$\begin{array}{ccc} \hat{\mathscr{O}}^{*}_{\mathbb{K},\mathfrak{p}} & \xrightarrow{\lambda_{\mathbb{K},\mathfrak{p}}} & Z_{\mathbb{K},\mathfrak{p}} \\ \downarrow \varphi & & \downarrow \Phi \\ \hat{\mathscr{O}}^{*}_{\mathbb{L},\varphi(\mathfrak{p})} & \xrightarrow[\lambda_{\mathbb{L},\varphi(\mathfrak{p})}]{} & Z_{\mathbb{L},\varphi(\mathfrak{p})} \end{array}$$

- Fix rational prime p totally split in $\mathbb{K}$ (hence unramified) $\Rightarrow$ arithm equiv: p tot split in $\mathbb{L}$
- Set $\mathbb{Z}_{(p\Delta)}$ integers coprime to $p\Delta$ with $\Delta = \Delta_{\mathbb{K}} = \Delta_{\mathbb{L}}$ discriminant
- map $\varpi_{\mathbb{K},\mathfrak{p}} : \mathbb{Z}_{(p\Delta)} \hookrightarrow \hat{\mathscr{O}}^*_{\mathbb{K},\mathfrak{p}} \to Z_{\mathbb{K},\mathfrak{p}}$ with $\varpi_{\mathbb{K},\mathfrak{p}} : a \mapsto [(1, a \cdot 1_{\mathfrak{p}})]$
- $a = \mathfrak{p}_1 \ldots \mathfrak{p}_r$ rational prime unramified $\Rightarrow$ permute factors $\alpha_x((a)) = \mathfrak{p}_{\sigma(1)} \ldots \mathfrak{p}_{\sigma(r)}$ so $\alpha_x((a)) = (a)$ fixes ideals $(a) \in J^+_{\mathbb{Q}}$

$$\Phi(\varpi_{\mathbb{K},\mathfrak{p}}(a)) = \Phi((a)*\mathbf{1}_{\mathfrak{p}}) = \alpha_{\mathbf{1}_{\mathfrak{p}}}((a))*\Phi(\mathbf{1}_{\mathfrak{p}}) = (a)*\mathbf{1}_{\varphi(\mathfrak{p})} = \varpi_{\mathbb{L},\varphi(\mathfrak{p})}(a)$$

- so $\varphi : \hat{\mathscr{O}}^*_{\mathbb{K},\mathfrak{p}} \stackrel{\sim}{\to} \hat{\mathscr{O}}^*_{\mathbb{L},\varphi(\mathfrak{p})}$ constant on $\mathbb{Z}_{(p\Delta)}$

- As above fix ational prime p totally split in $\mathbb{K}$ (hence in $\mathbb{L}$) and $\mathfrak{p} \in J^+_{\mathbb{K}}$ above p with $f(\mathfrak{p}\,|\mathbb{K}) = 1$ (hence $f(\varphi(\mathfrak{p})|\mathbb{L}) = 1$)
- Use $\varphi : \hat{\mathcal{O}}^*_{\mathbb{K},\mathfrak{p}} \xrightarrow{\sim} \hat{\mathcal{O}}^*_{\mathbb{L},\varphi(\mathfrak{p})}$ to get multiplicative map of residue fields by Teichmüller lift $\tau_{\mathbb{K},p}\colon \overline{\mathbb{K}}^*_p \cong \mathbb{F}^*_p \hookrightarrow \hat{\mathcal{O}}^*_{\mathbb{K},\mathfrak{p}} \cong \mathbb{Q}^*_p$
- Show its extension by zero additive (hence identity map $\tilde{\varphi}\colon \mathbb{F}^*_p \to \mathbb{F}^*_p$) by extending $\tau_{\mathbb{K},p}\colon \hat{\mathcal{O}}^*_{\mathbb{K},\mathfrak{p}} \to \hat{\mathcal{O}}^*_{\mathbb{K},\mathfrak{p}}$ with $x \mapsto \lim_{n\to+\infty} x^{p^n}$
- for $\tilde{a}$ residue class in $\overline{\mathbb{K}}^*_{\mathfrak{p}} \cong \mathbb{F}_p$, choose integer a congruent to $\tilde{a}$ mod $\mathfrak{p}$ and coprime to discriminant Δ (Chinese remainder thm)

$$\varphi(\tau_{\mathbb{K},p}(a)) = \varphi\left(\lim_{n\to+\infty} a^{p^n}\right) = \lim_{n\to+\infty} \varphi(a)^{p^n} = \tau_{\mathbb{L},p}(\varphi(a)) = \tau_{\mathbb{L},p}(a)$$

$$\tilde{\varphi}(\tilde{a}) = \varphi(\tau_{\mathbb{K},p}(a)) \bmod \varphi(\mathfrak{p}) = \tau_{\mathbb{L},p}(a) \bmod \varphi(\mathfrak{p}) = \tilde{a} \bmod \varphi(\mathfrak{p})$$

- So φ identity mod any tot split prime, so for any $x, y \in \mathcal{O}_{\mathbb{K},+}$

$$\varphi(x+y) = \varphi(x) + \varphi(y) \bmod \varphi(\mathfrak{p})$$

- totally split primes of arbitrary large norm (Chebotarev)

$\Rightarrow \varphi$ additive