CYCLOTOMY AND ENDOMOTIVES

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ABSTRACT. We compare two different models of noncommutative geometry of the cyclotomic tower, both based on an arithmetic algebra of functions of roots of unity and an action by endomorphisms, the first based on the Bost-Connes (BC) quantum statistical mechanical system and the second on the Habiro ring, where the Habiro functions have, in addition to evaluations at roots of unity, also full Taylor expansions. Both have compatible endomorphisms actions of the multiplicative semigroup of positive integers. As a higher dimensional generalization, we consider a crossed product ring obtained using Manin's multivariable generalizations of the Habiro functions and an action by endomorphisms of the semigroup of integer matrices with positive determinant. We then construct a corresponding class of multivariable BC endomotives, which are obtained geometrically from self maps of higher dimensional algebraic tori, and we discuss some of their quantum statistical mechanical properties. These multivariable BC endomotives are universal for (torsion free) Λ-rings, compatibly with the Frobenius action. Finally, we discuss briefly how Habiro's universal Witten–Reshetikhin–Turaev invariant of integral homology 3-spheres may relate invariants of 3-manifolds to gadgets over $\mathbb{F}_1$ and semigroup actions on homology 3-spheres to endomotives.

1. INTRODUCTION

Several of the different current approaches to algebraic geometry over the "field with one element" are based on identifying a class of additional data to be imposed over a ring, or a scheme over $\mathbb{Z}$, for it to be defined over the "underlying" $\mathbb{F}_1$. Among these approaches, the one developed by Soulé [52] is based on finiteness and universality conditions on the structure of cyclotomic points, whereas another approach recently developed by Borger [7] is based on the structure of Λ-ring as a descent condition from $\mathbb{Z}$ to $\mathbb{F}_1$. In the recent work [16], it was shown that, in the Soulé approach to $\mathbb{F}_1$-geometry, the tower of extensions $\mathbb{F}_{1^n}$ with their Galois action are nicely encoded by a noncommutative space, the Bost–Connes (BC) endomotive. Endomotives are a class of noncommutative spaces introduced in [14] based on projective limits of Artin motives with semigroup actions, and morphisms given by correspondences. The BC algebra of [11] is a natural example in this class.

In this paper we show that the BC endomotive, and some direct generalizations, are also natural objects from the point of view of Borger's approach to geometry over $\mathbb{F}_1$, since the endomorphism action that defines the structure of endomotive is also a consistent lift of Frobenius morphisms in the sense of Λ-rings.

The results obtained in this paper fall under the common theme of comparing different models of noncommutative algebras associated to the cyclotomic tower with the action of the positive integers by endomorphisms. As argued in [21], [22], and §3 of [19], the Bost–Connes algebra can be regarded as the noncommutative ring of functions of a noncommutative space naturally associated to the zero-dimensional Shimura variety (the cyclotomic tower), when one considers on it the action of the semigroup of positive integers by endomorphisms. The latter corresponds to imposing the equivalence relation of commensurability on 1-dimensional $\mathbb{Q}$-lattices up to scaling, in the formulation given in [20]. Here we also consider another noncommutative algebra of functions associated to the same geometric object, which instead

of using the group algebra of $\mathbb{Q}/\mathbb{Z}$, or equivalently the algebra of continuous functions on $\hat{\mathbb{Z}}$, to describe functions on roots of unity, is based on the Habiro ring of "analytic functions of roots of unity" and on the same family of endomorphisms. Both the Habiro ring and the abelian part of the Bost-Connes algebra describe different aspects of the same underlying geometry and this serves as a very useful guideline to obtain various results on generalizations of the BC endomotive. For instance, the multivariable generalizations of the Habiro ring constructed by Manin in [44] also give rise to a family of noncommutative algebras related to actions of endomorphisms of higher dimensional algebraic tori. The resulting noncommutative spaces also admit a description in terms of endomotives in the sense of [14] and this leads to the construction of multivariable BC endomotives. We show that these multivariable BC algebras are closely related to the theory of Λ-rings, in the formulation given in [9], as another approach to the theory of geometry over the "field with one element". We show that the (multivariable) BC endomotives are universal (torsion free) Λ-rings, in the sense that any such Λ-ring that is of finite rank over $\mathbb{Z}$ and has no nilpotents can be embedded in a product of BC endomotives, compatibly with the Frobenius action.

In order to construct quantum statistical mechanical systems for the multivariable BC endomotives, one needs to correct for the fact that the natural time evolution that generalizes the one-variable case has infinite multiplicities in the spectrum of the Hamiltonian coming from the presence of an $\mathrm{SL}_n(\mathbb{Z})$-symmetry. The problem is completely analogous to the one encountered in [20] in the case of 2-dimensional $\mathbb{Q}$-lattices, and one can use essentially the same method used in [20] to circumvent the problem by using the trace of a type II_1 factor associated to the $\mathrm{SL}_n(\mathbb{Z})$-action. We show that multiple zeta values of cones define classes of states on the multivariable BC algebras.

The Habiro ring of functions was originally introduced (see [30] and [31]) as the natural receptacle for a universal invariant of integral homology 3-spheres that specializes to the Witten–Reshetikhin–Turaev invariant $\mathfrak{t}_\zeta(M)$ at each root of unity ζ and whose Taylor expansion at $\zeta = 1$ gives the Ohtsuki series $\mathfrak{t}^O(M)$. In view of the relation between the Habiro ring and the noncommutative geometry of the cyclotomic tower, it is then natural to try to relate the universal Witten–Reshetikhin–Turaev invariant itself to quantum statistical mechanics and noncommutative geometry, on one side, and to the geometry of gadgets over the "field with one element" in the sense of [52], on the other. We concentrate on these questions in the last part of the paper, where we consider the question of constructing convolution algebras associated to surgery presentations of 3-manifolds with actions of $\mathbb{N}$ by endomorphisms, and of possible relations to endomotives via the WRT invariants.

2. The Bost–Connes system and Habiro analytic functions

We recall here briefly the main properties of the Bost–Connes (BC) quantum statistical mechanical system introduced in [11], in the formulation given in [20] and in §3 of [19]. We then also recall the main properties of the Habiro ring of analytic functions of roots of unity, and we show that the same semigroup of endomorphisms used to for the crossed product BC algebra acts on the Habiro ring and gives rise to a similar crossed product construction. We

then show that using Habiro functions together with evaluations at roots of unity one can recover the Hilbert space representations of the BC algebra, as well as versions where the full Taylor expansions at roots of unity is used.

2.1. The Bost–Connes endomotive, the cyclotomic tower, and $\mathbb{F}_1$.

The cyclotomic tower is defined by the direct system of inclusions of rings $\mathbb{Z}[\zeta_n] \hookrightarrow \mathbb{Z}[\zeta_m]$, for primitive roots of unity with $n|m$, with covering groups $\mathrm{GL}_1(\hat{Z}) \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. Under the identification $\bar{\mathbb{Q}}^*_{tors} \cong \mathbb{Q}/\mathbb{Z}$, one can identify the algebra of functions of roots of unity with

$$(2.1) \qquad C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\hat{Z}),$$

where the isomorphism is Pontrjagin duality between $\mathbb{Q}/\mathbb{Z}$ and $\hat{Z}$. More precisely, when one views it as the group ring $C^*(\mathbb{Q}/\mathbb{Z})$ this is an algebra of convolution of functions with finite support, and only after passing to $C(\hat{Z})$ one has an algebra of functions with pointwise product. In addition to the automorphisms of the cyclotomic tower given by the action of $\hat{Z}^*$, there are endomorphisms given by the multiplicative semigroup of positive integers, acting on the generators $e(r)$ of the group algebra $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ by

$$(2.2) \qquad \rho_n(e(r)) = \frac{1}{n} \sum_{ns=r} e(s).$$

As shown in [20], when one identifies the algebra $C(\hat{Z})$ with functions on the set of 1-dimensional $\mathbb{Q}$-lattices up to scaling, the action (2.2) implements the equivalence relation of commensurability.

The noncommutative BC algebra is explicitly given in terms of generators and relations as the algebra (over $\mathbb{Q}$) generated by elements $e(r)$, with $r \in \mathbb{Q}/\mathbb{Z}$, and μ_n, with $n \in \mathbb{N}$, subject to the relations

$$(2.3) \qquad \begin{aligned} &(\mu_n^*)^* = \mu_n \\ &\mu_n\mu_m = \mu_{nm} \qquad \forall n,m \in \mathbb{N} \\ &\mu_n\mu_m^* = \mu_m^*\mu_n \quad \text{when } (n,m) = 1 \\ &\mu_n^*\mu_n = 1 \end{aligned}$$

and

$$(2.4) \qquad e(r+s) = e(r)e(s), \quad e(0) = 1$$

and the relation

$$(2.5) \qquad \mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s),$$

which implements the semigroup action (2.2) and identifies the resulting algebra with the semigroup crossed product

$$(2.6) \qquad \mathcal{A}_{\mathbb{Q},BC} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes_\rho \mathbb{N}.$$

The algebra $\mathcal{A}_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$ has as C^*-algebra completion the crossed product algebra

$$(2.7) \qquad \mathcal{A}_{BC} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N} = C(\hat{Z}) \rtimes \mathbb{N}.$$

One can also consider "categorifications" of the BC system [47] by describing the BC algebra as the convolution algebra associated to a small category. We take up this point of view in the last section of this paper, when we discuss algebras and categories associated to surgery presentations of 3-manifolds, *cf.* also [46].

As argued in [21], [22], the BC algebra is (up to Morita equivalence) the algebra of functions on the noncommutative zero-dimensional Shimura variety $C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^*$, whose set of classical points is the usual zero-dimensional Shimura variety $\mathbb{A}_f^*/\mathbb{Q}_+^* = \hat{Z}^*$ associated to the

cyclotomic tower. This classical space is recovered from the noncommutative space as the set of low temperature extremal KMS states of the BC quantum statistical mechanical system. The action of $\mathbb{Q}_+^*$ on the finite adeles $\mathbb{A}_f$ corresponds to a partially defined action on $\hat{\mathbb{Z}}$, which in turn corresponds to the semigroup action (2.2). This point of view based on towers and Shimura varieties was generalized to 2-dimensions in [20] and [21], [22], and to more general Shimura varieties in [28].

The quantum statistical mechanical procedure of associating to a noncommutative space with a natural time evolution a classical space given by the set of low temperature extremal KMS states was formulated in greater generality in [14] in the context of a theory of *endomotives*. The latter are noncommutative spaces obtained from towers of Artin motives and semigroups of endomorphisms. They carry a Galois action and a time evolution to which quantum statistical mechanical techniques can be applied.

The original case of the BC system is obtained in [14] as a particular case of a general procedure that constructs endomotives from self maps of pointed algebraic varieties, via a projective system of Artin motives obtained as inverse images of the fixed point under the given family of self-maps. In the case of the BC endomotive, the variety is $\mathbb{G}_m$, the multiplicative group, and the self maps are the semigroup $S = \mathbb{N}$ of morphisms

$$(2.8) \qquad s_k : P(t, t^{-1}) \mapsto P(t^k, t^{-k}), \qquad k \in \mathbb{N},$$

on $\mathbb{Q}[t, t^{-1}]$. The point 1 is fixed by all these maps and the inverse image under the map $t \mapsto t^k$ is $X_k = \mathrm{Spec}(\mathbb{Q}[t, t^{-1}]/(t^k - 1))$. The BC endomotive is obtained, at the algebraic level, by taking the projective limit $X = \varprojlim_k X_k$ or equivalently the direct limit of the corresponding algebras $A = \varinjlim_k A_k$, with $A_k = \mathbb{Q}[\mathbb{Z}/k\mathbb{Z}]$ and the crossed product $\mathcal{A}_{\mathbb{Q}, BC} = A \rtimes S = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$.

The endomorphisms (2.8) correspond (*cf.* [16]) to the endomorphisms of $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ given by

$$(2.9) \qquad \sigma_n(e(r)) = e(nr), \qquad \forall n \in \mathbb{N}, \ \forall r \in \mathbb{Q}/\mathbb{Z}.$$

These satisfy

$$(2.10) \qquad \sigma_n \rho_n(x) = x, \quad \text{and} \quad \rho_n \sigma_n(x) = e_n x, \quad \forall x \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}],$$

with the ρ_n as in (2.2). Here e_n is an idempotent in $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ that gives the projection onto the range of ρ_n. In particular, one has $\sigma_n(x) = \mu_n^* x \mu_n$.

It is useful to recall here more in detail the properties of the endomorphisms σ_n, ρ_n, and the idempotents e_n, because this will constitute an important difference between the model of the noncommutative geometry of the cylotomic tower based on the BC algebra and the one based on the Habiro ring. The idempotent e_n is given by

$$(2.11) \qquad e_n = \mu_n \mu_n^*,$$

and it satisfies the identity

$$(2.12) \qquad e_n = \frac{1}{n} \sum_{nr=0} e(r),$$

where one can see easily that the right-hand-side does not depend on the choice of a solution $r \in \mathbb{Q}/\mathbb{Z}$ of $nr = 0$. Thus, in particular, one has $e_n \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$.

We also recall one more property of the BC algebra, namely its Hilbert space representations associated to embeddings in $\mathbb{C}$ of the roots of unity, which we need in the following.

As shown in [11], the choice of an element $\rho \in \hat{\mathbb{Z}}^*$ determines a representation π_ρ of the BC algebra as bounded operators on the Hilbert space $\ell^2(\mathbb{N})$ with

$$(2.13) \qquad \mu_n \epsilon_k = \epsilon_{nk}$$

on the canonical basis $\{\epsilon_k\}$ of $\ell^2(\mathbb{N})$ and

(2.14) $$\pi_\rho(e(r))\,\epsilon_k = \zeta_r^k\,\epsilon_k,$$

where $\zeta_r = \rho(e(r))$ is the root of unity obtained using the chosen $\rho \in \hat{\mathbb{Z}}^*$ to embed the abstract roots of unity $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{C}$.

Finally, we also recall the fact that an integer model for the BC algebra was constructed in [16], as the ring $\mathcal{A}_{\mathbb{Z},BC}$ generated by elements $e(r)$, with $r \in \mathbb{Q}/\mathbb{Z}$ and $\tilde{\mu}_n$ and μ_n^*, subject to the same relations (2.4), but with the relations (2.3) replaced by the analogous relations

(2.15)
$$\begin{aligned}
\tilde{\mu}_n\tilde{\mu}_m &= \tilde{\mu}_{nm} & \forall n,m \in \mathbb{N} \\
\mu_n^*\mu_m^* &= \mu_{nm}^* & \forall n,m \in \mathbb{N} \\
\mu_n^*\tilde{\mu}_n &= n & \forall n \in \mathbb{N} \\
\tilde{\mu}_n\mu_m^* &= \mu_m^*\tilde{\mu}_n & \forall (n,m) = 1.
\end{aligned}$$

The endomorphisms σ_n of (2.9) are still defined on $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and they satisfy relations replacing (2.5) and (2.10), in the form

(2.16) $$\mu_n^*x = \sigma_n(x)\mu_n^* \quad \text{and} \quad x\tilde{\mu}_n = \tilde{\mu}_n\sigma_n(x). \quad \forall x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}].$$

The $\tilde{\mu}_n$ can be realized as bounded operators on $\ell^2(\mathbb{N})$ acting by

$$\tilde{\mu}_n\epsilon_m = n\epsilon_{nm}.$$

They no longer are adjoints to the μ_n^* but they satisfy $\tilde{\mu}_n = n\mu_n$. One then sets ([16])

(2.17) $$\tilde{\rho}_n(x) = \tilde{\mu}_n\,x\,\mu_n^*,$$

to replace the original relation $\rho_n(x) = \mu_n\,x\,\mu_n^*$, which involves denominators. The $\tilde{\rho}_n$ are no longer ring homomorphisms, but only correspondences, so that $\mathcal{A}_{\mathbb{Z},BC}$ is not a semigroup crossed product ring.

It was also shown in [16] that the Bost–Connes endomotive has a model over the "field with one element" $\mathbb{F}_1$, in the sense that it is obtained from a projective limit of affine varieties $\mu^{(n)}$ over $\mathbb{F}_1$ (in the sense of Soulé [52], as well as in the sense of [17]), with the endomorphisms σ_n of (2.9) defining morphisms of "gadgets" over $\mathbb{F}_1$ (again in the terminology of [52]). In fact, the abelian part of the Bost–Connes endomotive corresponds to the inductive system of "extensions" $\mathbb{F}_{1^n}$ of $\mathbb{F}_1$, and the endomorphisms σ_n induce the Frobenius correspondence on all the associated positive characteristic reductions of the abelian part of the Bost–Connes algebra $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \otimes_{\mathbb{K}} \mathbb{K}$, with $\mathrm{char}(\mathbb{K}) = p > 0$.

2.2. The Habiro ring, endomorphisms, and Taylor series expansions.

We now recall the basic properties of the Habiro ring of "analytic functions on roots of unity" constructed in [30]. We then show that the same family of endomorphisms σ_n acting on the abelian part of the BC algebra also act on the Habiro ring in a compatible way, where the compatibility is given by the evaluations of Habiro functions at roots of unity. We introduce a semigroup crossed product ring obtained using the action of $\mathbb{N}$ on the Habiro ring, that provides an analog of the BC algebra. We discuss the effect of the action of the endomorphisms σ_n on the Taylor expansions of Habiro functions at roots of unity.

As in [30], [31], [44], we consider the ring $\widehat{\mathbb{Z}[q]}$ defined as the inverse limit

(2.18) $$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q)_n),$$

where

(2.19) $$(q)_n = (1-q)(1-q^2)\cdots(1-q^n).$$

These are ordered by divisibility, since $(q)_k|(q)_n$ for $k \leq n$, with corresponding inclusions $((q)_n) \subseteq ((q)_k)$ of ideals and homomorphisms $\mathbb{Z}[q]/((q)_n) \twoheadrightarrow \mathbb{Z}[q]/((q)_k)$.

Thus, one can think of elements in the Habiro ring as series $\sum_n P_n(q)$, with $P_n(q)$ in $((q)_n)$. Notice that can be seen as functions of roots of unity, with the pointwise product, instead of the convolution product of the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$.

Let $\mathcal{Z}$ denote the set of all roots of unity in $\mathbb{C}$. By the results of [30], [31], for any $\zeta \in \mathcal{Z}$, there exists an evaluation map

$$(2.20) \qquad\qquad ev_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta]$$

that is a surjective ring homomorphisms. Moreover, assembling these maps for all roots of unity one obtains an *injective*

$$(2.21) \qquad\qquad ev : \widehat{\mathbb{Z}[q]} \to \prod_{\zeta \in \mathcal{Z}} \mathbb{Z}[\zeta],$$

that is, elements of $\widehat{\mathbb{Z}[q]}$ are determined uniquely by their evaluations at roots of unity. This is proved in [30], Theorem 6.3, along with a characterization of the rings of coefficients R for which the corresponding Habiro ring $\widehat{R[q]}$ has the property that $ev : \widehat{R[q]} \to \prod_\zeta R[\zeta]$ is injective (see also §2 of [44]).

2.2.1. *Endomorphisms.* We then have the following result on the action of $\mathbb{N}$ by endomorphisms of the Habiro ring.

Proposition 2.1. *The semigroup $\mathbb{N}$ of positive integers acts by endomorphisms of $\widehat{\mathbb{Z}[q]}$ by*

$$(2.22) \qquad\qquad \sigma_n(f)(q) = f(q^n).$$

Proof. First notice that the homomorphism $\sigma_n : \mathbb{Z}[q] \to \mathbb{Z}[q]$ given by $\sigma_n(f)(q) = f(q^n)$ descends to a homomorphism $\sigma_n : \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}$. In fact, notice that $(q)_m|\sigma_n(q)_m$ for all n and m, hence the homomorphism σ_n induces compatible homomorphisms $\sigma_n : \mathbb{Z}[q]/(q)_m \to \mathbb{Z}[q]/(q)_m$. We still denote by σ_n the map on the inverse limit. $\qquad\square$

Suppose then that ζ is an m-th root of unit . The evaluation map ev_ζ is in this case defined on $\mathbb{Z}[q]/((q)_m)$ and one sees that the σ_n then give compatible maps $\mathbb{Z}[q]/((q)_m) \to \mathbb{Z}[q]/((q)_{m/n})$, for $n|m$, lifting the map $\mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta^n]$ given by $\sigma_n(f)(\zeta) = f(\zeta^n)$. See Corollary 2.11 below.

2.2.2. *Semigroup crossed product.* By analogy with the BC system, we would then like to form a semigroup crossed product of the Habiro ring by the action of $\mathbb{N}$ by endomorphisms. This requires introducing an analog of the ρ_n and of the isometries μ_n implementing them as additional generators of the crossed product ring.

Definition 2.2. *In the following we let $\widehat{\mathbb{Z}[q]}_\infty$ denote the direct limit of the system of maps $\sigma_n : \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}$.*

We can then define a ring generated by the functions of the Habiro ring $\widehat{\mathbb{Z}[q]}$ and the action of the endomorphisms σ_n in the following way.

Definition 2.3. *Let $\mathcal{A}_{\mathbb{Z},q}$ denote the ring generated by the Habiro ring $\widehat{\mathbb{Z}[q]}$ together with additional generators μ_n and μ_n^*, for $n \in \mathbb{N}$, subject to the relations (2.3) and the additional relations*

$$(2.23) \qquad\qquad \mu_n \sigma_n(f) = f\mu_n, \qquad \mu_n^* f = \sigma_n(f)\mu_n^*,$$

for all $f \in \widehat{\mathbb{Z}[q]}$ and all $n \in \mathbb{N}$, with $\sigma_n(f)(q) = f(q^n)$, as above. We also require that the elements $\mu_n \mu_n^*$ belong to the direct limit $\widehat{\mathbb{Z}[q]}_\infty$.

We can identify the ring $\mathcal{A}_{\mathbb{Z},q}$ with a crossed product ring. To this purpose, we make the following preliminary observation.

Lemma 2.4. *Let $\hat{R}_n$ denote the range of σ_n acting on $\widehat{\mathbb{Z}[q]}$, that is, $\hat{R}_n = \{f \in \widehat{\mathbb{Z}[q]} \mid \exists h \in \widehat{\mathbb{Z}[q]}, f = \sigma_n(h)\}$. There is a ring homomorphism $\eta_n : \hat{R}_n \to \widehat{\mathbb{Z}[q]}$, given by $\eta_n(f) = h$, which satisfies $\sigma_n \circ \eta_n = id|_{\hat{R}_n}$ and $\eta_n \circ \sigma_n = id|_{\widehat{\mathbb{Z}[q]}}$.*

Proof. It suffices to see that $\hat{R}_n$ is indeed a subring of $\widehat{\mathbb{Z}[q]}$ and that $\eta_n(f) = h$ gives a well defined ring homomorphism $\hat{R}_n \to \widehat{\mathbb{Z}[q]}$. To this purpose it suffices to observe that, if we can represent in two different ways $f = \sigma_n(h_1) = \sigma_n(h_2)$, then the image of $h_1 - h_2$ in $\mathbb{Z}[q]/((q)_N)$ lies in the ideal $\mathcal{I}_N$ generated by $(q)_N$. Thus, the map $\eta_n(f) = h$ is well defined in $\widehat{\mathbb{Z}[q]}$. $\square$

The statement above is just giving the *injectivity* of the maps σ_n. In fact, the condition $\mathrm{Ker}\sigma_n = 0$ follows from the fact that an element in the kernel would evaluate at zero on all the ζ^n. The injectivity of the evaluation (2.21) together with the divisibility of the group of roots of unity give the result. This should be contrasted with the situation of the BC system, where the σ_n are surjective and not injective.

We cannot directly extend the partial inverse η_n of σ_n to a homomorphism on the whole of $\widehat{\mathbb{Z}[q]}$, since, unlike what happens in the case of the Bost–Connes algebra, we do not have an idempotent in $\widehat{\mathbb{Z}[q]}$ analogous to the idempotent $\mu_n \mu_n^* = e_n \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$. However, we can give a more detailed description of the ring $\mathcal{A}_{\mathbb{Z},q}$ and we find that it is indeed a crossed product ring. We need some preliminary results.

Lemma 2.5. *Consider the rings $A_N = \widehat{\mathbb{Z}[q]}_N$ generated by all the elements of the form $\mu_N f \mu_N^*$, for $f \in \widehat{\mathbb{Z}[q]}$, with $\widehat{\mathbb{Z}[q]}_1 = \widehat{\mathbb{Z}[q]}$. These subrings $A_N = \widehat{\mathbb{Z}[q]}_N \subset A_\infty = \cup_N A_N$ satisfy the property that $A_N \cdot A_M \subset A_{NM/(N,M)}$. Moreover, the endomorphisms σ_n of $A_1 = \widehat{\mathbb{Z}[q]}$ extend to endomorphisms $\sigma_n : A_N \to A_N$ when $n \nmid N$ and $\sigma_n : A_N \to A_{N/n}$ when $n|N$.*
There are also well defined endomorphisms $\rho_n(a) = \mu_n a \mu_n^*$ mapping $\rho_n : A_N \to A_{nN}$. These satisfy

$$(2.24) \qquad \rho_n(\sigma_n(a)) = e_n a e_n, \qquad \sigma_n(\rho_n(a)) = a$$

where the idempotent $e_n = \mu_n \mu_n^*$ maps A_1 by $f \mapsto e_n f e_n$ onto the subring $\mu_n \hat{R}_n \mu_n^* \subset A_n$, and more generally it maps

$$A_N \to \mu_{nN/(n,N)} \hat{R}_{n/(n,N)} \mu_{nN/(n,N)}^* \subset A_{nN/(n,N)}, \qquad a_N \mapsto e_n a_N e_n,$$

where the $\hat{R}_n$ are as in Lemma 2.4.

Proof. Each $A_N = \widehat{\mathbb{Z}[q]}_N$ is a ring, by the property that $\mu_N^* \mu_N = 1$. Moreover, the relations (2.15) also imply the statement about multiplication in $\mathcal{A}_{\mathbb{Z},q}$ of two elements $a_N = \mu_N f \mu_N^* \in A_N$ and $a_M = \mu_M h \mu_M^* \in A_M$. In fact, this gives $a_N \cdot a_M = \mu_N f \mu_N^* \mu_M h \mu_M^*$. If $(N, M) = 1$ this is

$$\mu_N f \mu_N^* \mu_M h \mu_M^* = \mu_N f \mu_M \mu_N^* h \mu_M^* = \mu_N \mu_M \sigma_M(f) \sigma_N(h) \mu_N^* \mu_M^* \in A_{NM}.$$

If instead $N = aN'$ and $M = aM'$ with $(N', M') = 1$, we find, using $\mu_a^* \mu_a = 1$,

$$\mu_N f \mu_N^* \mu_M h \mu_M^* = \mu_N f \mu_{M'} \mu_{N'}^* h \mu_M^* = \mu_N \mu_{M'} \sigma_{M'}(f) \sigma_{N'}(h) \mu_{N'}^* \mu_M^*$$

$$= \mu_{aN'M'} \sigma_{M'}(f) \sigma_{N'}(h) \mu_{aN'M'}^* \in A_{aN'M'}.$$

Clearly $\rho_n(a) = \mu_n a \mu_n^*$ defines an endomorphism since $\mu_n^* \mu_n = 1$. Consider the restriction of ρ_n to $\hat{R}_n \subset A_1$. Elements of $\hat{R}_n$ are of the form $\sigma_n(h)$ for some $h \in A_1$. By the relations between σ_n and μ_n and μ_n^*, we have $\sigma_n(h) = \mu_n^* h \mu_n$, so that the action of ρ_n on elements of

this form is given by $\mu_n\mu_n^* h\mu_n\mu_n^*$. The relations (2.24) follow from the fact that $\sigma_n(a) = \mu_n^* a\mu_n$ and $\rho_n(a) = \mu_n a\mu_n^*$.

Moreover, for $a \in A_N$, $a = \mu_N f\mu_N^*$ for some $f \in A_1$, we have

$$\rho_n(\sigma_n(a)) = \mu_{nN/(n,N)}\sigma_{n/(n,N)}(f)\mu_{nN/(n,N)}^* = e_n a e_n,$$

which shows that the ring $e_n A_N e_n$, obtained by compressing A_N with the idempotent $e_n = e_n^2$, is the subring $\mu_{nN/(n,N)}\hat{R}_{n/(n,N)}\mu_{nN/(n,N)}^*$ of $A_{nN/(n,N)}$. In the case of elements $f \in \hat{R}_n \subset A_1$, the identification $e_n A_1 e_n = \mu_n \hat{R}_n \mu_n^*$ is induced by $\rho_n(h) = e_n\eta_n(h)e_n$, with $\eta_n : \hat{R}_n \to A_1$ the homomorphism of Lemma 2.4. $\qquad\square$

We then consider then the ring $A_\infty = \cup_N A_N$ and the induced maps σ_n and ρ_n. We have the following result.

Lemma 2.6. *The ring $A_\infty = \cup_N A_N$ is the direct limit $\widehat{\mathbb{Z}[q]}_\infty$ of the inductive system given by the endomorphisms $\sigma_n : \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}$. The induced morphisms $\sigma_n : \widehat{\mathbb{Z}[q]}_\infty \to \widehat{\mathbb{Z}[q]}_\infty$ are automorphisms with inverses the ρ_n.*

Proof. Let $\widehat{\mathbb{Z}[q]}_\infty$ be the direct limit of the system of injective homomorphisms $\sigma_n : \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}$. The rings A_N are arranged by inclusions $j_{M,N} : A_N \hookrightarrow A_M$, when $N|M$ with the embedding given by identifying $a = \mu_N f\mu_N^*$ with the same element written as $a = \mu_M \sigma_k(f)\mu_M^*$, with $k = M/N$. Thus, we can identify A_∞ with the direct limit of the A_N under these inclusions. The ring homomorphisms $\rho_N : A_1 \to A_N$ with $f \mapsto \rho_N(f) = \mu_N f\mu_N^*$ define maps of direct systems, since by construction $\rho_M\sigma_k(f) = j_{M,N}\rho_N(f)$ for $M = kN$. This induces a morphism on the direct limits $\widehat{\mathbb{Z}[q]}_\infty \to A_\infty$. We see that this is in fact an isomorphism, since one can construct an inverse to this map using the $\sigma_N : A_N \to A_1$ and their compatibility with the direct systems. In fact, using the assumption that the elements $e_n = \mu_n\mu_n^*$ belong to the direct limit $\widehat{\mathbb{Z}[q]}_\infty$, upon writing the idempotent $e_n = 1 - p_n$ for some projector p_n and using the relations induced on $\widehat{\mathbb{Z}[q]}_\infty$ by (2.23), we obtain $\sigma_n(p_n) = \mu_n^*(1 - \mu_n\mu_n^*)\mu_n = 0$, hence by the injectivity of the σ_n we have $e_n = 1$. This in fact shows that the σ_n become automorphisms on the direct limit $\widehat{\mathbb{Z}[q]}$ with inverses the ρ_n. One can see this also from the fact that one can write the direct limit $\widehat{\mathbb{Z}[q]}_\infty$ as sequences of elements $f_\ell \in \widehat{\mathbb{Z}[q]}$ where $f_{k\ell} = \sigma_k(f_\ell)$ and one sees in this way that the maps induced by the σ_k on the direct limit are surjective as well as injective. $\qquad\square$

The fact that the endomorphisms $\sigma_n : \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}$ induce automorphisms of the ring $\widehat{\mathbb{Z}[q]}_\infty$ can also be seen from the fact that, dually, one is taking a projective limit of a system of surjective maps of spaces. The map induced on the projective limit is then injective as well as surjective, as one sees already in the case of a single surjective map $s : X \to X$, where the projective limit is the space of sequences (x_n) with $x_n = s(x_{n+1})$ and the map $(x_n) \mapsto (s(x_n))$ is then injective since $(s(x_n)) = (s(y_n))$ implies $x_{n-1} = y_{n-1}$ for all n.

We then obtain a description of the ring $\mathcal{A}_{\mathbb{Z},q}$ as a crossed product as follows.

Theorem 2.7. *Let $\mathcal{A}_{\mathbb{Z},q}$ be, as in Definition 2.3, the ring generated by the elements of $\widehat{\mathbb{Z}[q]}$ and the elements μ_n, μ_n^* subject to the relations (2.3) and (2.23). Then $\mathcal{A}_{\mathbb{Z},q}$ is the crossed product ring*

$$(2.25) \qquad\qquad\qquad \mathcal{A}_{\mathbb{Z},q} = \widehat{\mathbb{Z}[q]}_\infty \rtimes \mathbb{Q}_+^*$$

of the action of the group $\mathbb{Q}_+^*$ on the ring $\widehat{\mathbb{Z}[q]}_\infty = \varinjlim_n (\sigma_n : \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]})$.

Proof. We have seen in Lemma 2.6 above that the endomorphisms σ_n become invertible in the direct limit $A_\infty = \widehat{\mathbb{Z}[q]}_\infty$ with inverses given by the $\rho_n(a) = \mu_n a \mu_n^*$. Thus the crossed product of the action of the multiplicative semigroup $\mathbb{N}$ on A_∞ extends to a group crossed product by the multiplicative group $\mathbb{Q}_+^*$. In other words, the action of $r = m/n \in \mathbb{Q}_+^*$, with $(m,n) = 1$, on A_∞ that gives the crossed product $A_\infty \rtimes \mathbb{Q}_+^*$ is given by $\rho_r(a) = \mu_m \mu_n^* a \mu_n \mu_m^*$. To see that there is an isomorphism between $A_\infty \rtimes \mathbb{Q}_+^*$ and $\mathcal{A}_{\mathbb{Z},q}$ we first show that they are isomorphic as $\mathbb{Z}$-modules and then that the ring structures also agree. As a $\mathbb{Z}$-module $\mathcal{A}_{\mathbb{Z},q}$ is spanned by elements of the form $\mu_N f \mu_M^*$, for $f \in \widehat{\mathbb{Z}[q]}$ and $N, M \in \mathbb{N}$. In fact, using the relations (2.3) and $\sigma_n(f) = \mu_n^* f \mu_n$, which follows from (2.23), one can see that products of monomials of the form $\mu_N f \mu_M^*$ is still of the same form. We then show that elements of $A_\infty \rtimes \mathbb{Q}_+^*$ can also be always written in this form. In fact, such elements are in the span of elements $a\mu_n$ for $a \in A_\infty$ and their adjoints $\mu_n^* a$. Any element of the form $\mu_N f \mu_N^* \mu_n$ can be written as

$$\mu_N f \mu_{n/(n,N)} \mu_{N/(n,N)}^* = \mu_{Nn/(n,N)} \sigma_{n/(N,n)}(f) \mu_{N/(n,N)}^*.$$

Conversely, any element of the form $\mu_N f \mu_M^*$ with $(N, M) = 1$ can be written equivalently as $a_N \mu_N \cdot \mu_M^*$, with $a_N = \mu_N f \mu_N^*$, which is the product of two elements $a_N \mu_N$ and $\mu_M^* = (\mu_M)^*$ in $A_\infty \rtimes \mathbb{Q}_+^*$. The product in $A_\infty \rtimes \mathbb{Q}_+^*$ is defined by $a\mu_n b\mu_m = a\rho_n(b)\mu_{nm}$, which, by the relations $\rho_n(b) = \mu_n b \mu_n^*$ and $\mu_n^* \mu_n = 1$, is the product of $a\mu_n$ and $b\mu_m$ in $\mathcal{A}_{\mathbb{Z},q}$. $\qquad\square$

This again should be compared with the different situation of the BC algebra, where one has a semigroup crossed product $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$. We can give a concrete realization of the ring $\widehat{\mathbb{Z}[q]}_\infty$ with the action of $\mathbb{Q}_+^*$ in terms of functions of rational powers of q.

Proposition 2.8. *Let $\mathcal{P}_\mathbb{Z} = \mathbb{Z}[q^r; r \in \mathbb{Q}_+^*]$ denote the ring of polynomials in rational powers of the variable q. Let $\hat{\mathcal{P}}_\mathbb{Z}$ denote the completion*

$$(2.26) \qquad \hat{\mathcal{P}}_\mathbb{Z} = \varprojlim_N \mathcal{P}_\mathbb{Z}/\mathfrak{I}_N,$$

where $\mathfrak{I}_N$ is the ideal generated by the $(q^r)_N = (1 - q^r)\cdots(1 - q^{rN})$ for $r \in \mathbb{Q}_+^*$. Then sending an element $a = \rho_n(f) = \mu_n f \mu_n^*$, for $f \in \widehat{\mathbb{Z}[q]}$, to the function $f(q^{1/n})$ gives an isomorphism

$$(2.27) \qquad \Phi : \widehat{\mathbb{Z}[q]}_\infty \xrightarrow{\cong} \hat{\mathcal{P}}_\mathbb{Z}.$$

The action of $\mathbb{Q}_+^*$ on $\widehat{\mathbb{Z}[q]}_\infty$ corresponds, under this identification, to the action on $\hat{\mathcal{P}}_\mathbb{Z}$ given by $\rho_r(f)(q) = f(q^r)$.

Proof. First notice that the map that sends $\mu_n f \mu_n^*$ to $f(q^{1/n})$ is well defined as a ring homomorphism $\widehat{\mathbb{Z}[q]}_\infty \to \hat{\mathcal{P}}_\mathbb{Z}$. In fact, if $f_1 - f_2 \in ((q)_N)$ we see that $f_1(q^{1/n}) - f_2(q^{1/n})$ is in the ideal $\mathfrak{I}_N$. In other words, if we write elements in the Habiro ring as series $f(q) = \sum_m P_m(q)$, with $P_m(q)$ in $((q)_m)$, the map Φ gives $\Phi(\rho_n(f))(q) = \sum_m P_m(q^{1/n})$ where $P_m(q^{1/n})$ is in $((q^{1/n})_m) \subset \mathfrak{I}_n$.

By construction, all functions in $\hat{\mathcal{P}}_\mathbb{Z}$ can be described as combinations of functions obtained from elements in the Habiro ring by changes of variable $q \mapsto q^r$ with $r \in \mathbb{Q}_+^*$. Thus, the morphism Φ is surjective. To see that is is also injective, notice that if $\Phi(\mu_n f \mu_n^*) \in \mathfrak{I}_N$ for some N, then there exists a sufficiently large $M \in \mathbb{N}$, such that $n|M$, and such that $f(q^{M/n}) \in ((q)_m)$ for some $m \in \mathbb{N}$. This implies that, for $k = M/n$, we have $\sigma_k(f) \in ((q)_m)$, but the injectivity of the σ_k implies that $f = 0$ in the Habiro ring $\widehat{\mathbb{Z}[q]}$. Thus, Φ is an isomorphism. It is then clear by construction that the action of $\mathbb{Q}_+^*$ on $\widehat{\mathbb{Z}[q]}_\infty$ takes the form $\rho_r(f)(q) = f(q^r)$ on $\hat{\mathcal{P}}_\mathbb{Z}$. $\qquad\square$

2.2.3. *Taylor expansions at roots of unity.* We first recall how the Taylor expansion at roots of unity for Habiro functions is defined. For any choice of a root of unity $\zeta \in \mathcal{Z}$, one has an embedding of rings $\mathbb{Z}[q] \hookrightarrow \mathbb{Z}[\zeta, q]$. Moreover, suppose given $f_N \in \mathbb{Z}[q]/((q)_N)$. Since for $N > i \operatorname{ord}(\zeta)$ we have $(q - \zeta)^i | (q)_N$, we have a corresponding ring homomorphism

$$(2.28) \qquad \mathsf{t}_\zeta^{(i)} : \mathbb{Z}[q]/((q)_N) \to \mathbb{Z}[\zeta, q]/((q - \zeta)^i), \quad f \mapsto \mathsf{t}_\zeta^{(i)}(f)$$

with $\mathsf{t}_\zeta^{(0)} = ev_\zeta$. These maps are compatible with the projective system, hence they define, for an element $f \in \widehat{\mathbb{Z}[q]} = \varprojlim_N \mathbb{Z}[q]/((q)_N)$ a Taylor expansion at $\zeta \in \mathcal{Z}$, given by the resulting ring homomorphism

$$(2.29) \qquad \mathsf{t}_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q - \zeta]].$$

It was shown in [30] that the homomorphism (2.29) is injective. The following result is already contained in [30], [31], but we reproduce it here explicitly for later use.

Lemma 2.9. *For $f \in \mathbb{Z}[q]/((q)_N)$ the class of a polynomial $P \in \mathbb{Z}[q]$, and for a choice of i and ζ such that $i \operatorname{ord}(\zeta) < N$, we have*

$$(2.30) \qquad \mathsf{t}_\zeta^{(i)}(f)(q) = \sum_{k=0}^{i} \frac{1}{k!} P^{(k)}(\zeta) (q - \zeta)^k.$$

Proof. For $P(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_\ell q^\ell$ we write

$$P(q) = a_0 + a_1(\zeta + (q - \zeta)) + \cdots + a_\ell(\zeta + (q - \zeta))^\ell =$$

$$\sum_{j=0}^{\ell} a_j \sum_{k=0}^{j} \binom{j}{k} \zeta^{j-k} (q - \zeta)^k =$$

$$a_0 + a_1 \zeta + \cdots + a_\ell \zeta^\ell +$$

$$(a_1 + 2a_2 \zeta + \cdots + \ell a_\ell \zeta^{\ell-1})(q - \zeta) +$$

$$(a_2 + \binom{3}{2} a_3 \zeta + \cdots + \binom{\ell}{2} \zeta^{\ell-2})(q - \zeta)^2 +$$

$$\cdots + a_\ell (q - \zeta)^\ell =$$

$$P(\zeta) + P'(\zeta)(q - \zeta) + \frac{1}{2} P''(\zeta)(q - \zeta)^2 + \cdots + \frac{1}{\ell!} P^{(\ell)}(\zeta)(q - \zeta)^\ell,$$

so that, for $k < i$, the coefficient of $(q - \zeta)^k$ in $\mathsf{t}_\zeta^{(i)}(f)$ is given by the expression

$$\frac{1}{k!} P^{(k)}(\zeta) = \sum_{j=k}^{\ell} a_j \binom{j}{k} \zeta^{j-k}.$$

This gives (2.30) by taking the above expression for $P(q)$ modulo $(q - \zeta)^i$. $\square$

The endomorphisms σ_n of the Habiro ring induce endomorphisms of the rings $\mathbb{Z}[\zeta][[q - \zeta]]$ via the Taylor expansion at roots of unity.

Lemma 2.10. *The endomorphisms $\sigma_n : \widehat{\mathbb{Z}[q]} \to \widehat{\mathbb{Z}[q]}$ of the Habiro ring given by $\sigma_n(f)(q) = f(q^n)$ induce an action of the multiplicative semigroup $\mathbb{N}$ on the Taylor expansions at roots of unity $\sigma_n(\mathsf{t}_\zeta(f)) = \mathsf{t}_\zeta(\sigma_n(f))$, where the Taylor expansion $\mathsf{t}_\zeta(\sigma_n(f))$ is given by*

$$(2.31) \qquad \mathsf{t}_\zeta(\sigma_n(f)) = \sum_{k} \frac{1}{k!} (P \circ \sigma_n)^{(k)}(\zeta) (q - \zeta)^k,$$

for f represented by a polynomial $P(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_\ell q^\ell$ and $(P \circ \sigma_n)(q) = a_0 + a_1 q^n + \cdots + a_\ell q^{n\ell}$.

Proof. The Taylor expansion of Habiro functions at a given root of unity $\zeta \in \mathcal{Z}$ is an injective ring homomorphism $\mathsf{t}_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q-\zeta]]$. Thus, the map $\mathsf{t}_\zeta(f) \mapsto \mathsf{t}_\zeta(\sigma_n(f))$ is a well defined ring homomorphism from the image of $\widehat{\mathbb{Z}[q]}$ in $\mathbb{Z}[\zeta][[q-\zeta]]$ to itself. One then checks directly that, given $f \in \widehat{\mathbb{Z}[q]}$ represented by a polynomial $P(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_\ell q^\ell$, the Taylor expansion of $\sigma_n(f)$ is obtained by considering the polynomial

$$P(q^n) = a_0 + a_1(\zeta + (q - \zeta))^n + \cdots + a_\ell(\zeta + (q - \zeta))^{n\ell}.$$

The coefficient of $(q - \zeta)^k$ in this expansion is of the form

$$\sum_j a_j \binom{jn}{k} \zeta^{jn-k} = \frac{1}{k!}(P \circ \sigma_n)^{(k)}(\zeta),$$

So that we get

$$\mathsf{t}_\zeta(\sigma_n(f)) = \sum_{k \geq 0} \frac{1}{k!}(P \circ \sigma_n)^{(k)}(\zeta)(q - \zeta)^k.$$

$\square$

Notice, however, the following simple remark on the endomorphisms σ_n and the Taylor expansion.

Corollary 2.11. *The evaluations ev_ζ of Habiro functions satisfy the property*

(2.32) $$ev_\zeta(\sigma_n(f)) = ev_{\zeta^n}(f).$$

However, on the full Taylor expansion this no longer holds and one finds

(2.33) $$\mathsf{t}_\zeta(\sigma_n(f)) \neq \mathsf{t}_{\zeta^n}(f).$$

Proof. The case of the evaluations ev_ζ follows directly from Proposition 2.1. On the full Taylor expansion we have

$$\mathsf{t}_\zeta(\sigma_n(f)) = \sum_{k \geq 0} \frac{1}{k!}(P \circ \sigma_n)^{(k)}(\zeta)(q - \zeta)^k,$$

in the notation of Lemma 2.10, while we have

$$\mathsf{t}_{\zeta^n}(f) = \sum_{k \geq 0} \frac{1}{k!} P^{(k)}(\zeta^n)(q - \zeta^n)^k,$$

where

$$\frac{1}{k!} P^{(k)}(\zeta^n) = \sum_j a_j \binom{j}{k} \zeta^{n(j-k)}.$$

$\square$

Notice, however, that while we do not have the simple equality (2.33), one can still compare the Taylor expansion of f at ζ^n and the one of $\sigma_n(f)$ at ζ in the following way. If J_ζ denotes the ideal $\mathrm{Ker}(ev_\zeta)$, then, as one sees from Lemma 2.9, the Taylor expansion of f at ζ comes from the filtration in powers of J_ζ. The identity (2.32) gives $\sigma_n^{-1}(J_\zeta) = J_{\zeta^n}$ and the same holds for powers of these ideals. This means that there are operators $\mathcal{T}_{\zeta,n}$, which can be put in the form of triangular matrices, that relates the power series $\mathsf{t}_\zeta(\sigma_n(f))$ and $\mathsf{t}_{\zeta^n}(f)$.

2.2.4. *Compatibility of endomorphism actions.* To show in what sense the action on the Habiro ring of the endomorphisms σ_n is "compatible" with the action (2.9) of the BC algebra, we show how to obtain the C^*-algebra of operators (2.14) on the Hilbert space $\ell^2(\mathbb{N})$ using Habiro functions together with all their evaluations at roots of unity. Notice that this construction does not give a representation of the Habiro ring on $\ell^2(\mathbb{N})$, because we are going to associate to a Habiro function all of its evaluations at roots of unity. While through each evaluation at a given $\zeta \in \mathcal{Z}$ one obtains a surjective homomorphism of the Habiro ring onto a piece of the abelian part of the BC algebra factoring through the ring $\mathbb{Z}[\zeta]$, these do not assemble to give a representation of the product $\prod_\zeta \mathbb{Z}[\zeta]$. However, the construction given here, which is based on these partial representations, suffices to explain in what sense one has a compatibility, through evaluations at roots of unity, between the actions of σ_n on the Habiro ring and on the abelian part of the BC algebra.

Lemma 2.12. *For a given $\zeta \in \mathcal{Z}$, the map $P \mapsto E_P$ with*

(2.34) $$E_P \epsilon_n = P(\zeta^n) \epsilon_n$$

defines a representation of the ring $\mathbb{Z}[\zeta]$ as bounded operators on the Hilbert space $\ell^2(\mathbb{N})$. This defines a norm $\|f\|_{\mathcal{Z}}$ for elements $f \in \widehat{\mathbb{Z}[q]}$.

Proof. For $\zeta \in \mathcal{Z}$ and $P \in \mathbb{Z}[\zeta]$, $P(\zeta) = a_0 + a_1\zeta + \cdots + a_r\zeta^r$, we have $|P(\zeta^k)| \leq \sum_i |a_i|$, for all $k \in \mathbb{N}$. Thus, we have

$$\|P\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \sup_{v \neq 0} \frac{\|Pv\|}{\|v\|} \leq \sum_i |a_i| < \infty.$$

$\square$

Thus, we can consider the abelian C^*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{N}))$ generated by all the operators $E_{ev_\zeta(f)}$ as in (2.34), for $f \in \widehat{\mathbb{Z}[q]}$ and $\zeta \in \mathcal{Z}$.

Definition 2.13. *Let $C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]})$ denote the abelian C^*-algebra generated by all the operators in $\mathcal{B}(\ell^2(\mathbb{N}))$ of the form*

(2.35) $$E_{\zeta,f}\, \epsilon_n = ev_{\zeta^n}(f)\, \epsilon_n,$$

for $\zeta \in \mathcal{Z}$ and $f \in \widehat{\mathbb{Z}[q]}$.

We have the following result relating the algebra constructed above to the abelian part of the Bost–Connes algebra.

Proposition 2.14. *Any element $\rho \in \hat{\mathbb{Z}}^*$ determines an isomorphism between the C^*-algebra $C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]})$ and the abelian part $C^*(\mathbb{Q}/\mathbb{Z})$ of the Bost–Connes algebra.*

Proof. The choice of an element $\rho \in \hat{\mathbb{Z}}^*$ can be viewed as the choice of an embedding of the roots of unity in $\mathbb{C}$. In particular, any $\rho \in \hat{\mathbb{Z}}^*$ determines a representation of the Bost–Connes algebra on $\ell^2(\mathbb{N})$, where the action of the abelian part $C^*(\mathbb{Q}/\mathbb{Z})$ is given by the action of the generators $e(r)$, for $r \in \mathbb{Q}/\mathbb{Z}$ as

(2.36) $$\pi_\rho(e(r))\epsilon_n = \zeta_r^n\, \epsilon_n,$$

where $\zeta_r = \rho(r)$ is the root of unity corresponding to the generator $e(r)$. Moreover, given two such elements $\rho, \rho' \in \hat{\mathbb{Z}}^*$, one has

$$\pi_\rho(C^*(\mathbb{Q}/\mathbb{Z})) = \pi_{\rho'}(C^*(\mathbb{Q}/\mathbb{Z})) \subset \mathcal{B}(\ell^2(\mathbb{N})),$$

since, for a given generator $e(r)$ with $r \in \mathbb{Q}/\mathbb{Z}$, one has

$$\pi_{\rho'}(e(r)) = \pi_\rho(e(\gamma(r))),$$

where the automorphism γ of $\mathbb{Q}/\mathbb{Z}$ is given by $\gamma = \rho^{-1}\rho' \in \hat{\mathbb{Z}}^*$. We use here the identification $\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$.

We see that, for any given choice of $\rho \in \hat{\mathbb{Z}}^*$, we have an inclusion of the algebra $\pi_\rho(C^*(\mathbb{Q}/\mathbb{Z})) \subset \mathcal{B}(\ell^2(\mathbb{N}))$ in $C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]}) \subset \mathcal{B}(\ell^2(\mathbb{N}))$. In fact, it suffices to show that, for all the generators $e(r)$ of $C^*(\mathbb{Q}/\mathbb{Z})$, with $r \in \mathbb{Q}/\mathbb{Z}$ the operator $\pi_\rho(e(r))$ is in the algebra $C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]})$. This follows by writing $\pi_\rho(e(r))$ equivalently as the operator $ev_{\rho(r)}(f)$, for $f(q) = q$. In fact, this acts like

$$E_{\rho(r),f}\epsilon_n = \zeta_r^n\,\epsilon_n,$$

with $\zeta_r = \rho(r) \in \mathcal{Z}$.

To see that the reverse inclusion also holds, consider an element $ev_\zeta(f) \in \mathbb{Z}[\zeta]$, acting on $\ell^2(\mathbb{N})$ as in (2.35). Suppose given an element $\rho \in \hat{\mathbb{Z}}^*$. Then there exists an $r \in \mathbb{Q}/\mathbb{Z}$ such that $\zeta = \rho(r)$. If $ev_\zeta(f) = P(\zeta) = a_0 + a_1\zeta + \cdots + a_n\zeta^n$, we can then write the operator $E_{\zeta,f}$ as

$$E_{\zeta,f} = a_0 + a_1\pi_\rho(e(r)) + \cdots + a_n\pi_\rho(e(nr)),$$

so that we obtain $C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]}) \subset \pi_\rho(C^*(\mathbb{Q}/\mathbb{Z}))$.

We have shown in this way that a choice of $\rho \in \hat{\mathbb{Z}}^*$ determines an isomorphism

$$\iota_\rho : C^*(\mathbb{Q}/\mathbb{Z}) \to C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]})$$

given on generators by

(2.37) $$\iota_\rho(e(r)) = E_{\rho(r),f}, \quad \text{for} \quad f(q) = q.$$

$\square$

Notice that the C^*-algebra $C^*_{\mathcal{Z}}(\widehat{\mathbb{Z}[q]})$ is not a representation of the Habiro ring $\widehat{\mathbb{Z}[q]}$, hence in particular (2.37) is *not* a ring homomorphism between the abelian part of the BC algebra and the Habiro ring, though it gives a convenient parameterization of the operators in terms of Habiro functions and roots of unity.

The action of $\hat{\mathbb{Z}}^*$ by automorphisms of of $C^*(\mathbb{Q}/\mathbb{Z})$ is then given by

(2.38) $$\gamma(E_{\zeta,f}) = E_{\theta_\gamma(\zeta),f},$$

where $\theta : \hat{\mathbb{Z}}^* \to \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ is the class field theory isomorphism.

The "compatibility" between the action of the endomorphisms σ_n of the Habiro ring and of the abelian part of the BC algebra can then be expressed as the property that

(2.39) $$\sigma_n(E_{\zeta,f}) = E_{\zeta^n,f} = E_{\zeta,f^n} = E_{\zeta,\sigma_n(f)}.$$

2.3. **Hopf algebra and endomotive.** As we have seen above, we do not have a direct relation between the Habiro ring and the crossed product ring $\mathcal{A}_{\mathbb{Z},q}$ and the BC algebra in terms of Hilbert space representations. The main difficulty in relating them directly is the fact that the maps σ_n have very different properties in the two cases. There is, however, some further structure of the Habiro ring, which makes it possible to obtain from the endomorphisms σ_n a construction similar to the endomotive description of the BC system.

First recall that, as introduced in [14], endomotives are a category of noncommutative spaces constructed out of projective limits of Artin motives with semigroup actions, with morphisms that are also constructed out of correspondences between Artin motives, with a suitable compatibility condition with respect to the semigroup actions. We refer the reader to [14] and also to §4 of [19] for details. In particular, the BC endomotive is obtained by considering the

projective system of Artin motives given by the preimages $s_k^{-1}(1)$ under the system of endo-morphisms $s_k : \mathbb{G}_m \to \mathbb{G}_m$ of the multiplicative group that correspond to the endomorphisms of the ring $\mathbb{Z}[t, t^{-1}]$ given by $s_k(P)(t) = P(t^k)$.

It is known that the Habiro ring has the structure of a Hopf algebra. The argument repro-duced here below is due to Habiro: I learned of it from Morava [47].

Lemma 2.15. *The coproduct* $\Delta(q) = q \otimes q$, *counit* $\varepsilon(q) = 1$, *and antipode* $S(q) = q^{-1}$ *on the ring* $\mathbb{Z}[q, q^{-1}]$ *induce a coproduct, counit, and antipode on* $\widehat{\mathbb{Z}[q]}$ *with respect to the completed tensor product*

$$(2.40) \qquad \widehat{\mathbb{Z}[q]} \hat{\otimes} \widehat{\mathbb{Z}[q]} = \varprojlim_n (\mathbb{Z}[q] \otimes \mathbb{Z}[q]) / ((q)_n \otimes 1, 1 \otimes (q)_n).$$

This makes the Habiro ring into a Hopf algebra.

Proof. The compatibility of the coproduct with the quotients defining the Habiro ring follows from the fact that

$$\Delta(1 - q^N) = (1 - q^N) \otimes 1 + q^N \otimes (1 - q^N)$$

so that

$$\Delta((1 - q^N)^{2M}) \in ((1 - q^N)^M) \otimes (1) + (1) \otimes ((1 - q^N)^M).$$

One then uses an equivalent description of the Habiro ring as projective limit

$$\widehat{\mathbb{Z}[q]} = \varprojlim_{N,M} \mathbb{Z}[q] / ((1 - q^{N!})^M),$$

to see that the coproduct is well defined on the projective limit, with the corresponding completion, obtained using the ideals $((1 - q^{N!})^M \otimes 1, 1 \otimes (1 - q^{N!})^M)$. For the antipode the argument is similar, since $(1 - q^{-N!})^M \in (1 - q^{N!})^M \; \widehat{\mathbb{Z}[q]}$. $\qquad \square$

The endomorphisms σ_n of the Habiro ring are induced by the endomorphisms $s_k(P)(q) = P(q^k)$ of the polynomial ring $\mathbb{Z}[q, q^{-1}]$, which in turn define endomorphisms of the multi-plicative group $\mathbb{G}_m$. They in turn determine endomorphisms of the affine group scheme $\hat{\mathbb{G}}$ dual to the Hopf algebra $\widehat{\mathbb{Z}[q]}$, with $\hat{\mathbb{G}}(R) = \mathrm{Hom}(\widehat{\mathbb{Z}[q]}, R)$, for a given commutative ring R. Let $\mathfrak{s}_k : \hat{\mathbb{G}} \to \hat{\mathbb{G}}$ be the morphism induced dually by σ_n. We then let $\mathbb{X}_k = \mathfrak{s}_k^{-1}(1)$ with the projective system of maps $\mathbb{X}_n \to \mathbb{X}_m$ given by $x \mapsto \mathfrak{s}_k(x)$ for $n = mk$. Then the projective limit $\mathbb{X} = \varprojlim_n \mathbb{X}_n$ with the induced action gives an endomotive associated to the Habiro ring and its dual group $\hat{\mathbb{G}}$, which parallels the construction of the BC endomotive from the multiplicative group $\mathbb{G}_m$.

2.4. **Operators from Taylor expansions.** The content of this subsection is a side remark with respect to the main theme of the paper and will not be needed in what follows, so it can be skipped by the reader interested in moving on directly to the next section.

We construct here Hilbert space operators associated to the data of Taylor expansions $\mathfrak{t}_\zeta(f)$ of the Habiro functions at all $\zeta \in \mathcal{Z}$. These will in general give unbounded operators, except in the case where only finitely many terms of the expansion are nonzero. We consider a time evolution that generalizes the BC dynamics and recovers the extremal KMS states of the BC system at low temperature from the values at roots of unity, as well as the higher terms in the Taylor expansion from suitable limits of positive temperature Gibbs states.

Let $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N} \cup \{0\})$ with the canonical basis $\epsilon_{n,m} = \epsilon_n \otimes \epsilon_m$. We set

$$(2.41) \qquad T_{\zeta,f}^{(i)} \epsilon_{n,m} := \sum_{k=0}^{i} \frac{1}{k!} (P \circ \sigma_n)^{(k)}(\zeta) \, \epsilon_{n,m+k}.$$

We also introduce shift operators δ_k defined as

(2.42) $$\delta_k \epsilon_{n,m} = \epsilon_{n,m+k}$$

so that we write

(2.43) $$T_{\zeta,f}^{(i)} = \sum_{k=0}^{i} \frac{1}{k!} ev_\zeta((f \circ \sigma_n)^{(k)}) \delta_k,$$

where the notation $ev_\zeta((f \circ \sigma_n)^{(k)})$ is understood in the sense of (2.41), according to the definition of the Taylor expansions at roots of unity of the Habiro functions discussed above.

Lemma 2.16. *The operators* (2.41) *are bounded operators on* $\mathcal{H}$ *for all* $\zeta \in \mathcal{Z}$ *for all* $f \in \mathbb{Z}[q]/((q)_N)$ *with* $N > \mathrm{iord}(\zeta)$. *These operators determine operators* $T_{\zeta,f}$ *on* $\mathcal{H}$ *for all* $f \in \widehat{\mathbb{Z}[q]}$, *satisfying* $T_{\zeta,fh} = T_{\zeta,f} T_{\zeta,h}$, *for each* $\zeta \in \mathcal{Z}$ *and* $f,h \in \widehat{\mathbb{Z}[q]}$. *The operators* $T_{\zeta,f}$ *are typically unbounded, except in the case where only finitely many terms of the series are nonzero.*

Proof. Only finitely many terms are involved in the sum (2.43), hence they are bounded operators. Consider the the operators of the form

(2.44) $$T_{\zeta,f} \, \epsilon_{n,m} = \sum_{k \geq 0} \frac{1}{k!} ev_\zeta((f \circ \sigma_n)^{(k)}) \epsilon_{n,m+k}.$$

To see the multiplicative property for fixed $\zeta \in \mathcal{Z}$, notice that $\mathfrak{t}_\zeta(fh) = \mathfrak{t}_\zeta(f)\mathfrak{t}_\zeta(h)$, since $\mathfrak{t}_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q - \zeta]]$ is a ring homomorphism. The property then follows using $\mathfrak{t}_\zeta(f) = \sum_{k \geq 0} \frac{1}{k!} f^{(k)}(\zeta)(q - \zeta)^k$, together with the fact that the homomorphisms σ_n of $\widehat{\mathbb{Z}[q]}$ induce homomorphisms of the Taylor series expansions as in Lemma 2.10. To see that the operators (2.44) are typically unbounded, consider elements in the Habiro ring defined by series $f(q) = \sum_m P_m(q)$ with $P_m \in ((q)_m)$ of the form $P_m(q) = n_m(q)_m$, for an arbitrary sequence (n_m) of non-negative integers. Taking $\zeta = 1$ one sees that the Taylor expansion of f has coefficients that are all positive integers, since $(q)_m$ is a polynomial with positive coefficients in the variable $t = q - 1$. Thus, when acting with $T_{1,f}$ on a basis vector $\epsilon_{1,m}$ one finds $T_{1,f}\epsilon_{1,m} = \sum_{k \geq 0} \frac{1}{k!} ev_1(f^{(k)}) \epsilon_{1,m+k}$, with $\frac{1}{k!} ev_1(f^{(k)}) \geq n_k$. Thus, the element $T_{1,f}\epsilon_{1,m}$ is in $\mathcal{H}$ only if all but finitely many of the n_k are zero. More generally, for any element $f = \sum_m P_m(q) \in \widehat{\mathbb{Z}[q]}$, the coefficients $\frac{1}{k!} ev_1(f^{(k)})$ are integers, hence $T_{1,f}$ is bounded only if all but finitely many of these terms are zero, which implies that f is in fact a polynomial, since the Taylor expansion at $\zeta = 1$ is an injective homomorphism. $\qquad\square$

We just write $\mathbb{Z}[q] \subset \widehat{\mathbb{Z}[q]}$ for the functions $f = \sum_m P_m(q) \in \widehat{\mathbb{Z}[q]}$ that are in fact polynomial, as above. We let $C_{\mathfrak{t},\mathcal{Z}}^*(\mathbb{Z}[q])$ denote the C^*-subalgebra of the algebra of bounded operators on $\mathcal{H} = \mathcal{B}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}_{\geq 0}))$ generated by the operators $T_{\zeta,f}$, for $f \in \mathbb{Z}[q] \subset \widehat{\mathbb{Z}[q]}$ and $\zeta \in \mathcal{Z}$. Moreover, we denote as above with μ_n the isometries $\mu_n \epsilon_{k,m} = \epsilon_{nk,m}$. The μ_n satisfy the relations (2.3) and the commutation relations

(2.45) $$[\mu_n, \delta_k] = [\mu_n, \delta_k^*] = [\mu_n^*, \delta_k] = [\mu_n^*, \delta_k^*] = 0, \quad \forall\, n \in \mathbb{N}, \forall k \in \mathbb{Z}_{\geq 0},$$

with the shift operators δ_k of (2.42). We denote by $\mathcal{A}_{\mathfrak{t},\mathcal{Z}}$ the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by the $T_{\zeta,f}$ as above, together with the operators μ_n, μ_n^*.

Proposition 2.17. *The C^*-algebra $\mathcal{A}_{\mathfrak{t},\mathcal{Z}}$ is a semigroup crossed product*

(2.46) $$\mathcal{A}_{\mathfrak{t},\mathcal{Z}} = C_{\mathfrak{t},\mathcal{Z}}^*(\mathbb{Z}[q]) \rtimes_\rho \mathbb{N},$$

where $\rho_n(T_{\zeta,f}) = \mu_n T_{\zeta,f} \mu_n^*$. The endomorphisms ρ_n satisfy

$$(2.47) \qquad \rho_n(T_{\zeta,f}) = e_n T_{\zeta,\eta_n(f)} e_n = \frac{1}{n} \sum_{\xi^n = \zeta} ev_\xi(u) T_{\zeta,\eta_n(f)},$$

with $e_n = \mu_n \mu_n^*$ and with $u \in \widehat{\mathbb{Z}[q]}$ the class of $u(q) = q$.

Proof. The semigroup crossed product of $C^*_{\mathfrak{t},\mathcal{Z}}(\mathbb{Z}[q]) \rtimes_\rho \mathbb{N}$ is implemented by the action $\rho_n(T_{\zeta,f})\epsilon_{m,\ell} = \mu_n T_{\zeta,f} \mu_n^* \epsilon_{m,\ell} = e_n \sum_{k \geq 0} \frac{1}{k!} (f \circ \sigma_{m/n})^{(k)}(\zeta) \, e_n \, \epsilon_{m,\ell+k}$, where $e_n = \mu_n \mu_n^*$ acts as $e_n \epsilon_{m,\ell} = 0$ if $n \nmid m$ and as $e_n \epsilon_{m,\ell} = \epsilon_{m,\ell}$ if $n | m$. Thus, we obtain (2.47), where the right hand side is well defined because it is nonzero only on the elements $\epsilon_{m,\ell}$ where $n | m$, where the operator $T_{\zeta,f}$ of $\mathfrak{t}_\zeta(f)$ acts as $\mathfrak{t}_\zeta(h \circ \sigma_{m/n})$ for $h = f \circ \sigma_n \in \hat{R}_n$, so that it makes sense to apply the morphism η_n. Notice then that the action of e_n on the given Hilbert space coincides with the action of the element of $C^*_{\mathfrak{t},\mathcal{Z}}(\mathbb{Z}[q])$ given by $\sum_{\xi^n = \zeta} ev_\xi(u)$. In fact, we have

$$\frac{1}{n} \sum_{\xi^n = \zeta} ev_\xi(u)\epsilon_{m,\ell} = \frac{1}{n} \sum_{\xi^n = \zeta} \xi^m \epsilon_{m,\ell} = \left\{ \begin{array}{ll} 1 & n | m \\ 0 & n \nmid m. \end{array} \right.$$

One then argues as in Theorem 2.7 to show that both $\mathcal{A}_{\mathfrak{t},\mathcal{Z}}$ and $C^*_{\mathfrak{t},\mathcal{Z}}(\mathbb{Z}[q]) \rtimes_\rho \mathbb{N}$ are linearly generated by elements of the form $\mu_N T_{\zeta,f} \mu_M^*$, for some $N, M \in \mathbb{N}$, and for some $f \in \mathbb{Z}[q] \subset \widehat{\mathbb{Z}[q]}$ and some $\zeta \in \mathcal{Z}$, and that the product in $\mathcal{A}_{\mathfrak{t},\mathcal{Z}}$ agrees with the product given by the semigroup action by the relation $\rho_n(T_{\zeta,f}) = \mu_n T_{\zeta,f} \mu_n^*$. $\qquad\square$

In the following, we denote by $\mathcal{A}_{\mathfrak{t},\mathcal{Z},\delta}$ the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{A}_{\mathfrak{t},\mathcal{Z}}$ together with the shift operators δ_ℓ and δ_ℓ^* defined by (2.42). We also use of the shorthand notation

$$(2.48) \qquad \mathfrak{t}_{\zeta,k}(f) := \frac{1}{k!} ev_\zeta((f \circ \sigma_n)^k).$$

Lemma 2.18. *The algebra $\mathcal{A}_{\mathfrak{t},\mathcal{Z},\delta}$ is generated by the $T_{\zeta,f}$, the μ_n, μ_n^* and the δ_ℓ, δ_ℓ^*, subject to all the relations previously described for the algebra $\mathcal{A}_{\mathfrak{t},\mathcal{Z}}$, with the additional relations (2.45) and $\delta_\ell T_{\zeta,f} = T_{\zeta,f} \delta_\ell$ and*

$$(2.49) \qquad [\delta_\ell^*, T_{\zeta,f}] = Y_{\zeta,f,\ell},$$

where, for $f \in \mathbb{Z}[q] \subset \widehat{\mathbb{Z}[q]}$, the operator $Y_{\zeta,f,\ell}$ is the bounded operator on $\mathcal{H}$ acting as

$$(2.50) \qquad Y_{\zeta,f,\ell}\epsilon_{n,m} = \sum_{k \geq 0} \frac{1}{(k+\ell)!} ev_\zeta((f \circ \sigma_n)^{k+\ell})\epsilon_{n,m+k}.$$

Proof. The only statement that does not follow immediately from the definitions is the last. We check

$$\delta_\ell^* T_{\zeta,f} \, \epsilon_{n,m} = \left\{ \begin{array}{ll} \sum_{k \geq 0} \mathfrak{t}_{\zeta,k}(f \circ \sigma_n)\epsilon_{n,k+m-\ell} & k + m - \ell \geq 0 \\ 0 & k + m - \ell < 0, \end{array} \right.$$

while

$$T_{\zeta,f} \delta_\ell^* \, \epsilon_{n,m} = \left\{ \begin{array}{ll} \sum_{k \geq 0} \mathfrak{t}_{\zeta,k}(f \circ \sigma_n)\epsilon_{n,k+m-\ell} & m - \ell \geq 0 \\ 0 & m - \ell < 0, \end{array} \right. .$$

This gives $(\delta_\ell^* T_{\zeta,f} - T_{\zeta,f} \delta_\ell^*) \, \epsilon_{n,m} = \sum_{k \geq \ell-m} \mathfrak{t}_{\zeta,k}(f \circ \sigma_n) \, \epsilon_{n,m+k-\ell}$. $\qquad\square$

We now construct a time evolution on the algebra $\mathcal{A}_{\mathfrak{t},\mathcal{Z},\delta}$ introduced above, defined by analogy to the case of the BC algebra.

Proposition 2.19. *Fix a choice of a parameter $\hbar$ with $0 < \hbar < 1$, such that the only solution $(r, x) \in \mathbb{Q}_+^* \times \mathbb{Z}$ of the equation $\log r - x \log \hbar = 0$ is given by the pair $r = 1$ and $x = 0$. Then setting*

$$(2.51) \qquad \sigma_t(\delta_k) = \hbar^{-ikt}\delta_k, \qquad \sigma_t(\mu_n) = n^{it}\mu_n,$$

$$(2.52) \qquad \sigma_t(T_{\zeta,f})\epsilon_{n,m} = \sigma_t\left(\sum_{k\geq 0} \mathfrak{t}_{\zeta,k}(f \circ \sigma_n)\delta_k\right)\epsilon_{n,m} = \sum_{k\geq 0}\frac{1}{k!}(P \circ \sigma_n)^{(k)}(\zeta)\,\hbar^{-ikt}\,\delta_k\epsilon_{n,m}.$$

defines a time evolution on the algebra $\mathcal{A}_{\mathfrak{t},\mathcal{Z},\delta}$. The operator H on $\mathcal{H}$ given by

$$(2.53) \qquad H\,\epsilon_{n,m} = (\log(n) - m\log(\hbar))\,\epsilon_{n,m}$$

is a Hamiltonian for the time evolution σ_t. The partition function is of the form

$$(2.54) \qquad Z_\hbar(\beta) = \frac{\zeta(\beta)}{1 - \hbar^\beta},$$

where $\zeta(\beta)$ is the Riemann zeta function.

Proof. To see that (2.51) and (2.52) define a time evolution $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}_{\mathfrak{t},\mathcal{Z},\delta})$ we need to check the compatibility with the relations in the algebra. The action (2.52) is compatible with (2.51) and both are compatible with (2.47) and (2.49). To see that H is a Hamiltonian we check that $e^{itH}\mu_k e^{-itH}\epsilon_{n,m} = (nk)^{it}\hbar^{-itm}n^{-it}\hbar^{itm}\epsilon_{nk,m} = k^{it}\mu_k\epsilon_{n,m} = \sigma_t(\mu_k)\,\epsilon_{n,m}$ and that $e^{itH}\delta_k e^{-itH}\epsilon_{n,m} = n^{it}\hbar^{-it(m+k)}n^{-it}\hbar^{itm}\epsilon_{n,m+k} = \hbar^{-itk}\delta_k\epsilon_{n,m} = \sigma_t(\delta_k)\epsilon_{n,m}$. Similarly, we have $e^{itH}T_{\zeta,f}e^{-itH}\epsilon_{n,m} = \sum_{k\geq 0}\hbar^{-it(m+k)}\mathfrak{t}_{\zeta,k}(f \circ \sigma_n)\hbar^{itm}\epsilon_{n,m+k} = \sum_{k\geq 0}\hbar^{-ikt}\mathfrak{t}_{\zeta,k}(f \circ \sigma_n)\delta_k\epsilon_{n,m} = \sigma_t(T_{\zeta,f})\epsilon_{n,m}$. Notice that, since $k \geq 1$ and $0 < \hbar < 1$, we have $\log(k) - m\log(\hbar) \geq 0$ and it is equal to zero only for the pair $k = 1$ and $m = 0$. Moreover, for all $a \in \mathbb{R}_+$, the equation $\log(k) - m\log(\hbar) = a$ has at most one solution with $k \in \mathbb{N}$ and $m \in \mathbb{Z}_{\geq 0}$, else we would obtain a nontrivial solution $(r, x) \in \mathbb{Q}_+^* \times \mathbb{Z}$ of $\log(r) - x\log(\hbar) = 0$. This implies that the Hamiltonian has positive energy and all the eigenvalues of H have multiplicity one. The partition function is then given by

$$Z_\hbar(\beta) = \mathrm{Tr}(e^{-\beta H}) = \sum_{n,m}e^{-\beta(\log(k)-m\log(\hbar))} = \left(\sum_{n\geq 1}n^{-\beta}\right)\left(\sum_{m\geq 0}\hbar^{\beta m}\right) = \zeta(\beta)\frac{1}{1-\hbar^\beta}.$$

$\square$

Since the partition function $Z_\hbar(\beta)$ converges absolutely for $\beta > 1$, in this low temperature range of the thermodynamic parameter we have KMS states that are of the Gibbs form

$$(2.55) \qquad \varphi_\beta(a) = \frac{\mathrm{Tr}(ae^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}.$$

These satisfy the following property.

Proposition 2.20. *In the zero temperature limit the Gibbs states (2.55) converge weakly to KMS_∞ states satisfying*

$$(2.56) \qquad \varphi_\infty(T_{\zeta,f}) = \lim_{\beta\to\infty}\varphi_\beta(T_{\zeta,f}) = ev_\zeta(f).$$

Moreover, one obtains limits

$$(2.57) \qquad \lim_{\beta\to\infty}\frac{\varphi_\beta(\delta_\ell^* T_{\zeta,f})}{\hbar^{\beta\ell}} = \mathfrak{t}_{\zeta,\ell}(f),$$

that is, the ℓ-th coefficient of the Taylor expansion at ζ of the Habiro function f.

Proof. First notice that, for an element in the algebra of the form $\delta_\ell^* T_{\zeta,f}$, we have

$$(2.58) \qquad \varphi_\beta(\delta_\ell^* T_{\zeta,f}) = \hbar^{\beta\ell}(1 - \hbar^\beta)\zeta(\beta)^{-1} \sum_n \mathfrak{t}_{\zeta,\ell}(f \circ \sigma_n) n^{-\beta},$$

since $\varphi_\beta(\delta_\ell^* T_{\zeta,f}) = Z_\hbar(\beta)^{-1} \sum_{n,m}\langle \epsilon_{n,m}, \delta_\ell^* T_{\zeta,f}\epsilon_{n,m}\rangle n^{-\beta} \hbar^{\beta m}$ which gives $Z_\hbar(\beta)^{-1} \sum_n \mathfrak{t}_{\zeta,\ell}(f \circ \sigma_n) n^{-\beta} \hbar^{\beta\ell} = \hbar^{\beta\ell}(1 - \hbar^\beta)\zeta(\beta)^{-1} \sum_n \mathfrak{t}_{\zeta,\ell}(f \circ \sigma_n)n^{-\beta}$. In particular, from (2.58) we have as a special case

$$(2.59) \qquad \varphi_\beta(T_{\zeta,f}) = (1 - \hbar^\beta)\zeta(\beta)^{-1} \sum_n ev_\zeta(f \circ \sigma_n)\ n^{-\beta}.$$

One then sees from this that in the weak limit as $\beta \to \infty$ one obtains $\lim_{\beta\to\infty} \varphi_\beta(\delta_\ell^* T_{\zeta,f}) = \lim_{\beta\to\infty} \hbar^{\beta\ell}(1-\hbar^\beta)\zeta(\beta)^{-1} \sum_n \mathfrak{t}_{\zeta,\ell}(f\circ\sigma_n)n^{-\beta}$. This limit is zero if $\ell \neq 0$, while for $\ell = 0$ it gives the projection onto the kernel of the Hamiltonian H, which is the span of the vacuum vector $\epsilon_{1,0}$, hence KMS$_\infty$ states are of the form $\langle \epsilon_{1,0}, T_{\zeta,f}\epsilon_{1,0}\rangle$, which gives (2.56). Similarly, the limits of (2.57) give $\lim_{\beta\to\infty}(1 - \hbar^\beta)\zeta(\beta)^{-1} \sum_n \mathfrak{t}_{\zeta,\ell}(f \circ \sigma_n)n^{-\beta} = \langle \epsilon_{1,0}, \delta_\ell^* T_{\zeta,f}\epsilon_{1,0}\rangle = \mathfrak{t}_{\zeta,\ell}(f)$. $\square$

We then consider the action of $\hat{\mathbb{Z}}^* \cong \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$ on the algebra $\mathcal{A}_{\mathfrak{t},\mathcal{Z}}$ given by

$$\alpha : T_{\zeta,f} \mapsto T_{\alpha(\zeta),f},$$

with $\alpha(\mu_n) = \mu_n$ and $\alpha(\mu_n^*) = \mu_n^*$. We recover the intertwining property of symmetries and Galois action as in the BC system in the case of KMS$_\infty$ states evaluated at elements of the rational subalgebra $\mathcal{A}_{\mathfrak{t},\mathcal{Z},\mathbb{Q}}$, by

$$(2.60) \qquad \varphi_\infty(\alpha(T_{\zeta,f})) = ev_{\alpha(\zeta)}(f) = \alpha(ev_\zeta(f)) = \alpha(\varphi_\infty(T_{\zeta,f})).$$

As $\langle \epsilon_{1,0}, \delta_\ell^* T_{\zeta,f}\epsilon_{1,0}\rangle = \mathfrak{t}_{\zeta,\ell}(f)$ also holds in the case of unbounded $T_{\zeta,f}$, these last observations apply to non-polynomial $f \in \widehat{\mathbb{Z}[q]}$.

3. Multivariable Bost–Connes systems

In [44], Manin introduced multivariable versions of the Habiro ring. We recall here the construction and we show that these also admit semigroups of endomorphisms coming from self maps of higher dimensional algebraic tori. This then serves as a guideline for a construction we present in this section of multivariable BC endomotives, obtained from iterations of self-maps of higher dimensional tori.

3.1. Multivariable Habiro rings and endomorphisms.
The multivariable versions of the Habiro ring are defined in [44] as projective limits, as in the one variable case, by setting

$$(3.1) \qquad \mathbb{Z}\widehat{[q_1, \ldots,} q_n] = \varprojlim_N \mathbb{Z}[q_1, \ldots, q_n]/I_{n,N},$$

where $I_{n,N}$ is the ideal

$$(3.2) \qquad I_{n,N} = \left((q_1 - 1)(q_1^2 - 1)\cdots(q_1^N - 1), \ldots, (q_n - 1)(q_n^2 - 1)\cdots(q_n^N - 1)\right).$$

As shown in [44], like in the single variable case, the rings (3.1) have evaluations at roots of unity. Namely, given a vector $Z = (\zeta_1, \ldots, \zeta_n)$ in $\mathcal{Z}^n$, there is a ring homomorphism

$$(3.3) \qquad ev_Z : \mathbb{Z}\widehat{[q_1, \ldots,} q_n] \to \mathbb{Z}[\zeta_1, \ldots, \zeta_n].$$

Moreover, the functions in the multivariable Habiro rings have a Taylor expansion at points $Z = (\zeta_1, \ldots, \zeta_n)$ in $\mathcal{Z}^n$. The Taylor expansion is an *injective* ring homomorphism

$$(3.4) \qquad \mathfrak{t}_{n,Z} : \mathbb{Z}\widehat{[q_1, \ldots,} q_n] \to \mathbb{Z}[\zeta_1, \ldots, \zeta_n][[q_1 - \zeta_1, \ldots, q_n - \zeta_n]].$$

We introduce the following notation. Given an integer $n \times n$-matrices with positive determinant $\alpha \in M_n(\mathbb{Z})^+$, and given $q = (q_1, \ldots, q_n)$, we write q^α for

$$(3.5) \qquad q^\alpha = (q_i^\alpha)_{i=1,\ldots,n}, \quad \text{with} \quad q_i^\alpha = \prod_j q_j^{\alpha_{ij}}.$$

We show now that the semigroup $S_n = M_n(\mathbb{Z})^+$ acts by endomorphisms on the multivariable Habiro ring.

Proposition 3.1. *For $\alpha \in M_n(\mathbb{Z})^+$, the ring homomorphisms*

$$\sigma_\alpha : \mathbb{Z}[q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}] \to \mathbb{Z}[q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}]$$

given by

$$(3.6) \qquad \qquad \qquad \sigma_\alpha : q \mapsto q^\alpha,$$

with q^α defined as in (3.5), induces a ring homomorphism

$$\sigma_\alpha : \mathbb{Z}[\widehat{q_1, \ldots,} q_n] \to \mathbb{Z}[\widehat{q_1, \ldots,} q_n]$$

of the multivariable Habiro ring.

Proof. First notice that an equivalent presentation of the Habiro ring can be given as

$$(3.7) \qquad \begin{aligned} \mathbb{Z}[\widehat{q_1, \ldots,} q_n] &= \varprojlim_N \mathbb{Z}[q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}] / \mathcal{J}_{n,N} \\ &= \varprojlim_N \mathbb{Z}[q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}] / \mathcal{I}_{n,N}, \end{aligned}$$

where $\mathcal{J}_{n,N}$ is the ideal generated by the polynomials $(q_i - 1) \cdots (q_i^N - 1)$, for $i = 1, \ldots, n$ and $(q_i^{-1} - 1) \cdots (q_i^{-N} - 1)$ and $\mathcal{I}_{n,N}$ is the ideal generated by the $(q_i - 1) \cdots (q_i^N - 1)$, for $i = 1, \ldots, n$.
For $\alpha \in M_n(\mathbb{Z})^+$, the map

$$(3.8) \qquad q \mapsto \sigma_\alpha(q) = \sigma_\alpha(q_1, \ldots, q_n) = (q_1^{\alpha_{11}} q_2^{\alpha_{12}} \cdots q_n^{\alpha_{1n}}, \ldots, q_1^{\alpha_{n1}} q_2^{\alpha_{n2}} \cdots q_n^{\alpha_{nn}}) = q^\alpha$$

defines a ring homomorphism of $\mathbb{Z}[q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}]$. We want to show that it induces a morphism of the projective limits of the quotients by the ideals $\mathcal{I}_{n,N}$. To this purpose it is useful to make another preliminary observation, namely that, as shown in [30], the ring $\widehat{\mathbb{Z}[q]}$ can equivalently be described as

$$(3.9) \qquad \qquad \widehat{\mathbb{Z}[q]} = \varprojlim_{f \in \Phi^*} \mathbb{Z}[q]/(f(q)),$$

where Φ^* is the multiplicative subset of $\mathbb{Z}[q]$ generated by the cyclotomic polynomials $\Phi_N(q)$. Similarly, we can describe the multivariable versions of the Habiro ring also in terms of the cyclotomic polynomials instead of the polynomials $(q)_N$ by

$$(3.10) \qquad \mathbb{Z}[\widehat{q_1, \ldots,} q_n] = \varprojlim_{f_i \in \Phi^*} \mathbb{Z}[q_1, \ldots, q_n]/(f_1(q_1), \ldots, f_n(q_n)).$$

The image under σ_α of the polynomials $(q_i - 1) \cdots (q_i^N - 1)$ is given by the polynomials

$$P_i^\alpha(q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}) = (\prod_{j=1}^n q_j^{\alpha_{ij}} - 1) \cdots (\prod_{j=1}^n q_j^{N\alpha_{ij}} - 1).$$

We show that there are $N_i \in \mathbb{N}$ such that there is an inclusion of ideals

$$(3.11)$$
$$(\Phi_{N_1}(q_1), \ldots, \Phi_{N_n}(q_n)) \supset (P_1^\alpha(q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}), \ldots, P_n^\alpha(q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1})),$$

so that we can view the map σ_α as inducing a homomorphism

$$\sigma_\alpha : \mathbb{Z}[q_1,\dots,q_n]/\mathcal{I}_{n,N} \to \mathbb{Z}[q_1,\dots,q_n]/(\Phi_{N_1}(q_1),\dots,\Phi_{N_n}(q_n)).$$

Let ζ_i be a solution of $\Phi_{N_i}(q_i) = 0$. Then $\zeta_i^{N_i} = 1$ and all these roots appear with multiplicity one, which is the advantage of using the cyclotomic polynomials as opposed to the $(q)_N$ here. Substituting in the polynomial P_i^α we obtain the expressions

$$\prod_{k=1}^{N}(\prod_{j=1}^{n} \zeta_j^{\alpha_{ij}k} - 1),$$

for $i = 1,\dots,n$. Thus, if we choose each N_j in such a way that $N_j | \alpha_{ij} k_i$ for some $k_i \in \{1,\dots,N\}$ and for all α_{ij}, we see that the ζ_i that are solutions of $\Phi_{N_i}(q_i) = 0$ are also solutions of $P_i^\alpha(q_1,\dots,q_n,q_1^{-1},\dots,q_n^{-1}) = 0$. $\qquad\square$

It is then possible to consider, as in the one-variable case, an associated non commutative ring obtained as the (semi)group crossed product

$$(3.12)\qquad\qquad \mathcal{A}_{\mathbb{Z},q,n} = \mathbb{Z}[\widehat{q_1,\dots,q_n}]_\infty \rtimes \mathrm{GL}_n(\mathbb{Z})^+,$$

where $\mathcal{A}_{\mathbb{Z},q,n}$ is the ring generated by the multivariable Habiro functions $f \in \mathbb{Z}[\widehat{q_1,\dots,q_n}]$, together with additional generators μ_α and μ_α^* subject to relations

$$(3.13)\qquad \begin{aligned} (\mu_\alpha^*)^* &= \mu_\alpha && \forall\alpha \in M_n(\mathbb{Z})^+ \\ \mu_\alpha\mu_\beta &= \mu_{\alpha\beta}, && \forall\alpha,\beta \in M_n(\mathbb{Z})^+ \\ \mu_\alpha^*\mu_\beta &= \mu_\gamma && \text{for } \beta = \alpha\gamma \in M_n(\mathbb{Z})^+. \end{aligned}$$

In particular, the μ_α are isometries satisfying $\mu_\alpha^*\mu_\alpha = 1$ for all $\alpha \in M_n(\mathbb{Z})^+$. We also have in $\mathcal{A}_{\mathbb{Z},q,n}$ the additional relation

$$(3.14)\qquad\qquad \mu_\alpha^* f = \sigma_\alpha(f)\,\mu_\alpha^*, \quad \text{and} \quad f\,\mu_\alpha = \mu_\alpha\,\sigma_\alpha(f),$$

for all $f \in \mathbb{Z}[\widehat{q_1,\dots,q_n}]$ and $\alpha \in M_n(\mathbb{Z})^+$.

The semigroup crossed product structure (3.12) is given as in Theorem 2.7 for the one-variable case, with the ring $A_{\infty,n} = \mathbb{Z}[\widehat{q_1,\dots,q_n}]_\infty$ the inductive limit of the system of the $\sigma_\alpha : \mathbb{Z}[\widehat{q_1,\dots,q_n}] \to \mathbb{Z}[\widehat{q_1,\dots,q_n}]$. As in the 1-dimensional case, one obtains induced homomorphisms $\sigma_\alpha : \mathbb{Z}[\widehat{q_1,\dots,q_n}]_\infty \to \mathbb{Z}[\widehat{q_1,\dots,q_n}]_\infty$ that are now invertible with inverses implementing the group action in (3.12), given by

$$(3.15)\qquad\qquad \rho_\alpha(x) = \mu_\alpha\,x\,\mu_\alpha^*, \quad \text{for } x \in A_{\infty,n}.$$

As in Proposition 2.8, this can also be seen by considering the ring $\mathcal{P}_{\mathbb{Z},n}$ of polynomials in n variables $(q_1^{r_1},\cdots,q_n^{r_n})$ that are fractional powers of the variables q_i, with $r_i \in \mathbb{Q}_+^*$, and the projective limit

$$\hat{\mathcal{P}}_{\mathbb{Z},n} = \varprojlim_N \mathcal{P}_{\mathbb{Z},n}/\mathfrak{I}_{n,N},$$

with respect to the ideals

$$\mathfrak{I}_{n,N} = ((q_1^{r_1} - 1)\cdots(q_1^{r_1 N} - 1),\dots,(q_n^{r_n} - 1)\cdots(q_n^{r_n N} - 1)).$$

for $r_i \in \mathbb{Q}_+^*$, endowed with the action $\rho_\alpha(f)(q^r) = f(q^{\alpha^{-1}r})$ and $\sigma_\alpha(f)(q^r) = f(q^{\alpha r})$, with $q^r = (q_1^{r_1},\dots,q_n^{r_n})$, where for $\alpha \in \mathrm{GL}_n(\mathbb{Q})^+$ we write

$$q^{\alpha r} = (q_1^{\alpha_{11}r_1} q_2^{\alpha_{12}r_2}\cdots q_n^{\alpha_{1n}r_n},\dots,q_1^{\alpha_{n1}r_1} q_2^{\alpha_{n2}r_2}\cdots q_n^{\alpha_{nn}r_n}).$$

3.2. **Multivariable BC endomotives.** The construction described above using the multivariable Habiro rings immediately suggests the existence of an interesting class of "multivariable" generalizations of the BC endomotive. Just like the original BC endomotive is obtained in [14] using self-maps of the multiplicative group $\mathbb{G}_m$, as recalled in §2.1 above, the multivariable versions are similarly associated to self-maps of higher rank algebraic tori.

Unlike the case of the action on the Habiro rings, in this case as in the case of the original BC system, we find semigroup actions of $M_n(\mathbb{Z})^+$ that are implemented by isometries hence they do not extend to group actions of $\mathrm{GL}_n(\mathbb{Q})^+$.

We let $\mathbb{T}^n = (\mathbb{G}_m)^n$ be an n-dimensional (split) algebraic torus. We then consider the multiplicative semigroup of self-maps of $\mathbb{T}^n$ that is the analog of the semigroup $\mathbb{N}$ acting by (2.8) on the 1-dimensional torus $\mathbb{G}_m$.

Lemma 3.2. *The semigroup $M_n(\mathbb{Z})^+$ acts by endomorphisms of the torus $\mathbb{T}^n = (\mathbb{G}_m)^n$. The elements of $\mathrm{SL}_n(\mathbb{Z}) \subset M_n(\mathbb{Z})^+$ act by automorphisms. The point $t_0 = (1, 1, \ldots, 1) \in \mathbb{T}^n$ is a fixed point of all the endomorphisms in the semigroup $S_n = M_n(\mathbb{Z})^+$.*

Proof. This can be seen directly using the exponential map

$$0 \to \mathbb{Z} \to \mathbb{C} \stackrel{\exp(2\pi i \cdot)}{\to} \mathbb{C}^* \to 1$$

and the action of $M_n(\mathbb{Z})^+$ on $\mathbb{C}^n$, preserving the lattice $\mathbb{Z}^n$ and the orientation. The action of $\alpha \in M_n(\mathbb{Z})^+$ on the algebra $\mathbb{C}[t_i, t_i^{-1}]$ of the n-torus is given by

$$(3.16) \qquad s_\alpha : P(t_i, t_i^{-1}) \mapsto P(\exp(2\pi i \sum_j \alpha_{ij} u_j), \exp(-2\pi i \sum_j \alpha_{ij} u_j)),$$

where $t_i = \exp(2\pi i u_i)$ for $i = 1, \ldots, n$. Since all the endomorphisms are induced by linear maps on $\mathbb{C}^n$, the point t_0 is a fixed point for all $\alpha \in S_n$. $\qquad \square$

We define as above t^α to be the result of applying $\alpha \in M_n(\mathbb{Z})^+$ to the coordinates $t = (t_i)$ of $\mathbb{T}^n$,

$$(3.17) \qquad t^\alpha := (\exp(2\pi i \sum_j \alpha_{ij} u_j))_{j=1,\ldots,n}, \quad \text{for } t_i = \exp(2\pi i u_i),$$

that is, equivalently

$$(3.18) \qquad t^\alpha = (t_i^\alpha)_{i=1,\ldots,n} \quad \text{with} \quad t_i^\alpha = \prod_j t_j^{\alpha_{ij}}.$$

Under multiplication in the semigroup $M_n(\mathbb{Z})^+$ this satisfies

$$(3.19) \qquad t^{\alpha\beta} = (t^\beta)^\alpha.$$

For example, in the case of $\alpha \in M_2(\mathbb{Z})^+$ this gives

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad t = (t_1, t_2) \mapsto t^\alpha = (t_1^a t_2^b, t_1^c t_2^d).$$

The action (3.16) is then given equivalently as the endomorphisms of the rings $\mathbb{Q}[t_i, t_i^{-1}]$ determined by setting

$$(3.20) \qquad s_\alpha : t \mapsto t^\alpha,$$

for $t = (t_1, \ldots, t_n)$ and t^α as in (3.18).

Notice that, unlike in the setting of endomotives defined in [14], here we allow semigroups of endomotives that are non-abelian, like our $S_n = M_n(\mathbb{Z})^+$. This contains interesting abelian subsemigroups, in particular the diagonal subsemigroup $S_{n,diag} = \mathbb{N}^n$ acting by

$$(3.21) \qquad s_k : P(t_i, t_i^{-1}) \mapsto P(t_i^{k_i}, t_i^{-k_i}), \quad i = 1, \ldots, n, \quad k = (k_1, \ldots, k_n) \in \mathbb{N}^n.$$

on the ring of functions of $\mathbb{T}^n$. We can similarly restrict to other interesting abelian subsemigroups of $M_n(\mathbb{Z})^+$. For example, one can consider the set of n-tuples

$$(3.22) \qquad S_{n,ord} := \{(k_1, \ldots, k_n) \in \mathbb{N}^n \,|\, k_1 \geq \cdots \geq k_n\}.$$

For $\alpha \in M_n(\mathbb{Z})^+$, we denote by X_α the preimage

$$(3.23) \qquad X_\alpha = \{t = (t_1, \ldots, t_n) \in \mathbb{T}^n \,|\, s_\alpha(t) = t_0\}.$$

The X_α form an inverse system as in [14] with the maps

$$(3.24) \qquad \xi_{\alpha,\beta} : X_\beta \to X_\alpha, \quad t \mapsto t^\gamma,$$

as in (3.17), whenever $\alpha = \beta\gamma \in M_n(\mathbb{Z})^+$. We denote by $X = \varprojlim_\alpha X_\alpha$ the inverse limit of this family and by $A = \varinjlim_\alpha A_\alpha$ the corresponding commutative algebra, with $X = \mathrm{Spec}(A)$. We have

$$(3.25) \qquad X_\alpha = \mathrm{Spec}(\mathbb{Q}[t_i, t_i^{-1}]/(t^\alpha - t_0)),$$

with t^α as in (3.18).

The multivariable analog of the BC endomotive is then given by the action on X of the multiplicative semigroup $S_n = M_n(\mathbb{Z})^+$. We have the following more concrete description of the space $X(\bar{\mathbb{Q}})$.

Lemma 3.3. *Any choice of an element $\rho \in GL_n(\hat{\mathbb{Z}})$ determines an identification between the set of algebraic points $X(\bar{\mathbb{Q}})$ of the abelian part $X = \mathrm{Spec}(A)$ of the multivariable Bost–Connes endomotive and the set*

$$(3.26) \qquad \rho(X(\bar{\mathbb{Q}})) = \mathcal{Z}^n = \underbrace{\mathcal{Z} \times \cdots \times \mathcal{Z}}_{n-\text{times}},$$

where, as above, $\mathcal{Z}$ is the set of all roots of unity in $\mathbb{C}$.

Proof. We show that the projective limit X computed over the system of maps $\xi_{\alpha,\beta}$ of (3.24) is in fact the same as computing it over the smaller system given by considering only the diagonal maps of (3.21). Since $S_{n,diag}$ is a subsemigroup of $M_n(\mathbb{Z})^+$, we have a surjective map of the projective limits

$$X = \varprojlim_{\alpha \in M_n(\mathbb{Z})^+} X_\alpha \to X' = \varprojlim_{k \in S_{n,diag}} X_k.$$

On the other hand, to see that we also have a map in the opposite direction, notice that, for any given $\alpha \in M_n(\mathbb{Z})^+$ we can find a $k = k(\alpha) \in S_{n,diag}$ such that there is a map in the projective system (3.24)

$$(3.27) \qquad \xi_{\alpha,k(\alpha)} : X_{k(\alpha)} \to X_\alpha.$$

Let $k(\alpha) = (k_1, \ldots, k_n)$ be such that $k_i = \gcd\{|\alpha_{ij}|\}_{j=1,\ldots,n}$. Then there exists $\beta \in M_n(\mathbb{Z})^+$ such that $\alpha = k(\alpha)\beta$. This implies that there is a map in the projective system of the X_α of the form (3.27). The $X_{k(\alpha)}$ form an inverse system, since for $\alpha = \beta\gamma \in M_n(\mathbb{Z})^+$ we have $\alpha k(\beta)^{-1} k(\gamma)^{-1} \in M_n(\mathbb{Z})^+$ hence $(k(\beta)k(\gamma))_i | \alpha_{ij}$ for all $j = 1, \ldots, n$, which means that $(k(\beta)k(\gamma))_i | k(\alpha)_i$ hence there is some $k \in S_{n,diag}$ such that $k(\alpha) = k(\beta)k$, which gives a map

$$(3.28) \qquad \xi_{k(\alpha),k(\beta)} : X_{k(\beta)} \to X_{k(\alpha)}.$$

This shows that we also have a map of inverse limits $X' \to X$ and that therefore the inverse limits agree.

The inverse system (3.28) of the X_k for $k \in S_{n,diag}$ corresponds dually to a direct system of algebras $A_k = \mathbb{Q}[t_i, t_i^{-1}]/(t^{k_i} - 1)$, with $k \in \mathbb{N}^n$, whose direct limit is then just $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$. We then use an element of $\mathrm{GL}_n(\hat{\mathbb{Z}})$ to give an identification of $(\mathbb{Q}/\mathbb{Z})^n$ with $\mathscr{Z}^n \subset \mathbb{C}^n$, as one does with $\hat{\mathbb{Z}}^*$ in the case of the one-dimensional BC endomotive. $\square$

The result of Lemma 3.3 is simply showing that the set $X(\bar{\mathbb{Q}})$ that defines the analytic endomotive of the multivariable Bost–Connes system consists of all the torsion points of the n-torus $\mathbb{T}^n$. In particular, it follows that the algebra $C(X(\bar{\mathbb{Q}}))$ has an explicit presentation with generators $e(r_1) \otimes \cdots \otimes e(r_n)$, for $r_i \in \mathbb{Q}/\mathbb{Z}$, where the $e(r_i)$ are the generators of the abelian part of the 1-dimensional BC algebra and satisfying the same relations.

To construct the noncommutative crossed product algebra for the multivariable BC endomotives, we consider, as in the one-variable case, isometries implementing the action of the semigroup of endomorphisms. As in the previous discussion for the multivariable Habiro ring, we consider isometries μ_α as in (3.13). Notice that these can be represented as isometries on the Hilbert space $\mathcal{H}_n = \ell^2(M_n(\mathbb{Z})^+)$ by setting

$$(3.29) \qquad \mu_\alpha \epsilon_\beta = \epsilon_{\alpha\beta}.$$

with the adjoints acting as

$$(3.30) \qquad \mu_\alpha^* \epsilon_\beta = \begin{cases} \epsilon_\gamma & \beta = \alpha\gamma \in M_n(\mathbb{Z})^+ \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have representations of the abelian part $C(X(\bar{\mathbb{Q}}))$ of the algebra on the same Hilbert space by the following.

Lemma 3.4. *The choice of an element $\rho \in \mathrm{GL}_n(\hat{\mathbb{Z}}^*)$ determines a representation of the algebra $C(X(\bar{\mathbb{Q}}))$ on $\ell^2(M_n(\mathbb{Z})^+)$.*

Proof. This is given on the generators $e(r_1) \otimes \cdots \otimes e(r_n)$ by

$$(3.31) \qquad (e(r_1) \otimes \cdots \otimes e(r_n)) \, \epsilon_\beta = \left(\prod_{i=1}^n \zeta_i^{\tilde{\beta}} \right) \epsilon_\beta,$$

where $\zeta = (\zeta_{r_1}, \ldots, \zeta_{r_n}) = \rho(r_1, \ldots, r_n) \in \mathscr{Z}^n$, with the transpose $\tilde{\beta} = \beta^\dagger \in M_n(\mathbb{Z})^+$ and $\zeta^{\tilde{\beta}} = (\zeta_i^{\tilde{\beta}})$ is defined as in (3.18). $\square$

Let σ_α for $\alpha \in M_n(\mathbb{Z})^+$ denote the endomorphisms of the algebra $C(X(\bar{\mathbb{Q}}))$ induced by the s_α of (3.16). The relation between the isometries μ_α and the endomorphisms σ_α is as follows.

Lemma 3.5. *The endomorphisms σ_α of the multivariable BC endomotive defined by the torus maps (3.16) are given by*

$$(3.32) \qquad \sigma_\alpha(e(\underline{r})) = \mu_\alpha^* \, e(\underline{r}) \, \mu_\alpha,$$

for $\underline{r} = (r_1, \ldots, r_n) \in (\mathbb{Q}/\mathbb{Z})^n$.

Proof. For $e(\underline{r}) = e(r_1) \otimes \cdots \otimes e(r_n)$, the operator $\mu_\alpha^* \, e(\underline{r}) \, \mu_\alpha$ acts on a basis element ϵ_β by

$$\mu_\alpha^* \, e(\underline{r}) \, \mu_\alpha \, \epsilon_\beta = \left(\prod_{i=1}^n \zeta_i^{\widetilde{\alpha\beta}} \right) \epsilon_\beta = \left(\prod_{i=1}^n \zeta_i^{\tilde{\beta}\tilde{\alpha}} \right) \epsilon_\beta = \left(\prod_{i=1}^n (\zeta^{\tilde{\alpha}})_i^{\tilde{\beta}} \right) \epsilon_\beta = \left(\prod_{i=1}^n \xi_i^{\tilde{\beta}} \right) \epsilon_\beta,$$

where $\xi = \zeta^{\tilde{\alpha}}$. $\square$

Similarly, the following result gives the direct analog of the relation

$$\mu_k e(r) \mu_k^* = \frac{1}{k} \sum_{ks=r} e(s)$$

of the 1-dimensional case.

Lemma 3.6. *The operators μ_α and $e(\underline{r}) = e(r_1) \otimes \cdots \otimes e(r_n)$ satisfy the relation*

$$(3.33) \qquad\qquad \mu_\alpha \, e(\underline{r}) \, \mu_\alpha^* = \frac{1}{\det \alpha} \sum_{\alpha(\underline{s})=\underline{r}} e(\underline{s}).$$

for $\underline{r} = (r_1, \dots, r_n)$ and $\alpha(\underline{s}) = (\alpha(\underline{s})_i) \in (\mathbb{Q}/\mathbb{Z})^n$.

Proof. We first show that the number of solutions to $\alpha(\underline{s}) = \underline{r}$ is equal to $\det(\alpha) \in \mathbb{N}$. In fact, the number of such solutions is equal to the number of solutions of the equation $\alpha(\underline{s}) = 0$. Any $\underline{s} = (s_i)$ such that all the s_i satisfy $\det(\alpha)s_i = 0$ is a solution, since we can write $\alpha = \det(\alpha)\hat{\alpha}$ so that if $\det(\alpha)\underline{s} = 0$ then also $\alpha(\underline{s}) = 0$. Conversely, a solution of $\alpha(\underline{s}) = 0$ will also satisfy $\det(\alpha)s_i = 0$. In fact, the entries s_i are torsion of orders that divide $\det(\alpha)$. This one can see easily in the case of $\alpha \in M_2(\mathbb{Z})^+$, where

$$\begin{cases} as_1 + bs_2 = 0 \\ cs_1 + ds_2 = 0 \end{cases} \Rightarrow (ad - bc)s_1 = 0 \quad \text{and} \quad (ad - bc)s_2 = 0.$$

The general case is completely analogous. We then have

$$\mu_\alpha \, e(\underline{r}) \, \mu_\alpha^* \, \epsilon_\beta = 0$$

if $\alpha^{-1}\beta \notin M_n(\mathbb{Z})^+$, while otherwise

$$\mu_\alpha \, e(\underline{r}) \, \mu_\alpha^* \, \epsilon_\beta = \left(\prod_{i=1}^n \zeta_i^{\widetilde{\alpha^{-1}\beta}} \right) \epsilon_\beta = \left(\prod_{i=1}^n \zeta_i^{\tilde{\beta}\widetilde{\alpha^{-1}}} \right) \epsilon_\beta = \left(\prod_{i=1}^n (\zeta^{\widetilde{\alpha^{-1}}})_i^{\tilde{\beta}} \right) \epsilon_\beta = \left(\prod_{i=1}^n \xi_i^{\tilde{\beta}} \right) \epsilon_\beta,$$

where ξ is such that $\xi^{\tilde{\alpha}} = \zeta$. Notice that, if ξ_1 and ξ_2 both satisfy $\xi_k^{\tilde{\alpha}} = \zeta$, for $k = 1, 2$, we have

$$\zeta^{\widetilde{\alpha^{-1}\beta}} = (\xi_k^{\tilde{\alpha}})^{\widetilde{\alpha^{-1}\beta}} = (\xi_k^{\tilde{\alpha}})^{\tilde{\beta}\widetilde{\alpha^{-1}}} = \xi_k^{\tilde{\beta}}.$$

Thus, the expression above for $\mu_\alpha e(\underline{r})\mu_\alpha^*$ does not depend on the choice of a ξ with $\xi^\alpha = \zeta$, *i.e.* of an $\underline{s}$ with $\alpha(\underline{s}) = \underline{r}$, so that it can be written equivalently in the form (3.33). $\square$

The noncommutative algebra of the multivariable BC endomotive is the rational semigroup crossed product algebra

$$(3.34) \qquad\qquad \mathcal{A}_{\mathbb{Q},BC,n} = A \rtimes_\rho S_n \cong \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n} \rtimes_\rho M_n(\mathbb{Z})^+,$$

and the C^*-algebra that gives the analytic endomotive is obtained as

$$(3.35) \qquad\qquad \mathcal{A}_{BC,n} = C(X(\bar{\mathbb{Q}})) \rtimes_\rho S_n \cong C^*(\mathbb{Q}/\mathbb{Z})^{\otimes n} \rtimes_\rho M_n(\mathbb{Z})^+,$$

where the semigroup action is given by $\rho_\alpha(e(\underline{r})) = \mu_\alpha e(\underline{r}) \mu_\alpha^*$, as in (3.33).

It is natural to consider on the algebra $\mathcal{A}_{BC,n}$ a time evolution and quantum statistical mechanical properties that generalize the corresponding ones of the BC system. However, as we discuss briefly here and more in detail in §3.3 below, the natural generalization of the BC dynamics runs into a problem coming from the presence of infinite multiplicities in the spectrum of the Hamiltonian. We show in §3.3 that this problem can be resolved following the same method used in the case of 2-dimensional $\mathbb{Q}$-lattices in [20].

Lemma 3.7. *The multivariable Bost–Connes algebra $\mathcal{A}_{BC,n} = C^*(\mathbb{Q}/\mathbb{Z})^{\otimes n} \rtimes_\rho M_n(\mathbb{Z})^+$ has a time evolution $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}_n)$ defined by setting*

$$(3.36) \qquad \sigma_t(\mu_\alpha) = \det(\alpha)^{it}\, \mu_\alpha, \quad \forall \alpha \in M_n(\mathbb{Z})^+, \quad and \quad \sigma_t(e(\underline{r})) = e(\underline{r}) \quad \forall \underline{r} \in (\mathbb{Q}/\mathbb{Z})^n.$$

Let H be the Hamiltonian H that generates this time evolution in a representation π_ρ of $\mathcal{A}_{BC,n}$ on $\mathcal{H}_n$, with $\rho \in \mathrm{GL}_n(\hat{\mathbb{Z}})$. The operator H has eigenvalues $\log\det(\alpha)$ with infinite multiplicities.

Proof. It is easy to check that (3.36) defines a 1-parameter family $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}_{BC,n})$ of automorphisms. Recall that a self-adjoint positive (unbounded) linear operator H on $\mathcal{H}_n$ is a Hamiltonian for the time evolution σ_t, in a representation π_ρ of the algebra $\mathcal{A}_{BC,n}$, if it satisfies

$$\pi_\rho(\sigma_t(a)) = e^{itH}\, \pi_\rho(a)\, e^{-itH}, \quad \forall a \in \mathcal{A}_{BC,n}, \quad \forall t \in \mathbb{R}.$$

Let H be the operator on $\mathcal{H}_n$ defined by

$$(3.37) \qquad\qquad\qquad H\,\epsilon_\beta = \log\det(\beta)\,\epsilon_\beta.$$

We have

$$e^{itH}\, \mu_\alpha e^{-itH} \epsilon_\beta = \det(\beta)^{-it}\det(\alpha\beta)^{it}\epsilon_{\alpha\beta} = \det(\alpha)^{it}\epsilon_{\alpha\beta} = \sigma_t(\mu_\alpha)\epsilon_\beta.$$

Similarly

$$e^{itH}\, \pi_\rho(e(\underline{r}))\, e^{-itH}\epsilon_\beta = \pi_\rho(e(\underline{r}))\epsilon_\beta = \pi_\rho(\sigma_t(e(\underline{r})))\epsilon_\beta.$$

Thus H is a Hamiltonian for the time evolution in all the representations π_ρ with $\rho \in \mathrm{GL}_n(\hat{\mathbb{Z}})$. Any other choice of Hamiltonian differs by a constant, $H' = H + \log\lambda$, for some $\lambda \in \mathbb{R}_+^*$. The Hamiltonian has infinite multiplicities in the spectrum, since all basis vectors $\epsilon_{\gamma\beta}$ and $\epsilon_{\beta\gamma}$ with $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ are eigenvalues with the same eigenvector $\log\det(\beta)$. $\square$

3.3. **Multivariable BC endomotive and automorphisms.**

One new feature of the multivariable case is the presence of a large subgroup of automorphisms in the semigroup of endomorphisms defining the endomotive, here given by $\mathrm{SL}_n(\mathbb{Z}) \subset M_n(\mathbb{Z})^+$. We have seen above how this creates a problem of infinite multiplicities in the spectrum of the Hamiltonian generating the time evolution. The same phenomenon occurred already in the construction of the quantum statistical mechanical system for 2-dimensional $\mathbb{Q}$-lattices in [20] and we will treat it here in a similar manner. Namely, instead of working with the C^*-algebra $C(X(\bar{\mathbb{Q}})) \rtimes M_n(\mathbb{Z})^+$, for the purpose of quantum statistical mechanics we can replace it with a convolution algebra where we mod out the automorphisms.

The algebra, representations, and time evolution we describe here are a minor variant over the one introduced in [20], as "determinant part of the GL_2-system". Here we deal with the higher rank case of GL_n and with vectors in $\hat{\mathbb{Z}}^n$ instead of matrices in $M_n(\hat{\mathbb{Z}})$, but we make use of the same technique of passing to a type II_1 factor and corresponding trace to eliminate the infinite multiplicities, as in [20], §1.7.

3.3.1. Groupoid, convolution algebra, and type II_1 factor. We let

$$(3.38) \qquad\qquad \mathcal{U} = \{(\alpha, \rho) \in \mathrm{GL}_n(\mathbb{Q})^+ \times \hat{\mathbb{Z}}^n \mid \alpha(\rho) \in \hat{\mathbb{Z}}^n\}.$$

The set $\mathcal{U}$ is a groupoid with source and target maps $s(\alpha, \rho) = \rho$ and $t(\alpha, \rho) = \alpha(\rho)$ and with composition law $(\alpha_2, \rho_2) \circ (\alpha_1, \rho_1) = (\alpha_2\alpha_1, \rho_1)$ if $\rho_2 = \alpha_1(\rho_1)$. The inverse of the arrow (α, ρ) in the groupoid $\mathcal{U}$ is the arrow $(\alpha^{-1}, \alpha(\rho))$.

We also define a 2-groupoid $\mathcal{U}_\Gamma$ where the objects and 1-morphisms are those of the groupoid $\mathcal{U}$ and the 2-morphisms are given by the action

$$(\gamma_1, \gamma_2) : (\alpha, \rho) \mapsto (\gamma_1\alpha\gamma_2^{-1}, \gamma_2(\rho)),$$

of $\Gamma \times \Gamma$, with $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. This 2-groupoid is a way to represent the quotient

$$(3.39) \qquad \{(\alpha, \rho) \in \Gamma \backslash \mathrm{GL}_n(\mathbb{Q})^+ \times_\Gamma \hat{\mathbb{Z}}^n \,|\, \alpha(\rho) \in \hat{\mathbb{Z}}^n\},$$

in a way that avoids actually taking the quotient by the $\Gamma \times \Gamma$-action, which would not lead to a good quotient in the ordinary sense. The 2-groupoid $\mathcal{U}_\Gamma$ provides a version of the multivariable Bost–Connes algebras that eliminates the infinite multiplicities due to the action of Γ. In the case of a 2-groupoid, as in the case of more general 2-categories (see also §6.3 below and [46]) one can construct an associated convolution algebra by considering compactly supported continuous functions $f : \mathcal{U}_\Gamma \to \mathbb{C}$, *i.e.* functions $f(\alpha, \rho, \gamma_1, \gamma_2)$ with the two convolution products of vertical and horizontal composition of 2-morphisms.

Following the analogous case of [20], on can instead work with the groupoid $\mathcal{U}$ itself, and eliminate the infinite multiplicities in the spectrum of the Hamiltonian caused by the Γ-symmetry by working with an associated type II$_1$ factor and the corresponding trace instead of the ordinary trace.

The convolution algebra $C_c(\mathcal{U})$ with the product dictated by the groupoid composition law

$$(3.40) \qquad (f_1 \star f_2)(\alpha, \rho) = \sum_{(\alpha,\rho)=(\alpha_1,\rho_1)\circ(\alpha_2,\rho_2)\in\mathcal{U}_\Gamma} f_1(\alpha_1, \rho_1) f_2(\alpha_2, \rho_2),$$

is just of the form

$$(3.41) \qquad (f_1 \star f_2)(\alpha, \rho) = \sum_{\beta\in\mathrm{GL}_n(\mathbb{Q})^+ \,:\, \beta\rho\in\hat{\mathbb{Z}}^n} f_1(\alpha\beta^{-1}, \beta(\rho)) f_2(\beta, \rho),$$

and with the adjoint $f^*(\alpha, \rho) = \overline{f(\alpha^{-1}, \alpha(\rho))}$. This algebras has a family of representations determined by the choice of an element $\rho \in \hat{\mathbb{Z}}^n$, the set of units of the groupoid $\mathcal{U}$, by setting

$$(3.42) \qquad (\pi_\rho(f)\xi)(\alpha) = \sum_{\beta\in\Gamma\backslash\mathrm{GL}_n(\mathbb{Q})^+ \,:\, \beta\rho\in\hat{\mathbb{Z}}^*} f(\alpha\beta^{-1}, \beta(\rho))\xi(\beta).$$

on the Hilbert space $\ell^2(G_\rho)$, where $G_\rho = \{\alpha \in \mathrm{GL}_n(\mathbb{Q})^+ \,|\, \alpha(\rho) \in \hat{\mathbb{Z}}^n\}$. The algebra $C_c(\mathcal{U})$ can be completed with respect to the norm

$$\|f\| = \sup_{\rho\in\hat{\mathbb{Z}}^n} \|\pi_\rho(f)\|_{\mathcal{B}(\ell^2(G_\rho))}$$

and one denotes by $C^*(\mathcal{U})$ the resulting C^*-algebra. This gives an equivalent description of the multivariable BC endomotive. The time evolution described above corresponds to

$$(3.43) \qquad \sigma_t(f)(\alpha, \rho) = \det(\alpha)^{it} f(\alpha, \rho).$$

In fact, it is clear from (3.40) that $\sigma_t(f_1 \star f_2) = \sigma_t(f_1) \star \sigma_t(f_2)$ and that $\sigma_t(f^*) = \sigma_t(f)^*$. We consider as before the representations π_ρ of the algebra of the multivariable BC system, described here as the groupoid algebra $C^*(\mathcal{U})$ on the Hilbert space $\ell^2(M_n(\mathbb{Z})^+)$, where $\rho \in (\hat{\mathbb{Z}}^*)^n$. The group $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ acts on the right on the same Hilbert space by $\epsilon_\alpha \mapsto \epsilon_{\alpha\gamma^{-1}}$, so that this action commutes with the action of $C^*(\mathcal{U})$. This action of Γ determines a type II$_1$ factor given by the image of the group algebra $C^*(\Gamma)$ in the algebra of bounded operators $\mathcal{B}(\ell^2(M_n(\mathbb{Z})^+))$. Thus, instead of the usual trace, we can use as in [20] the trace Tr_Γ associated to this type II$_1$ factor, which eliminates the infinite multiplicities in the spectrum of the Hamiltonian. The same argument given in §1.7 of [20] gives then the following result.

Lemma 3.8. *In the representation π_ρ on $\ell^2(M_n(\mathbb{Z})^+)$ the time evolution is generated by the Hamiltonian*

$$(3.44) \qquad H\,\epsilon_m = \log\det(m)\,\epsilon_m, \quad \text{for } m \in M_n(\mathbb{Z})^+/\Gamma.$$

The partition function, computed with respect to the type II_1-trace, is then of the form

$$(3.45) \qquad Z(\beta) = \mathrm{Tr}_\Gamma(e^{-\beta H}) = \sum_{m \in M_n(\mathbb{Z})^+/\Gamma} \det(m)^{-\beta} = \prod_{k=0}^{n-1} \zeta(\beta - k).$$

Proof. The argument is completely analogous to that of Lemma 1.18 of [20] for the GL_2-case. $\qquad\square$

Notice that, in these representations the Hamiltonian has positive energy, since on $M_n(\mathbb{Z})^+$ we have $\log \det(m) \geq 0$.

In particular, at sufficiently low temperature, KMS states of Gibbs type can be constructed as in Proposition 1.19 of [20]. Roughly, they will be of the form

$$(3.46) \qquad \varphi_{\beta,\rho}(f) = Z(\beta)^{-1} \mathrm{Tr}_\Gamma(\pi_\rho(f)e^{-\beta H}) = Z(\beta)^{-1} \sum_{m \in M_n(\mathbb{Z})^+/\Gamma} f(1, m(\rho)) \det(m)^{-\beta},$$

parameterized by the elements $\rho \in (\hat{\mathbb{Z}}^*)^n$, though more precisely one first defines them on functions that depend on $\hat{\mathbb{Z}}^n$ through a projection $p_N : \hat{\mathbb{Z}} \to \mathbb{Z}/N\mathbb{Z}$ and then extend them to the algebra $C^*(\mathcal{U})$. At zero temperature these give rise, by weak limits, to KMS states of the form

$$(3.47) \qquad \varphi_{\infty,\rho}(f) = f(1, \rho).$$

3.4. **The dual system and zeta functions.** As in the case of the original Bost–Connes system, one can consider for the multivariable cases the dual system obtained by taking the crossed product with the time evolution. As shown in [14] and [15], as well as in Connes' earlier work on the Riemann zeta function [12], the dual system of the Bost–Connes system is the natural noncommutative space that supports the spectral realization of the zeros of the Riemann zeta function, on a "noncommutative motive" describing the complement of the classical points of this space (see [15] for a detailed analysis). It is then natural to ask what the corresponding procedure (called "cooling and distillation" in [14]) applied to the multivariable Bost–Connes systems introduce here will give in terms of L-functions. It is natural to expect that something like the Kurokawa L-function [40] should appear in this setting. These motivically should correspond to the conjectural tensor product over $\mathrm{Spec}\mathbb{Z}$ as explained in Manin's [45]. This is beyond the scope of the present paper and we hope to return to this question in future work.

In particular, as explained in [15] (see also Chapter 4 of [19]), there is a well developed dictionary of analogies between the Weil proof of the Riemann hypothesis for function fields and the approach for number fields via noncommutative geometry developed in [12], [14], [15]. The action of Frobenius of étale cohomology is replaced by the scaling action on the cyclic homology of a cokernel (in the abelian category of cyclic modules defined in [13]) of the restriction map from the dual system $\mathcal{A}_{BC} \rtimes_\sigma \mathbb{R}$ to its "space of classical points".

Geometrically, if one hopes to transpose the main ideas in the Weil proof from the function field to the number field case, one needs a good analog for number fields of the geometry of the curve C for function fields $\mathbb{K} = \mathbb{F}_q(C)$, and in particular an analog of correspondences on $C \times_{\mathrm{Spec}(\mathbb{F}_q)} C$. Possible analogs are identified in [15] in terms of the noncommutative geometry of the adèle class space.

The idea of developing geometry over $\mathbb{F}_1$ is also aimed at a similar purpose, and that's why identifying precisely its relation to noncommutative geometry via the BC endomotive is an important ingredient of the general picture. From the point of view of geometry over $\mathbb{F}_1$, one would like to view $\mathrm{Spec}(\mathbb{Z})$ as a direct analog of the curve C. However, as already observed in [44] and [52], $\mathrm{Spec}(\mathbb{Z})$ is not of finite type over $\mathbb{F}_1$. The geometric space where a possible

analog of the Weil proof would be taking place should then be $\mathrm{Spec}(\mathbb{Z}) \otimes_{\mathrm{Spec}(\mathbb{F}_1)} \mathrm{Spec}(\mathbb{Z})$. Although this object does not have a good definition within $\mathbb{F}_1$-geometry, some of its expected properties were identified by Manin in [45] in terms of the zeta functions of Kurokawa [40]. Kurokawa's approach to L-functions that would be associated to spaces $\mathrm{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \times \cdots \times_{\mathbb{F}_1} \mathrm{Spec}(\mathbb{Z})$ is by sums over zeros of the Riemann zeta function. More precisely, the Kurokawa tensor product of zeta functions corresponds to regularized infinite products of the form (*cf.* [44] (2.30)-(2.31))

$$(3.48) \qquad \prod_{\lambda \in \Xi}(s-\lambda)^{m_\lambda} \otimes \prod_{\mu \in \Theta}(s-\mu)^{n_\mu} := \prod_{(\lambda,\mu)}(s-\lambda-\mu)^{m_\lambda+n_\mu},$$

where the infinite products are computed as

$$(3.49) \qquad \prod_{\lambda \in \Xi}(s-\lambda)^{m_\lambda} = \exp(-\frac{d}{dz}(\Gamma(z)^{-1}\int_0^\infty \sum_\lambda m_\lambda e^{(s-\lambda)t}t^{z-1}\,dt)|_{z=0}).$$

In particular, for ρ ranging over the critical zeros of the Riemann zeta function one has (see [44], (1.5))

$$(3.50) \qquad \prod_\rho \frac{s-\rho}{2\pi} = \frac{s(s-1)}{4\pi^2}2^{-1/2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Kurokawa uses a splitting into the product ([44], §5.2)

$$\xi_\pm(s) := \prod_{\mathrm{sign}\Im(\rho)=\pm 1}(1-\frac{s}{\rho})e^{s/\rho}$$

to deal with the infinite products and defined the tensor product of the completed Riemann zeta function $\xi(s) = s(s-1)\Gamma_\mathbb{R}(s)\zeta(s)$ to be

$$(3.51) \qquad \xi(s)^{\otimes n} = \xi_+(s)^{\otimes n}\left(\xi_-(s)^{\otimes n}\right)^{(-1)^{n-1}}.$$

In the noncommutative geometry approach to the Riemann hypothesis developed in [12] the zeros of the Riemann zeta function arise naturally from the scaling action on the cokernel of the restriction map from the noncommutative adeles class space to the classical spaces of ideles classes. In the formulation of [14], [15], this action is on the cyclic homology of the cyclic module describing this cokernel, with the adeles class space arising from the dual system of the BC quantum statistical mechanical system. It is therefore natural to view the multivariable versions of the BC system considered here as a possible candidate for endomotives where the sums over zeros of zeta used in the construction of the Kurokawa zeta functions should arise by following the general procedure of "cooling and distillation" of endomotives described in [14]. We intend to return to a more detailed analysis of this problem in future work.

Another source of evidence for the fact that the multivariable BC endomotives introduced here should be related to the missing geometry of the products $\mathrm{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \times \cdots \times_{\mathbb{F}_1} \mathrm{Spec}(\mathbb{Z})$ comes from a different source, namely the relation between BC endomotives and Λ-rings, which we discuss in detail in §4 here below. In fact, recently, an approach to geometry over $\mathbb{F}_1$ in terms of Λ-rings proposed in [7], [8], [9] suggests that the missing $\mathrm{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \times \cdots \times_{\mathbb{F}_1} \mathrm{Spec}(\mathbb{Z})$ should be related to the space of big Witt vectors $W(\mathrm{Spec}(\mathbb{Z}))$.

4. Endomotives and Λ-rings

The notion of Λ-ring was introduced by Grothendieck [27] to formalize the properties of the exterior power operations on Grothendieck groups and K-theory. They provide a useful notion in the context of characteristic classes and the Riemann–Roch theorem, see [24].

It was shown recently in Borger–de Smit [9] that the study of abstract properties of Λ-rings has interesting number theoretic implications. The formulation of the Λ-ring structure adopted in [9] is based on an equivalent reformulation given in [55] of the original Grothendieck definition.

In particular, as argued in greater detail in [7], the structure of Λ-ring can be thought of as a descent condition for a ring from $\mathbb{Z}$ to $\mathbb{F}_1$, in the sense that it imposes a strong compatibility condition on the Frobenius at all the different reductions modulo primes. This gives a different approach to defining varieties and schemes over $\mathbb{F}_1$, where the compatibility is expressed in terms of a semigroup action, rather than through the cyclotomic points as in the Soulé approach. One of the main points we want to stress in this paper is the fact that the BC endomotive encodes the compatibility between these two approaches, in the sense that not only it fits the Soulé definition of affine (pro)varieties over $\mathbb{F}_1$ as proved in [16], but it also fits the Λ-ring approach to the definition of varieties over $\mathbb{F}_1$ in the sense of [7], [9]. In fact, it is a universal object both for the Soulé approach, where it encodes the tower of extensions $\mathbb{F}_{1^n}$ of [37] [52], with their Frobenius actions, and also in the sense of [9], because (together with its multivariable versions introduced here) it gives universal Λ rings into which any Λ-ring that is torsion free and finite over $\mathbb{Z}$ embeds.

We are going to show here that the integral model of the BC endomotive constructed in [16] is a an *integral Λ model* in the sense of [9]. Moreover, it is universal in the sense that any integral Λ model admits a map of Λ-rings to a (multivariable) integral BC endomotive.

Following [9], we give the following definition of Λ-structure on a commutative ring and on finite dimensional reduced $\mathbb{Q}$-algebra.

Definition 4.1. *Let R be a commutative ring such that the underlying abelian group is torsion free. An (integral) Λ-ring structure on R is an action of the abelian multiplicative semigroup $\mathbb{N}$ generated by the primes as endomorphisms of the ring R, with the property that the endomorphism σ_p lifts the Frobenius map modulo p, i.e.*

$$(4.1) \qquad s_p(x) - x^p \in pR, \quad \forall x \in R.$$

A map of (torsion free) Λ-rings is a ring homomorphism compatible with the semigroup actions, i.e. satisfying $f \circ s_k = s_k \circ f$ for all $k \in \mathbb{N}$.

A typical example of a Λ structure on a ring is the following case.

Example 4.2. *Consider the ring $R = \mathbb{Z}[t, t^{-1}]/(t^n - 1)$ with the endomorphisms $s_k(P)(t, t^{-1}) = P(t^k, t^{-k})$ for $k \in \mathbb{N}$, as in (2.8). It is a Λ-ring in the sense of Definition 4.1, satisfying (4.1).*

This will be the basic example we are interested in to establish the connection between Λ-rings and the multivariable BC endomotives.

As in [9], after tensoring with $\mathbb{Q}$, one obtains a $\mathbb{Q}$-algebra $A = R \otimes \mathbb{Q}$, with an action of the semigroup $\mathbb{N}$ by endomorphisms, now without the condition (4.1). We assume here, as in [9], to be working with Λ rings that are torsion free and finite over $\mathbb{Z}$. As in the case of endomotives, these correspond to zero-dimensional geometries.

Definition 4.3. *Let A be a finite dimensional reduced algebra over $\mathbb{Q}$. Then A has a Λ-ring structure if it has an action of the multiplicative group $\mathbb{N}$ by endomorphisms. This Λ-ring structure has an integral model if $A = R \otimes \mathbb{Q}$, where R is a Λ ring as in Definition 4.1 and the inclusion of R as a subring of A is a morphism of Λ-rings.*

We also give the following definition that will be useful in comparing Λ-rings and endomotives.

Definition 4.4. *A direct limit of Λ-rings is a direct system of $\mathbb{Q}$-algebras A_α such that all the maps in the direct system are morphisms of Λ-rings.*

This induces a Λ-ring structure (*i.e.* an action of $\mathbb{N}$ by endomorphisms) on the direct limit $A = \varinjlim A_\alpha$. Consider then the BC endomotive. We have the following result.

Lemma 4.5. *The Bost–Connes endomotives is a direct limit of Λ-rings with compatible integral models.*

Proof. The abelian algebra of the BC endomotive is a direct limit $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] = \varinjlim_n A_n$, where $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$. At each finite level the algebra $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ has a Λ-ring structure given by the semigroup action $\sigma_k : A_n \to A_n$ is the one given by (2.9), which correspond to the s_k of (2.8). The actions σ_k are compatible with the maps of the direct system since we have

$$\xi_{m,n} \circ \sigma_k = \sigma_k \circ \xi_{m,n}, \quad \text{for } n|m \text{ and } \forall k \in \mathbb{N},$$

for s_k as in (2.8) and with $\xi_{m,n} : X_n \to X_m$ the maps of the inverse system with $X_n = \mathrm{Spec}(\mathbb{Q}[t]/(t^n - 1))$.

The integral model of the BC endomotive constructed in [16] is obtained by taking as abelian part the ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] = \varinjlim \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ and the action of $\mathbb{N}$ by endomotives given by the same σ_n as above. (Unlike the ρ_n these do not involve denominators and are therefore defined over $\mathbb{Z}$.) Thus, at each finite level the $A_n = R_n \otimes \mathbb{Q}$, with $R_n = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ are Λ-rings with integral models. The compatibility with the maps in the direct system give the result for $A = R \otimes \mathbb{Q}$ with the resulting $\mathbb{N}$ action by endomorphisms. To see that the integral model of the BC endomotive defines, at each finite level R_n an integral Λ-ring in the sense of Definition 4.1, we only need to check that the compatibility with the Frobenius is satisfied for all primes p. The compatibility of the action of σ_{p^ℓ} on the integral model of the BC system with the Frobenius action follows from the case of Example 4.2. $\square$

The fact that the endomorphisms action of $\mathbb{N}$ on the integral model of the abelian part of the BC algebra is compatible with the action of the Frobenius in all the reductions modulo primes is discussed in detail in [16], where this compatibility is an important part of the structure in viewing the BC endomotive as defined over $\mathbb{F}_1$, with the abelian part providing the system of extensions $\mathbb{F}_{1^n}$ and the endomorphisms σ_p providing a compatible action that induces the Frobenius correspondences in characteristic $p > 0$. In fact, the compatibility with Frobenius action as formulated for the BC system in [16] is stronger than the condition (4.1) used in the context of Λ-rings. It would be interesting to investigate whether stronger versions of (4.1) of the kind considered in [16], involving reductions mod p of the endomotive, may also be implemented for Λ-rings and what kind of structure they give rise to.

We also have the following general result that produces endomotives from Λ-rings.

Lemma 4.6. *A direct limit of Λ-rings defines an endomotive over $\mathbb{Q}$. This has an integral model if at each finite level the Λ-rings have an integral model in the sense of Definition 4.3.*

Proof. We have a direct system of finite dimensional reduced algebras A_α over $\mathbb{Q}$. Consider the crossed product algebra $A \rtimes \mathbb{N}$, where the commutative A is the direct limit $A = \varinjlim_\alpha A_\alpha$ and the crossed product with $\mathbb{N}$ is implemented by

$$\rho_n(a) = \mu_n \, a \, \mu_n^*,$$

where the μ_n are isometries satisfying

$$\mu_n^* \, a \, \mu_n = \sigma_n(a).$$

Unlike the σ_n, the ρ_n do not preserve levels A_α. This is the noncommutative algebra of an algebraic endomotive over $\mathbb{Q}$, according to the definition given in [14]. Notice moreover that

Λ-rings given by finite-dimensional reduced commutative algebras over $\mathbb{Q}$ have a continuous action of the Galois group $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on

$$\mathcal{X}_\alpha = \mathrm{Hom}(A_\alpha, \bar{\mathbb{Q}}),$$

as observed in [9]. The induced action on $\mathcal{X} = \mathrm{Hom}(A, \bar{\mathbb{Q}})$ is the action of G defined for endomotives in [14] given by composition

$$A \xrightarrow{\chi} \bar{\mathbb{Q}} \xrightarrow{\gamma} \bar{\mathbb{Q}},$$

for $\chi \in \mathcal{X}$ and $\gamma \in G$. The analytic endomotive is then obtained as the C^*-algebra $C(\mathcal{X}) \rtimes \mathbb{N}$ with the induced action of G by automorphisms.

If the Λ rings A_α have an integral model R_α, then the compatibility of the action σ_n with the maps of the injective system implies that we have an induced action on $R = \varinjlim R_\alpha$. Now the noncommutative algebra for the integral model can be defined by adding generators μ_n^* and $\tilde{\mu}_n$ with the relations (2.15) and

$$\mu_n^* \, a = \sigma_n(a) \, \mu_n^* \quad \text{and} \quad a \, \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(a),$$

for all $a \in R$ and all $n \in \mathbb{N}$. This is an integral model for the algebraic endomotive algebra $A \rtimes \mathbb{N}$. $\qquad\square$

It would be interesting to investigate more closely what this relation between Λ-rings and endomotives can say in the original case of Λ-rings in the context of Chern classes as introduced by Grothendieck in [27].

An interesting characterization is given in [9] of when a Λ-ring defined over $\mathbb{Q}$ has an integral model in the sense of Definition 4.3. Namely, this is the case if and only if the action of $G \times \mathbb{N}$ on $\mathcal{X} = \mathrm{Hom}(A, \bar{\mathbb{Q}})$ factors through a (multiplicative) action of $\hat{\mathbb{Z}}$. In the case of the BC endomotive, this property can be seen explicitly as follows.

Lemma 4.7. *The multiplicative action of $\hat{\mathbb{Z}}$ on $\mathrm{Hom}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \bar{\mathbb{Q}})$ is given by the symmetries of the BC system, together with the action of $\mathbb{N}$ by the endomorphisms σ_n.*

Proof. For $\alpha \in \hat{\mathbb{Z}}$ we set

$$(4.2) \qquad\qquad \alpha : e(r) \mapsto e(\alpha(r)), \quad \forall r \in \mathbb{Q}/\mathbb{Z},$$

where we identify $\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. This gives a multiplicative action of $\hat{\mathbb{Z}}$ on the abelian part of the BC endomotive, $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ and its integral model $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. For $\alpha = n \in \mathbb{N}$ this action reduces to the action of the endomorphisms σ_n described above, while for $\alpha \in \hat{\mathbb{Z}}^*$ it agrees with the action of the automorphisms group of the Bost–Connes system. This action of $\hat{\mathbb{Z}}^*$ corresponds to the Galois action of $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ as the latter factorizes through the action of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ given by the cyclotomic character, *i.e.* through the action of $\hat{\mathbb{Z}}^*$ of the form (4.2), under the class field theory isomorphism $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^*$. $\qquad\square$

Thus, the action of $\hat{\mathbb{Z}}$ is given by the action of the symmetries $\hat{\mathbb{Z}}^*$ of the BC algebra (which preserve the integral model) combined with the action of the endomorphisms $\mathbb{N}$. The following observation then links the notion of integral structure of a Λ-ring to Haran's proposal [34] for a notion of Frobenius over the field with one element.

Corollary 4.8. *For the BC endomotive, the action of $\hat{\mathbb{Z}}$ on $\mathrm{Hom}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \bar{\mathbb{Q}})$ that determines the integral structure on the inductive limit of Λ-rings agrees with the "Frobenius over $\mathbb{F}_{1^\infty}$" of [34].*

Proof. As observed in §6.1 of [16], the action (4.2) corresponds to the notion of Frobenius over $\mathbb{F}_{1^\infty}$ as defined in [34] as follows. Given a set with a free action of roots of unity (regarded as a "vector space over $\mathbb{F}_{1^\infty}$"), and an element $\alpha \in \hat{\mathbb{Z}}$, one obtains a new action on the same set by

$$(4.3) \qquad \qquad \zeta : x \mapsto \zeta^\alpha x,$$

which corresponds in fact to the action (4.2). $\qquad \square$

In order to relate the multivariable BC endomotives to Λ-rings, we first observe that, as in the one-variable case, these admit integral models constructed as in [16]. Namely, the integer model of the multivariable BC endomotive algebra $\mathcal{A}_{BC,n}$ is the ring $\mathcal{A}_{\mathbb{Z},BC,n}$ generated by the elements $e(\underline{r})$ of $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ and by generators $\tilde{\mu}_\alpha$ and μ_α^*, with the relations

$$(4.4) \qquad \begin{aligned} \tilde{\mu}_\alpha \tilde{\mu}_\beta &= \tilde{\mu}_{\alpha\beta}, & \forall \alpha, \beta \in M_n(\mathbb{Z})^+ \\ \mu_\alpha^* \mu_\beta^* &= \mu_{\beta\alpha}^*, & \forall \alpha, \beta \in M_n(\mathbb{Z})^+ \\ \mu_\alpha^* \tilde{\mu}_\alpha &= \det(\alpha) & \forall \alpha \in M_n(\mathbb{Z})^+ \\ \mu_\alpha^* \tilde{\mu}_\beta &= \det(\alpha)\tilde{\mu}_\gamma & \text{for } \beta = \alpha\gamma \in M_n(\mathbb{Z})^+. \end{aligned}$$

replacing the relations (3.13), and with the additional relations

$$(4.5) \qquad \qquad \mu_\alpha^* x = \sigma_\alpha(x) \mu_\alpha^*, \quad \text{and} \quad x \tilde{\mu}_\alpha = \tilde{\mu}_\alpha \sigma_\alpha(x).$$

for all $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ and all $\alpha \in M_n(\mathbb{Z})^+$. As in the one-variable case, the $\tilde{\mu}_\alpha$ are no longer adjoints to the μ_α^*: they satisfy $\tilde{\mu}_\alpha = \det(\alpha)\mu_\alpha$.

We can then prove that the results of [9], when rephrased in our setting, show that the BC endomotive and its multivariable analogs considered in the previous section are universal with respect to the (torsion free) Λ-rings of Definition 4.1 that are finite over $\mathbb{Z}$.

Theorem 4.9. *Let $A = \varinjlim_\alpha A_\alpha$ be a direct limit of Λ-rings, with an integral model $R = \varinjlim_\alpha R_\alpha$, where the R_α have finite rank as abelian groups an have no non-zero nilpotent elements. Then the associated endomotive A with the semigroup action of $\mathbb{N}$ that gives the Λ-ring structure embeds in a product of copies of the BC endomotive $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ compatibly with the action of $S_{n,diag} \subset M_n(\mathbb{Z})^+$.*

Proof. Rephrased in our setting, Corollary 0.3 of [9] shows that every torsion free finite rank Λ ring embeds in a Λ-ring that is a product of copies of $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, which is the integral model of the abelian part of a multivariable BC endomotive. The fact that this is an embedding of Λ-rings implies that the semigroup action of $\mathbb{N}$ on R is compatible with the action of the semigroup $S_{n,diag} \subset M_n(\mathbb{Z})^+$ of the multivariable BC endomotive, so that one obtains an induced embedding of the crossed product algebras compatible with the integer model $\mathcal{A}_{\mathbb{Z},BC,n}$ described above. $\qquad \square$

This result shows how the (multivariable) BC endomotives bridge between the Soulé approach to varieties over $\mathbb{F}_1$, with respect to which the BC endomotive embodies the tower of $\mathbb{F}_{1^n}$ as in [16], and the approach of Borger via Λ-rings, where they provide universal Λ-rings in the sense of Theorem 4.9 above.

The examples considered in this paper of crossed product constructions both in the case of Habiro rings and of the original BC system and its multivariable versions suggests that, more generally, one should be able to associate an endomotive to a Λ-ring, in the form of a crossed product, which we loosely write here as $A \rtimes \mathbb{N}$. While we do not give here a general construction for such crossed products, one can see that the statement of Theorem 4.9 should then give a natural map, in the case of the crossed product of a Λ-ring that is finite rank as

an abelian groups an has no non-zero nilpotent elements, from the crossed product algebra $A \rtimes \mathbb{N}$ to a multivariable BC algebra, compatibly with the integral models.

As recalled in §4.1 of [44] (see also [8] and [10], §2) the affine ring scheme of big Witt vectors is defined by considering the polynomial ring $\mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots]$ in the elementary symmetric functions $\lambda_1 = x_1 + x_2 + \cdots$, $\lambda_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots$, etc. This can be equivalently written in the set of generators $\mathbb{Z}[u_1, u_2, \ldots, u_n, \ldots]$

$$(4.6) \qquad \psi_n = \sum_{d|n} d \, u_d^{n/d},$$

where the ghost coordinates are $\psi_n = x_1^n + x_2^n + \cdots$ and the u_n are determined by (4.6). For a commutative ring R, the ring of big Witt vectors is $W(R) = \prod_{n \geq 1} R$, with the addition and multiplication given by componentwise addition and multipication in the ghost coordinates. The truncated Witt scheme $W^{(N)}$ is obtained by considering the subring $\mathbb{Z}[u_1, \ldots, u_N]$. As shown in §4.1 of [44] the $W^{(N)}$ define gadgets $\mathcal{W}_{\mathbb{Z}}^{(N)}$ over $\mathbb{F}_1$ in the sense of Soulé [52], by setting $\mathcal{W}_{\mathbb{Z}}^{(N)}(R)$ to be the points of $W^{(N)}(R)$ with ghost coordinates that are either zero or roots of unity. The Λ-ring structure is given by the Adams operations ψ_n, which by (4.6) satisfy the requirement (4.1). Thus, in particular, one can use the result of Theorem 4.9 to relate the gadgets $\mathcal{W}_{\mathbb{Z}}^{(N)}$ to the gadgets over $\mathbb{F}_1$ associated to the multivariable BC endomotives in the same way as done in [16] for the original BC algebra, with compatible Λ-ring structure. This may provide the compatibility between the idea of realizing the missing $\mathrm{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \cdots \times_{\mathbb{F}_1} \mathrm{Spec}(\mathbb{Z})$ in terms of the quantum statistical mechanics of multivariable BC endomotives, as sketched in §3.4 above and the idea of [7] to realize them via the spaces of big Witt vectors.

In this light, the multivariable BC endomotives constructed here should also be regarded as basic building blocks for a more general class of noncommutative spaces with quantum statistical mechanical properties that can be constructed from the data of toric varieties and possibly of other varieties over $\mathbb{F}_1$ in the sense of [17], [18], [43]. We intend to return to this topic in future work.

5. QUANTUM CHANNELS AND ENDOMOTIVES

This section contains a quick side remark on the general quantum statistical mechanical formalism of endomotives, aimed at showing that, while one usually concentrates on studying equilibrium (KMS) states for quantum statistical mechanical systems such as the BC system and its generalizations, one can also get some interesting information by considering states that are not equilibrium states under the dynamics, but which are defined by particular density operators. Given endomotives consisting of an abelian part of the algebra and a semigroup of isometries acting on it, it is interesting to see how density matrices of states transform under the semigroup action.

In quantum computing a formalism that describes the phenomenon of decoherence, the evolution of pure states into mixed states, is obtained by replacing the group action by unitaries that gives the evolution of density matrices by an evolution given not by a semigroup action by isometries, which is given in the operator-sum representation or Kraus representation in the form $\rho \mapsto \sum_{i=1}^N \mu_i \rho \mu_i^*$, where the μ_i are isometries satisfying $\sum_i \mu_i^* \mu_i = 1$ (see *e.g.* [49]). The case of a single μ with $\mu^* \mu = \mu \mu^* = 1$ gives back the unitary evolution. In the terminology of quantum computing such data are said to describe a quantum channel. We show here how a similar formalism can be applied in the context of endomotives using the semigroup action and the induced action on states to obtain a transformation of the convex space of

density operators. It is important to distinguish the finite dimensional case, where a quantum channel realized by a single isometry would have to be given by a unitary, and the infinite dimensional case we are considering here, where it is possible to have a single isometry with $\mu^*\mu = 1$ but with $\mu\mu^* = e = e^2 \neq 1$ a projector. We consider here the isometries in the family μ_s with $s \in S$ a semigroup acting by endomorphisms on an abelian algebra of observables and we consider the quantum channels $\rho \mapsto \mu_s\rho\mu_s^*$ determined by these isometries.

Given a semigroup S acting by endomorphisms on the abelian part A of an endomotive in the sense of [14], consider the action of endomorphisms on states given as in [20] by

$$(5.1) \qquad (s^*\varphi)(a) = \frac{\varphi(s(a))}{\varphi(s(1))},$$

where $s(a) = \mu_s\, a\, \mu_s^*$ and $\mu_s\mu_s^* = e_s$ is a non-zero idempotent in the algebra A, $e_s^2 = e_s = e_s^*$, such that $\varphi(e_s) \neq 0$. Suppose that we consider states on A of the form

$$(5.2) \qquad \varphi(a) = \frac{\mathrm{Tr}(a\varrho)}{\mathrm{Tr}(\varrho)},$$

where ϱ is a positive density operator, *i.e.* a positive trace class operator. Then we have

$$\frac{\varphi(\mu_s\, a\, \mu_s^*\varrho)}{\varphi(e_s\, \varrho)} = \frac{\mathrm{Tr}(a\, \mu_s^*\, \varrho\, \mu_s)}{\mathrm{Tr}(\varrho)}\, \frac{\mathrm{Tr}(\varrho)}{\mathrm{Tr}(\mu_s^*\varrho\mu_s)}.$$

This means that we have

$$(s^*\varphi)(a) = \frac{\mathrm{Tr}(a\varrho_s)}{\mathrm{Tr}(\varrho_s)}, \qquad \text{with } \varrho_s = \mu_s^*\varrho\mu_s.$$

Notice that, while the action $a \mapsto \mu_s^* a\mu_s$ is an algebra homomorphism only on the compressed algebra $a = e_s x e_s$, with $x \in A$ and $e_s = \mu_s\mu_s^*$ the idempotent that gives the projection onto the range of μ_s, so that $(\mu_s^* a\mu_s)(\mu_s^* b\mu_s) = \mu_s^* a e_s b\mu_s = \mu_s^* ab\mu_s$, for $a = e_s x e_s$ and $b = e_s y e_s$ with $x, y \in A$, using $e_s^2 = e_s$. However, when considering an action on density operators, one does not need the compatibility with the multiplicative structure, as the only operation one performs on density operators is that of taking convex linear combinations. Thus, one obtains a semigroup action of S on the convex space of density operators by

$$(5.3) \qquad \varrho \mapsto \mu_s^*\varrho\mu_s.$$

We now sketch some examples of how this point of view may be useful.

Recall that a rational convex cone $\mathcal{C}$ is a convex subset of $\mathbb{R}^n$ is the span

$$(5.4) \qquad \mathcal{C} = \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_r, \qquad \text{with} \quad v_i \in \mathbb{Q}^n,\ i = 1, \ldots, r.$$

The cone is simplicial if the v_i are linearly independent. Terasoma [53] constructed multiple zeta values for rational convex cones by setting

$$(5.5) \qquad \zeta_{\mathcal{C}}(\ell_1, \ldots, \ell_k, \chi) = \sum_{v \in \mathcal{C}^0 \cap \mathbb{Z}^n} \frac{\chi(v)}{\ell_1(v) \cdots \ell_k(v)},$$

where the ℓ_i are $\mathbb{Q}$-linear forms on $\mathbb{Q}^n$ that are strictly positive on the interior $\mathcal{C}^0$ of the cone $\mathcal{C}$, $\ell_i(v) > 0$ for all $v \in \mathcal{C}^0$, and $\chi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^*)$ is a character of $\mathbb{Z}^n$.

Let then $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ be the abelian part of one of the multivariable Bost–Connes systems considered earlier in this paper. Let $\mathcal{C}$ be a rational convex cone in $\mathbb{R}^n$ with the property that $\mathcal{C}^0 \cap \mathbb{Z}^n$ is preserved by the action of a subsemigroup $S_{\mathcal{C}} \subset M_n(\mathbb{Z})^+$. We consider special cases of states of the form (5.2) where we set

$$(5.6) \qquad \mathrm{Tr}(a\varrho) = \sum_{v \in \mathcal{C}^0 \cap \mathbb{Z}^n} \frac{\chi_a(v)}{\ell_1(v) \cdots \ell_k(v)},$$

where the density matrix ϱ is defined as the operator on the Hilbert space $\ell^2(\mathcal{C}^0 \cap \mathbb{Z}^n)$ given by

$$\varrho\, \epsilon_v = \frac{1}{\ell_1(v) \cdots \ell_k(v)} \epsilon_v,$$

and the character χ_a is defined by the choice of an element $\alpha \in \mathrm{GL}_n(\hat{\mathbb{Z}})$ that maps $(\mathbb{Q}/\mathbb{Z})^n$ to $\mathcal{Z}^n$ and then defining, for $a = e(r_1) \otimes \cdots \otimes e(r_n)$,

$$\chi_a(v) = \zeta_1^{k_1} \cdots \zeta_n^{k_n},$$

for $v = (k_1, \ldots, k_n) \in \mathcal{C}^0 \cap \mathbb{Z}^n$ and $\zeta_i = \alpha(e(r_i)) \in \mathcal{Z}$.

Under the action of (5.3), for $s \in S_{\mathcal{C}}$, we have

$$(\ell_1, \ldots, \ell_k) \mapsto (\ell_1 \circ s, \ldots, \ell_k \circ s)$$

so that

$$\mathrm{Tr}(a\varrho_s) = \sum_{v \in \mathcal{C}^0 \cap \mathbb{Z}^n} \frac{\chi_a(v)}{\ell_1(s(v)) \cdots \ell_k(s(v))}.$$

A quantum channels $\varrho \mapsto \varrho_s$ thus gives a "change of variables" in the multiple zeta values of cones, which preserves relations, in the sense that if a polynomial relation of the form

$$(5.7) \qquad \mathcal{R}(\zeta_{\mathcal{C}}(\ell_{1,i}, \ldots, \ell_{k_i,i}, \chi_{a_i})) = \sum_I \alpha_I \prod_{i \in I} \mathrm{Tr}(a_i \varrho_i) = 0$$

holds for some $a_i \in (\mathbb{Q}/\mathbb{Z})^n$, and if $a_i = s_i(b_i)$ for some $s_i \in S_{\mathcal{C}}$ then

$$(5.8) \qquad \mathcal{R}(\zeta_{\mathcal{C}}(\ell_{1,i} \circ s_i, \ldots, \ell_{k_i,i} \circ s_i, \chi_{b_i})) = 0.$$

This follows immediately from the fact that, for $a_i = s_i(b_i) = \mu_{s_i}^* b_i \mu_{s_i}$, we have $\mathrm{Tr}(a_i \varrho_i) = \mathrm{Tr}(b_i \varrho_{s_i,i})$ with $\varrho_{s_i,i} = \mu_{s_i}^* \varrho_i \mu_{s_i}$.

6. Homology 3-spheres

We now discuss briefly how Habiro's universal Witten–Reshetikhin–Turaev (WRT) invariant may be used to lift some of the results described in the previous sections to the world of 3-manifolds. This section will mostly contain some questions about 3-manifolds motivated by the circle of ideas considered in the previous sections.

The idea of relating certain categories of 3-manifolds to convolution algebras and quantum statistical mechanical systems was developed, in a different context, in [46], where instead of relying on the description of 3-manifolds in terms of surgeries on links in the 3-sphere, which one uses to define WRT invariants, one works with the description as branched coverings of the 3-sphere. While the considerations we outline here are different from the setting of [46], the underlying principle and philosophy are similar.

6.1. **Habiro's universal WRT invariant.** The WRT invariants of 3-manifolds were originally introduced by Witten [57] in terms of Chern–Simons path integrals, and then defined rigorously by Reshetikhin and Turaev [50] using representations of quantum groups at roots of unity. Restricting to the case of 3-manifolds that are integral homology 3-spheres, one has a family of WRT invariants parameterized by roots of unity

$$(6.1) \qquad \tau(M) : \mathcal{Z} \to \mathbb{C}, \quad \zeta \mapsto \tau_\zeta(M),$$

with consistency conditions, such as Galois equivariance

$$\tau_{\alpha(\zeta)}(M) = \alpha(\tau_\zeta(M)), \quad \forall \alpha \in \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}), \ \forall \zeta \in \mathcal{Z}.$$

Moreover, the Ohtsuki series provides an expansion near $\zeta = 1$ of the WRT invariant, with coefficients that in turn are invariants of 3-manifolds,

$$(6.2) \qquad \tau^O(M) = 1 + \sum_{n=1}^{\infty} \lambda_n(M)(q-1)^n.$$

The recent work of Habiro [31] provides a universal formulation for the WRT invariants, in terms of a single functions $J_M(q)$ in the Habiro ring

$$(6.3) \qquad J_M(q) \in \widehat{\mathbb{Z}[q]},$$

with the properties that its evaluations at roots of unity recover the usual WRT invariants for 3-dimensional integral homology spheres,

$$(6.4) \qquad ev_\zeta(J_M) = \tau_\zeta(M),$$

and such that one has an analog of the Ohtsuki series at every root of unity, given by the Taylor expansion of Habiro functions, which recovers the Ohtsuki series at $\zeta = 1$,

$$(6.5) \qquad \mathfrak{t}_1(J_M) = \tau^O(M),$$

where $\mathfrak{t}_\zeta(f)$ denotes, as before, the Taylor expansion of $f \in \widehat{\mathbb{Z}[q]}$ at $\zeta \in \mathcal{Z}$.

The universal WRT invariant is obtained in [31] by defining the invariant for an algebraically split link L in S^3 with framing ± 1. Then one uses the fact that any integral homology 3-sphere has a presentation as surgery on an algebraically split link in the 3-sphere with framing ± 1, where two such presentations give rise to the same homology 3-spheres if and only is they differ by Fenn–Rourke moves. One then shows the invariance under Fenn–Rourke moves to obtain a well defined J_M.

Let $\mathbb{Z}HS$ denote the free abelian group generated by the orientation-preserving homeomorphism classes of integral homology 3-spheres. This is a ring with the product given by the connected sum $M_1 \# M_2$.

The universal WRT invariant defines a ring homomorphism

$$(6.6) \qquad J : \mathbb{Z}HS \to \widehat{\mathbb{Z}[q]},$$

since it satisfies the properties

$$(6.7) \qquad J_{M_1 \# M_2}(q) = J_{M_1}(q) J_{M_2}(q), \quad J_{S^3}(q) = 1, \quad J_{-M}(q) = J_M(q^{-1}).$$

Moreover, to express the fact that the WRT invariants have some "finite type" properties, one can consider, as in [31], the Ohtsuki filtration on $\mathbb{Z}HS$,

$$(6.8) \qquad \mathbb{Z}HS = F_0 \supset F_1 \supset \cdots \supset F_k \supset \cdots,$$

where F_k is the $\mathbb{Z}$-submodule spanned by the alternating sums

$$(6.9) \qquad [M, L_1, \ldots, L_k] = \sum_{L' \subset \{L_1, \ldots, L_k\}} (-1)^{|L'|} M_{L'},$$

with $|L|$ the number of components and the L_i algebraically split links of framing ± 1. This is generalized in [31] to a filtration by submodules F_d, for $d : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ finitely supported, where now $L = \{L_1, \ldots L_k\}$ is an algebraically split link with framings in the set $\{\pm 1/m \mid m \in \mathbb{Z} \smallsetminus \{0\}\}$ and in the expressions (6.9) there are $d(n)$ components of the link that have framing with $m = \pm n$.

Habiro conjectures in [31] that the universal WRT invariant will induce a ring homomorphism

$$(6.10) \qquad J : \widehat{\mathbb{Z}HS} \to \widehat{\mathbb{Z}[q]},$$

where

$$(6.11) \qquad \widehat{\mathbb{Z}HS} = \varprojlim_d \mathbb{Z}HS/F_d.$$

Notice that the universal WRT invariant takes values in the single-variable Habiro ring and one may wonder whether there are similar 3-manifolds or link invariants that give rise naturally to functions in the multivariable Habiro rings of [44]. The multivariable link invariants constructed in [26] may provide such objects.

6.2. **Homology 3-spheres and the field with one element.** We recall the notion of *truc* or *gadget* over the field with one element $\mathbb{F}_1$, as defined by Soulé in [52].

Definition 6.1. *A gadget over $\mathbb{F}_1$ is the datum $(X, \mathcal{A}_X, e_{x,\sigma})$ of a covariant functor $X : \mathcal{R} \to \mathcal{S}$ from the category of commutative finite flat rings $\mathcal{R}$ to the category $\mathcal{S}$ of sets, a complex algebra $\mathcal{A}_X$ and evaluation maps given by $\mathbb{C}$-algebra homomorphisms $e_{x,\sigma} : \mathcal{A}_X \to \mathbb{C}$, for all $x \in X(R)$ and all homomorphisms $\sigma : R \to \mathbb{C}$, such that, for any given ring homomorphism $f : R' \to R$,*

$$(6.12) \qquad e_{f(y),\sigma} = e_{y,\sigma \circ f}.$$

Affine varieties $V_{\mathbb{Z}}$ over $\mathbb{Z}$ give rise to associated gadgets $X = G(V_{\mathbb{Z}})$ over $\mathbb{F}_1$, by taking $X(R) = \mathrm{Hom}(O(V), R)$ and $\mathcal{A}_X = O(V) \otimes \mathbb{C}$.

According to the definition of Soulé [52], affine variety over $\mathbb{F}_1$ are gadget over $\mathbb{F}_1$ where all the $X(R)$ are finite and for which there exists a variety $X_{\mathbb{Z}}$ over $\mathbb{Z}$ and a morphism of gadgets

$$X \to G(X_{\mathbb{Z}})$$

such that all morphisms of gadgets $X \to G(V_{\mathbb{Z}})$ come from morphisms of varieties $X_{\mathbb{Z}} \to V_{\mathbb{Z}}$ over $\mathbb{Z}$.

In particular, in the Soulé approach to $\mathbb{F}_1$ geometry, it is a consistency condition on the cyclotomic points of a scheme over $\mathbb{Z}$ that determines whether it comes from an underlying $\mathbb{F}_1$ structure. This replaces the consistency condition on lifts of Frobenius that is used in the Λ-ring approach we discussed in §4 above. In particular, as observed in §1.7 of [44], one can probe the presence of an $\mathbb{F}_1$ structure on a $\mathbb{Z}$-scheme, in the Soulé setting, by restricting to the case where the rings R are group rings of finite abelian groups, such as the $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$.

For example, as discussed in [16], the affine varieties $\mu^{(n)}$ over $\mathbb{F}_1$ given by roots of unity assemble as a direct limit, with $\mathcal{A}_X = C(S^1)$, to give the multiplicative group $\mathbb{G}_m$ over $\mathbb{F}_1$ (as in [52]) or as an inverse limit, with $\mathcal{A}_X = C^*(\mathbb{Q}/\mathbb{Z})$, the abelian part of the BC algebra, to give the structure of (pro)variety over $\mathbb{F}_1$ to the BC endomotive.

A generalization of this notion of variety over $\mathbb{F}_1$ that covers certain important classes of non-affine cases was recently given in [17].

We now show how to use integral homology 3-spheres and the universal Witten–Reshetikhin–Turaev invariant to construct a gadget over $\mathbb{F}_1$ in the sense described above.

Lemma 6.2. *For $R \in \mathcal{R}$ let $\mathrm{Hom}(\mathbb{Z}HS, R)$ denote the set of ring homomorphisms and define*

$$(6.13) \qquad X_{\mathbb{Z}HS}(R) := \{\phi \in \mathrm{Hom}(\mathbb{Z}HS, R) \,|\, \exists \tilde{\phi} : \widehat{\mathbb{Z}[q]} \to R, \ \phi = \tilde{\phi} \circ J\}.$$

Then $X_{\mathbb{Z}HS}$ defines a covariant functor $X_{\mathbb{Z}HS} : \mathcal{R} \to \mathcal{S}$.

Proof. Let $f : R' \to R$ be a ring homomorphism, for $R, R' \in \mathcal{R}$. Let $X_{\mathbb{Z}HS}(f) : X_{\mathbb{Z}HS}(R') \to X_{\mathbb{Z}HS}(R)$ be given by $X_{\mathbb{Z}HS}(f)(\psi) = f \circ \psi$. If there exists $\tilde{\psi} : \widehat{\mathbb{Z}[q]} \to R'$ such that $\psi = \tilde{\psi} \circ J$, then setting $\tilde{\phi} = f \circ \tilde{\psi}$ satisfies $\phi = \tilde{\phi} \circ J$ for $\phi = f \circ \psi$. This suffices to see that $X_{\mathbb{Z}HS}$ defines a covariant functor as described. $\square$

The set $X_{\mathbb{Z}HS}(R)$ defined as in (6.13) describes the set of all R-valued invariants of integral homology 3-spheres that are coarser than the Witten–Reshetikhin–Turaev invariant, in the sense that they factor through this invariant. This means that they do not distinguish homology 3-spheres that are not already distinguished by the WRT invariant.

We define the complex algebra $\mathcal{A}_{\mathbb{Z}HS}$ as follows. Consider the complex algebra $\mathbb{Z}HS \otimes \mathbb{C}$ and let $\mathcal{A}_{\mathbb{Z}HS}$ be the C^*-completion with respect to the norm $\|M\| = \|J_M\|_{\mathcal{Z}}$, where the latter is the norm defined as in Lemma 2.12, using the evaluations $ev_\zeta(J_M) \in \mathbb{Z}[\zeta]$ viewed as complex numbers after embedding $\mathcal{Z} \subset \mathbb{C}$. The supremum is taken over $\mathcal{Z}$ identified with $\hat{\mathbb{Z}}$, so that it is a compact set. With the involution defined by $M \mapsto -M$, the algebra $\mathcal{A}_{\mathbb{Z}HS}$ is an abelian C^*-algebra.

Lemma 6.3. *Given $\phi \in X_{\mathbb{Z}HS}(R)$ and $\sigma : R \to \mathbb{C}$, then there exists a unique C^*-algebra homomorphism*

$$(6.14) \qquad\qquad e_{\phi,\sigma} : \mathcal{A}_{\mathbb{Z}HS} \to \mathbb{C},$$

that extends $\sigma \circ \phi$ to $\mathbb{Z}HS \otimes \mathbb{C}$ and to $\mathcal{A}_{\mathbb{Z}HS}$. This satisfies the property (6.12).

Proof. Suppose given a ring hoomorphism $\sigma : R \to \mathbb{C}$. Then $\sigma \circ \phi$ determines a homomorphism $\Xi = \sigma \circ \tilde{\phi} : \widehat{\mathbb{Z}[q]} \to \mathbb{C}$. Notice that any homomorphism $\Xi : \widehat{\mathbb{Z}[q]} \to \mathbb{C}$ factors through the evaluations at roots of unity. Namely, assigning such a homomorphism is equivalent to assigning a family of homomorphisms $\Xi_m : \mathbb{Z}[q]/((q)_m) \to \mathbb{C}$, compatible with the maps of the projective system. A homomorphism $\Xi_m : \mathbb{Z}[q] \to \mathbb{C}$ such that $\Xi_m((q)_m) = 0$ has the property that $\Xi_m(q) = \zeta \in \mathbb{C}$ is a root of unity, since

$$\Xi_m((1 - q) \cdots (1 - q^m)) = (1 - \zeta) \cdots (1 - \zeta^m) = 0.$$

Thus, Ξ_m factors through ev_ζ and so, by the compatibility of the Ξ_m with the maps of the projective system, does Ξ. One can then define $e_{\phi,\sigma} : \mathcal{A}_{\mathbb{Z}HS} \to \mathbb{C}$ to be given by $e_{\phi,\sigma}(M) = ev_\zeta(J_M)$. By the choice of the norm $\|M\| = \|J_M\|_{\mathcal{Z}}$ on $\mathcal{A}_{\mathbb{Z}HS}$ we see that these are continuous maps. Moreover, they satisfy by construction the property (6.12). $\qquad\square$

This means that one uses the WRT invariants as cyclotomic coordinates on the gadget $X_{\mathbb{Z}HS}$. The results of Lemma 6.2 and 6.3 directly imply the following.

Proposition 6.4. *The data $(X_{\mathbb{Z}HS}, \mathcal{A}_{\mathbb{Z}HS}, e_{\phi,\sigma})$ define a gadget over $\mathbb{F}_1$.*

It seems then reasonable to ask whether, as in the Habiro conjecture recalled above, the analogous data $(X_{\mathbb{Z}HS/F_d}, \mathcal{A}_{\mathbb{Z}HS/F_d}, e_{\phi,\sigma})$ constructed using the quotients of the Ohtsuki filtration, can be used to define a directed system of *affine varieties* over $\mathbb{F}_1$, which gives rise to a direct limit $(X_{\widehat{\mathbb{Z}HS}}, \mathcal{A}_{\widehat{\mathbb{Z}HS}}, \hat{e}_{\phi,\sigma})$ as varieties over $\mathbb{F}_1$.

6.3. **Semigroup actions on homology 3-spheres.** It is also natural in this context to look for semigroup actions of the multiplicative semigroup $\mathbb{N}$ of positive integers on the ring $\mathbb{Z}HS$ of integral homology 3-spheres. The question is whether one can produce an endomotive, constructed out of the ring $\mathbb{Z}HS$ (or of the $\mathbb{Z}HS/F_d$ and $\widehat{\mathbb{Z}HS}$) in such a way that a suitable invariant $\alpha : \mathbb{Z}HS \to \widehat{\mathbb{Z}[q]}$, obtained from the universal WRT invariant, gives an algebra homomorphism of the semigroup crossed product to the Bost–Connes algebra.

The construction of the universal WRT invariants given in [31] is based on the description of homology 3-spheres via a special kind of surgery presentations, as in [32]. Using a surgery presentation of integral homology 3-spheres as surgeries $M = S^3_{(L,\mathfrak{r})}$, with $L = L_1 \cup \cdots \cup L_\ell$ an algebraically split link in S^3 and framings $\mathfrak{r} = (1/m_i)_{i=1,\ldots,\ell}$, $m_i \in \mathbb{Z}$.

In the case of integer framings, it is well known that different surgery presentations that give rise to the same 3-manifold up to an orientation preserving homeomorphism differ by
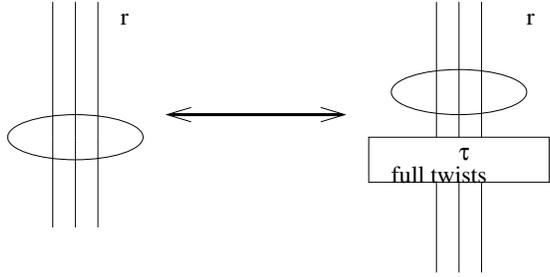
FIGURE 1. Rolfsen move for links with rational framings.

Fenn–Rourke moves or equivalently by the two Kirby moves (see §VI.19 of [48]). In the case of rational instead of integer framings, the analog of these Kirby moves are the Rolfsen moves described in [51]. The main result of [51] states that two surgery presentations with links L and L' and rational framings r and r' give rise to orientation-preserving homeomorphic 3-manifolds if and only if the surgery data are related by a sequence of moves of the following types (Rolfsen moves):

- Introduce or delete a component of the link with framing ∞.
- Replace a linked component L_1 with framing r_1 with a number τ of full twists of the remaining components with new framings

(6.15)
$$\begin{cases} r'_1 = & \frac{1}{r_1^{-1}+\tau} \\ r'_i = & r_i + \tau\ell(L_1, L_i)^2, \quad i \neq 1, \end{cases}$$

where $\ell(L_1, L_i)$ is the linking number. This move is illustrated in Figure 1.

It was conjectured by Hoste in [35] that this should in fact be possible, that is, that two surgery presentations by algebraically split links with framings in the set $\{1/m \,|\, m \in \mathbb{Z}, m \neq 0\}$ give rise to orientation-preserving homeomorphic homology 3-spheres if and only if they are related by a sequence of Rolfsen moves *through links of this same type*. The conjecture was then proved by Habiro in [32], Corollary 5.2. More precisely, Habiro defines an *admissible* link to be an algebraically split link with framings in the set $\{1/m \,|\, m \in \mathbb{Z}\}$ and defines a *rational Hoste move* as a Rolfsen move between a pair of admissible links (see Figure 2). It is shown in Corollary 5.2 of [32] that two admissible links yield a pair of orientation-preserving diffeomorphic homology 3-spheres if and only if they are related by a sequence of rational Hoste moves. In these moves a component L_1 with framing $1/m$ surrounding a number of other components L_i, arranged in such a way that $\ell(L_i, L_1) = 0$ as in the figure, is replaced by $-m$ full twists of the remaining pairs of strands. The framings of the remaining components are unchanged by (6.15) with $\tau = -m$ and $\ell(L_i, L_1) = 0$.

We can then formulate the following question, suggested by the constructions described in the previous sections in terms of endomotives and Habiro rings.

Question 6.5. *Is there any natural (semi)group action*

(6.16)
$$M \mapsto \sigma_n(M), \quad \sigma_n(M_1 \# M_2) = \sigma_n(M_1) \# \sigma_n(M_2)$$

of the multiplicative semigroup $\mathbb{N}$ or of the group $\mathbb{Q}^*_+$, by symmetries of the ring $\mathbb{Z}HS$? If so, is it then possible to obtain a suitable modification Υ_M of the WRT invariant J_M such
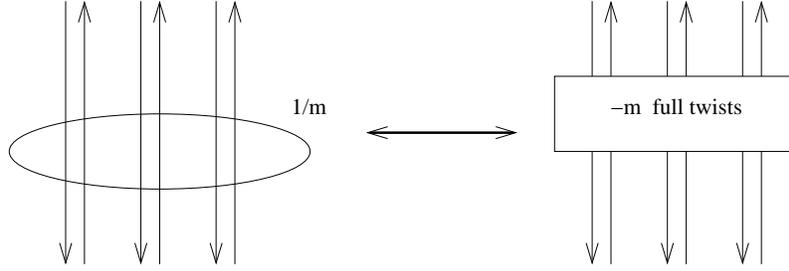
FIGURE 2. Rational Hoste move between admissible links.

that it is still a ring homomorphism $\Upsilon : \mathbb{Z}HS \to \widehat{\mathbb{Z}[q]}$, *or possibly a ring homomorphism* $\Upsilon : \widehat{\mathbb{Z}HS} \to \widehat{\mathbb{Z}[q]}$, *with the property*

(6.17) $$\Upsilon_{\sigma_n(M)} = \sigma_n(\Upsilon_M) \in \widehat{\mathbb{Z}[q]}$$

that it intertwines the semigroup actions? More generally, are there semigroup actions (not necessarily by $\mathbb{N}$*) on 3-manifolds that induce corresponding actions on the WRT invariants?*

In the more general form stated above, one knows that there are such examples, at least for suitable classes of 3-manifolds. For instance, in the case of mapping cylinders and Seifert 3-manifolds (see [1], [33], [41]) an action of the mapping class group on the surface translates into a linear action on the corresponding space of invariants. In the more restrictive form of the question formulated above, that is, whether one can construct an action of the multiplicative semigroup $\mathbb{N}$, and whether this can be done in a way related to the action on the Habiro ring, the question is more complicated to address.

More precisely, the problem of defining an action $M \mapsto \sigma_n(M)$ can be formulated as the question of defining an action on link diagrams, which is well defined under the equivalence via Reidemeister moves and in addition preserves the Hoste moves, so that it induces a well defined action on the homology 3-spheres.

In view of the result of [31], §15.3, on the universal WRT invariant of $1/m$ surgery on a knot in a homology 3-sphere, and of the effect of changes of framings on the WRT invariants as analyzed in [39], it would seem that an ingredient in the construction of a semigroup action of the form (6.16) may be given by a multiplicative change of the framings, that is, a semigroup action on the set of admissible framed links of the form $(L, \mathfrak{r}) \mapsto (L, n^{-1}\mathfrak{r})$, where $L = L_1 \cup \cdots, L_\ell$ with framings $\mathfrak{r} = (1/m_1, \ldots, 1/m_\ell)$, $m_i \in \mathbb{Z}$, $m_i \neq 0$, and $n^{-1}\mathfrak{r} = (1/(m_1 n), \ldots, 1/(m_\ell n))$. However, while changing the framings multiplicatively as in gives a well defined action on the set of admissible links, the Hoste move shows that this does not translate directly into a well defined semigroup action on the set of homology 3-spheres unless one operates also on the full twists. In fact, different surgery presentations for the same homology 3-sphere can replace a component with $1/m$ framings by $-m$ full twists of the pairs of strands linked by the components, and the operations can only become independent of the surgery presentation if the number of full twists is also changed multiplicatively to $-mn$ full twists. However, the number of twists is a property of a planar projection of a link and can be changed by Reidemeister moves: by a Reidemeister move one can change a crossing into a full twist followed by a crossing in the opposite direction. (This simple observation leads to the description of crossings changes via Hoste moves.)

One can, in fact, approach the problem of Question 6.5 in a slightly different way. In the formulation we gave of Question 6.5 we considered ring homomorphisms $\mathbb{Z}HS \to \mathfrak{R}$, in particular with ring of values $\mathfrak{R} = \widehat{\mathbb{Z}[q]}$ the Habiro ring. The generators of $\mathbb{Z}HS$ are homology

3-spheres up to oriented homeomorphisms, or equivalently admissible links up to Reidemeister and Hoste moves, and the multiplication operation is connected sums, or equivalently disjoint unions of links. Instead of working with equivalence classes of admissible links, both under Reidemeister and Hoste moves, one can follow the usual method for handling equivalence relations in noncommutative geometry, which is to keep the equivalence relations explicitly in the form of a groupoid or a category, instead of passing to the set of equivalence classes. In this sense, the situation resembles more closely the approach followed in [46] using 3-manifolds as coverings and then cobordisms between them, assembled first in a 2-category and then into an associative algebra.

In terms of admissible links with equivalences given by Reidemeister and Hoste moves, one can proceed in the following way.

Definition 6.6. *Let $\mathcal{CL}^{(2)}$ be the 2-category whose objects are admissible framed link diagrams $(D, \mathfrak{r})$, the 1-morphisms are sequences of Hoste moves $(D, \mathfrak{r}) \xrightarrow{H} (D', \mathfrak{r}')$ between diagrams, including the identity move, and 2-morphisms $H_1 \xrightarrow{\mathcal{R}} H_2$ are sequences of pairs of Reidemeister moves $\mathcal{R} = (R, R')$ between the resulting diagrams so that the squares commute*

$$(6.18) \qquad \begin{array}{ccc} (D_1, \mathfrak{r}_1) & \xrightarrow{H_1} & (D_1', \mathfrak{r}_1') \\ \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R'} \\ (D_2, \mathfrak{r}_2) & \xrightarrow{H_2} & (D_2', \mathfrak{r}_2') \end{array}$$

There is an associated convolution algebra of functions $\mathcal{A}_{\mathcal{CL}^{(2)}}$ given by functions with compact support $f : Mor^{(2)}(\mathcal{CL}^{(2)}) \to \mathbb{C}$ with two different associative products corresponding, respectively, to the horizontal and vertical composition of 2-morphisms,

$$(6.19) \qquad (f_1 \circ f_2)(\mathcal{R}) = \sum_{\mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2} f_1(\mathcal{R}_1) f_2(\mathcal{R}_2),$$

$$(6.20) \qquad (f_1 \bullet f_2)(\mathcal{R}) = \sum_{\mathcal{R} = \mathcal{R}_1 \bullet \mathcal{R}_2} f_1(\mathcal{R}_1) f_2(\mathcal{R}_2),$$

where the horizontal composition is given by $\mathcal{R}_1 = (R, R')$ and $\mathcal{R}_2 = (R', R'')$ with

$$\begin{array}{ccccc} (D_1, \mathfrak{r}_1) & \xrightarrow{H_1} & (D_1', \mathfrak{r}_1') & \xrightarrow{H_1'} & (D_1'', \mathfrak{r}_1'') \\ \downarrow{\scriptstyle R_1} & & \downarrow{\scriptstyle R_1'} & & \downarrow{\scriptstyle R_1''} \\ (D_2, \mathfrak{r}_2) & \xrightarrow{H_2} & (D_2', \mathfrak{r}_2') & \xrightarrow{H_2'} & (D_2'', \mathfrak{r}_2'') \end{array}$$

and the vertical composition is given by $\mathcal{R}_1 = (R_1, R_1')$ and $\mathcal{R}_2 = (R_2, R_2')$ with

$$\begin{array}{ccc} (D_1, \mathfrak{r}_1) & \xrightarrow{H_1} & (D_1', \mathfrak{r}_1') \\ \downarrow{\scriptstyle R_1} & & \downarrow{\scriptstyle R_1'} \\ (D_2, \mathfrak{r}_2) & \xrightarrow{H_2} & (D_2', \mathfrak{r}_2') \\ \downarrow{\scriptstyle R_2} & & \downarrow{\scriptstyle R_2'} \\ (D_3, \mathfrak{r}_3) & \xrightarrow{H_3} & (D_3', \mathfrak{r}_3') \end{array}$$

The algebra $\mathcal{A}_{\mathcal{CL}^{(2)}}$ has an involution $f^*(\mathcal{R}) = \overline{f(\mathcal{R}^{-1})}$, where $\mathcal{R}^{-1} = (R'^{-1}, R^{-1})$ is defined by the diagram

$$
\begin{array}{ccc}
(D'_2, \mathfrak{r}'_2) & \xrightarrow{H_2^{-1}} & (D_2, \mathfrak{r}_2) \\
\downarrow{\scriptstyle R'^{-1}} & & \downarrow{\scriptstyle R^{-1}} \\
(D'_1, \mathfrak{r}'_1) & \xrightarrow{H_1^{-1}} & (D_1, \mathfrak{r}_1)
\end{array}
$$

This involution satisfies

$$(f_1 \circ f_2)^\vee = f_2^\vee \circ f_1^\vee, \quad \text{and} \quad (f_1 \bullet f_2)^\vee = f_2^\vee \bullet f_1^\vee.$$

One can pass from the 2-category $\mathcal{CL}^{(2)}$ to a category $\mathcal{CL}$ where one "collapses the 2-morphisms", that is, one uses the 2-morphisms to generate an equivalence relation on objects and 1-morphisms. The category one obtains in this way is the following.

Definition 6.7. *Let $\mathcal{CL}$ be the category whose objects are admissible framed link types $(L, \mathfrak{r})$ in S^3 and whose morphisms are sequences of Hoste moves between link types (that is, Hoste moves up to Reidemeister moves), $(L, \mathfrak{r}) \xrightarrow{H} (L', \mathfrak{r}')$. The corresponding convolution algebra $\mathcal{A}_{\mathcal{CL}}$ is given by functions with compact support $f : Mor(\mathcal{CL}) \to \mathbb{C}$ with the convolution product*

$$(6.21) \qquad (f_1 \circ f_2)(H) = \sum_{H = H_1 \circ H_2} f_1(H_1) f_2(H_2),$$

where the sum is over the compositions of arrows given by Hoste moves:

$$(L, \mathfrak{r}) \xrightarrow{H} (L', \mathfrak{r}') = ((L, \mathfrak{r}) \xrightarrow{H_1} (\tilde{L}, \tilde{\mathfrak{r}})) \circ ((\tilde{L}, \tilde{\mathfrak{r}}) \xrightarrow{H_2} (L', \mathfrak{r}')).$$

One can, for instance, construct a time evolution on the algebra $\mathcal{A}_{\mathcal{CL}}$ by setting

$$(6.22) \qquad \sigma_t(f)(H) = e^{imt} f(H), \quad \text{with} \quad m = \ell - \ell',$$

if H is a sequence of Hoste moves between links $L = L_1 \cup \cdots \cup L_\ell$ and $L' = L'_1 \cup \cdots \cup L'_{\ell'}$. This manifestly satisfies $\sigma_f(f_1 \circ f_2) = \sigma_t(f_1) \circ \sigma_t(f_2)$.

Let $\mathcal{S}_{(L,\mathfrak{r})}$ be the set of sequences of Hoste moves H transforming some link types $(\tilde{L}, \tilde{\mathfrak{r}})$ into $(L, \mathfrak{r})$. Let $\mathcal{H}_{(L,\mathfrak{r})} = \ell^2(\mathcal{S}_{(L,\mathfrak{r})})$ and define the action of $\mathcal{A}_{\mathcal{CL}}$ on $\mathcal{H}_{(L,\mathfrak{r})}$ by setting

$$(6.23) \qquad (\pi_{(L,\mathfrak{r})}(f)\xi)((\tilde{L}, \tilde{\mathfrak{r}}) \xrightarrow{H} (L, \mathfrak{r})) = \sum_{H = H_1 \circ H_2} f((\tilde{L}, \tilde{\mathfrak{r}}) \xrightarrow{H_1} (L'', \mathfrak{r}'')) \, \xi((L'', \mathfrak{r}'') \xrightarrow{H_2} (L, \mathfrak{r})).$$

In this representation the time evolution (6.22) is generated by a Hamiltonian $\mathfrak{H}$ of the form

$$(6.24) \qquad \mathfrak{H}\delta_H = m\, \delta_H, \quad H \in \mathcal{S}_{(L,\mathfrak{r})},$$

where $(\tilde{L}, \tilde{\mathfrak{r}}) \xrightarrow{H} (L, \mathfrak{r})$ with $m = \tilde{\ell} - \ell$ the difference between the number of components of $\tilde{L}$ and L. Thus, $\mathrm{Spec}(\mathfrak{H}) = \mathbb{Z}$ and the eigenspace E_m of $\mathfrak{H}$ is spanned by all the sequences of Hoste moves that have final result $(L, \mathfrak{r})$ and change the total number of components by m. To see that $\mathfrak{H}$ is the Hamiltonian in the representation $\pi_{(L,\mathfrak{r})}$, notice that we have

$$(\pi_{(L,\mathfrak{r})}(\sigma_t(f))\xi)(H) = \sum_{H = H_1 \circ H_2} e^{im_1 t} f(H_1) \xi(H_2)$$

which agrees with

$$(e^{i\mathfrak{H}t} \pi_{(L,\mathfrak{r})}(f) e^{-i\mathfrak{H}t} \xi)(H) = e^{imt} \sum_{H = H_1 \circ H_2} f(H_1) e^{-im_2 t} \xi(H_2).$$

In [46] we introduced the notion of horizontal and vertical time evolutions on convolution algebras associated to 2-categories, *i.e.* time evolutions with respect to either the horizontal or the vertical convolution product. Since Reidemeister moves do not change the number of components, the time evolution σ_t of (6.22) can be lifted to a horizontal time evolution on the algebra $\mathcal{A}_{\mathcal{CL}^{(2)}}$. Vertical time evolutions can be obtained similarly, by considering, instead of the number of components, quantities such as the self-linking number with respect to the blackboard framing of the link diagram, which is altered along Reidemeister moves.

In view of passing from links and Hoste moves to 3-manifolds, we can introduce a refinement of the algebras $\mathcal{A}_{\mathcal{CL}^{(2)}}$ and $\mathcal{A}_{\mathcal{CL}}$ considered above. In fact, the convolution products of these algebras are designed to account for the compositions of Hoste and Reidemeister moves but not for the product of links given by disjoint union, which in turn will determine the product in the ring $\mathbb{Z}HS$ of integral homology 3-spheres given by the connected sum. One can introduced variants $\mathcal{A}_{\mathcal{CL}^{(2)},\cup}$ and $\mathcal{A}_{\mathcal{CL},\cup}$ where the convolution product is modified to include also the product operation of disjoint union. We explain in detail how this can be done in the simpler case of $\mathcal{A}_{\mathcal{CL},\cup}$, while the case of $\mathcal{A}_{\mathcal{CL}^{(2)},\cup}$ is similar.

We define $\mathcal{A}_{\mathcal{CL},\cup}$ to be the algebra of functions with finite support $f : Mor(\mathcal{CL}) \to \mathbb{C}$ with the modified product

$$(6.25) \qquad (f_1 \star f_2)(L \xrightarrow{H} L') = \sum_{H=(H_{1,1}\cup H_{1,2})(H_{2,1}\cup H_{2,2})} f_1(L_1 \xrightarrow{H_{1,1}} \tilde{L}_1)f_2(\tilde{L}_2 \xrightarrow{H_{2,2}} L'_2),$$

where one sums over all possible splittings into connected components as well as over compositions of Hoste moves, with

$$L = L_1 \cup L_2 \xrightarrow{H=H_1\cup H_2} L' = L'_1 \cup L'_2.$$

Notice in particular, that in the case where $H = id$ one obtains in this way the product

$$(f_1 \star f_2)(L) = \sum_{id=(H_{1,1}\cup H_{1,2})(H_{2,1}\cup H_{2,2})} f_1(H_1)f_2(H_2),$$

and the sum splits into a term of the form $\sum_{L=L_1\cup L_2} f_1(L_1)f_2(L_2)$, corresponding to the case where all the terms have $H_{i,j} = id$, and a term containing nontrivial Hoste moves and their inverses $H_{1,1} = H_{2,1}^{-1}$ and $H_{1,2} = H_{2,2}^{-1}$. Thus, this can be seen as a modification of the group ring product

$$(f_1 * f_2)(L) = \sum_{L=L_1\cup L_2} f_1(L_1)f_2(L_2)$$

of the group of links with the product $\cup$ given by disjoint union.

Lemma 6.8. *The product* (6.25) *is associative.*

Proof. Consider a triple composition of morphisms $L_j \xrightarrow{H^{(j,j+1)}} L_{j+1}$, $j = 1, 2, 3$, with corresponding possible splittings into connected components of the form

$$\cup_i L_{1,i} \xrightarrow{\cup H_i^{(12)}} \cup_i L_{2,i} \xrightarrow{\cup H_i^{(23)}} \cup_i L_{3,i} \xrightarrow{\cup_i H_i^{(34)}} \cup_i L_{4,i},$$

for $i = 1, 2, 3$. We then have

$$(f_1 \star (f_2 \star f_3))(L_1 \xrightarrow{H} L_4) = \sum_{H=H^{(12)}\circ H^{(24)}} f_1(H_1^{(12)})\ (f_2 \star f_3)(H_2^{(24)}),$$

where $H^{(24)} = H^{(23)} \circ H^{(34)}$ with $L_{2,2} \cup L_{2,3} \overset{H^{(24)}_{23}}{\Rightarrow} L_{4,2} \cup L_{4,3}$. This gives then

$$(f_1 \star (f_2 \star f_3))(L_1 \overset{H}{\to} L_4) = \sum_{H=H^{(12)}\circ H^{(23)}\circ H^{(34)}} f_1(H_1^{(12)}) \, f_2(H_2^{(23)}) \, f_3(H_3^{(34)}),$$

where

$$L_{2,2} \cup L_{2,3} \overset{H^{(24)}_{23}}{\Rightarrow} L_{4,2} \cup L_{4,3} = L_{2,2} \cup L_{2,3} \overset{H^{(23)}_2 \cup H^{(23)}_3}{\to} L_{3,2} \cup L_{3,3} \overset{H^{(34)}_2 \cup H^{(34)}_3}{\to} L_{4,2} \cup L_{4,3}.$$

On the other hand, we have

$$((f_1 \star f_2) \star f_3)(L_1 \overset{H}{\to} L_4) = \sum_{H=H^{(13)}\circ H^{(34)}} (f_1 \star f_2)(H_{12}^{(23)}) \, f_3(H_3^{(34)})$$

with $L_{1,1} \cup L_{1,2} \overset{H^{(13)}_{12}}{\Rightarrow} L_{3,1} \cup L_{3,2}$, so that we have

$$((f_1 \star f_2) \star f_3)(L_1 \overset{H}{\to} L_4) = \sum_{H=H^{(12)}\circ H^{(23)}\circ H^{(34)}} f_1(H_1^{(12)}) \, f_2(H_2^{(23)}) \, f_3(H_3^{(34)}),$$

with

$$L_{1,1} \cup L_{1,2} \overset{H^{(13)}_{12}}{\Rightarrow} L_{3,1} \cup L_{3,2} = L_{1,1} \cup L_{1,2} \overset{H^{(12)}_1 \cup H^{(12)}_2}{\to} L_{2,1} \cup L_{2,2} \overset{H^{(23)}_1 \cup H^{(23)}_2}{\to} L_{3,1} \cup L_{3,2}.$$

Thus, we still have an associative product.   □

One can similarly refine the algebra $\mathcal{A}_{\mathcal{CL}^{(2)}}$ to a version $\mathcal{A}_{\mathcal{CL}^{(2)},\cup}$ that incorporates the product of link diagrams by disjoint union. Then, in this setting, an analog of Question 6.5 can be formulated in terms of homomorphisms from one of the algebras $\mathcal{A}_{\mathcal{CL}}$, $\mathcal{A}_{\mathcal{CL},\cup}$, $\mathcal{A}_{\mathcal{CL}^{(2)}}$, $\mathcal{A}_{\mathcal{CL}^{(2)},\cup}$ to a suitable ring $\mathfrak{R}$ of values (*e.g.* the Habiro ring), viewed as "categorified" versions of invariants of homology 3-spheres usually defined as ring homomorphisms $\mathbb{Z}HS \to \mathfrak{R}$.

One can introduce other variants of the convolution algebras of link diagrams, Hoste and Reidemeister moves. We discuss here a variant that is related to the possibility of defining an action on links by a multiplicative change of framings.

We recall that given a link L with ℓ components and a vector $\mathfrak{n} = (n_i) \in \mathbb{N}^\ell$, the cabling $c_\mathfrak{n}(L)$ has each component of L replaced by n_i parallel components, where parallel is defined with respect to the Seifert framing of the link, *i.e.* the one obtained using a Seifert surface Σ for L and moving the link components to parallel copies lying on a collar neighborhood of the boundary of the Seifert surface. Notice that if L is an admissible link, so that all the linking numbers are trivial, $lk(L_i, L_j) = 0$, then $c_\mathfrak{n}(L)$ is also an admissible link. In fact, the Seifert framing has the property that the parallel components $L_{i,k}$ for $k = 1, \ldots, n_i$ have zero linking numbers $lk(L_{i,k}, L_{i,k'}) = 0$, while for $i \neq j$ the linking numbers remain $lk(L_{i,k}, L_{j,k'}) = lk(L_i, L_j) = 0$.

In the variant we describe here below, we also impose an orientation on the Hoste moves, so that an oriented Hoste move is directed towards reducing the number of components of the link. We still consider the identity move as part of the oriented Hoste moves. Working with morphisms that are oriented Hoste moves will give rise to a convolution algebra that is only a semigroupoid and not a groupoid algebra and is no longer involutive.

Definition 6.9. *The category of cabled links $\mathcal{CCL}$ has objects of the form $(L, \mathfrak{r}, \mathfrak{n})$ where L is an admissible link with components $L = L_1 \cup \cdots \cup L_\ell$, and framings $\mathfrak{r} = (1/m_i)_{i=1,\ldots,\ell}$, and with $\mathfrak{n} \in \mathbb{N}^\ell$ the datum of a cabling $c_\mathfrak{n}(L)$ of L obtained as above, where all the components $L_{i,k}$ of $c_\mathfrak{n}(L)$ are given the same framing $1/m_i$. The morphisms are given by sequences of*

"cabled oriented Hoste moves". These are oriented Hoste moves $c_{\mathfrak{n}}(L) \xrightarrow{H} c_{\mathfrak{n}'}(L')$ that come from an oriented Hoste move $L \xrightarrow{\tilde{H}} L'$ in the sense that one has a commutative diagram

$$
\begin{array}{ccc}
c_{\mathfrak{n}}(L) & \xrightarrow{\ H\ } & c_{\mathfrak{n}'}(L') \\
\Big\uparrow{\scriptstyle c_{\mathfrak{n}}} & & \Big\uparrow{\scriptstyle c_{\mathfrak{n}'}} \\
L & \xrightarrow{\ \tilde{H}\ } & L'
\end{array}
$$

The corresponding convolution algebra $\mathcal{A}_{\mathcal{CCL}}$ is given by functions with finite support on the objects of this category with the associative product

$$
(f_1 \star f_2)(H) = \sum_{H = H_1 \circ H_2} f_1(H_1) f_2(H_2)
$$

where the sum is over composition of cabled oriented Hoste moves.

Notice that we can also modify the product to obtain an associative algebra $\mathcal{A}_{\mathcal{CCL}, \cup}$ where the product also involves the product by disjoint union of links as in (6.25). In particular, notice that in this case, since we work with oriented Hoste moves, if $H = id$ and $H = H_1 H_2$ also $H_1 = H_2 = id$. Thus, in $\mathcal{A}_{\mathcal{CCL}, \cup}$, when we restrict to $H = id$ we recover the group ring product

$$
(f_1 \star f_2)(L, \mathfrak{n}, \mathfrak{r}) = \sum_{L = L_1 \cup L_2} f_1(L_1, \mathfrak{n}_1, \mathfrak{r}_1) f_2(L_2, \mathfrak{n}_2, \mathfrak{r}_2)
$$

of the group of (cabled) framed links with the disjoint union operation.

We then make the following observation.

Lemma 6.10. *Suppose given an admissible framed link $L = L_1 \cup \cdots \cup L_\ell$, and a positive integer $n \geq 1$ such that $n | m_i$ for all the integers m_i such that L_i has framing $1/m_i$. Then if a link L' is obtained from L by applying a sequence of oriented Hoste moves, one still has $n | m_i'$ for all the framings $1/m_i'$ of the components of L'.*

Proof. This is an immediate consequence of the fact that an oriented Hoste move eliminates a component, inserting a number of full twists of the remaining components, but does not alter the framings of all the remaining components. $\square$

Notice that the same property clearly does not work for Hoste moves with the opposite orientation, namely those that replace a number of full twists with an additional component. In fact, even if $n | m_i$ for all components of L' this does not imply that the same condition will be satisfied for L because, depending on the number of twists on the components of L', the extra component of L may have a framing $1/m$ with $n \nmid m$.

Lemma 6.11. *Let $(L, \mathfrak{r})$ be an admissible framed link. Suppose given $n \geq 1$ such that $n | m_i$ for all the framings $1/m_i$ of the components L_i. Let L_n denote the cabling $c_{\mathfrak{n}}(L)$ of the framed link L with $\mathfrak{n} = (n, \ldots, n)$. Then, if there is an oriented Hoste move H transforming $(L, \mathfrak{r})$ into another framed link $(L', \mathfrak{r}')$, there exists an oriented cabled Hoste move H_n relating the link $(L_n, n\mathfrak{r})$ to the link $(L_n', n\mathfrak{r}')$.*

Proof. It suffices to take the case of two links that differ by a single Hoste move as in Figure 2, with L having one more component than L', since the Hoste move is oriented. If we denote by L_0 the component of L that is removed in the Hoste move, with framing m_0, then the remaining components of the link acquire $-m_0$ full twists by effect of the Hoste move.
By Lemma 6.10 $n | m_i$ for all components of both L and L'. In the link $(L_n, n\mathfrak{r})$, the component L_0 of L is replaced by n parallel components, all with framing n/m_0. On the other hand,

the link $(L'_n, n\mathfrak{r}')$ consists of the n-cabled L'_n with components of framings $n\mathfrak{r}' = (n/m_i)$. Notice then that, if we apply a Hoste move to each of the n components parallel to L_0 in $(L_n, n\mathfrak{r})$ with framing n/m_0 we obtain a link where we replace each of these components by $-m_0/n$ full twists. After performing this operation on each of the components parallel to L_0 we therefore obtain $-m_0$ full twists on the n-cabled version of the remaining components of $L \smallsetminus L_0$, with framings n/m_i, which is $(L'_n, n\mathfrak{r}')$.   □

We can now define a transformation on the algebra $\mathcal{A}_{\mathcal{CCL},\cup}$ given by

(6.26)
$$\rho_n(f)(H) = \begin{cases} f(H_n) & n|m_i, \ \forall i, \\ 0 & \text{otherwise,} \end{cases}$$

where H is a cabled oriented Hoste move

$$(L, \mathfrak{n}, \mathfrak{r}) \xrightarrow{H} (L', \mathfrak{n}', \mathfrak{r}'),$$

the framing of L is $\mathfrak{r} = (1/m_i)$, and H_n is the cabled oriented Hoste move obtained as in Lemma 6.11 with

$$(L, n\mathfrak{n}, n\mathfrak{r}) \xrightarrow{H_n} (L', n\mathfrak{n}', n\mathfrak{r}').$$

Notice that $n\mathfrak{r}'$ is still an admissible framing by Lemma 6.10.

Proposition 6.12. *The ρ_n of* (6.26) *give an action of $\mathbb{N}$ by endomorphisms of $\mathcal{A}_{\mathcal{CCL},\cup}$.*

Proof. The product $(f_1 \star f_2)(H)$ in $\mathcal{A}_{\mathcal{CCL},\cup}$ is given by a sum over compositions $H = H_1 \circ H_2$ and splittings into disjoint unions of components $H_1 = H_{1,1} \cup H_{2,1}$ and $H_2 = H_{2,1} \cup H_{2,2}$. Consider a composition

$$(L, \mathfrak{n}, \mathfrak{r}) \xrightarrow{H_1} (\tilde{L}, \tilde{\mathfrak{n}}, \tilde{\mathfrak{r}}) \xrightarrow{H_2} (L', \mathfrak{n}', \mathfrak{r}').$$

We show that $\rho_n(f_1 \star f_2) = \rho_n(f_1) \star \rho_n(f_2)$. One first can observe that $\rho_n(f_1 \star f_2)$ is nonzero if and only if L has the property that $n|m_i$ for all the component. This is the same property that determines whether $\rho_n(f_1)$ is non-zero. If $\rho_n(f_1) \neq 0$ then Lemma 6.10 implies that $\tilde{L}$ also has the property that $n|\tilde{m}_i$ hence $\rho_n(f_2) \neq 0$. One then sees that, when non-zero, the product $\rho_n(f_1) \star \rho_n(f_2)$ gives

$$(\rho_n(f_1) \star \rho_n(f_2))(H) = \sum f_1(H_{n,1,1})f_2(H_{n,2,2}),$$

where the sum is over the splittings

$$H_n = H_{n,1} \circ H_{n,2} = (H_{n,1,1} \cup H_{n,1,2}) \circ (H_{n,2,1} \cup H_{n,2,2}),$$

while $\rho_n(f_1 \star f_2)$ gives

$$\rho_n(f_1 \star f_2)(H) = \sum f_1(\tilde{H}_{1,1})f_2(\tilde{H}_{2,2}),$$

where here the sum is over all splittings

$$H_n = \tilde{H}_1 \circ \tilde{H}_2 = (\tilde{H}_{1,1} \cup \tilde{H}_{1,2}) \circ (\tilde{H}_{2,1} \cup \tilde{H}_{2,2}).$$

Since we use *cabled* oriented Hoste moves as morphisms, we know that $\tilde{H}_{i,j} = H_{n,i,j}$ must be obtained as the n-cabling of a Hoste move $H_{i,j}$, so that the two sums agree.   □

This gives a partial answer, in this more general setting of convolution algebras of links and Hoste moves, to the question of defining an action of $\mathbb{N}$ by endomorphisms of a ring generalizing $\mathbb{Z}HS$. It would be interesting to construct actions of $\mathbb{N}$ on the other algebras $\mathcal{A}_{\mathcal{CL}}$, $\mathcal{A}_{\mathcal{CL},\cup}$, $\mathcal{A}_{\mathcal{CL}^{(2)}}$, $\mathcal{A}_{\mathcal{CL}^{(2)},\cup}$ described above.

We discuss briefly another possible approach towards constructing endomorphisms actions on 3-manifolds, using closed braid representations of links and endomorphisms of braid groups. We only sketch the construction, but we do not fully develop these ideas in the present paper. The problem can once again be summarized as trying to construct an action on (framed) links, which is well defined under the usual equivalence of planar diagrams via Reidemeister moves, and which can be constructed locally at crossings, and which moreover preserves the Fenn–Rourke moves (or Rolfsen moves in the case of rational framings), so that it induces a well defined action at the level of the 3-manifolds, mapping equivalent surgery presentations to equivalent surgery presentations.

First recall quickly the relation between links and braids (see the references in [4]). It is well known by a classical result of Alexander that all link types can be represented by closed N-braids, for some N. A closed braid representation of a link is obtained by a choice of a braid axis A and an open braid is obtained from the link by cutting it open along a half-plane H_θ originating from the axis A. Link types correspond to equivalences of braid representations via conjugation and Markov stabilization/destabilization operations. These give a reformulation in braid terms of the usual equivalence of link diagrams via Reidemeister moves. It is known by a result of Morton that an equivalence class of closed braid diagrams corresponding to a link type consists of an infinite family of conjugacy classes in braid groups B_N, for varying N, or in B_∞, the union of the B_N. The main result of [4] extracts from this set of conjugacy classes a finite number of elements of *minimal complexity*, with respect to a complexity function. The complexity function constructed in [4] is obtained from geometric data, which in turn admit a combinatorial formulation. Namely, given a link L and a choice of an axis A, one considers an oriented (in general not connected) embedded Seifert surface of maximal Euler characteristic with boundary $\partial\Sigma = L$ and a fibration H of $\mathbb{R}^3 \smallsetminus A$ with fibers H_θ, with good assumptions on how tubular neighborhoods of A and the fibers H_θ intersect Σ. The complexity function is then defined as the triple $(\#(L \cap H_\theta), \#(A \cap \Sigma), \#T(H, \Sigma))$, where $T(H, \Sigma)$ is the set of singular points of the foliation induced on Σ by H. The latter is known as the *braid foliations*. The complexity function defined in this way takes values in $\mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+$ ordered lexicographically, and it only depends on the conjugacy class in B_N determined by the data (L, A), with $N = \#(L \cap H_\theta)$. The advantage of using representatives of minimal complexity is that one can avoid Markov stabilization moves, which is the basis for the more recent "Markov's theorem without stabilization" of [5]. We discuss below the reason why this approach may be useful to our problem.

We first need to recall also some properties of braid groups and their endomorphisms. It is known by [2] that braid groups (for $N \geq 4$) are Hopfian but not co-Hopfian, where the Hopfian property follows from the fact that braid groups are linear (see [3]) and it means that a surjective endomorphism of a braid group B_N is in fact an automorphism. Automorphisms are rather easy to describe, as they are combinations of the inner ones and the inversion that sends each generator to its inverse. The fact that the braid groups are not co-Hopfian means that there are injective endomorphisms that are not automorphisms. In fact, a complete description of injective endomorphisms is given in [2]. They come from homeomorphisms $h : D_N \to D_N$ of the N-punctured disk, by setting

$$(6.27) \qquad \rho_{h,m}(s_a) = s_{h(a)}^{\epsilon(h)} T_N^m,$$

where $\epsilon(h) = \pm 1$ according to the orientation preserving/reversing property of h, while s_a are the half-twists switching two punctures along an arc a, which generate the group B_N, and T_N is the generator of the center of B_N, namely the full twist of the n strands, $T_N = (s_1 \cdots s_{N-1})^N$.

Suppose then that one defines an operation on closed braid diagrams of links by considering a simple class of such endomorphisms of braid groups, namely those of the form $\rho(s_a) = s_a T_N^m$, for some $m \in \mathbb{Z}$. This means that one obtains a new braid diagram from a given one by replacing the open braid $\gamma \in B_N$ by $\rho(\gamma) \in B_N$ and closing the diagram again. A different opening and closing of the braid diagram gives rise of an element in the same conjugacy class $C(\gamma)$ in B_N, which will in turn be mapped to an element in the conjugacy class $C(\rho(\gamma))$ so that the action is well defined on conjugacy classes. In fact, even more precisely, since the element T_N is in the center, one has $\rho(\alpha\gamma\alpha^{-1}) = \alpha\rho(\gamma)\alpha^{-1}$, since $\rho(\alpha) = \alpha T_N^{k(\alpha)}$ with $k(\alpha) = m\ell(\alpha)$, where $\ell(\alpha)$ is the length of α defined as the image under the homomorphism $\ell : B_N \to \mathbb{Z}$ that sends all the generators s_i to 1.

Thus, we can define in this way an operation of $\rho \in \text{End}(B_N)$ on conjugacy classes in B_N. For it to induce an action on the link types we would need it to be compatible with the Markov equivalence between braid representatives, realized by conjugations and Markov de/stabilizations. It is easy to see that the operation is well defined under conjugation, or equivalently under the second Reidemeister move. In fact, if two braids γ and γ' differ by a twist s_i and an opposite twist s_i^{-1}, their images $\rho(\gamma)$ and $\rho(\gamma')$ will again differ in the same way, again because of the fact that T_N is in the center. However, the case of Markov de/stabilizations is more delicate, since these change the number of braids and the extra crossing that corresponds to the first Reidemeister move has the effect that, if $\gamma \in B_N$ and $\gamma' \in B_{N+1}$ differ by a Markov stabilization, that is $\gamma' = \gamma s_N$, then the images satisfy

$$\rho_{N+1}(\gamma') = \rho_{N+1}(\gamma)\rho_{N+1}(s_N) = \rho_{N+1}(\gamma)s_N T_{N+1}^{m_{N+1}}$$

which is off from a Markov stabilization by m_{N+1} full twists.

This is where the existence of minimal complexity representatives may become useful in trying to make this kind of action well defined on link types. Selecting the finitely many minimal complexity representative conjugacy classes, one eliminates the need for Markov de/stabilizations. However, iterating the transformation defined by ρ on these classes requires the image of a minimal complexity representative to be still of minimal complexity for the link type obtained after applying the transformations. The new link is in each case obtained from the old one simply by inserting (it does not matter where, since it is an element in the center of B_N) a number $m_N \ell(\gamma)$ of full twists.

A first question is the following: if γ and γ' in B_N represent the same link type, do the corresponding elements $\gamma T_N^{m_N \ell(\gamma)}$ and $\gamma' T_N^{m_N \ell(\gamma')}$ also represent the same link type? Notice that a positive answer to this question would follow, if we knew that two equivalent γ and γ' will have the same length. This is certainly not the case if one is allowed to use Markov de/stabilizations. However, it is a well known conjecture of Jones (see [36] p.357) that the algebraic crossing number (or writhe) of a minimal braid (with braid index N equal to the minimum value for the given knot type) is a link invariant, hence if γ and γ' in B_N are braid representatives with N minimal, then $\ell(\gamma) = \ell(\gamma')$. Evidence for this conjecture is given, for instance, in [38].

If we assume the Jones conjecture that minimal braids have a unique writhe, then it is clear that, if γ and γ' are two such braid representatives of a given link type, then the transformed $\gamma T_N^{m_N \varpi}$ and $\gamma' T_N^{m_N \varpi}$ also represent the same link type, where ϖ is the writhe of both γ and γ'. In fact, both are obtained from equivalent braids by composing them with the same number $m_N \varpi$ of full twists of the N strands, which still yields equivalent braids. Minimal complexity braid representatives are in particular minimal braids, so if we apply our transformation to these elements only and we assume the Jones conjecture holds, we do get a positive answer to the question above, which in turn implies that the link type of the image is well defined

regardless of which minimal complexity representative we choose among the finitely many available for a given link type.

We can then ask the next question, which is needed in order to be able to iterate the transformation we defined by applying the braid group endomorphism ρ to braids that are of minimal complexity in the sense of [4].

Question 6.13. *For what classes of links does the operation $\gamma \mapsto \gamma T_N^{m_N \ell(\gamma)}$ preserve the minimal complexity property?*

To understand the answer to this question one needs to understand how inserting full twists in a given minimal complexity braid diagram alters the properties of the Seifert surface Σ and of the braid foliation determined on it by the fixed fibration H. Notice that neither the choice of the axis A nor the number N are changed by the operation above. However, the fact that N is unchanged does not a priori imply that it will still be minimal for the new link type. We do not at present know of an answer to Question 6.13. Thus, we can, for the sake of this discussion, define $\mathcal{L}_{\min}$ to be the class of link types that admit a minimal complexity braid representative for which the answer to the question is positive. Notice that the set $\mathcal{L}_{\min}$ is non-empty. In fact, it contains at least the torus knots $T(a, b)$, which have a unique minimal braid conjugacy class, which is determined by the braid $(s_1 \cdots s_{a-1})^b$. On these minimal braids the action described above is of the form

$$\rho : (s_1 \cdots s_{a-1})^b \mapsto (s_1 \cdots s_{a-1})^b T_a^{mb(a-1)} = (s_1 \cdots s_a)^{b(1+ma(a-1))},$$

which has the effect of transforming $T(a, b)$ into $T(a, b(1 + ma(a - 1)))$.

Thus, modulo obtaining a better answer to Question 6.13, we will be restricting our attention to the subring $\mathcal{L}_{\min}$ of $\mathcal{L}$. One can use the complexity arguments of [6] to check that the ring multiplication, which is given by the disjoint union of links, is well defined on this subset. Notice then that if we have two endomorphisms of the same type

$$\rho_{N,n_i}(\gamma) = \gamma T_N^{n_i \ell(\gamma)}, \quad i = 1, 2,$$

they satisfy

$$\rho_{N,n_2} \rho_{N,n_1}(\gamma) = \gamma T_N^{(n_1 + n_2 + n_1 n_2 N(N-1))\ell(\gamma)} = \rho_{N,n_1} \rho_{N,n_2}(\gamma).$$

Thus, upon choosing an endomorphism ρ_{N,n_p} for each generator p of the multiplicative semigroup $\mathbb{N}$ one obtains an action of the abelian semigroup $\mathbb{N}$ on the links of $\mathcal{L}_{\min}$.

We are then left with the question of implementing the surgery equivalence of links. Since we are working with closed braid representatives, it is best to think of the equivalence relation generated by the Fenn–Rourke moves (let us restrict to the original case of integer framings for simplicity here) in terms of braid groups. This was done in [42], see also [25] and [29]. We give here the formulation of [29], which is closer to the form we need. In this form, the set of surgery equivalence classes of N-strand braid representatives is the quotient of the braid group B_N by its normal subgroup $P(N)_3$, which is the third term in the lower central series of the pure braid group $P(N)$ defined by the exact sequence

$$1 \to P(N) \to B_N \to S_N \to 1,$$

which maps the braid group to the symmetric group acting as permutations of the endpoints of the braid. This point of view has the advantage of formulating the equivalence relation in a way that is manifestly compatible with the braid group structure and compositions of braids. Thus, in particular, if our transformation is given by the braid group endomorphism $\rho_N : \gamma \mapsto \gamma T_N^{m_N \ell(\gamma)}$, then if γ and γ' are surgery equivalent so that $\gamma^{-1}\gamma' \in P(N)_3$, we have $\rho_N(\gamma)^{-1}\rho_N(\gamma') = \gamma^{-1}\gamma' T_N^{m_N(\ell(\gamma')-\ell(\gamma))}$. In particular, if γ and γ' are surgery equivalent and have $\ell(\gamma) = \ell(\gamma')$ then $\rho_N(\gamma^{-1}\gamma') = \gamma^{-1}\gamma' \in P(N)_3$, so their images under ρ_N are also surgery

equivalent. This only takes care of implementing surgery equivalence among representatives with the same number of strands and with the same writhe. Since γ and γ' here belong to different link types, the Jones conjecture for their minimal braid representatives need no longer imply that $\ell(\gamma) = \ell(\gamma')$ even if they have the same N. Thus, this argument alone does not suffice to implement surgery equivalence. A more refined argument along these lines may be obtained, in the case of algebraically split links, in terms of the Milnor invariants obtained from triple intersections of Seifert surfaces as in [42].

A different way to approach the question of implementing the surgery moves is to proceed as in [54], where one works with multiplicative invariants of links and obtains from these invariants of 3-manifolds only after a suitable averaging over repeated cabling operations and a thermodynamical limit. We only make some preliminary observations on what questions one needs to address for this approach to be indeed compatible with the action on links defined using the braid group endomorphisms and the minimal complexity braid representatives. Essentially, we are again asking a question similar to Question 6.13 above, on whether certain operations, in this case cabling, preserve the minimal complexity property. There are some indications to this effect, at least for what concerns the first entry of the complexity function of [4], that is, the braid index. In fact, it is shown in [56] (see also [38]) that the braid index of the n-cabling of a link L is n-times the braid index of L, so the n-cabling of a minimal braid for L is a minimal braid for the n-cabling of L. Also, by [38], if the Jones conjecture holds for L, it also holds for its cabled versions. The question of whether this property of the braid index extends to the rest of the complexity function of [4] depends on the Seifert surface of the cabled link and its braid foliations. Since this will locally look like n copies of the original Seifert surface and foliations, it seems likely that the rest of the complexity function should also behave well under the cabling operations, but this requires a more careful investigation. If one can indeed extend in this way the action from a link to all its n-cabled version, one can take the resulting action as being the one that induces the action on the 3-manifold invariants, when the latter are constructed according to the procedure of [54].

One can also, in this approach based on braid representatives of links, introduce the same kind of categorical constructions discussed in the first part of this section, with categories of braid representatives, 1-morphisms given by surgery equivalences and and 2-morphisms by Markov de/stabilizations and conjugations. For a categorical framework using braid representatives of links see for instance [58].

At this stage it is not clear how far one can develop this point of view. It would be interesting to interpret the universal WRT invariant as a morphism from a suitable "endomotive of homology 3-spheres", which means a crossed product $\mathcal{A} \rtimes \mathbb{N}$ by an action of $\mathbb{N}$ on an algebra $\mathcal{A}$ of admissible links and Hoste moves implementing the surgery equivalence between them, to the endomotive constructed out of the one-variable Habiro ring earlier in this paper. Such a construction would make it possible to import the techniques of noncommutative geometry developed in [14] for the theory of endomotives to the context of 3-manifold invariants.

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