Geometry and Physics of Numbers

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What are numbers?

• natural numbers: 1, 2, 3, 4, 5, ...
can add and multiply

• integer numbers: ..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...
can add and subtract and multiply (but not divide)

• rational numbers: include fractions like $\frac{2}{3}$, $\frac{1}{2}$, ...
can add, subtract, multiply and divide

• real numbers: include things like $\sqrt{2}$ or π not rational
same operations

• Other types of numbers: complex numbers, p-adic numbers, algebraic numbers, ...

Note: natural numbers look simple, but they are already mysterious

Natural numbers and Prime numbers

Among the natural numbers some special ones: primes
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, ...

- primes are basic building blocks for natural numbers: any natural number is a product of prime numbers
- a prime number is only divisible by itself and by 1: it cannot be further simplified

Addition: can get *any* natural number by adding 1 to itself enough times: $1+1=2$, $1+1+1=3$, $1+1+1+1=4$, ... *only one "building block"* is needed: the number 1

Multiplication: much more difficult: $1\times 1=1$ don't get anything else; $2\times 2=4$, $2\times 2\times 2=8$, $2\times 2\times 2\times 2=16$: get very few numbers...

To obtain all numbers by multiplication need *infinitely many building blocks*: the primes

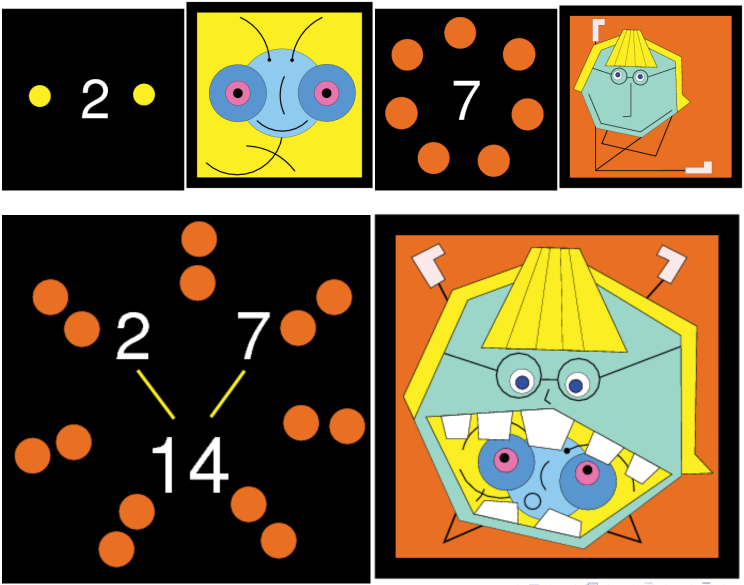
Erathostenes sieve

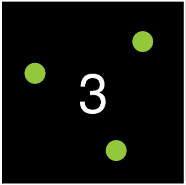
Identify primes and numbers obtained multiplying primes:

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Prime decomposition

Rick Schwartz's children book: "You can count on monsters"

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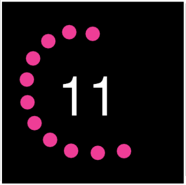
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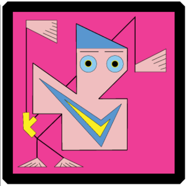

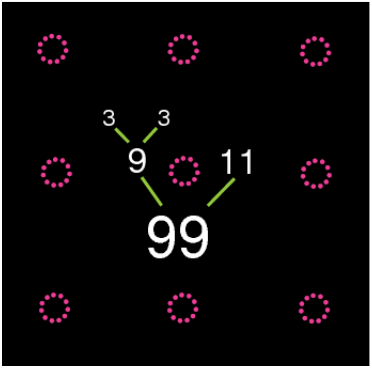
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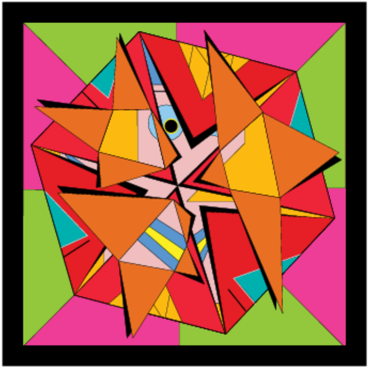


Large primes

- There are infinitely many prime numbers, so there are arbitrarily *large* ones
- A *very large* Marsenne prime number $2^{257885161} - 1$, discovered January 25th, 2013 by the Great Internet Mersenne Prime Search (GIMPS)
- Large primes and cryptography Public-key cryptography based on large primes and factorization into primes: *product of two very large prime numbers* very difficult to factorize even with very fast (non-quantum) computers

Small technical note: how it actually works: choose randomly two very large primes p, q; compute the product $n = p \cdot q$, this is part of the *public key*, together with $1 < e < \phi(n) = (p-1)(q-1)$ with $(e, \phi(n)) = 1$; *private key* is d with $d \cdot e = 1 \mod \phi(n)$; p and q are secret: can be used to compute d (RSA algorithm)

The mystery of prime numbers

Question: which natural numbers are prime? how are they distributed among natural numbers?

We don't know how to predict where the prime numbers are!

"Prime numbers grow like weeds among the natural numbers, seeming to obey no other law than that of chance but also exhibit stunning regularity" (Don Zagier, number theorist)

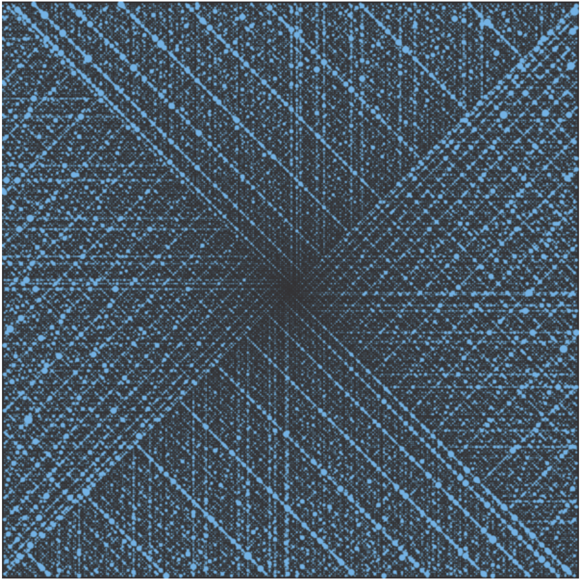
The Ulam spiral

Strange patterns appear that seem to indicate regularity, but are not predictable by any simple rule

37	36	35	34	33	32	31
38	17	16	15	14	13	30
39	18	5	4	3	12	29
40	19	6	1	2	11	28
41	20	7	8	9	10	27
42	21	22	23	24	25	26
43	44	45	46	47	48	49...

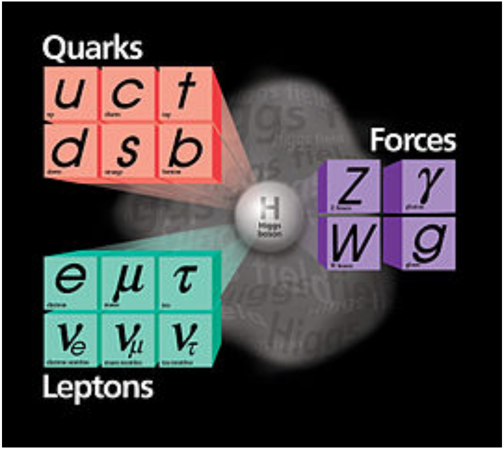
Ulam spiral: write natural number in a grid, spiraling out, then circle the primes... diagonal patterns appear and disappear

Ulam spiral:



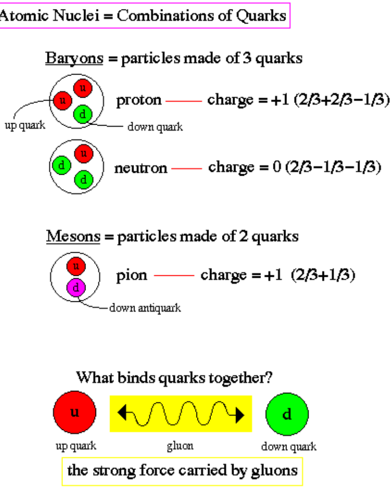
Primes are *basic building blocks* of natural numbers

Analogy: Elementary particles and basic building blocks of matter



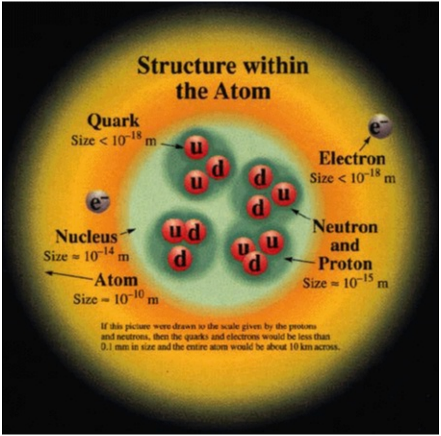
Question: can we use this analogy to understand more about the primes?

Elementary particles combine to form more complicated (composite) particles, such as protons and neutrons (forces)



Primes combine to form natural numbers (multiplication)

Metaphor: breaking a number into its prime factors is *like* breaking an atom into its constituent elementary particles

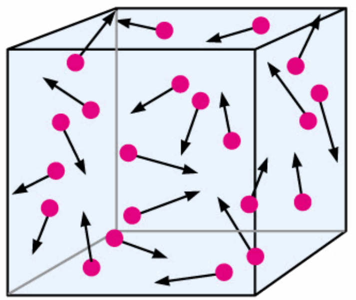


Moral: metaphors are crucially important in mathematics, just as they are in poetry and art

Maxwell–Boltzmann statistics (Gibbs distribution)

A gas of particles at varying temperature: $\beta = 1/(k_B T)$

$$\frac{N_i}{N} = \frac{m_i \, e^{-\beta E_i}}{Z(\beta)}, \qquad Z(\beta) = \sum_j m_j e^{-\beta E_j}$$



- at higher temperatures, higher energy levels activated
- at low temperatures, system freezes on ground state
- (small technical note: in classical limit both Bose–Einstein and Fermi–Dirac distributions approximate Maxwell–Boltzmann)

The *primon gas* (or Riemann gas)
Imagine that numbers are *like* a gas of particles, with energies

$$E_n = \log n, \quad n = p_1 \cdots p_k \quad E_n = \log(p_1) + \cdots + \log(p_k)$$

measuring the sum of the *lengths* (in number of digits) of the prime constituents

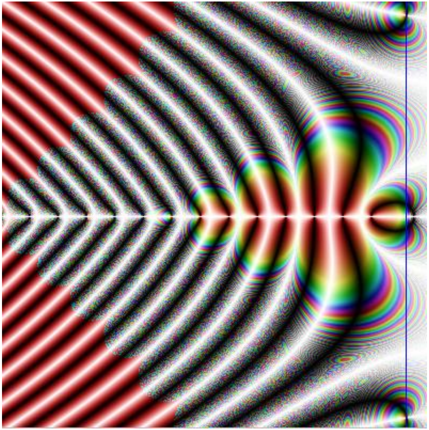
• $n = 1$ is the *ground state*, with lowest energy $\log(1) = 0$

• the largest the primes involved in the factorization of n the more n is at a higher energy state

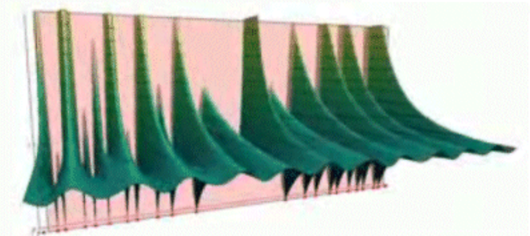
Maxwell–Boltzmann distribution for the primon gas

$$Z(\beta) = \sum_{n\geq 1} e^{-\beta E_n} = \sum_{n\geq 1} e^{-\beta \log(n)} = \sum_{n\geq 1} n^{-\beta} = \zeta(\beta)$$

Riemann zeta function



Riemann zeros: the location of the zeros of the Riemann zeta function is related to the distribution of the prime numbers



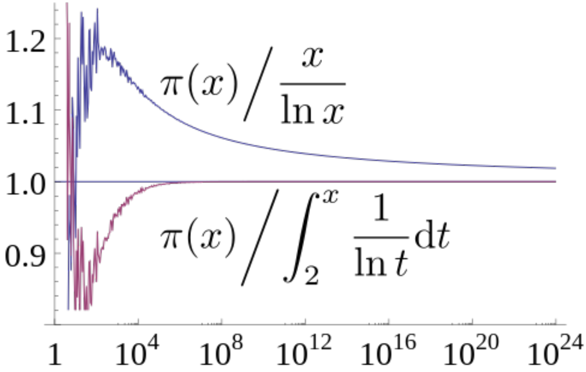
A 3-D representation of Riemann's zeta function. All the points where the function has zero value seem to lie on the same "critical" line (the black downward spikes), an observation known as the Riemann Hypothesis.

Riemann (1859): number of primes less than a given number expressed in terms of a sum over zeros of the Riemann zeta function

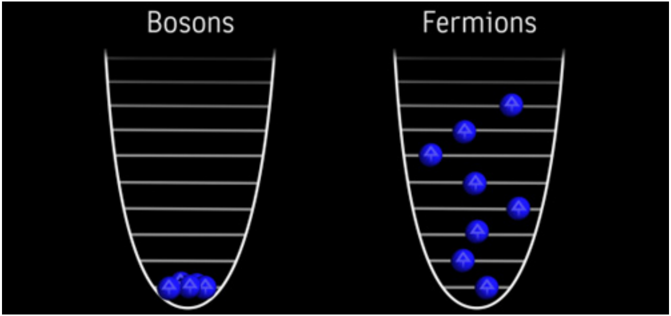
Riemann hypothesis: besides the "trivial zeros" at $-2, -4, -6, \ldots$ the zeta function only has zeros on the critical line $1/2 + it$.

The mystery of Riemann's zeros

- The Riemann hypothesis is one of the most famous unsolved problem in mathematics
- If the Riemann hypothesis is true then "most regular" possible distribution of prime numbers (prime number theorem)



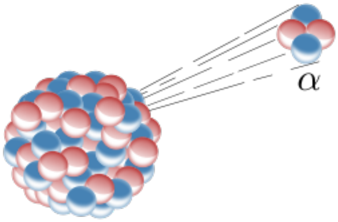
Elementary particles come in two families: Bosons and Fermions



Bosons: can all occupy the same quantum state at once

Fermions: no two of them in the same state at the same time

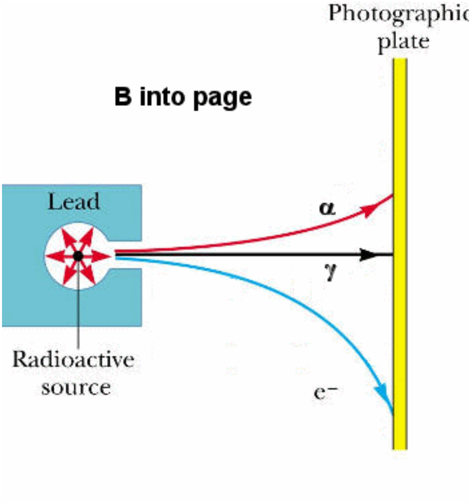
A particle made of an *even* number of Fermions is a Boson



- Example: α-rays (Helium nuclei = 2 protons + 2 neutrons)

Conclusion: a *sign* $(-1)^{\text{number of fermions}}$ distinguishes if the resulting particle is a Fermion or a Boson: $(-1) \times (-1) = +1$

- γ-rays are also Bosons (photons)
- β-rays are fermions (electrons)



Primon gas with Supersymmetry

If think of primes as *fermions* then can only form numbers with no repeated factors ($6 = 2 \times 3$ is OK but $4 = 2 \times 2$ is not allowed)

The Möbius function

$$\mu(n) = \begin{cases} 0 & \text{repeated prime factors} \\ +1 & \text{even number of prime factors (no repetitions)} \\ -1 & \text{odd number of prime factors (no repetitions)} \end{cases}$$

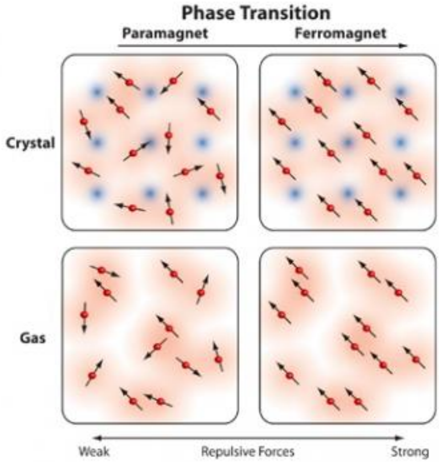
Supersymmetry: even number of fermions = bosons

Partition function (with supersymmetry)

$$\sum_n \mu(n) n^{-\beta} = \frac{1}{\zeta(\beta)}$$

Zeros of zeta as *phase transitions*

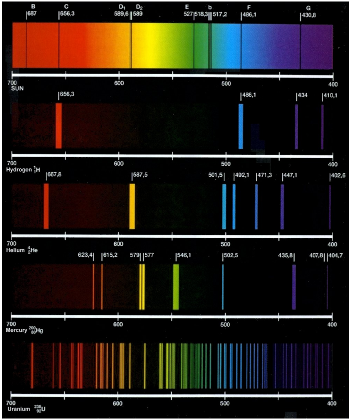
Phase transitions in physical systems



Critical temperatures at which behavior of the system changes from an ordered to a disordered phase or viceversa

Atomic spectra and the zeta function

• Modeling spectra of heavy atoms in nuclear physics

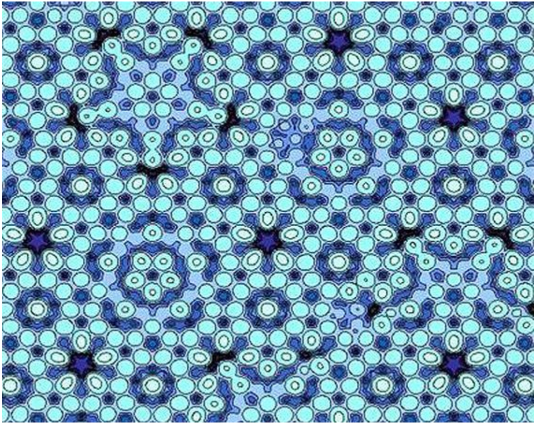


• Idea: spacings between the energy levels of a heavy atom like spacings between eigenvalues of a random matrix

• Observation: also distribution of zeros of Riemann zeta function *looks like* random matrices (heavy atoms spectra)

Quasi-crystals and the zeta function

quasicrystals are aperiodic solids (Penrose tilings)



- The Riemann hypothesis is equivalent to the fact that the zeros of the Riemann zeta function form a 1-dimensional quasi-crystal (Dyson, 2009)