EXPOSITIONES
MATHEMATICAE

Expo. Math 15 (1997), 229-249

Gauge Groups and Characteristic Classes

Matilde Marcolli & Mauro Spreafico

Conjugacy of the gauge groups provides a weaker notion of equivalence of bundles, which we call *fundamental equivalence*. Two vector bundles are fundamentally equivalent iff they are obtained from one another by tensoring with a line bundle. We present a survey of topological properties of fundamentally equivalent bundles. In particular we prove that two fundamentally equivalent vector bundles whose structure group has a discrete centre will have the same Chern classes. Moreover, we investigate the relation between stable and fundamental equivalence, and possible generalisations in a K-theoretic context.

1. Introduction

A fibre bundle $\xi = (E, p, B)$ with structure group G has a large (in fact infinite-dimensional) group of symmetries, the gauge group $\mathcal{G}(\xi)$ of all bundle automorphisms. The gauge group acts on sections of the bundle and, when working in the smooth category, on various other geometric objects: connections and curvature forms.

The studying of such objects modulo the action of the gauge group, subject to the constraint given by some non-linear differential equations, has lead to the development of gauge theory (among the very many references avilable we would point the reader to the work of Atiyah and Bott [3]). Nowadays gauge theory is an extremely active field that achieved striking results in the construction of invariants of smooth manifolds.

This whole field of research has its origin in gauge theory as a model of basic interactions of matter and fields in theoretical physics. The bibliography on the mathematical physics aspects of gauge theory is enormous, hence we shall not attempt to summarise it here. However, the reader should be aware of the rich interplay of topology, geometry, and physics, and of the fact that the abstract topological investigation of the properties of gauge groups can provide new insight in the realm of physical modelling as well.

It is important to mention that gauge groups have also been the object of a parallel yet different direction of study in algebraic topology. In fact, the gauge group can be considered as a topological object of

0723-0869/97/030229-0249$5.70

its own interest. Topological properties of groups of self equivalences of bundles and more generally of fibrations have been of interest for the algebraic topologists since the development of a unified theory of bundles and fibrations [15]. The homotopy type of the space of gauge equivalences has been studied in [10] (see also [4], [5], [20]).

The two approaches are not totally disjoint, in as the topological properties of the gauge group play a role in the differential geometric world of gauge theory as well (see for instance [3], [8]).

However, the point of view that we want to stress in this paper is a rather different one. Following the general idea underlying the whole development of algebraic topology, we want to associate an algebraic object to a topological bundle and via some algebraic properties of the first deduce topological properties of the latter.

It is clear that the gauge group encodes interesting information on the topology of the bundle. Thus, it is a natural idea to try to recover this topological information from the algebraic properties of the group.

A direct analysis and classification of these infinite dimensional gauge groups as algebraic objects is complicated, although some attempts in this direction have been made [21]. However, it is less complicated to compare gauge groups of different bundles inside some larger group and deduce from the algebraic relation some topological properties of the bundles.

Suppose we consider principal G bundles over a certain base space B. There is a large group in which gauge groups of different bundles over B live. In fact, under mild hypotheses we can assume that all bundles are trivialized over the same open covering of B. In this case we can form an infinite dimensional group, the local gauge group, which only depends on the open covering and on the structure group G and in which all gauge groups $\mathcal{G}(\xi)$ embed as topological subgroups.

Inside the local gauge group it is therefore possible to compare gauge groups by the algebraic relation of being conjugate subgroups. Thus the question is what kind of topological information on the corresponding G bundles is encoded by this algebraic relation of the automorphism groups.

Conjugacy relation of the gauge groups inside the local gauge group was first introduced as an interesting notion of weaker topological equivalence of bundles in [18]. In the context of this paper we shall refer to this kind of equivalence as "fundamental equivalence", in as it corresponds to the actual topological equivalence of the associated fundamental bundles.

The purpose of this paper is to illustrate the meaning of this algebraic relation in the particular case of vector bundles in the smooth

category. In this case conjugacy of the gauge groups has a clear geometric interpretation: in fact it corresponds to the action of a line bundle via tensor product.

Based on this geometric interpretation we develop some characterization of fundamentally equivalent bundles and we investigate whether this notion of equivalence can be extended to virtual bundles for which a notion of gauge group is not defined.

Preliminary notions and notation

We shall consider mainly differentiable vector bundles, hence, if no other hypotesis is assumed, the structure group G will be a Lie group (indeed, a subgroup of $Gl(n, \mathbf{R})$ or $Gl(n, \mathbf{C})$). The base space B will be a connected smooth paracompact manifold endowed with a Riemannian metric, and all the maps will be smooth.

In order to simplify notations, we are going to describe connections, curvatures, and gauge transformations of a vector bundle ξ as those of the associated principal bundle: the corresponding objects on ξ can be recovered from these.

We shall write $Vect_F(B)$ for the semiring of equivalence classes of vector bundles over B, with F being the real or complex field; $Vect_F^r(B)$ will denote the set of classes of vector bundles of a fixed rank r. $K_F(B)$ will denote the ring completion of $Vect_F(B)$ (see [11], or [17]).

Given a principal G bundle $\xi = (E, p, B)$, the gauge group $\mathcal{G}(\xi)$ is the group of self equivalences of the bundle, namely the group of continuous maps

$$\lambda_\alpha : U_\alpha \to G$$

$$\lambda_\beta = g_{\beta\alpha}\lambda_\alpha g_{\alpha\beta},$$

where the bundle is trivial over U_α, and has transition functions $g_{\alpha\beta}$. Gauge groups of principal G bundles over B are viewed as subgroups of the local gauge group

$$\prod_\alpha \mathcal{M}(U_\alpha, G).$$

Their conjugacy classes have been studied in [18] and [14]. As shown in [18], we have the following.

Theorem 1 *If two given principal G bundles ξ and ξ' over B satisfy the following conditions*[1]*:*

[1] More precisely, in [18] it is also required that all points of B are non degenerate, i.e., $\{b\} \to B$ is a cofibration $\forall b \in B$. This is ensured in our case since B is a manifold.

(i) the map $\eta : \mathcal{G}(\xi) \to G$, that sends $\{\lambda_\alpha\} \mapsto \lambda_\alpha(b)$ for a fixed base point $b \in U_\alpha \subset B$, is an epimorphism for all $b \in B$,
(ii) all the sets U_α have the property that every map $\bar{\psi}_\alpha : U_\alpha \to G/ZG$ can be lifted to a map $\psi_\alpha : U_\alpha \to G$;
then the three statements below are equivalent:
a) the fundamental bundles $A(\xi)$ and $A(\xi')$ are equivalent (see [18] for the definition),
b) the gauge groups $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi')$ are conjugate,
c) the following relation between the transition functions of the two bundles is satisfied

$$g'_{\alpha\beta} = \lambda_\alpha^{-1} c_{\alpha\beta} g_{\alpha\beta} \lambda_\beta, \tag{1}$$

with $\lambda_\alpha : U_\alpha \to G$, $c_{\alpha\beta} : U_\alpha \cap U_\beta \to ZG$, where ZG is the centre of G.

Note that, since two smooth bundles are equivalent if and only if they are smoothly equivalent, we can provide smooth maps λ_α that satisfy (1) as shown in [17], pg. 40.

In the present setting, since the base space B is a manifold, condition (ii) is certainly satisfied. In fact, by the riemannian structure, we may take geodesically convex neighbourhoods U_α that are contractible. On the other hand, since we consider vector bundles, the structure group G can always be reduced to $U(r)$ or $O(r)$. In the first case, condition (i) also is satisfied. The map η is always a fibration. In the second case, when $G = O(r)$, η is also onto the path connected component of $O(r)$ containing the unit (i.e. $SO(r)$), and this is a sufficient condition for the equivalence of the three statements above (since the centraliser of $SO(r)$ in $O(r)$ is $ZO(r)$, compare with [18] pg. 236-237).

Thus, we can state that for the principal bundles associated to any pair of vector bundles over B the statements $a)$, $b)$, $c)$ above are equivalent, and we introduce the following.

Definition 2 *We say that two vector bundles $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ of rank r are fundamentally equivalent (f-equivalent) if one of the three conditions a), b), c) above holds.*

In the following, unless otherwise assumed, the structure group will be the whole $U(r)$ or $O(r)$.

In particular, for classes of vector bundles fundamental equivalence can be expressed in a more convenient coordinate-free way, as in the following.

Lemma 3 *Two vector bundles ξ, $\xi' \in Vect_F^r(B)$ are fundamentally equivalent iff*

$$\xi' = \xi \otimes l, \tag{2}$$

with l a line bundle.

Proof. The lemma is an immediate consequence of the relation (1) that holds between the transition functions: the maps $c_{\alpha\beta} : U_\alpha \cap U_\beta \to ZG$, with $ZG = \mathbf{Z}_2$ in the real and $ZG = U(1)$ in the complex case, define a line bundle (they satisfy the cocycle rule). Thus, up to equivalence of bundles, fundamental equivalence is the same as the condition described above. QED

It is natural therefore to investigate the relation between characteristic classes induced by fundamental equivalence: some results will be given in the next section.

Moreover, in a following section we shall consider generalisations of fundamental equivalence for vector bundles of different ranks, and we shall introduce a proper notion that makes sense in the K-ring, even for virtual bundles, where gauge groups are not defined.

2. Characteristic Classes

In the following we shall consider Chern classes in the cohomology with coefficients in a field of characteristic 0, unless otherwise stated.

A known consequence of (2), in the case of complex bundles, is the following relation between the Chern classes (see e.g. [6], pg. 493, or [9], pg. 56).

Proposition 4 *If two complex bundles of rank r are related by (2) then the Chern classes are related by*

$$c_k(\xi') = \sum_{j=0}^{k} \binom{r-j}{k-j} c_j(\xi) c_1(l)^{k-j}. \tag{3}$$

We shall derive the same formula from a relation between the curvatures of fundamentally equivalent bundles, and we'll see that something more can be said if the structure group is reduced to a subgroup of $U(r)$ which has a discrete centre (e.g. $SU(r)$).

It is widely known (see [11], [12], or [16]) that, modulo torsion, the Chern classes of a complex bundle can be expressed by means of symmetric invariant polynomials in the curvature 2-form Ω.

A connection on the bundle ξ is specified by the local potentials $\mathcal{A}_\alpha \in L(G) \otimes T^*U_\alpha$, that transform according to the law

$$\mathcal{A}_\beta = g_{\beta\alpha}\mathcal{A}_\alpha g_{\alpha\beta} + g_{\beta\alpha} dg_{\alpha\beta}, \tag{4}$$

and the corresponding curvature has local fields $\Omega_\alpha = d\mathcal{A}_\alpha + \mathcal{A}_\alpha \wedge \mathcal{A}_\alpha$, which satisfy

$$\Omega_\beta = g_{\beta\alpha}\Omega_\alpha g_{\alpha\beta}. \tag{5}$$

The Chern classes correspond to the elementary symmetric polynomials in the curvature 2-form:

$$c(\xi) = \det(1 + \frac{i}{2\pi}\Omega) = 1 + c_1(\xi) + c_2(\xi) + \cdots + c_k(\xi). \tag{6}$$

With this notation explained, we can introduce the following.

Lemma 5 *The affine spaces of connections of two fundamentally equivalent complex bundles are isomorphic.*

<u>Proof.</u> Let ξ and ξ' be two such bundles, with transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ respectively. The proof is by construction. Indeed the isomorphism is given on the local potentials as follows:

$$\mathcal{A}'_\beta := \lambda_\beta^{-1}\mathcal{A}_\beta\lambda_\beta + \lambda_\beta^{-1}d\lambda_\beta + \theta_\beta, \tag{7}$$

$$\theta_\beta := \sum_\gamma \rho_\gamma c_{\beta\gamma} dc_{\gamma\beta},$$

where ρ_γ is a partition of unity subordinate to the cover $\{U_\gamma\}$.

A connection on the bundle ξ' satisfies

$$\mathcal{A}'_\alpha = g'_{\alpha\beta}\mathcal{A}'_\beta g'_{\beta\alpha} + g'_{\alpha\beta} dg'_{\beta\alpha}. \tag{8}$$

We want to show that if we assume $\mathcal{A}'_\alpha$ to satisfy (8), then it also satisfies (7).

We know that the transition functions satisfy (1), thus the right hand side of (8) becomes

$$\lambda_\alpha^{-1}c_{\alpha\beta}g_{\alpha\beta}\lambda_\beta(\lambda_\beta^{-1}\mathcal{A}_\beta\lambda_\beta + \lambda_\beta^{-1}d\lambda_\beta + \sum_\gamma \rho_\gamma c_{\beta\gamma}dc_{\gamma\beta})\lambda_\beta^{-1}c_{\beta\alpha}g_{\beta\alpha}\lambda_\alpha +$$

$$+(\lambda_\alpha^{-1}c_{\alpha\beta}g_{\alpha\beta}\lambda_\beta)d(\lambda_\beta^{-1}c_{\beta\alpha}g_{\beta\alpha}\lambda_\alpha) =$$

$$= \lambda_\alpha^{-1}g_{\alpha\beta}\mathcal{A}_\beta g_{\beta\alpha}\lambda_\alpha + \lambda_\alpha^{-1}g_{\alpha\beta}(d\lambda_\beta\lambda_\beta^{-1})g_{\beta\alpha}\lambda_\alpha +$$

$$+\lambda_\alpha^{-1} g_{\alpha\beta}\lambda_\beta(\sum_\gamma \rho_\gamma c_{\beta\gamma} dc_{\gamma\beta})\lambda_\beta^{-1} g_{\beta\alpha}\lambda_\alpha + \lambda_\alpha^{-1} g_{\alpha\beta}(\lambda_\beta d\lambda_\beta^{-1}) g_{\beta\alpha}\lambda_\alpha +$$

$$+\lambda_\alpha^{-1} g_{\alpha\beta}(c_{\alpha\beta} dc_{\beta\alpha}) g_{\beta\alpha}\lambda_\alpha + \lambda_\alpha^{-1} g_{\alpha\beta} dg_{\beta\alpha}\lambda_\alpha + \lambda_\alpha^{-1} d\lambda_\alpha.$$

Now, the second and fourth terms of the sum cancel due to the identity $\lambda_\beta d\lambda_\beta^{-1} + (d\lambda_\beta)\lambda_\beta^{-1} = 0$ and, since the adjoint action of any element of the group on the elements of $L(ZG)$ is trivial, we get

$$\lambda_\alpha^{-1}(g_{\alpha\beta}\mathcal{A}_\beta g_{\beta\alpha} + g_{\alpha\beta} dg_{\beta\alpha})\lambda_\alpha + \lambda_\alpha^{-1} d\lambda_\alpha + \sum_\gamma \rho_\gamma c_{\beta\gamma} dc_{\gamma\beta} + c_{\alpha\beta} dc_{\beta\alpha}.$$

But the last two summands, by the cocycle rule $c_{\gamma\beta} = c_{\gamma\alpha} c_{\alpha\beta}$, give exactly the term $\sum_\gamma \rho_\gamma c_{\alpha\gamma} dc_{\gamma\alpha}$. Therefore

$$\mathcal{A}'_\alpha = \lambda_\alpha^{-1}\mathcal{A}_\alpha\lambda_\alpha + \lambda_\alpha^{-1} d\lambda_\alpha + \sum_\gamma \rho_\gamma c_{\alpha\gamma} dc_{\gamma\alpha}.$$

QED

We can now state the following result.

Theorem 6 *Suppose that two complex bundles ξ and ξ' have structure group reduced to a subgroup H of $U(r)$ which has a discrete centre, and they are fundamentally equivalent with respect to the structure group H. Then they have the same Chern classes.*

Proof. The theorem holds true, since in this case the 1-forms

$$\theta_\alpha = \sum_\gamma \rho_\gamma c_{\alpha\gamma} dc_{\gamma\alpha}$$

defined in (7) vanish, the Lie algebra of ZH being trivial. Therefore the curvatures are related by

$$\Omega'_\alpha = \lambda_\alpha^{-1}\Omega_\alpha\lambda_\alpha,$$

and the Chern classes, which are computed as invariant polynomials, satisfy

$$c_i(Ad_{\lambda_\alpha}\Omega_\alpha) = c_i(\Omega_\alpha).$$

Hence we have for all i: $c_i(\xi') = c_i(\xi)$. QED

Note that the condition of being fundamentally equivalent with respect to the reduced structure group H is different than fundamental equivalence with respect to $U(r)$ in a way that will be explained more carefully in the last section. Actually, for $H = SU(r)$ it is a stronger

condition, as will be shown in lemma 27. In general, however, this might not be the case.

Note also that the above result does not follow from proposition 4, since the main step in the argument is to show that there is a flat connection on the line bundle l, which implies that its Chern class is a torsion element.

The converse of theorem 6 is false: in fact l has a flat connection if and only if it has locally constant transition functions, as in [12] pg. 6; but it is easy to construct a line bundle with locally constant transition functions that are not contained in any finite order subgroup of $U(1)$.

Moreover, as a consequence of lemma 5 we can also prove the following, which is the same statement as proposition 4:

Theorem 7 *Given two fundamentally equivalent complex bundles, ξ and ξ', the relation (3) between their Chern classes holds in $H^*(B;\mathbf{R})$, with l the complex line bundle defined by the transition functions $c_{\alpha\beta}$.*

Proof. By lemma 5, we have the following relation between the local fields Ω'_α and Ω_α:

$$\Omega'_\alpha = \lambda_\alpha^{-1}\Omega_\alpha\lambda_\alpha + \Theta_\alpha,$$

$$\Theta_\alpha := \sum_\gamma d\rho_\gamma \wedge c_{\alpha\gamma} dc_{\gamma\alpha}. \tag{9}$$

Again, since Chern classes are computed from invariant polynomials, we have that $c_i(\Omega'_\alpha) = c_i(\Omega_\alpha + \Theta_\alpha)$.

We know by equation (7) that formally Chern classes are completely determined by the eigenvalues of the matrix Ω_α,

$$c(\Omega_\alpha) = \det\left(1 + \frac{i}{2\pi} diag\{x_{\alpha j}\}\right) = \prod_{j=1}^{r}\left(1 + \frac{i}{2\pi}x_{\alpha j}\right).$$

Now consider the same expression for ξ'

$$c(\Omega'_\alpha) = \prod_{j=1}^{r}\left(1 + \frac{i}{2\pi}x'_{\alpha j}\right) = \prod_{j=1}^{r}\left(1 + \frac{i}{2\pi}(x_{\alpha j} + \Theta_\alpha)\right).$$

The result follows by expanding the two products[2], where $c_1(l) = 1 + \Theta_\alpha$. The latter equality is true since the local forms Θ_α introduced

[2]The r.h.s. gives: $\prod_{j=1}^{r}(1 + x_j + c) = \sum_{h=0}^{r} e_{r-h}(x_1,\ldots,x_r)(1+c)^h = \sum_{k=0}^{r}\sum_{h=k}^{r}\binom{h}{k} c^k e_{r-h}(x_1,\ldots,x_r)$, where $e_{r-h}(x_1,\ldots,x_r)$ is the elementary symmetric function of degree $r-h$ in $x_1,\ldots,x_r$.

in equation (9) are obtained by covariant differentiation of the family of one forms θ_α,

$$\Theta_\alpha = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha.$$

Moreover, the θ_α's define a family of local potentials for the line bundle l over B, with the transition functions

$$c_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1),$$

since

$$\theta_\beta = c_{\beta\alpha}\theta_\alpha c_{\alpha\beta} + c_{\beta\alpha} dc_{\alpha\beta} = \theta_\alpha + c_{\beta\alpha} dc_{\alpha\beta}.$$

Thus, the local fields Θ_α define a curvature on the same bundle. Note that, since the structure group is abelian, this curvature is indeed defined globally on the whole base space, i.e. $\Theta_\alpha = \Theta_\beta = \Theta$. QED

An obvious consequence of theorem 6 is that, in the same hypothesis, the Chern character and the Todd class will also coincide.

Another result that follows from theorem 6 concerns the case of real vector bundles. Since Pontrjagin classes[3] can also be computed as invariant polynomials in the curvature form, which in this case has coefficients in the Lie algebra of $O(r)$, we have the following.

Corollary 8 *Two fundamentally equivalent real bundles have the same Pontrjagin classes.*

Again, as a trivial consequence, they will have the same Hirzebruch L-polynomial and the same $\hat{A}$-class.

Remark In theorems 6 and 7 we have assumed the characteristic classes to be elements of the de Rham cohomology ring. If we consider the Chern and Pontrjagin classes in $H^*(B; \mathbf{Z})$, where there may be torsion elements, then in the case of a discrete centre fundamental equivalence would no longer imply the same characteristic classes, as the following example shows.

Example 9 *The real bundle over $\mathbf{R}P^4$ $\xi = \eta \oplus \eta$, where η is the canonical line bundle, is fundamentally equivalent to the trivial bundle, but it has non-vanishing top Pontrjagin class with integer coefficients (see also example 2 of [14]).*

[3]This definition of Pontrjagin classes is equivalent to the better known $p_j(\xi) = (-1)^j c_{2j}(\xi \otimes \mathbf{C})$.

Proof. The complexified bundle satisfies $\eta_{\mathbf{C}} = \bar{\eta}_{\mathbf{C}}$. Therefore $c_1(\eta_{\mathbf{C}})$ is a non-zero element of order 2 in $H^2(B;\mathbf{Z})$. Hence the top Pontrjagin class of ξ is also a non-zero element of order 2. QED

Note that in the above remark we pointed out one of the possible obstructions in having the converse of theorem 7. In fact the converse is false both because of having considered the cohomology in a field of characteristic 0, and also because of the fact that, even with integer cohomology, the Chern classes do not specify the bundle completely. The following example deals with this kind of obstruction.

In order to construct two non-equivalent bundles that have the same Chern classes (with integer coefficients) it is sufficient to find two non homotopic maps from B into the Grassmanian $G_r(\mathbf{C}^m)$, that induce the same homomorphism in cohomology.

Example 10 *Let $B = S^5$. Since $\pi_5(BU(2)) = \mathbf{Z}_2$, the generator of this homotopy group gives rise to a non-trivial class of 2-plane bundles over S^5. On the other hand, since $H^{2i}(S^5) = 0$ for $i > 0$, any such bundle has trivial Chern classes.*

Proof. $\pi_5(BU(2)) = \pi_4(U(2)) = \pi_4(SU(2)) = \pi_4(S^3) = \mathbf{Z}_2$.

Note, moreover, that the above bundle can be described in terms of one transition function $g : S^4 \to U(2)$ that is the suspension of the Hopf map $h : S^3 \to S^2$. QED

Note also that, since $H^2(S^5;\mathbf{Z}) = 0$, f-equivalence is the same as equivalence, thus the above example gives a counterexample to the converse of theorem 7. A counterexample for the real case will be given by example 18.

Torsion elements become the major obstruction when the problem is rephrased in a K-theoretic context: this is a consequence of the fact that the Chern character is an isomorphism modulo torsion, i.e. between $K_{\mathbf{C}}(B) \otimes \mathbf{Q}$ and $H^*(B;\mathbf{Q})$. This remark will be made more precise in section 4.

A result analogous to (3) is known to hold for Stiefel–Whitney (SW) classes of real bundles, as shown in [6], pg. 497.

Proposition 11 *Two f-equivalent real bundles of rank r have Stiefel–Whitney classes related by the following:*

$$w_k(\xi') = \sum_{j=0}^{k} \binom{r-j}{k-j} w_j(\xi) w_1(l)^{k-j}, \tag{10}$$

where coefficients are taken mod 2.

A more geometric interpretation of equation (10) can be given in the case $k = 1, 2$.

Recall that the Čech cohomology is isomorphic to the De Rham cohomology and computable by means of a good cover (every manifold has a good cover [7]). It is well known how to describe the first and second Stiefel–Whitney classes in terms of Čech cocycles.

Theorem 12 *Two real even dimensional bundles which are fundamentally equivalent have the same first Stiefel-Whitney class.*

Proof. We use the notation with multiplicative $\mathbf{Z}_2$ for the first and second Stiefel-Whitney classes. Since the structure group is $O(2m)$, we have $det(c_{\alpha\beta}) = 1$. Therefore by (1)

$$det(g'_{\alpha\beta}) = det(g_{\alpha\beta})det(\lambda_\alpha)^{-1}det(\lambda_\beta).$$

Hence, the cocycles that define the first Stiefel-Whitney class differ by a coboundary

$$f'_1(\alpha, \beta) = f_1(\alpha, \beta)(\delta f_0)(\alpha, \beta).$$

QED

Clearly this corresponds to (10) computed for $k = 1$ and r even, with $\mathbf{Z}_2$ denoted additively.

Theorem 13 *Suppose given two real orientable even dimensional bundles which are fundamentally equivalent. If the bundle $\oplus_r l$, defined by the fundamental equivalence, has trivial second Stiefel-Whitney class, then they have the same second Stiefel-Whitney class.*

Proof. Now we are considering two real orientable vector bundles of even dimension $r = 2m$. $w_1(\xi) = w_1(\xi') = 1$ in the multiplicative $\mathbf{Z}_2$, hence the transition functions take values in the reduced structure group $SO(r)$. Suppose $w_2(\xi) = 1$. We can choose the $\{U_\alpha\}$ to be a good cover and we can lift locally the maps $c_{\alpha\beta}$, and λ_α to the universal covering $SPIN(r)$ of $SO(r)$,

$$\tilde{c}_{\alpha\beta} : U_\alpha \cap U_\beta \to SPIN(r),$$

$$\tilde{\lambda}_\alpha : U_\alpha \to SPIN(r).$$

Thus, we can write a lifting for $g'_{\alpha\beta}$ as

$$\tilde{g}'_{\alpha\beta} = \tilde{\lambda}_\alpha^{-1}\tilde{c}_{\alpha\beta}\tilde{g}_{\alpha\beta}\tilde{\lambda}_\beta.$$

Therefore, since we have assumed that $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = 1$, we get

$$\tilde{g}'_{\alpha\beta}\tilde{g}'_{\beta\gamma}\tilde{g}'_{\gamma\alpha} = \tilde{c}_{\alpha\beta}\tilde{c}_{\beta\gamma}\tilde{c}_{\gamma\alpha},$$

which implies that $w_2(\xi') = 1$. QED

This corresponds to (10), with $k = 2$, r even, and $w_1(\xi) = 0$ in the additive $\mathbf{Z}_2$.

As an example of a possible application of (3), we can obtain a particular case of theorem 3 (in the version of corollary 1) of [14].

Theorem 14 *Suppose that B is a manifold of finite type, and consider a polyhedral decomposition. Let ξ and ξ' be two f-equivalent complex bundles of rank r over B. We can write $\xi' = \xi \otimes l$. Assume that $rc_1(l) \neq 0$ in the integral cohomology. If the bundles are equivalent when restricted over the 2-skeleton, then they are equivalent.*

Proof. By (3) with integer coefficients, we have that

$$c_1(\xi') = c_1(\xi) + rc_1(l).$$

By assumption, $i^*(rc_1(l)) = 0$ in $H^2(B^{(2)}; \mathbf{Z})$, with $i : B^{(2)} \hookrightarrow B$ the inclusion of the 2–skeleton. But $i^* : H^2(B) \to H^2(B^{(2)})$ is a monomorphism: in fact, in the exact cohomology sequence of the pair $(B, B^{(2)})$ we have $H^2(B, B^{(2)}) = H^2(B/B^{(2)}) = 0$, since this is the second cohomology group of a complex with no cells in dimension less than 3. QED

The condition on the Chern class of l is necessary, as shown by the following example.

Example 15 *Consider the trivial complex 2-plane bundle ϵ_2 over $\mathbf{R}P^4$, and the bundle $\eta_{\mathbf{C}}$ of example 9. $\eta_{\mathbf{C}} \oplus \eta_{\mathbf{C}}$ is f-equivalent to ϵ_2 (example 9); moreover, they are equivalent when restricted to $\mathbf{R}P^2 \hookrightarrow \mathbf{R}P^4$ ([1], pg.620-622). However, they are not equivalent over $\mathbf{R}P^4$ (example 9).*

Remark The condition on $c_1(l)$ not being torsion of order r is related to a more general fact. The correspondence between the action of a line bundle via tensor product, $\xi' = \xi \otimes l$, and the conjugacy classes of gauge groups may have a non-trivial kernel. This problem will be discussed briefly in the following section.

3. The Problem of Classifying f-Equivalent Bundles

As a consequence of characterising fundamental equivalence as in lemma 3, it is natural to ask whether it is possible to classify, with the

choice of the line bundle l in (2), the elements of an f-equivalence class up to bundle equivalence.

It is well known how to classify real or complex line bundles over B, namely

Lemma 16 *The equivalence classes of line bundles are given by the cohomology groups $H^1(B;\mathbf{Z}_2)$ in the real case and by $H^2(B;\mathbf{Z})$ in the complex case.*

However, as already pointed out in theorem 14 and example 15, the action of a line bundle l on vector bundles ξ can have "isotropy": there may be a bundle ξ such that $\xi \otimes l$ and ξ are equivalent, though l is a non-trivial line bundle. An example where this situation occurs is the following.

Example 17 *Let ξ be the trivial real 4-plane bundle over $\mathbf{R}P^2$, and l be the line bundle η of example 9 restricted over $\mathbf{R}P^2 \hookrightarrow \mathbf{R}P^4$. Then $\xi \otimes l = \xi$, as already described in example 15.*

In this situation the classification of line bundles differs from the classification of f-equivalent bundles. The problem of investigating the action of line bundles by tensor product, and possible conditions under which the cohomology groups $H^1(B;\mathbf{Z}_2)$ or $H^2(B;\mathbf{Z})$ represent topological obstructions to equivalence of f-equivalent vector bundles, is certainly worth a more detailed studying, and will be considered elsewhere.

Note that the same problem can be reformulated in the more general case of topological G bundles satisfying conditions (i) and (ii) of theorem 1.

In fact, if we want to classify the elements in the f-equivalence class of a principal G bundle ξ that has transition functions $g_{\alpha\beta}$, we are led to consider a condition of the form

$$\lambda_\alpha^{-1} c_{\alpha\beta} g_{\alpha\beta} \lambda_\beta = \tilde{c}_{\alpha\beta} g_{\alpha\beta}. \tag{11}$$

The remark on the presence of isotropy in the case of vector bundles is reflected here in the fact that, if $c_{\alpha\beta} = c_\alpha^{-1} \tilde{c}_{\alpha\beta} c_\beta$, with c_α that takes values in ZG, then (11) is certainly satisfied. But there may be other ZG-bundles (defined by the transition functions $c_{\alpha\beta}$ and $\tilde{c}_{\alpha\beta}$) that are not equivalent but that still satisfy (11).

A possible approach to this problem might be the following. Consider the gauge group $\mathcal{G}(\xi)$ of a principal G bundle ξ. An element $\{f_\alpha\} \in \mathcal{G}(\xi)$ satisfies by definition

$$f_\beta = g_{\beta\alpha} f_\alpha g_{\alpha\beta}.$$

If ξ' is equivalent to ξ, the transition functions satisfy

$$g'_{\alpha\beta} = \lambda_\alpha^{-1} g_{\alpha\beta} \lambda_\beta.$$

Thus, $\mathcal{G}(\xi')$ is conjugate to $\mathcal{G}(\xi)$ inside the local gauge group via the following:

$$\lambda_\beta f_\beta \lambda_\beta^{-1} = g_{\beta\alpha}(\lambda_\alpha f_\alpha \lambda_\alpha^{-1}) g_{\alpha\beta}.$$

We might look for conditions under which, in the above, the group $\mathcal{G}(\xi')$ will not coincide with $\mathcal{G}(\xi)$ unless the functions λ_α take values in ZG.

If that is the case, we can fix the group $\mathcal{G}(\xi)$ inside the local gauge group and identify all possible principal G bundles that have this same gauge group with the set of all the transition functions of the form

$$g'_{\alpha\beta} = c_{\alpha\beta} g_{\alpha\beta},$$

with $c_{\alpha\beta}$ that take values in ZG.

But the maps $c_{\alpha\beta} : U_\alpha \cap U_\beta \to ZG$ define a class in the sheaf cohomology $H^1(B; ZG \otimes \mathcal{C})$,

$$[c_{\alpha\beta}] \in H^1(B; ZG \otimes \mathcal{C}),$$

since $c_{\alpha\beta} c_{\beta\gamma} c_{\gamma\alpha} = 1$ is a cocycle. Here $\mathcal{C}$ denotes the sheaf of non-vanishing continuous functions in the topological setting, or smooth functions in the differentiable setting.

We expect that a more detailed investigation of this problem might identify conditions under which the elements of $H^1(B; ZG \otimes \mathcal{C})$ represent obstructions to equivalence of f-equivalent G-bundles.

4. Stable Equivalence[4]

In the following we investigate the possibility of extending the notion of f-equivalence for bundles of different ranks. It turns out that a direct natural generalisation can not be found.

However, a suitable notion of fundamental equivalence can be introduced for classes of stably equivalent bundles and in the K-ring.

This notion, though, does not reduce to the usual f-equivalence when restricted to bundles of the same rank. A reason why this is the case is the fact that for bundles of the same rank f-equivalence does not imply stable equivalence, since in general f-equivalent bundles may have different Chern classes. Also the converse implication is false, as the following example shows.

[4] In this section, the base space B is assumed to be a finite dimensional CW complex.

Example 18 *Given the $SO(4)$ bundles $\xi = S^4 \times \mathbf{R}^4$ and $\xi' = TS^4$, there is a trivial line bundle ϵ such that*

$$\xi \oplus \epsilon = \xi' \oplus \epsilon;$$

but their gauge groups are not conjugate.

Proof. In particular, if the gauge groups were conjugate, they would have the same topological structure as subgroups of the topological group

$$\prod_\alpha \mathcal{M}(U_\alpha, SO(4));$$

but, using a result of [13], in the case of the trivial bundle ξ we have that

$$\pi_1(\mathcal{G}(\xi)) = \pi_1(\mathcal{M}(S^4, SO(4))) = \pi_1(SO(4)) \oplus \pi_5(SO(4)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2,$$

while a result of [19] gives that $\pi_1(\mathcal{G}(\xi')) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. QED

Note that the claim of example 18 can be proved directly using lemma 3, but we prefer to give a proof which enlightens the relation among the gauge groups.

We can even state something stronger about the relation between stable and fundamental equivalence.

Theorem 19 *Suppose given two vector bundles of the same rank that are stably and fundamentally equivalent. We can write $\xi' = \xi \otimes l$ by lemma 3. If $rc_1(l) \neq 0$ in the integral cohomology, then the two bundles are equivalent.*

Proof. By stable equivalence they have the same Chern classes in the cohomology with integer coefficients (Stiefel-Whitney classes in the real case). But fundamental equivalence implies that the Chern classes are related by (3), which holds true also with integer coefficients [6] (SW classes related by (10)). QED

Remark: Note that the use of formula (3) for higher Chern classes can give further information and, in many cases, refine the result of theorem 19.

The first approach that one would consider in order to compare gauge groups of bundles of different rank would be to embed all the gauge groups in a unique large local one, like[5] $\prod_\alpha \mathcal{M}(U_\alpha, U(\infty, F))$. This can be done in two different ways.

[5] $U(\infty, F) = dirlim_n U(n, F)$, where $U(n, \mathbf{C}) = U(n)$, and $U(n, \mathbf{R}) = O(n)$.

The first is to consider the groups $U(r,F)$ for different r's as subgroups of $U(\infty,F)$, in such a way that any fixed vector bundle ξ can be considered as having structure group $U(\infty,F)$, regardless of the rank, i.e. identified with the bundle $\xi \oplus \epsilon_\infty$.

Thus the gauge group $\mathcal{G}_\infty(\xi)$ is defined with respect to $G = U(\infty,F)$. It is the space $\mathcal{G}(\xi \oplus \epsilon_\infty)$ of maps $f_\alpha : U_\alpha \to U(\infty,F)$ such that $f_\beta = g_{\alpha\beta} f_\alpha g_{\beta\alpha}$, with the transition functions also considered as taking values in $U(\infty,F)$. Note that in this case conditions (i) and (ii) are satisfied (see [18]).

This approach leads to the following notion of equivalence.

Definition 20 *Two bundles ξ and ξ' of ranks r and r' are ∞-equivalent if, considered as bundles with structure group $U(\infty,F)$, their gauge groups are conjugate inside the local gauge group $\prod_\alpha \mathcal{M}(U_\alpha, U(\infty,F))$.*

Since $ZU(\infty,F) = 1$, this means that two such bundles are equivalent with respect to $U(\infty,F)$, i.e. their transition functions are related by

$$g'_{\alpha\beta} = \lambda_\alpha^{-1} g_{\alpha\beta} \lambda_\beta,$$

with all maps taking values in $U(\infty,F)$.

Two properties are evident by this definition. Stable equivalence implies ∞-equivalence, and ∞-equivalent bundles have the same Chern classes (with integer coefficients). The latter property implies that an analogous of theorem 19 holds in this case too, namely

Theorem 21 *If two vector bundles of the same rank r are fundamentally and ∞-equivalent, and $rc_1(l) \neq 0$ with integer coefficients, then they are equivalent.*

However, definition 20 is not directly generalisable to virtual vector bundles, for which there is not a gauge group.

Another approach would be to consider the bundle ξ with structure group $U(r,F)$, and the gauge group embedded in $\prod_\alpha \mathcal{M}(U_\alpha, U(\infty,F))$ via the map $i_r : U(r,F) \to U(\infty,F)$.

The conjugacy of these gauge groups gives a possible notion of equivalence. Locally it is written as

$$g'_{\alpha\beta} = \lambda_\alpha^{-1} \gamma_{\alpha\beta} g_{\alpha\beta} \lambda_\beta,$$

where $\gamma_{\alpha\beta} \in Z_{U(\infty,F)}(U(n,F))$, the centraliser of $U(n,F)$ in $U(\infty,F)$.

But, since $Z_{U(\infty,F)}(U(n,F))$ is not contained in $ZU(\infty,F)$, this does not induce an equivalence relation between the transition functions.

Anyway, this construction can provide examples of bundles with isomorphic gauge groups that are not conjugate (with respect to $U(r, F)$). This would be another way to answer a problem discussed in [18] and [14].

Equivalently, one might wish to extend lemma 3 for bundles of different ranks, where they are viewed as having structure group $U(k, F)$ for some large fixed k instead of $U(\infty, F)$. That would lead to introduce relations of the form:

$$\xi' \oplus \epsilon_p = l \otimes (\xi \oplus \epsilon_q). \tag{12}$$

It is clear that such a notion would have the advantage of being related to stable equivalence somehow similarly to how f-equivalence, introduced for bundles of the same rank, is related to bundle equivalence.

Unfortunately, as in the previous case, (12) does not define an equivalence relation.

Hence, if we want a notion of f-equivalence which is somehow compatible with stable equivalence, we have to use a different approach.

For the K-ring, we can introduce the following:

Definition 22 *Let $\xi - \eta$ and $\xi' - \eta'$ be elements of $K_F(B)$; we say that they are stably-fundamentally equivalent (sf-equivalent) if there exist a line bundle $l - 0$ in $K_F(B)$ (i.e. l is a line bundle over B) such that*

$$\xi' - \eta' = (l - 0).(\xi - \eta),$$

where the dot is the product in $K_F(B)$.

We claim that this is a good generalisation of f-equivalence to the K-ring. In fact, we get the following

Theorem 23 *Suppose ξ and ξ' are two bundles over a base B, of rank r and r' respectively; then, $\xi' - 0 \sim_{sf} \xi - 0$ iff $r = r'$ and there exists a line bundle l over B such that $\xi' \sim_s l \otimes \xi$.*

<u>Proof</u> If $\xi' - 0 \sim_{sf} \xi - 0$, then, there exists a line bundle l over B such that $\xi' - 0 = (l - 0).(\xi - 0)$; hence, $\xi' - 0 = l \otimes \xi - 0$ in $K_F(B)$, and this means that there exists a bundle c such that $\xi' \oplus c = l \otimes \xi \oplus c$. But, by assumption, a bundle c' exists, with $c \oplus c' = \epsilon_p$, for some p, and thus, $\xi' \oplus \epsilon_p = l \otimes \xi \oplus \epsilon_p$, i.e. $\xi' \sim_s l \otimes \xi$. Conversely, if $r = r'$ and $\xi' \sim_s l \otimes \xi$, by definition there exists q such that $\xi' \oplus \epsilon_q = l \otimes \xi \oplus \epsilon_q$, i.e. $\xi' - 0 = (l - 0).(\xi - 0)$ in $K_F(B)$. QED

Moreover, with definition 22, the result about characteristic classes stated in theorem 7 extends to $K_F(B)$. Also, as we have already pointed out discussing example 9, the following is true:

Corollary 24 *If two complex bundles ξ and ξ' have the Chern classes related by*

$$c_k(\xi') = \sum_{j=0}^{k} \binom{r-j}{k-j} c_j(\xi) c_1(l)^{k-j},$$

then, modulo torsion, they are fundamentally equivalent in the K-ring. This means that in $K_{\mathbf{C}}(B) \otimes \mathbf{Q}$

$$\xi' = \xi \otimes l,$$

for some line bundle l.

Proof. This is a consequence of the fact that the Chern character

$$ch : K_{\mathbf{C}}(B) \otimes \mathbf{Q} \to H^*(B; \mathbf{Q})$$

is an isomorphism, and that the Chern character is determined by the Chern classes. QED

Note that, unlike the notion that we gave for bundles of the same rank, this notion of fundamental equivalence behaves well with respect to stable equivalence. In fact stable equivalence implies sf-equivalence, since the quotient of $Vect_F(B)$ with respect to stable equivalence is the reduced K-ring $\tilde{K}_F(B)$ ([11], [17]), and elements of $\tilde{K}_F(B)$ can be represented by classes of the form $\xi - rank(\xi)$.

With this definition understood, it would be now interesting to investigate whether there are topological invariants that might be associated to the sf-equivalence classes in $K_F(B)$.

We give here an example of a cohomology class that provides a necessary condition for sf-equivalence in $K_F(B)$.

The following result is due to P. Aluffi and C. Faber [2].

Proposition 25 *Let ξ and η be complex bundles of rank r and r' respectively. Let l be a line bundle. Then the following relation holds*

$$c_{r-r'+1}(\xi \otimes l - \eta \otimes l) = c_{r-r'+1}(\xi - \eta).$$

This means that the $(r - r' + 1)$st Chern class is invariant under fundamental equivalence, for fixed r and r'.

The result is non trivial only in the context where virtual bundles appear, since in the case of bundles the $(r + 1)$st class would vanish.

5. Changing the Structure Group

To the purpose of studying fundamental equivalence, it is useful to investigate whether the conjugacy relation of the gauge groups is preserved under extension or reduction of the structure group.

It is clear that reductions of the structure group in general do not preserve f-equivalence. The following example shows that we can also find bundles that are f-equivalent in $U(2)$ but they are no longer such when the structure group is enlarged to $U(3)$.

Example 26 *The bundle $\xi = \eta \oplus \eta$ over $\mathbf{C}P^2$, where η is the canonical line bundle, is f-equivalent to a trivial bundle (f-trivial). However, $\xi \oplus \epsilon$ is not f-trivial if ϵ is a trivial line bundle.*

Proof.

In fact, $\xi \otimes \eta^* = \epsilon_2$, but there is not a line bundle l such that

$$(\xi \oplus \epsilon) \otimes l = \epsilon_3,$$

since a computation of the Chern classes by means of (2) shows that

$$c((\xi \oplus \epsilon) \otimes l) \neq 1.$$

In fact there are nonvanishing terms in the left hand side, since $c_1(\eta)$ is the generator of the cohomology ring of $\mathbf{C}P^2$. Thus, we have seen that extensions of the structure group in general do not preserve f-equivalence. QED

In particular, these two bundles are f-equivalent, but they are not ∞-equivalent consistently with theorem 21.

Nevertheless, there is a case when indeed f-equivalence with respect to a subgroup $H < G$ is a stronger condition than f-equivalence with respect to the structure group G.

We shall state the condition below in the general setting of topological (or differentiable) principal G bundles satisfying conditions (i) and (ii) of theorem 1.

Lemma 27 *Let ξ and ξ' be principal G bundles that admit a reduction of the structure group to $H \leq G$. Let H be such that $ZH \subseteq ZG$. Then, if the bundles are f-equivalent with respect to H, they are f-equivalent with respect to G.*

Proof. Lemma 27 just follows from the relation (1) between the transition functions. QED

We are deeply grateful to R. A. Piccinini: without his ideas and suggestions this paper wouldn't have been possible. We thank W. Sutherland for having pointed out some mistakes in our original arguments and for his valuable comments and suggestions. We also thank P. Aluffi and P. Salvatore for some useful remarks. The second author thanks J. P. May and the University of Chicago for the kind hospitality.

References

[1] J.F. Adams, *Vector Fields on Spheres*, Ann. of Math. 75 (1962) 603-632;

[2] P. Aluffi, C. Faber, *A Remark on the Chern Class of a Tensor Product*, to appear in Manuscripta Mathematica;

[3] M.F. Atiyah, R. Bott, *The Yang-Mills equations over Riemann surfaces* , Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523-615;

[4] P. Booth, P. Heath, C. Morgan, R. Piccinini, *H-spaces of self-equivalences of fibrations and bundles*, Proc. London Math. Soc. (3) 49 (1984), no. 1, 111-127;

[5] P. Booth, P. Heath, C. Morgan, R. Piccinini, *Remarks on the homotopy type of groups of gauge transformations*, C. R. Math. Rep. Acad. Sci. Canada 3 (1981), no. 1, 3-6;

[6] A. Borel, F. Hirzebruch, *Characteristic Classes and Homogeneous Spaces: I*, Am. J. Math. 80 (1958) 458-538;

[7] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Springer 1986;

[8] S. Fujiwara *On the topology of some gauge groups*, Math. Japon. 38 (1993), no. 3, 469-476;

[9] W. Fulton, *Intersection Theory*, Springer 1984;

[10] D.H. Gottlieb, *Applications of bundle map theory*, Trans. Amer. Math. Soc. 171 (1972), 23-50;

[11] D. Husemoller, *Fibre Bundles*, third edition, Springer 1994;

[12] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Princeton 1987;

[13] S.S. Koh, *Note on the homotopy properties of the components of the mapping space X^{S^p}*, Proc. Amer. Math. Soc. 11 (1960), 896-904;

[14] M. Marcolli, *Some Remarks on Conjugacy Classes of Bundle Gauge Groups*, to appear in Cahiers de Topologie;

[15] J.P. May, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc. 1 (1975), 1, no. 155;

[16] J.W. Milnor, J.D. Stasheff, *Characteristic Classes*, Princeton 1974;

[17] A.S. Mišenko, *Vektornye Rassloenija i ich Primenenija*, Nauka 1984;

[18] C. Morgan, R. Piccinini, *Conjugacy classes of groups of bundle automorphisms*, Manuscripta Math. 63 (1989), no. 2, 233-244;

[19] M. Salvetti, *Automorphisms of fibre bundles on S^n*, Boll. U.M.I. Algebra e Geometria, Serie VI, N.1 (1983), 99-112.

[20] W.A. Sutherland, *Function spaces related to gauge groups*, Proc. Roy. Soc. Edinburgh Sect. A 121 (1992), no. 1-2, 185-190;

[21] L.N. Vaserstein, *Normal subgroups of gauge groups*, Classical groups and related topics (Beijing, 1987), Contemp. Math., 82, Amer. Math. Soc. (1989) 199-220;

Matilde Marcolli
Department of Mathematics
University of Chicago
USA
E-mail: matilde@math.uchicago.edu

Mauro Spreafico
Dipartimento di Matematica
Universitá degli Studi and
INFN (National Institute for Nuclear Physics)
Milano
Italy
E-mail: spreafico@vmimat.mat.unimi.it

Received: 05.08.96