# Gamma Spaces and Information

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Ma148a: Geometry and Physics of Information
Caltech, Fall 2021

This lecture based on:

- M. Marcolli, *Gamma spaces and information*, J. Geom. Phys. 140 (2019), 26–55.
- Yu.I. Manin, M. Marcolli, *Homotopy Theoretic and Categorical Models of Neural Information Networks*, arXiv:2006.15136

## Finite probabilities and stochastic maps

- $\mathcal{FP}$ category:
  - Objects: pairs $(X, P)$ finite set $X$ with probability measure $P$
  - Morphisms:
    $$S \in \mathrm{Mor}_{\mathcal{FP}}((X, P), (Y, Q))$$
    are stochastic $(\#Y \times \#X)$-matrices:
    1. $S_{yx} \geq 0$, for all $x \in X$, $y \in Y$;
    2. $\sum_{y \in Y} S_{yx} = 1$ for all $x \in X$;
    3. the probability measures are related by $Q = S\,P$.
- can view stochastic matrix $S$ as multivalued function $f_S : X \to Y$ with $f_S(x_j) = \{y_i \in Y \,:\, S_{ij} > 0\}$ with measure preserving condition $Q_y = \sum_{x \in X} S_{yx} P_x$
- so think of this as correspondences version of category of finite probabilities and measure preserving functions
- category $\mathcal{FP}$ has zero object $(\{x\}, 1)$ (initial and terminal)

## Information loss on $\mathcal{FP}$

- information loss functional on $\mathcal{FP}$ is a continuous real valued map on the set of morphisms $\mathcal{H} : \mathrm{Mor}_{\mathcal{FP}} \to \mathbb{R}$ with:
  1. $\mathcal{H}(S) = 0$ on isomorphisms
  2. for $S \in \mathrm{Mor}_{\mathcal{FP}}((X,P),(Y,Q))$ and $S' \in \mathrm{Mor}_{\mathcal{FP}}((Y,Q),(Z,Q'))$
     $$\mathcal{H}(S' \circ S) = \mathcal{H}(S') + \mathcal{H}(S)$$
  3. $\lambda S \oplus (1-\lambda)S' \in \mathrm{Mor}_{\mathcal{FP}}((X \sqcup X', \lambda P \oplus (1-\lambda)P'),(Y,Q))$
     $$\mathcal{H}(\lambda S \oplus (1-\lambda)S') = \lambda \mathcal{H}(S) + (1-\lambda)\mathcal{H}(S') + \mathcal{H}(\hat{1}_{(\lambda,1-\lambda)}),$$
     with $\hat{1}_{(\lambda,1-\lambda)}$ the unique morphism from $(\{x,y\},(\lambda,1-\lambda))$ to the zero object
- Warning: disjoint union $X \sqcup X'$ with weighted sum $\lambda P \oplus (1-\lambda)P'$ is *not* the coproduct
- Shannon entropy: $\mathcal{H}(S) = H(Q) - H(P)$, with $H(P) = -\sum_{x \in X} P_x \log P_x$ satisfies properties

## Shannon entropy and information loss

- properties of information loss determine uniquely functional, up to multiplicative constant $C$,

$$\mathcal{H}(S) = C \cdot (H(Q) - H(P)),$$

with Shannon entropy $H(P) = -\sum_{x \in X} P_x \log P_x$

- composition of any $S \in \mathrm{Mor}_{\mathcal{FP}}((X,P),(Y,Q))$ with unique morphism $\hat{1}_{(Y,Q)} : (Y,Q) \to (\{x\},1)$

$$\hat{1}_{(X,P)} = \hat{1}_{(Y,Q)} \circ S : (X,P) \to (\{x\},1)$$

$$\mathcal{H}(S) = \mathcal{H}(\hat{1}_{(X,P)}) - \mathcal{H}(\hat{1}_{(Y,Q)})$$

- then show that $\tilde{H}(P) := -\mathcal{H}(\hat{1}_{(X,P)})$ satisfies Khinchin axioms so $=$ Shannon
- same argument as in J.C. Baez, T. Fritz, T. Leinster, *A characterization of entropy in terms of information loss*, Entropy 13 (2011) no. 11, 1945–1957.

## Probabilistic pointed sets

- $\mathcal{PS}_*$ category:
  - Objects: convex combinations of pointed sets

$$\Lambda X = \sum_i \lambda_i (X_i, x_i)$$

    with $\Lambda = (\lambda_i)$ wtih $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$
  - Morphisms: $\Phi \in \mathrm{Mor}_{\mathcal{PS}_*}(\Lambda X, \Lambda' X')$ consist pairs $\Phi = (S, F)$
    1. $S$ is a stochastic map with $S\Lambda = \Lambda'$
    2. $F = (F_{ji})$ is a collection of probabilistic pointed maps $F_{ji} : (X_i, x_i) \to (X'_j, x'_j)$
  - probabilistic pointed map $F_{ji}$ is a finite set $\{F_{ji,a}\}$ of pointed maps $F_{ji,a} : (X_i, x_i) \to (X'_j, x'_j)$ together with a set of probabilities $\mu_a^{(ji)}$ with $\sum_a \mu_a^{(ji)} = S_{ji}$

- probabilistic pointed maps $F = \{F_{ji,a}\}$: value $F(x)$ for $x \in (X_i, x_i)$ by choosing a map $F_{ji,a}$ with probability $\mu_a^{(ji)}$
- the set $F_{ji} = \{F_{ji,a}\}$ has probability $S_{ji}$
- composition: $S' \circ S$ product of stochastic matrices and $F' \circ F = \{(F' \circ F)_{ki}\}$ with $(F' \circ F)_{ki} = \{F'_{kj,a} \circ F_{ji,b}\}$ with probabilities $\mu_a^{(kj)} \mu_b^{(ji)}$ with $\sum_{a,b,j} \mu_a^{(kj)} \mu_b^{(ji)} = \sum_j S'_{kj} S_{ji} = (S' \circ S)_{ki}$
- category $\mathcal{PS}_*$ has zero object given by singleton $(\{x\}, 1)$
- category $\mathcal{PS}_*$ has coproduct inherited from the category of pointed sets

$$\Lambda X \amalg \Lambda' X' := \sum_{ij} \lambda_i \lambda'_j \, (X_i, x_i) \vee (Y_j, x'_j)$$

$$(X_i, x_i) \vee (Y_j, x'_j) = (X_i \sqcup Y_j / x_i \sim y_j, x_i \sim y_j)$$

## Probabilistic categories

- category $\mathcal{C}$ with a zero object 0 and categorical sum (coproduct) $\oplus$
- $\mathcal{PC}$ category:
  - Objects: formal finite convex combinations

$$\Lambda C = \sum_i \lambda_i C_i$$

    with $\Lambda = (\lambda_i)$ with $\sum_i \lambda_i = 1$ and $C_i \in \mathrm{Obj}(\mathcal{C})$
  - Morphisms: $\Phi : \Lambda C \to \Lambda' C'$ pairs $\Phi = (S, F)$
    1. $S$ stochastic matrix with $S\Lambda = \Lambda'$
    2. $F = \{F_{ab,r}\}$ finite collection of morphisms $F_{ab,r} : C_b \to C'_a$ with assigned probabilities $\mu_r^{ab}$
    3. $\sum_r \mu_r^{ab} = S_{ab}$
- wreath product $\mathcal{FP} \wr \mathcal{C}$ of the category $\mathcal{C}$ with the category $\mathcal{FP}$ of finite probabilities

## Information loss on probabilistic categories

- $\mathcal{PC} = \mathcal{FP} \wr \mathcal{C}$ probabilistic category as above, for $\mathcal{C}$ category with zero object and coproduct
- sets of morphisms $\mathrm{Mor}_{\mathcal{PC}}(\Lambda C, \Lambda' C')$ are convex sets: combination $S_\lambda = \lambda S + (1-\lambda) S'$ of stochastic matrices with $S_\lambda \Lambda = \Lambda'$ and $F_\lambda = \{F_{ab,r}\} \cup \{F'_{ab,r'}\}$ with probabilities $\lambda \mu^{ab}_r$ and $(1-\lambda)\nu^{ab}_{r'}$
- can also form convex combinations of objects of $\mathcal{PC}$ (already defined as convex combinations)

- information loss functional:

$$\mathcal{H} : \cup_{\Lambda C, \Lambda' C'} \mathrm{Mor}_{\mathcal{PC}}(\Lambda C, \Lambda' C') \to \mathbb{R}$$

1. $\mathcal{H}(\Phi) = 0$ if $\Phi$ isomorphism
2. additivity under composition: $\mathcal{H}(\Phi \circ \Phi') = \mathcal{H}(\Phi) + \mathcal{H}(\Phi')$
3. extensivity under convex combinations for objects and morphisms

$$\mathcal{H}(\lambda\Phi + (1-\lambda)\Phi') = \lambda\mathcal{H}(\Phi) + (1-\lambda)\mathcal{H}(\Phi') + \mathcal{H}(\hat{1}_{(\lambda,1-\lambda)})$$

where $\hat{1}_{(\lambda,1-\lambda)}$ unique morphism in $\mathcal{PC}$ from object $\Lambda 0$, (with $\Lambda = (\lambda, 1-\lambda)$ and 0 zero object of $\mathcal{C}$) to the zero object of $\mathcal{PC}$

- strong information loss functional if it also satisfies the property
  4. inclusion-exclusion on coproducts:

$$\mathcal{H}(\Phi \amalg_{\mathcal{PC}} \Phi') = \mathcal{H}(\Phi) + \mathcal{H}(\Phi') - \mathcal{H}(\widehat{\Sigma C''}),$$

  for $\Phi \in \mathrm{Mor}(\Lambda C, \Sigma C'')$, $\Phi' \in \mathrm{Mor}(\Lambda' C', \Sigma C'')$ and $\widehat{\Sigma C''}$ the unique morphism from the zero object of $\mathcal{PC}$ to $\Sigma C''$.
- embedding $\mathcal{J} : \mathcal{FP} \hookrightarrow \mathcal{PC}$ on objects $\Lambda = (\lambda_i) \mapsto \sum_i \lambda_i 0_i$ (sum of copies of zero object of $\mathcal{C}$) and on morphisms morphisms $S \mapsto \Phi = (S, 1)$
- information loss functional on $\mathcal{PC}$ with first three properties restricts as information loss $\mathcal{H}(\mathcal{J}(S)) = \kappa(H(\Lambda') - H(\Lambda))$, with $H(\Lambda) = -\sum_i \lambda_i \log \lambda_i$ Shannon entropy

- information loss functional on $\mathcal{PC}$ satisfying the first three properties must be

$$\mathcal{H}(\Phi) = \tilde{H}(\Lambda' C') - \tilde{H}(\Lambda C), \quad \text{with} \quad \tilde{H}(\Lambda C) := -\mathcal{H}(\hat{1}_{\Lambda C})$$

- for $\Phi \in \mathrm{Mor}_{\mathcal{FP}}(\Lambda C, \Lambda' C')$ and $\hat{1}_{\Lambda C}$ unique morphism in $\mathcal{PC}$ from $\Lambda C$ to zero object
- $\tilde{H}$ satisfies the extensivity property

$$\tilde{H}(\Lambda C) = \kappa(H(\Lambda) + \sum_i \lambda_i \tilde{H}(C_i))$$

with $\tilde{H}(C) = -\mathcal{H}(\hat{1}_C)$ for $\hat{1}_C$ unique morphism from $C$ to zero object of $\mathcal{C}$

- so induces a functional $\mathcal{H} : \mathrm{Mor}_{\mathcal{C}}(C, C') \to \mathbb{R}$ given by

$$\mathcal{H}(f) = \tilde{H}(C') - \tilde{H}(C)$$

1. $\mathcal{H}(f) = 0$ if $f$ is isomorphism
2. additivity on compositions: $\mathcal{H}(f \circ f') = \mathcal{H}(f) + \mathcal{H}(f')$

- if $\mathcal{H}$ strong information loss functional on $\mathcal{PC}$ then $\tilde{\mathcal{H}}$ also satisfies

3. inclusion-exclusion on coproducts:

$$\mathcal{H}(f \amalg_{\mathcal{C}} f') = \mathcal{H}(f) + \mathcal{H}(f') - \mathcal{H}(\hat{C}'')$$

for $f \in \mathrm{Mor}_{\mathcal{C}}(C, C'')$, $f' \in \mathrm{Mor}_{\mathcal{C}}(C', C'')$, and $\hat{C}''$ unique morphism in $\mathcal{C}$ from zero object to $C''$

- if category $\mathcal{C}$ also has product can consider special class of functionals $\mathcal{H}(f) = \tilde{H}(C') - \tilde{H}(C)$ with additional
  4. multiplicative property:

$$\tilde{H}(C \otimes C') = \tilde{H}(C) \cdot \tilde{H}(C')$$

- Note: does not imply multiplicativity for $\tilde{H} : \mathrm{Obj}(\mathcal{PC}) \to \mathbb{R}$ because
  - product on $\mathcal{C}$ does not extend to categorical product on $\mathcal{PC}$
  - Shannon entropy $H(\Lambda)$ additive on products of statistically independent measures, while $\tilde{H}(C)$ multiplicative
- also consider a log version: additive property

$$\tilde{H}(C \otimes C') = \tilde{H}(C) + \tilde{H}(C')$$

then both $\tilde{H}(C)$ and Shannon additive on independent systems in $\mathcal{PC}$

## Examples

- category $\mathcal{C}$ of pointed simplicial sets $\Delta_*$ or cubical sets $\square_*$
- strong information loss functional on $\mathcal{PC}$ satisfies the inclusion-exclusion relation

$$\tilde{H}(K \cup K') = \tilde{H}(K) + \tilde{H}(K') - \tilde{H}(K \cap K')$$

- if we also know $\tilde{H}(K)$ is a homotopy invariant: additivity on coproducts $\tilde{H}(K \vee K') = \tilde{H}(K) + \tilde{H}(K')$ (inclusion-exclusion) and value $\tilde{H}(\{x, \star\}, \star) = \kappa$ determine

$$\tilde{H}(K) = \kappa \cdot \tilde{\chi}(K)$$

  $\tilde{\chi}(K) = \chi(K) - 1$ reduced Euler characteristic
- for $\kappa = 1$ also multiplicative property as $\tilde{H}(K) = \tilde{\chi}(K)$ multiplicative under smash products

$$\tilde{\chi}(K \wedge K') = \tilde{\chi}(K)\tilde{\chi}(K')$$

- while $H(K) = \log \tilde{\chi}(K)$ additive $H(K \wedge K') = H(K) + H(K')$ but not strong (no inclusion-exclusion)

## Gamma spaces (G. Segal, 1973)

- $\Gamma$-space is a functor $F : \Gamma^0 \to \Delta_*$ from the category of pointed finite sets to the category of pointed simplicial sets
- Notation: use $\mathrm{sSets}_*$ for pointed simplicial sets, also use $\Delta_*$ when no confusion with $\Delta =$ simplex-category
- pointed sets $\Gamma^0$ objects $(X, x_0)$ and morphisms:
  - original Segal definition: preimages under maps of pointed sets

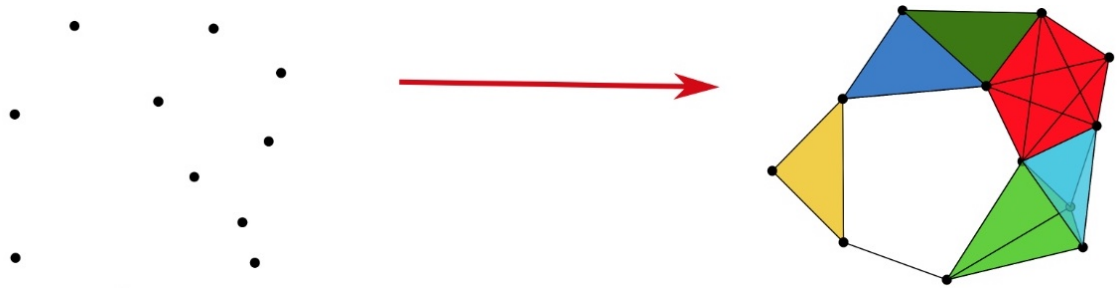
$$\phi : Y \to X, \quad \phi = \{A_y\}_{y \in Y} \quad X \supseteq A_y = f^{-1}(y)$$

$$\text{for some} \quad f : X \to Y, \quad f(x_0) = y_0$$

  - dual definition: morphisms just maps of pointed sets $f : X \to Y$, $f(x_0) = y_0$
- with second choice just previous $\Gamma^0 = \mathcal{F}_*$ (pointed finite sets)
- then $\Gamma$-spaces contravariant functors from opposite category (covariant functors of pointed maps)

## Segal's Gamma Spaces

- construction introduced in homotopy theory in the '70s: it provides a general construction of (connective) *spectra* (generalized homology theories)
- a Gamma space is a functor $\Gamma : \mathcal{F} \to \Delta$ from finite (pointed) sets to (pointed) simplicial sets



- Key construction: a category $\mathcal{C}$ with sum and zero-object determines a Gamma space $\Gamma_{\mathcal{C}} : \mathcal{F} \to \Delta$
  - for a finite set $X$ take category of summing functors $\Sigma_{\mathcal{C}}(X)$ and simplicial set given by nerve $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ of this category

## Gamma spaces from categories

- category $\mathcal{C}$ with zero object and categorical sum $\Rightarrow$ $\Gamma$-space $F_{\mathcal{C}} : \Gamma^0 \to \Delta_*$
- category $\Sigma_{\mathcal{C}}(X)$ of summing functors $\Phi_X : P(X) \to \mathcal{C}$
  - $P(X)$ category with objects the pointed subsets $A \subseteq X$ and morphisms the inclusions
  - functors $\Phi_X : P(X) \to \mathcal{C}$ satisfy summing properties:
    1. $\Phi_X(\star) = 0$, the base point of $X$ maps to the zero object of $\mathcal{C}$
    2. $\Phi_X(A \cup A') = \Phi_X(A) \amalg \Phi_X(A')$ for any two points sets with $A \cap A' = \{\star\}$ and with $\amalg$ the categorical sum of $\mathcal{C}$
- Morphisms of $\Sigma_{\mathcal{C}}(X)$ are invertible natural transformations
- the simplicial set $F_{\mathcal{C}}(X) = \mathcal{N}\Sigma_{\mathcal{C}}(X)$ is the nerve of the category of summing functors
- Note: need restriction to invertible natural transformations otherwise nerve $\mathcal{N}\Sigma_{\mathcal{C}}(X)$ would be contractible (because everything contracts to zero object)

## Nerve of a category

- $\mathcal{C}$ small category: nerve $\mathcal{N}\mathcal{C}$ simplicial set with
- one zero-simplex (vertex) for every object $C \in \mathrm{Obj}(C)$
- a 1-simplex (edge) from $C$ to $C'$ for each morphism $\phi \in \mathrm{Mor}_{\mathcal{C}}(C, C')$
- a 2-simplex for each composition of morphisms (commutative diagram)

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[dr, "\psi=\phi'\circ\phi"'] & C' \arrow[d, "\phi'"] \\ & C'' \end{tikzcd}$$

- $k$-simplex: $k$-tuple of composable morphisms

$$C_0 \xrightarrow{\phi_1} C_1 \xrightarrow{\phi_2} C_2 \to \cdots C_{k-1} \xrightarrow{\phi_k} C_k$$

- boundary maps $d_k : \mathcal{N}\mathcal{C}_k \to \mathcal{N}\mathcal{C}_{k-1}$ perform compositions: $d_k$ maps

$$C_0 \to \cdots C_{i-1} \xrightarrow{\phi_i} C_i \xrightarrow{\phi_{i+1}} C_{i+1} \to \cdots \to C_k$$

to the $k-1$ composable morphisms

$$C_0 \to \cdots C_{i-1} \xrightarrow{\phi_{i+1} \circ \phi_i} C_{i+1} \to \cdots \to C_k$$

- degeneracy maps $s_i : \mathcal{N}\mathcal{C}_k \to \mathcal{N}\mathcal{C}_{k+1}$ insert identity morphism $1_{C_i}$ at $i$-th place

$$C_0 \to \cdots C_{i-1} \xrightarrow{\phi_i} C_i \xrightarrow{1_{C_i}} C_i \xrightarrow{\phi_{i+1}} C_{i+1} \to \cdots \to C_k$$

## geometric realization of a simplicial set

- for a $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, the coend

$$\int^{C \in \mathcal{C}} F(C, C)$$

  is the initial cowedge (maps to other cowedges)

- a cowedge to an object $X$ in $\mathcal{C}$ is a family of morphisms $h_A : A \to X$, for each $A \in \mathcal{C}$, such that, for all morphisms $f : A \to B$ in $\mathcal{C}$ the following diagrams commute:

$$\begin{tikzcd} F(B,A) \arrow[r, "F(f,A)"] \arrow[d, "F(B,f)"] & F(A,A) \arrow[d, "h_A"] \\ F(B,B) \arrow[r, "h_B"] & X \end{tikzcd}$$

- realization functor $|\cdot| : \Delta \to \mathrm{Top}$ and $|\cdot| : sSets_* \to \mathrm{Top}_*$

- for standard simplex $\Delta^n = [n] \in \mathrm{Obj}(\Delta)$ geometric realization

$$|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1\}$$

- simplicial set $K : \Delta^{op} \to \mathrm{Sets}$
- geometric realization $|K|$ (prod instead of $\wedge$ if non-pointed)

$$|K| = \int^{[n]\in\Delta} |\Delta^n| \wedge K_n$$

- equivalently

$$|K| = (\bigsqcup_{n=0}^{\infty} |\Delta^n| \wedge K_n)/\sim$$

  with equivalence $(t, x) \sim (t', x')$ if $d^i t = t'$ and $d_i x' = x$ or $s^j t = t'$ and $s_j x' = x$ with $d^i, s^j$ induced by faces, degeneracies $d_i, s_j$ through functor $\Delta \to \mathrm{Top}$
- glue together standard simplices $|\Delta^n|$ according to combinatorial prescription given by $K_n = K([n])$

## Meaning of summing functors and nerve

- summing functors are like $\mathcal{C}$-valued measures on $X$
- can think of $\Phi \in \Sigma_{\mathcal{C}}(X)$ as a consistent assignment of objects in $\mathcal{C}$ to all (pointed) subsets of $(X, \star)$

$$\Phi(A) = \oplus_{x \in A} \Phi(\{x, \star\})$$

- nerve $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ of category of summing functors organizes all such assignments of $\mathcal{C}$-objects to $X$-subsets and their (invertible) transformations (classified up to equivalence)
- organized into a single topological structure that keeps track of equivalence relations between them (invertible natural transformations as morphisms of $\Sigma_{\mathcal{C}}(X)$ and all their possible compositions become simplices of the nerve)
- view $|\mathcal{N}(\Sigma_{\mathcal{C}}(X))|$ as topological space parameterizing space for all such consistent assignments of objects of $\mathcal{C}$ to subsets of $X$

## Spectra in homotopy theory

- smash product of pointed (simplicial) sets

$$(X, x_0) \wedge (Y, y_0) = (X \times Y)/((X \times \{y_0\}) \cup (\{x_0\} \times Y))$$

  with base point the point = collapsed subset
- spectrum (sequential spectrum): sequence of (pointed) simplicial sets (or CW complexes) $E = \{E_n\}$ with structure maps

$$\sigma_n : \Sigma E_n = S^1 \wedge E_n \to E_{n+1}$$

- sphere spectrum: $E_n = S^n = S^1 \wedge \cdots \wedge S^1$ with $\sigma_n = id$
- a space $X$ (CW complex, simplicial set) defines suspension spectrum $E_n = \Sigma^n X$
- Eilenberg-MacLane spectrum: abelian group $A$, spectrum $HA = \{K(A, 0), K(A, 1), K(A, 2), \ldots\}$
- loop space and suspension adjunction $[\Sigma X, Y] \simeq [X, \Omega Y]$

$$K(A, n-1) \simeq \Omega K(A, n) \Rightarrow \sigma_n : \Sigma K(A, n-1) \to K(A, n)$$

- spectrum $E$: associated generalized cohomology theory, homotopy classes of maps

$$h_E^n(X) = \varinjlim_{k\to\infty} [\Sigma^k X, E_{n+k}]$$

- Eilenberg-McLane case: $h_{HA}^n(X) = H^n(X; A)$
- homotopy groups of spectrum

$$\pi_n(E) = \varinjlim_{k\to\infty} \pi_{n+k}(E_k)$$

with maps $\pi_{n+k}(E_k) \to \pi_{n+k+1}(E_{k+1})$ induced by

$$E_n \xrightarrow{\Sigma} \Sigma E_n = S^1 \wedge E_n \xrightarrow{\sigma_n} E_{n+1}$$

- connective spectrum: $\pi_k(E) = 0$ for $k < 0$

## infinite loop spaces and delooping

- $X$ is a loop space if $X \simeq \Omega Y$ for some $Y$
- delooping: construction of such $Y$
- infinite loop space if admits infinite sequence of deloopings
- problem: explicit conditions for $X$ to be infinite loop space (May) and explicit construction of a delooping $Y = BX$ (May, Segal)
- $X_n$ finite pointed set in $\Gamma^0$ with $\#X_n = n + 1$
- given a $\Gamma$-space $F : \Gamma^0 \to \Delta_*$ such that $F(X_n) \to F(X_1) \wedge \cdots \wedge F(X_1)$ weak equivalences
- think of $F(X_1)$ as underlying space of the $\Gamma$-space and $F(X_n)$ as model for higher powers
- delooping $BF : \Gamma^0 \to \Delta_*$

$$BF(X) := \mathrm{hocolim}(Y \mapsto F(X \wedge Y))$$

- again a $\Gamma$-space so delooping operation can be repeated

## homotopy colimit

- a way of describing limits and colimits in categories: indexing category $\mathcal{I}$ and diagram $D : \mathcal{I} \to \mathcal{C}$, category $\mathcal{C}^{\mathcal{I}}$ of diagrams
- diagonal functor $\delta_0 : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ constant diagram
- limit = right adjoint of $\delta_0$; colimit = left adjoint of $\delta_0$
- $\mathcal{C} = \mathrm{Top}$ category of topological spaces (similar for simplicial sets, pointed/not-pointed)
- modify diagonal functor $\delta_0$ to $\delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ mapping object $X$ to diagram with $i$-th object

$$\delta(X)(i) = \mathrm{Mor}_{\mathcal{C}}(|\mathcal{N}(\mathcal{I}^{op}/i)|, X)$$

- $|\mathcal{N}(\mathcal{I}^{op}/i)|$ = realization of nerve of *slice category* $\mathcal{I}/i$ (objects arrows $j \to i$ in $\mathcal{I}^{op}$, morphisms comm diagrams of such arrows)
- homotopy colimit: right adjoint to $\delta$ functor
- idea: this takes colimits (similar for limits) in a homotopy category of diagrams

## Gamma spaces of categories and delooping

- take case where $\mathcal{C}$ is an abelian category, then (Quillen) the higher K-theory $K(\mathcal{C})$ is the K-theory of an infinite loop space
- the category of summing functors $\Sigma_{\mathcal{C}}(X)$ provides a delooping of this infinite loop space (Carlsson)
- more general categories: $\Sigma_{\mathcal{C}}(X)$ provides *delooping* of infinite loop space given by (a completion of) the classifying space of $\mathcal{C}$ (Carlsson)
- G.Carlsson, *Deloopings in Algebraic K-Theory*, in "Handbook of K-Theory" pp. 3–37, Springer, 2005

## Spectra from Gamma spaces

- extend $\Gamma$-space $F : \Gamma^0 \to \Delta_*$ to endofunctor $F : \Delta_* \to \Delta_*$
- $X_n$ finite pointed set in $\Gamma^0$ with $\#X_n = n + 1$
- define endofunctor as mapping simpliciat set $K \in \Delta_*$ to coend

$$F : K \mapsto \int^{X_n \in \Gamma^0} K_n \wedge F(X_n)$$

- natural assembly maps $K \wedge F(K') \to F(K \wedge K')$
- as in geom realization smash product $K_n \wedge F(X_n)$ attaches copies of the simplicial sets $F(X_n)$ according to combinatorial scheme given by $K_n = K([n])$
- coend ensures glued together correctly according to faces and degeneracies of simplicial set $K$
- take simplicial sets given by spheres $S^n = S^1 \wedge \cdots \wedge S^1$
- maps $K' \wedge F(K) \to F(K' \wedge K)$ give rise to structure maps $S^1 \wedge F(S^n) \to F(S^{n+1})$
- spectrum: $X_n = F(S^n)$ with these $\sigma_n : S^1 \wedge F(S^n) \to F(S^{n+1})$

## $\Gamma$-spaces from categories (more general)

- symmetric monoidal category: small category $\mathcal{C}$ with functor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and object 0
- Note: more commonly denoted with multiplicative notation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and object $\mathbb{I}$ (here use additive notation to generalize case of zero object and sum)
- with properties
  1. natural isomorphisms of functors (for $S_1, S_2, S_3 \in \mathrm{Obj}(\mathcal{C})$)

$$\alpha : (S_1 \oplus S_2) \oplus S_3 \xrightarrow{\sim} S_1 \oplus (S_2 \oplus S_3)$$

$$\lambda : 0 \oplus S_1 \xrightarrow{\sim} S_1$$

and

$$\gamma : S_1 \oplus S_2 \xrightarrow{\sim} S_2 \oplus S_1$$

2. satisfying $\gamma^2 = id$
3. with pentagon and hexagon commutative diagrams

$$\begin{tikzcd}
((S_1 \oplus S_2) \oplus S_3) \oplus S_4 \arrow[r, "\alpha \oplus S_4"] \arrow[d, "\alpha"'] & (S_1 \oplus (S_2 \oplus S_3)) \oplus S_4 \arrow[dd, "\alpha"] \\
(S_1 \oplus S_2) \oplus (S_3 \oplus S_4) \arrow[d, "\alpha"'] & \\
S_1 \oplus (S_2 \oplus (S_3 \oplus S_4)) & S_1 \oplus ((S_2 \oplus S_3) \oplus S_4) \arrow[l, "S_1 \oplus \alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
(S_1 \oplus S_2) \oplus S_3 \arrow[r, "\alpha"] \arrow[d, "\gamma \oplus S_3"'] & S_1 \oplus (S_2 \oplus S_3) \arrow[r, "\gamma"] & (S_2 \oplus S_3) \oplus S_1 \arrow[d, "\alpha"] \\
(S_2 \oplus S_1) \oplus S_3 \arrow[r, "\alpha"] & S_2 \oplus (S_1 \oplus S_3) \arrow[r, "S_2 \oplus \gamma"] & S_2 \oplus (S_3 \oplus S_1)
\end{tikzcd}$$

4. commutative diagram for unit 0:

$$\begin{tikzcd}
(0 \oplus S_1) \oplus S_2 \arrow[r, "\gamma \oplus S_2"] \arrow[dr, "\lambda \oplus S_2"'] & (S_1 \oplus 0) \oplus S_2 \arrow[r, "\alpha"] & S_1 \oplus (0 \oplus S_2) \arrow[dl, "S_1 \oplus \lambda"] \\
& S_1 \oplus S_2 &
\end{tikzcd}$$

5. symmetric monoidal functor $\Psi : (\mathcal{C}, \oplus, 0) \to (\mathcal{C}', \oplus', 0')$ is a triple $\Psi = (F, f, f')$ of functor $F : \mathcal{C} \to \mathcal{C}'$ and natural transformation $f : FS_1 \oplus FS_2 \to F(S_1 \oplus S_2)$ and $\tilde{f} : 0 \to F0$ with commutative diagrams

$$\begin{tikzcd}
(FS_1 \oplus FS_2) \oplus FS_3 \arrow[r, "f \oplus FS_2"] \arrow[d, "\alpha"] & F(S_1 \oplus S_2) \oplus FS_3 \arrow[r, "f"] & F((S_1 \oplus S_2) \oplus S_3) \arrow[d, "F\alpha"] \\
FS_1 \oplus (FS_2 \oplus FS_3) \arrow[r, "FS_3 \oplus f"] & FS_1 \oplus F(S_2 \oplus S_3) \arrow[r, "f"] & F(S_1 \oplus (S_2 \oplus S_3))
\end{tikzcd}$$

and

$$\begin{tikzcd}
FS_1 \oplus FS_2 \arrow[r, "f"] \arrow[d, "\gamma"] & F(S_1 \oplus S_2) \arrow[d, "F\gamma"] \\
FS_2 \oplus FS_1 \arrow[r, "f"] & F(S_2 \oplus S_1)
\end{tikzcd}
\qquad
\begin{tikzcd}
F0 \oplus FS \arrow[r, "f"] & F(0 \oplus S) \arrow[d, "F\lambda"] \\
0 \oplus FS \arrow[u, "\tilde{f} \oplus FS"] \arrow[r, "\lambda"] & FS
\end{tikzcd}$$

## Gamma spaces of categories and connective spectra

- a category $\mathcal{C}$ with zero object and categorical sum is also a symmetric monoidal category
- Thomason: Segal's construction of Gamma spaces for categories with zero object and sum generalizes to symmetric monoidal categories
- a Gamma space defines an associated spectrum, by extending the functor $\Gamma : \mathcal{F}_* \to \Delta_*$ to an endofunctor $\Gamma : \Delta_* \to \Delta_*$ and applying it to spheres
- when $\mathcal{C} = \mathcal{F}_*$ with $\Gamma_{\mathcal{F}_*} : \mathcal{F}_* \to \Delta_*$ get the sphere spectrum
- all connective spectra are obtained through $\Gamma$-spaces $F_{\mathcal{C}}$ for $\mathcal{C}$ a symmetric monoidal category (Thomason)
- hence the nerves $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ are topologically very nontrivial

## Cubical sets

- cube category $\mathfrak{C}$ has objects $\mathcal{I}^n$, the $n$-cube, for $n \geq 0$, morphisms generated by faces and degeneracies $\delta_i^a$ and $s_i$
- cubical set: functor $C : \mathfrak{C}^{op} \to \mathcal{S}$ to category of sets
- $C_n = C(\mathcal{I}^n)$
- enlarge category $\mathfrak{C}$ to category $\mathfrak{C}_c$ with additional degeneracy maps $\gamma_i : \mathcal{I}^n \to \mathcal{I}^{n-1}$ (called connections)
- cubical set with connection is a functor $C : \mathfrak{C}_c^{op} \to \mathcal{S}$ to the category of sets
- natural transformations as morphisms: $\alpha = (\alpha_n)$ morphisms $\alpha_n : C_n \to C'_n$ with compatibility $\alpha \circ \delta_i^a = \delta_i^a \circ \alpha$ and $\alpha \circ s_i = s_i \circ \alpha$ (case with connection $\alpha \circ \gamma_i = \gamma_i \circ \alpha$)
- pointed cubical set with connection: functor $K : \mathfrak{C}_c^{op} \to \mathcal{S}_*$

## Cubical nerve

- cubical nerve $\mathcal{N}_{\mathfrak{C}}\mathcal{C}$ of a category $\mathcal{C}$ cubical set

$$(\mathcal{N}_{\mathfrak{C}}\mathcal{C})_n = \mathrm{Fun}(\mathcal{I}^n, \mathcal{C})$$

- the $n$-cube $\mathcal{I}^n$ seen as a category with objects the vertices and morphisms generated by the 1-faces (edges)
- $\mathrm{Fun}(\mathcal{I}^n, \mathcal{C})$ set of functors from $\mathcal{I}^n$ to $\mathcal{C}$
- cubical nerve construction (with connections) is homotopy equivalent to usual simplicial one

## Cubical Gamma spaces

- $\square_*$ category of pointed cubical sets (with connection): objects are functors $K : \mathfrak{C}_c^{op} \to \mathcal{S}_*$ morphisms are natural transformations
- cubical $\Gamma$-space (with connection): functor $F : \Gamma^0 \to \square_*$
- as in simplical case: cubical $\Gamma$-spaces from categories with zero object and a categorical sum (and more generally from symmetric monoidal categories)
- $F_{\mathcal{C}}^{\mathfrak{C}} : \Gamma^0 \to \square_*$ assigns to a pointed finite set $X$ the pointed cubical set (with connection)

$$F_{\mathcal{C}}^{\mathfrak{C}}(X) = \mathcal{N}_{\mathfrak{C}}\Sigma_{\mathcal{C}}(X)$$

- $\mathcal{N}_{\mathfrak{C}_c}\Sigma_{\mathcal{C}}(X)$ is homotopy equivalent to the simplicial nerve $\mathcal{N}\Sigma_{\mathcal{C}}(X)$ so equivalent constructions

## Example: cubical Γ space of probabilistic $\mathcal{PF}_*$

- category $\mathcal{PF}_*$ of probabilistic pointed finite sets as before
- summing functors in $\Sigma_{\mathcal{PF}_*}(X)$ with $\#X = N + 1$ determined by choice of a point in cube

$$\Lambda = \{\lambda_x\}_{x \in X \setminus \{\star\}} \in |\mathcal{I}^N|$$

- summing functor $\Phi \in \Sigma_{\mathcal{PF}_*}(X)$: for $A \in P(X)$

$$\Phi_\Lambda(A) = \Lambda_A X_A$$

combination of $2^{N_A}$ pointed sets of cardinalities $N_A = \#A - 1$ with probabilities

$$\Lambda_A = \{(t_1, \ldots, t_{N_A}) \,:\, t_x \in \{\lambda_x, (1 - \lambda_x)\}\}$$

- morphisms are permutations of the $2^{N_A}$ sequences

## Stochastic Γ-spaces

- in example above $\mathcal{C} = \mathcal{P}\mathcal{F}_*$ is a stochastic category, but source and target of Γ space $F_{\mathcal{P}\mathcal{F}_*}$ are still usual categories $\Gamma^0 = \mathcal{F}_*$ and simplicial $\Delta_*$ or cubical $\square_*$
- can extend the notion of Gamma-space by making the source and target categories also probabilistic

$$F : \mathcal{P}\mathcal{F}_* \to \mathcal{P}\Delta_* \quad \text{or cubical} \quad F : \mathcal{P}\mathcal{F}_* \to \mathcal{P}\square_*$$

- probabilistic points simplicial or cubical sets $\mathcal{P}\Delta_*$ and $\mathcal{P}\square_*$ wreath products $\mathcal{F}\mathcal{P} \wr \Delta_*$ and $\mathcal{F}\mathcal{P} \wr \square_*$ of the category $\Delta_*$ or $\square_*$ with the category $\mathcal{F}\mathcal{P}$ of finite probabilities
- given category $\mathcal{C}$ with sum and zero object want associated $F_{\mathcal{C}} : \mathcal{P}\mathcal{F}_* \to \mathcal{P}\Delta_*$ (and cubical $F^{\mathfrak{C}}_{\mathcal{C}} : \mathcal{P}\mathcal{F}_* \to \mathcal{P}\square_*$)

## Probabilistic summing functors

- for an object $\Lambda X = \sum_i \lambda_i(X_i, x_i)$ in $\mathcal{PS}_*$ associated category $\mathcal{P}(\Lambda X)$ (replaces $P(X)$ in usual summing functors)
  - *Objects*: $\Lambda A = \sum_i \lambda_i(A_i, x_i)$ with $(A_i, x_i) \in P(X_i, x_i)$, where each $P(X_i, x_i)$ is category of pointed subsets and inclusions
  - *Morphisms* in $\mathcal{P}(\Lambda X)$ are morphisms $\Phi_{A,A''} : \Lambda A \to \Lambda A'$ where the stochastic matrix $S = 1$ is identity and $F_{A,A'} = \{F_{A_i,A'_i}\}$ is collection of pointed inclusions $F_{A_i,A'_i} : (A_i, x_i) \hookrightarrow (A'_i, x_i)$ applied with probability 1
- $\mathcal{C}$ with sum and zero object: probabilistic summing functor

$$\Theta : \mathcal{P}(\Lambda X) \to \mathcal{PC}$$

$$\Theta(\sum_i \lambda_i(A_i, x_i)) = \sum_i \lambda_i \Theta_i(A_i, x_i)$$

with $\Theta_i : P(X_i, x_i) \to \mathcal{PC}$ summing functors in $\Sigma_{\mathcal{PC}}(X_i, x_i)$

- category of probabilistic summing functors $\mathcal{P}\Sigma_{\mathcal{C}}(\Lambda X)$ has objects $\Theta : \mathcal{P}(\Lambda X) \to \mathcal{C}$ and morphisms invertible natural transformations

- cubical nerve $\mathcal{N}_{\mathfrak{C}}(\mathcal{P}\Sigma_{\mathcal{PC}}(\Lambda X))$ can be identified with probabilistic pointed cubical set $\sum_i \lambda_i \mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{PC}}(X_i, x_i))$
- cubical nerve is given by

$$(\mathcal{N}_{\mathfrak{C}}(\mathcal{P}\Sigma_{\mathcal{PC}}(\Lambda X))_n = \mathrm{Fun}(\mathcal{I}^n, \mathcal{P}\Sigma_{\mathcal{PC}}(\Lambda X))$$

- to each vertex $v \in \mathcal{I}^n$ one assigns a summing functor $\Theta_v = \sum_i \lambda_i \Theta_{v,i}$
- to each edge $e \in \mathcal{I}^n$ one assigns an invertible natural transformation $\eta_e = \{\eta_{e,i}\}$
- point in $\mathcal{N}_{\mathfrak{C}}(\mathcal{P}\Sigma_{\mathcal{PC}}(\Lambda X))$ corresponds to choice of points in the $\mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{PC}}(X_i, x_i))$

- functor $F_{\mathcal{PC}} : \mathcal{PS}_* \to \mathcal{P}\square_*$ assigns to an object $\Lambda X = \sum_i \lambda_i (X_i, x_i) \in \mathrm{Obj}(\mathcal{PS}_*)$ the probabilistic pointed cubical set $\sum_i \lambda_i \mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{PC}}(X_i, x_i))$
- morphism $\Phi = (S, F) \in \mathrm{Mor}_{\mathcal{PS}_*}(\Lambda X, \Lambda' X')$ with $S\Lambda = \Lambda'$ and $F = \{F_{ij,r}\}$ with probabilities with $\sum_r \mu_r^{ij} = S_{ij}$ maps to morphism of probabilistic pointed cubical sets with same stochastic matrix $S$ and corresponding $\mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{PC}}(X_j, x_j)) \to \mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{PC}}(X'_i, x'_i))$ same probabilities $\mu_r^{ij}$
- can extend the functors $F : \mathcal{PS}_* \to \mathcal{P}\square_*$ to endofunctors $F : \mathcal{P}\square_* \to \mathcal{P}\square_*$

- smash product of Gamma-spaces:
  - smash product functor $\wedge : \Gamma^0 \times \Gamma^0 \to \Gamma^0$, with $((X,x),(Y,y) \mapsto (X,x) \wedge (Y,y) = (X \times Y/(X \times \{y\} \cup \{x\} \times Y), \star)$
  - smash product $K \wedge K'$ of pointed (simplicial) sets
  - pair $F, F' : \Gamma^0 \to \Delta_*$ determines bi-$\Gamma$-space $F \tilde{\wedge} F' : \Gamma^0 \times \Gamma^0 \to \Delta_*$

$$(F \tilde{\wedge} F')((X,x),(Y,y)) = F(X,x) \wedge F'(Y,y)$$

  - then define

$$(F \wedge F')((X,x) := \mathrm{colim}_{(X_1,x_1)\wedge(X_2,x_2)\to(X,x)}(F \tilde{\wedge} F')((X_1,x_1),(X_2,x_2))$$

    where $(X_1,x_1) \wedge (X_2,x_2)$ is smash product $\wedge : \Gamma^0 \times \Gamma^0 \to \Gamma^0$
  - up to natural isomorphism, smash product associative and commutative and with unit $\Gamma$-space $\mathbb{S}$ (sphere spectrum), and category of $\Gamma$-spaces symmetric monoidal with this product
  - provides nice description of smash product of spectra

- smash product of probabilistic Gamma-spaces:
- for probabilistic cubical $\Gamma$-spaces $F, F' : \mathcal{PS}_* \to \mathcal{P}\Box_*$

$$(F\tilde{\wedge}F')(\Lambda X, \Lambda' Y) = F(\Lambda X) \wedge F'(\Lambda' Y)$$

with $\Lambda K \wedge \Lambda' K'$ in $\mathcal{P}\Box_*$ given by

$$\Lambda K \wedge \Lambda' K' = \sum_{i,j} \lambda_i \lambda_j K_i \wedge K'_j$$

$K_i \wedge K'_j$ the smash product in $\Box_*$

- then take

$$(F \wedge F')(\Lambda X) = \mathrm{colim}_{\Lambda_1 X_1 \wedge \Lambda_2 X_2 \to \Lambda X} (F\tilde{\wedge}F')((\Lambda_1 X_1), (\Lambda_2 X_2))$$

- morphisms $\Lambda_1 X_1 \wedge \Lambda_2 X_2 \to \Lambda X$ given by

$$S_{u,(a,a')} = \lambda_u^{-1} S_{ua} S'_{ua'} \quad \text{when} \quad \lambda_u \neq 0$$

and $S_{u,(a,a')} = S_{ua} + S'_{ua'}$ otherwise

- and $f \wedge f' = \{f_{ua,r} \wedge f'_{ua',r'}\}$ with probabilities $\lambda_u^{-1} \mu_r^{ua} \mu_{r'}^{ua'}$ or $M^{-1} \mu_r^{ua} + N^{-1} \mu_{r'}^{ua'}$, respectively when $\lambda_u \neq 0$ or $\lambda_u = 0$

## Information loss and probabilistic Gamma-spaces

- $\mathcal{H} : \mathcal{P}\Box_* \to \mathbb{R}$ information loss functional (not nec. strong)
- $F_{\mathcal{PC}} : \mathcal{PS}_* \to \mathcal{P}\Box_*$ probabilistic $\Gamma$-space associated to a probabilistic category $\mathcal{PC}$
- $\hat{F}_{\mathcal{PC}} : \mathcal{P}\Box_* \to \mathcal{P}\Box_*$ its extension to endofunctor
- compositions $\mathcal{H} \circ F_{\mathcal{PC}}$ and $\mathcal{H} \circ \hat{F}_{\mathcal{PC}}$ are also information loss functionals
- not strong even if $\mathcal{H}$ is

## extensivity property

- consider probabilistic $\Gamma$-spaces $F : \mathcal{PS}_* \to \mathcal{P}\square_*$ of the form

$$F(\Lambda X) = \Lambda' K' \wedge F_{\mathcal{PS}_*}(\Lambda X)$$

  for $\Lambda' K'$ a given stochastic pointed cubical set in $\mathcal{P}\square_*$
- probabilistic generalization of classical $\Gamma$-spaces $F : \Gamma^0 \to \Delta_*$ of the form $F(X) = K \wedge F_{\Gamma^0}(X)$ with $F_{\Gamma^0} : \Gamma^0 \hookrightarrow \Delta_*$
- these Gamma-spaces have spectrum the suspension spectrum of simplicial set $K$
- probabilistic version represents product of two statistically independent systems, $\Lambda' K'$ and $F_{\mathcal{PS}_*}(\Lambda X)$
- information loss behaves additively

$$\tilde{H}(F(\Lambda X)) = \tilde{H}(\Lambda' K') + \tilde{H}(F_{\mathcal{PS}_*}(\Lambda X))$$

## Probabilistic categories in quantum information

- category $\mathcal{FQ}$ of quantum states $(X, \rho_X)$ (fin dim Hilbert space $\mathcal{H}_X$) and morphisms quantum channels (completely positive trace preserving maps)
- category $\mathcal{C}$ with sum and zero object, quantum category $\mathcal{QC}$ (also has zero object and sum): objects $\rho C = ((C_a, C_b), \rho_{ab})_{ab}$ with $(C_a, C_b)$ finite collection of pairs of objects in $\mathcal{C}$, $a, b = 1, \ldots, N$, and $\rho = (\rho_{ab})$ density matrix
- morphisms
$$\Xi = \{(\phi_{ai,r}, \psi_{bj,r})\}, (S_{\Phi_r})_{\substack{ab\\ij}}\}$$
where $\sum_r S_{\Phi_r} = S_\Phi$ is the Choi matrix of a quantum channel $\Phi$ with $\Phi(\rho) = \rho'$
- composition $\Xi' \circ \Xi$

$$\Xi' \circ \Xi = \{(\phi_{ua,r'} \circ \phi_{ai,r}, \psi_{vb,r'} \circ \psi_{bj,r}), (S_{\Phi_r})_{\substack{ab\\ij}} (S_{\Phi'_{r'}})_{\substack{ij\\uv}}\} \quad \text{with}$$

$$\sum_{r,r',i,j} (S_{\Phi_r})_{\substack{ab\\ij}} (S_{\Phi_{r'}})_{\substack{ij\\uv}} = \sum_{i,j} (S_\Phi)_{\substack{ab\\ij}} (S_{\Phi'})_{\substack{ij\\uv}} = (S_{\Phi' \circ \Phi})_{\substack{ab\\uv}}$$

## Example: $\mathcal{QS}_*$ quantum pointed sets and associated $\Gamma$-space

- an object $\Theta$ in category of summing functors $\Sigma_{\mathcal{QS}_*}(X)$, with $\#X = N+1$, completely specified by the choice of a point

$$\alpha = \{\alpha_x\}_{x \in X \smallsetminus \{\star\}} \in \mathcal{I}^N$$

and, for each choice of $\alpha$, a set of complex numbers $\theta = \{\theta_x\}_{x \in X \smallsetminus \{\star\}}$ contained in the annuli

$$\theta_x \in \mathcal{A}_x = \{z \in \mathbb{C} \,:\, \alpha_x(1-\alpha_x) - \frac{1}{4} \leq |z|^2 \leq \alpha_x(1-\alpha_x)\}$$

or disks $\{|z|^2 \leq \alpha_x(1-\alpha_x)\}$ if $\alpha_x(1-\alpha_x) \leq 1/4$

- the summing functor $\Theta$ acts by

$$\Theta_{\alpha,\theta}(A) = \rho_A C_A$$

- $\rho_A C_A$ consists of a collection of $2^{N_A} \times 2^{N_A}$ pairs of pointed sets of cardinality $N_A + 1 = \#A$
- with $2^{N_A} \times 2^{N_A}$ density matrix

$$\rho_A = \otimes_{a \in A \smallsetminus \{\star\}} \rho^{(a)}$$

$$\rho^{(a)} = \begin{pmatrix} \alpha_a & \theta_a \\ \bar{\theta}_a & 1 - \alpha_a \end{pmatrix}$$

- constraints on choices of $\alpha_x$ and $\theta_x$ from normalization of trace $\mathrm{Tr}(\rho) = 1$
- this requires $\alpha_x \in [0, 1]$, hence

$$\{\alpha_x\}_{x \in X \smallsetminus \{\star\}} \in |\mathcal{I}^N| \quad \text{with} \quad N = \#X - 1$$

- also need to require $\rho_A \geq 0$, hence all $\rho^{(a)}$ have non-negative eigenvalues
- characteristic polynomial

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(\rho)\lambda + \det(\rho) = \lambda^2 - \lambda + \det(\rho)$$

  has non-negative eigenvalues when discriminant $\mathrm{Tr}(\rho)^2 - 4\det(\rho) = 1 - 4\det(\rho) \geq 0$ and $\det(\rho) \geq 0$ this gives the condition on annulus/disk
- morphisms in $\Sigma_{\mathcal{QS}_*}(X)$ consist of invertible natural transformations
- isomorphisms $\eta_A : \Theta(A) \to \Theta'(A)$ in $\mathcal{QS}_*$ compatible with the inclusions $j : A \hookrightarrow A'$, with

$$\eta_A \circ \Theta(j) = \Theta'(j) \circ \eta_A$$

- isomorphism $\eta : \rho_A X_A \to \rho'_A X'_A$ in $\mathcal{QS}_*$ consists of an invertible quantum channel mapping $\rho_A$ to $\rho'_A$ and a collection of isomorphisms of the pairs of pointed sets in the collections $X_A$ and $X'_A$
- invertible quantum channels are unitary transformations

$$\rho'_A = U_A \rho_A U_A^* \quad \text{with} \quad U_A \in \mathcal{U}(2^{N_A})$$

- compatibility with inclusions of subsets implies unitary transformation is product of unitary transformations of the $\rho^{(a)}$: unitaries $U_a \in \mathcal{U}(2)$

$$U_A = U_{a_1} \otimes \cdots \otimes U_{a_{N_A}}$$

- cubical nerve $K = \mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{QS}_*}(X))$ with $K_n = \mathrm{Fun}(\mathcal{I}^n, \Sigma_{\mathcal{QS}_*}(X))$ is given by the action groupoid of $\mathcal{U}(2)^{\otimes N}$ acting on the cubical set

$$\mathcal{Z}_N = \bigcup_{Z \in |\mathcal{I}^N|} \bigcup_{k=0}^{N} \mathcal{I}_Z^k \times \mathcal{A}_k$$

  with $\mathcal{A}_k$ a product of $N-k$ annuli (or disks) $\mathcal{A}_x$
- functor $\mathcal{I}^n \to \Sigma_{\mathcal{QS}_*}(X)$ assigns to each vertex $v \in \mathcal{I}^n$ a summing functor $\Theta_v \in \Sigma_{\mathcal{QS}_*}(X)$ (choice of $\{\lambda_x, \theta_x\}_{x \in X \setminus \{\star\}}$ as above)
- edges of $\mathcal{I}^n$ correspond to natural transformations between $\Theta_v$ and $\Theta_{v'}$ for $\partial(e) = \{v, v'\}$

- sequences $s_1 \dots s_n \in \{0,1\}^n$ labeling adjacent vertices of $\mathcal{I}^n$ differ at a single digit $s_k$ so corresponding $\otimes_x \rho^{(x)}$ differ by action of a single $U_k \in \mathcal{U}(2)$ relating density matrices $\rho_v^{(x_k)}$ and $\rho_{v'}^{(x_k)}$
- so all choices parameterized by action of $\mathcal{U}^{\otimes N}$ on the set $\mathcal{Z}_N$
- simplicial set $F_{\mathcal{QS}_*}(X) = \mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{QS}_*}(X))$ is homotopy equivalent to the Borel homotopy quotient $\mathcal{M}_G = EG \times_G \mathcal{Z}_N$, with $G = \mathcal{U}(2)^{\otimes N}$, for $N = \#X - 1$
- classifying space $B\mathcal{G}$ of the action groupoid $\mathcal{G} = \mathcal{Z}_N \rtimes \mathcal{U}(2)^{\otimes N}$
- because category $\Sigma_{\mathcal{QS}_*}(X)$ identified with action groupoid of $\mathcal{U}(2)^{\otimes N}$ acting on $\mathcal{Z}_N$, its nerve is classifying space
- action groupoid $\mathcal{G} = \mathcal{Z} \rtimes G$ of a Lie group action on a manifold is homotopy equivalent to the Borel construction of the homotopy quotient $\mathcal{Z}_G = EG \times_G \mathcal{Z}$

## Categories of gapped quantum systems and associated spectra

- two different constructions of categories $\mathcal{QS}_*^{\Delta}$ and $\mathcal{QS}_*[\mathcal{T}_{\Delta}^{-1}]$ of gapped quantum systems
- associated Gamma-spaces

$$\mathcal{F}_{\mathcal{QS}_*^{\Delta}} : \Gamma^0 \to \Delta_* \quad \text{and} \quad \mathcal{F}_{\mathcal{QS}_*[\mathcal{T}_{\Delta}^{-1}]} : \Gamma^0 \to \Delta_*$$

- extensions of these to endofunctors of $\Delta_*$ determine associated spectra
- in cubical form, explicit descriptions as above of cubical nerves

- subcategory $\mathcal{FQ}^\Delta$ of category $\mathcal{FQ}$ of finite quantum probabilities and quantum channels
- objects are systems $(X, H_X)$ with a Hamiltonian $H_X$ (Hilbert space $\mathcal{H}_X = \oplus_{x \in X} \mathcal{V}_x$: for simplicity just $\mathcal{V}_x = \mathbb{C}$) with
  - $H_X^* = H_X$
  - gap in the spectrum above the ground level: $0 \in \mathrm{Spec}(H_X)$ and $\mathrm{Spec}(H_X) \subset \{0\} \cup [\Delta, \infty)$
- subcategory by associating to pair $(X, H_X)$ the pair $(X, \rho_X)$ in $\mathcal{FQ}$ with

$$\rho_X = \frac{e^{-\beta H_X}}{\mathrm{Tr}(e^{-\beta H_X})}$$

  where $\beta > 0$ is a fixed inverse temperature parameter
- induced morphisms from $\mathcal{FQ}$
- category $\mathcal{FQ}^\Delta$ has zero object and coproduct
- obtain then other categories $\mathcal{QC}^\Delta$ (for $\mathcal{C}$ with zero object and sum) using $\mathcal{FQ}^\Delta$ instead of $\mathcal{FQ}$
- in particular $\mathcal{QS}_*^\Delta$ gapped quantum category of pointed sets

- category of summing functors $\Sigma_{\mathcal{QS}^{\Delta}_{*}}(X)$
- object $\Theta$ in $\Sigma_{\mathcal{QS}^{\Delta}_{*}}(X)$ is specified by
  1. choice of a point $\alpha = \{\alpha_x\}_{x \in X \setminus \{\star\}} \in \mathcal{I}^N_{\beta,\Delta}$, with $N = \#X - 1$
  2. an interval $\mathcal{I}_{\beta,\Delta} = [a_{\beta,\Delta}, b_{\beta,\Delta}] \subset [0,1]$
  3. choice of $\{\theta_x\}_{x \in X \setminus \{\star\}} \in T^N_{r(Z)}$, where $T^N = (S^1)^N$ is a torus
  4. subscript $r(Z)$ indicates that the $k$-th circle has a radius $r = r(\alpha_x, \beta, \Delta)$ uniquely determined by choice of $\alpha_x$ and by fixed values of $\Delta$ and $\beta$
- morphisms in $\Sigma_{\mathcal{QS}^{\Delta}_{*}}(X)$ are given by unitary transformations in $\mathcal{U}(2)^{\otimes N}$ and by collections of isomorphisms of pointed sets

- case of a $2 \times 2$ Hamiltonian matrix $H$: gap condition means eigenvalues 0 and $\Delta$
- density matrix $\rho = e^{-\beta H}/\mathrm{Tr}(e^{-\beta H})$ has

$$\mathrm{Spec}(\rho) = \{\frac{e^{-\beta\Delta}}{1+e^{-\beta\Delta}}, \frac{1}{1+e^{-\beta\Delta}}\}$$

- for $q = 1 - \alpha(1-\alpha) + |\theta|^2$ this gives

$$\frac{1}{2}(1-q^{1/2}) = \frac{e^{-\beta\Delta}}{1+e^{-\beta\Delta}}, \qquad \frac{1}{2}(1+q^{1/2}) = \frac{1}{1+e^{-\beta\Delta}}$$

$$\Rightarrow q^{1/2} = \frac{1-e^{-\beta\Delta}}{1+e^{-\beta\Delta}}$$

- this gives

$$|\theta|^2 = \left(\frac{1-e^{-\beta\Delta}}{1+e^{-\beta\Delta}}\right)^2 - 1 + \alpha(1-\alpha) = \frac{-4e^{-\beta\Delta}}{(1+e^{-\beta\Delta})^2} + \alpha(1-\alpha)$$

where $0 < 4e^{-\beta\Delta}/(1+e^{-\beta\Delta})^2 \leq 1$

- an interval of values $0 < e^{-\beta\Delta} \leq u_{\beta,\Delta}$ with $u_{\beta,\Delta} < 1$ such that discriminant of $\frac{-4e^{-\beta\Delta}}{(1+e^{-\beta\Delta})^2} + \alpha(1-\alpha) = 0$ (seen as an equation in $\alpha$) is non-negative
- then for $\alpha$ in interval $[a_{\beta,\Delta}, b_{\beta,\Delta}]$ between the two roots

$$\frac{-4e^{-\beta\Delta}}{(1+e^{-\beta\Delta})^2} + \alpha(1-\alpha) \geq 0$$

- for fixed value of gap $\Delta$, $\exists$ inverse temperature $\beta > 0$ sufficiently large so that $e^{-\beta\Delta} \leq u_{\beta,\Delta}$
- then solutions: any choice of $\alpha \in [a_{\beta,\Delta}, b_{\beta,\Delta}]$ and a circle of values of $\theta$ with radius $r = r(\alpha, \Delta, \beta)$
- nerve $\mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{QS}^{\Delta}_{*}}(X))$ is the action groupoid of the $\mathcal{U}(2)^{\otimes N}$ action on the cubical set

$$\mathcal{Z}_{N,\Delta} = \bigcup_{Z \in |\mathcal{I}^N_{\beta,\Delta}|} \bigcup_{k=0}^{N} \mathcal{I}^k_Z \times T^{N-k}_{r(Z)}$$

- similar to previous argument for $\mathcal{QS}_*$

## Other category of gapped systems

- instead of fixing gap $\Delta$ and restricting category $\mathcal{FQ}$ to subcategory $\mathcal{FQ}^{\Delta}$ of gapped systems and gap preserving quantum channels
- consider all the objects of $\mathcal{FQ}$ so there is no fixed gap
- but change morphisms so that regard all the quantum channels that preserve a gap $\Delta > 0$ as isomorphisms
- localization $\mathcal{QS}_*[\mathcal{T}_{\Delta}^{-1}]$ of the category $\mathcal{FQ}$ at a collection of morphisms $\mathcal{T}_{\Delta}$

## localization

- set of morphisms $\mathcal{T}_\Delta$ consists of all morphisms in $\mathcal{QS}_*$ with both source and target in $\mathcal{QS}_*^\Delta$
- path category $\mathcal{P}(\mathcal{QS}_*, \mathcal{T}_\Delta^{-1})$ has the same objects as $\mathcal{QS}_*$ and morphisms given by arbitrary concatenations $\Psi_1 \cdots \Psi_N$ with the target of $\Psi_i$ equal to the source of $\Psi_{i+1}$
- the $\Psi_i$ are either morphisms in $\mathcal{QS}_*$ or formal inverses of morphisms in $\mathcal{T}_\Delta$
- equivalence relation on $\mathcal{P}(\mathcal{QS}_*, \mathcal{T}_\Delta^{-1})$ identifies:
  - the empty string at a given object with the identity morphism
  - a string $\Psi_1\Psi_2$ where both $\Psi_i$ are morphisms in $\mathcal{QS}_*$ with the morphism $\Psi_2 \circ \Psi_1$
  - a string $\Phi^{-1}\Phi$ or $\Phi\Phi^{-1}$, with $\Phi \in \mathcal{T}_\Delta$ and $\Phi^{-1}$ its formal inverse, with the identity morphism on the source, respectively target, of $\Phi$

- localization is this quotient

$$\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}] = \mathcal{P}(\mathcal{QS}_*, \mathcal{T}_\Delta^{-1})/\sim$$

- localization functor $\mathcal{QS}_* \to \mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]$ maps morphisms in $\mathcal{T}_\Delta$ to isomorphisms
- category $\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]$ has zero object and categorical sum inherited from $\mathcal{QS}_*$
- $\Gamma$-spaces

$$F_{\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]} : \Gamma^0 \to \Delta_* \qquad \text{and} \qquad F^{\mathfrak{C}}_{\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]} : \Gamma^0 \to \Box_*$$

  associated to localization $\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]$
- summing functors $\Theta : P(X) \to \mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]$ are specifed by data $\{\alpha_x\}_{x \in X \setminus \{\star\}} \in |\mathcal{I}^N|$ and $\{\theta_x\} \in \mathcal{A}_x$ as for $\mathcal{QS}_*$

- morphisms in the category $\Sigma_{\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]}(X)$ of summing functors are isomorphisms in $\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]$ compatible with the inclusions of subsets in $P(X)$ and isomorphisms of pointed sets
- now isomorphisms are both unitary transformations $\mathcal{U}(2)^{\otimes N}$ and morphisms in $\mathcal{T}_\Delta$
- morphism in $\mathcal{T}_\Delta$ compatible with inclusions: quantum channels relating the density matrices $\rho^{(x)}$ of source and target, with both source and target in $\mathcal{QS}_*^\Delta$
- so sufficiently large $\beta > 0$, the morphisms in $\Sigma_{\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]}(X)$ are given by unitary transformations in $\mathcal{U}(2)^{\otimes N}$ and by quantum channels $\Phi^{(x)}$ with $\Phi^{(x)}\rho^{(x)} = \rho'^{(x)}$
- both $\rho^{(x)}$ and $\rho'^{(x)}$ have to satisfy relation of $\theta, \alpha, \Delta$ discussed above
- nerve $\mathcal{N}_{\mathfrak{C}}(\Sigma_{\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]}(X))$ homotopy quotient eq rel on $\mathcal{Z}_N$ generated by quantum channels that preserve the $\Delta$-gap
- equivalent objects in $\Sigma_{\mathcal{QS}_*[\mathcal{T}_\Delta^{-1}]}(X)$ under unitaries in $\mathcal{U}(2)^{\otimes N}$ and quantum channels in $\mathcal{T}_\Delta$ corresponds to system in the same topological phase

## Categories of Resources

- mathematical theory of resources
  - B. Coecke, T. Fritz, R.W. Spekkens, *A mathematical theory of resources*, Information and Computation 250 (2016), 59–86. [arXiv:1409.5531]
- Resources modelled by a symmetric monoidal category $(\mathcal{R}, \circ, \otimes, \mathbb{I})$
- (Note: switch back to more common multiplicative notation for symmetric monoidal categories)
- objects $A \in \mathrm{Obj}(\mathcal{R})$ represent resources, product $A \otimes B$ represents combination of resources, unit object $\mathbb{I}$ empty resource
- morphisms $f : A \to B$ in $\mathrm{Mor}_{\mathcal{R}}(A, B)$ represent possible conversions of resource $A$ into resource $B$
- convertibility of resources when $\mathrm{Mor}_{\mathcal{R}}(A, B) \neq \emptyset$

## Measuring semigroups of categories of resources (Coecke, Fritz, Spekkens)

- preordered abelian semigroup $(R, +, \succeq, 0)$ on set $R$ of isomorphism classes of $\mathrm{Obj}(\mathcal{R})$ with $A + B$ the class of $A \otimes B$ with unit 0 given by the unit object $\mathbb{I}$ and with $A \succeq B$ iff $\mathrm{Mor}_{\mathcal{R}}(A, B) \neq \emptyset$
- (same for category $\mathcal{C}$ with sum and zero object)
- maximal conversion rate $\rho_{A \to B}$ of resources

$$\rho_{A \to B} := \sup\{\frac{m}{n} \mid n \cdot A \succeq m \cdot B, \ m, n \in \mathbb{N}\}$$

  number of copies of resource $A$ are needed on average to produce $B$
- measuring semigroup: abelian semigroup with partial ordering and semigroup homomorphism $M : (R, +) \to (S, *)$ with $M(A) \geq M(B)$ in $S$ when $A \succeq B$ in $R$
- satisfy $\rho_{A \to B} \cdot M(B) \leq M(A)$

## Expressing constraints and optimization in categorical form

- limits and colimits in categories
  - diagram $F : \mathcal{J} \to \mathcal{C}$ and cone $N$, limit is "optimal cone" (dual version for colimits)

$$\begin{tikzcd}
 & N \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
 & L \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\
F(X) \arrow[rr, "F(f)"'] & & F(Y)
\end{tikzcd}$$

  - special cases of limits and colimits: equalizers, coequalizers
- Example: thin categories $(S, \leq)$ set of objects $S$ and one morphism $s \to s'$ when $s \leq s'$
  - diagram in $(S, \leq)$ is selection of a subet $A \subset S$
  - limits and colimits greatest lower bounds and least upper bounds for subsets $A \subseteq S$
- Key idea: functors compatible with limits and colimits describe constrained optimization

## From finite sets to networks: directed graphs

- category $\mathbf{2}$ has two objects $V, E$ and two morphisms $s, t \in \mathrm{Mor}(E, V)$
- $\mathcal{F}$ category of finite sets: objects finite sets, morphisms functions between finite sets
- a directed graph is a functor $G : \mathbf{2} \to \mathcal{F}$
  - $G(E)$ is the set of edges of the directed graph
  - $G(V)$ is the set of vertices of the directed graph
  - $G(s) : G(E) \to G(V)$ and $G(t) : G(E) \to G(V)$ are the usual source and target maps of the directed graph
- category of directed graphs $\mathrm{Func}(\mathbf{2}, \mathcal{F})$ objects are functors and morphisms are natural transformations

## Systems organized according to networks

- instead of finite set $X$ want a directed graph (network) and its subsystems
- directed graph as functor $G : \mathbf{2} \to \mathcal{F}$ and functorial assignment $X \mapsto \Sigma_{\mathcal{C}}(X)$
- $\Sigma_{\mathcal{C}}(E_G)$ summing functors $\Phi_E : \mathcal{P}(E_G) \to \mathcal{C}$ for sets of edges and $\Sigma_{\mathcal{C}}(V_G)$ summing functors $\Phi_V : \mathcal{P}(V_G) \to \mathcal{C}$ for sets of vertices
- source and target maps $s, t : E_G \to V_G$ transform summing functors $\Phi_E \in \Sigma_{\mathcal{C}}(E_G)$ to summing functors in $\Sigma_{\mathcal{C}}(V_G)$

$$\Phi^s_{V_G}(A) := \Phi_{E_G}(s^{-1}(A)) \quad \Phi^t_{V_G}(A) := \Phi_{E_G}(t^{-1}(A))$$

  assigns to a set of vertices $\mathcal{C}$-resources of in/out edges
- categorical statement: source and target maps $s, t : E_G \to V_G$ determine functors between categories $\Sigma_{\mathcal{C}}(E_G)$ and $\Sigma_{\mathcal{C}}(V_G)$ of summing functors, hence map between their nerves

## Conservation laws at vertices

- source and target functors $s, t : \Sigma_{\mathcal{C}}(E_G) \rightrightarrows \Sigma_{\mathcal{C}}(V_G)$
- equalizer category $\Sigma_{\mathcal{C}}(G)$ with functor $\iota : \Sigma_{\mathcal{C}}(G) \to \Sigma_{\mathcal{C}}(E_G)$ such that $s \circ \iota = t \circ \iota$ with universal property

$$\begin{tikzcd}
\Sigma_{\mathcal{C}}(G) \arrow[r, "\iota"] & \Sigma_{\mathcal{C}}(E_G) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \Sigma_{\mathcal{C}}(V_G) \\
\mathcal{A} \arrow[u, "\exists u"] \arrow[ur, "q"] & &
\end{tikzcd}$$

- this is category of summing functors $\Phi_E : P(E_G) \to \mathcal{C}$ with conservation law at vertives: for all $A \in P(V_G)$

$$\Phi_E(s^{-1}(A)) = \Phi_E(t^{-1}(A))$$

in particular for all $v \in V_G$ have inflow of $\mathcal{C}$-resources equal outflow

$$\oplus_{e:s(e)=v} \Phi_E(e) = \oplus_{e:t(e)=v} \Phi_E(e)$$

- another kind of conservation law expressed by coequalizer

## Gamma spaces for networks

- $\mathcal{E}_{\mathcal{C}} : \mathrm{Func}(\mathbf{2}, \mathcal{F}) \to \Delta$ with $\mathcal{E}_{\mathcal{C}}(G) = \mathcal{N}(\Sigma_{\mathcal{C}}(G))$ nerve of equalizer of $s, t : \Sigma_{\mathcal{C}}(E_G) \rightrightarrows \Sigma_{\mathcal{C}}(V_G)$ (equalizer of nerves)
- more general types of Gamma networks besides equalizers $\Sigma^{eq}_{\mathcal{C}}(G)$ (and coequalizers)
  - for $G \in \mathrm{Func}(\mathbf{2}, \mathcal{F})$ take category $P(G)$ with objects (pointed) subgraphs $G'_*$ of $G_*$ and morphisms (pointed) inclusions $\iota : G'_* \hookrightarrow G''_*$
  - category $\Sigma_{\mathcal{C}}(G)$ of summing functors $\Phi_G : P(G) \to \mathcal{C}$
  - now value of functor $\Phi_G \in \Sigma_{\mathcal{C}}(G)$ on a subnetwork $G' \subset G$ not just sum of values on edges in the subnetwork
  - possible more complicated dependence on network structure (beyond conservation at vertices): general inclusion-exclusion type properties
- focus on equalizer case for simplicity

## Example of categories of resources: category of binary codes

- $C$ be a $[n, k, d]_2$ binary code of length $n$ with $\#C = q^k$
- category $\mathrm{Codes}_{n,*}$ of pointed codes of length $n$
  - objects codes that contain 0-word $c_0 = (0, 0, \ldots, 0)$
  - exclude code consisting only of constant words $c_0 = (0, 0, \ldots, 0)$ and $c_1 = (1, 1, 1, \ldots, 1)$ (for reasons of non-trivial information)
  - morphisms $f : C \to C'$ functions mapping the 0-word to itself (don't require maps of ambient $\mathbb{F}_2^n$)
  - sum as for pointed sets $C \vee C'$ (glued along the zero-word)
  - zero-object: code consisting only of the zero word
  - role of zero-word is like reference point (for neural code, baseline when no activity detected)

• Note: in coding theory often other form of categorical sum (decomposable codes), but changes code length $n$

$$C \oplus C' := \{(c, c') \in \mathbb{F}_2^{n+n'} \mid c \in C,\ c' \in C'\}$$

## Example: neural codes

- $T > 0$ time interval of observation, subdivided into some basic units of time, $\Delta t$
- code length $n = T/\Delta t$: number of basic time intervals considered
- number of nontrivial code words: neurons oberved
- each code word: firing pattern of that neuron, digit 1 for each time intervals $\Delta t$ that contained a spike and 0 otherwise
- zero-word baseline of no activity (for comparison)
- a neural code for $N$ neurons is a sum $C_1 \vee \cdots \vee C_N$ with $C_i = \{c_0, c\}$ with zero-word $c_0$ and firing pattern $c$ of $i$-th neuron

## Category of weighted codes

- category of weighted binary codes $\mathcal{W}\mathrm{Codes}_{n,*}$
- objects pairs $(C, \omega)$ of a code $C$ of length $n$ containing zero-word $c_0$ and function $\omega : C \to \mathbb{R}$ assigning (signed) weight to each code word, with $\omega(c_0) = 0$
- morphisms $\phi = (f, \lambda) : (C, \omega) \to (C', \omega')$ with $f : C \to C'$ mapping the zero-word to itself and $f(\mathrm{supp}(\omega)) \subset \mathrm{supp}(\omega')$ and weights $\lambda_{c'}(c)$ for $c \in f^{-1}(c')$
- sum $(C, \omega) \oplus (C', \omega') = (C \vee C', \omega \vee \omega')$ with $\omega \vee \omega'|_C = \omega$ and $\omega \vee \omega'|_{C'} = \omega'$
- zero object $(\{c_0\}, 0)$

## Equalizer: linear model

- a summing functor $\Phi$ in the equalizer of source and target functors

$$\Sigma^{eq}_{\mathcal{W}\mathrm{Codes}_{n,*}}(G) := \mathrm{eq}(s, t : \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(E_G) \rightrightarrows \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(V_G))$$

  is a summing functor $\Phi \in \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(E_G)$ with conservation laws $\Phi(s^{-1}(A)) = \Phi(t^{-1}(A))$ for $A \subset V_G$
- If directed graph $G$ has a single outgoing edge at each vertex, $\{e \in E_G \mid s(e) = v\} = \{\mathrm{out}(v)\}$, then equalizer condition

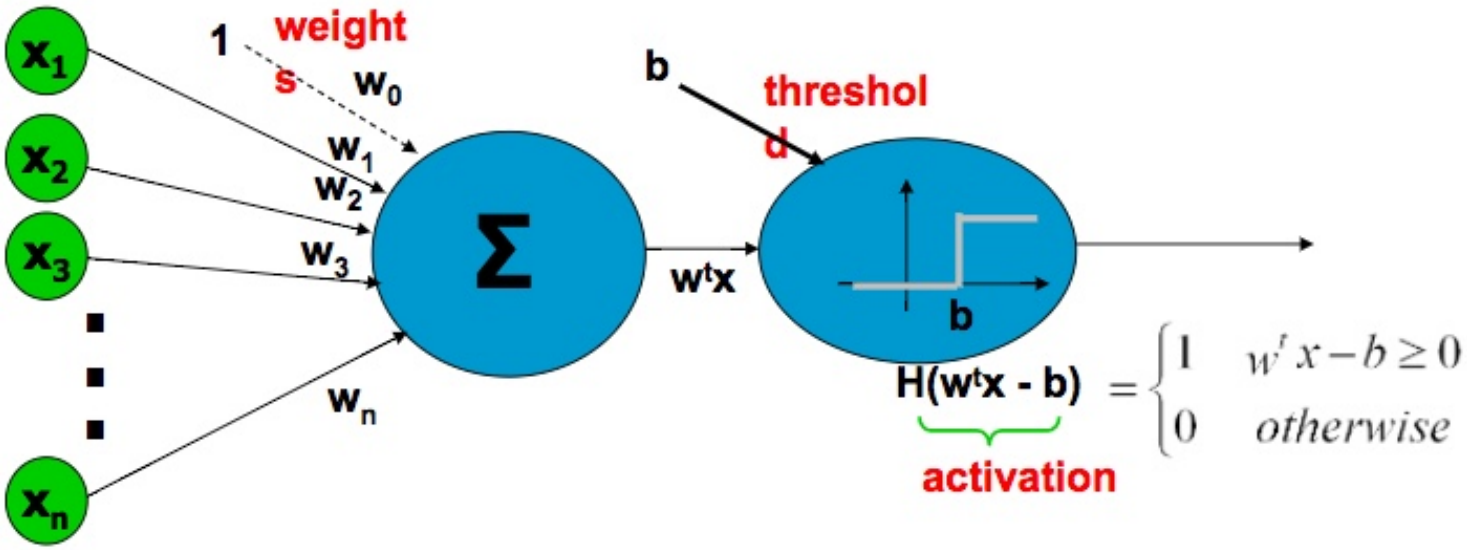
$$(C_{\mathrm{out}(v)}, \omega_{\mathrm{out}(v)}) = \oplus_{t(e)=v}(C_e, \omega_e),$$

- can be seen as a kind of categorical version of linear neuron model

## linear neuron and non-linearity threshold

- Cybernetics: "artificial neurons"

Warren McCulloch & Walter Pitts (1943)




need models of threshold dynamics in networks with resources (see discussion below on Hopfield dynamics in categories of summing functors)

## Keeping track of associated measures of informational complexity

### Information and codes

- probability distribution associated to neural codes through it firing rate
- word of length $n$ recording a digit 1 for each time interval $\Delta t$ that contains a spike and a 0 otherwise
- $(\Sigma_2^+, \mu_P)$ with $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ and $\mu_P$ Bernoulli measure
- $\mu_P(\Sigma_2^+(w_1, \ldots, w_n)) = p^{a_n(w)}(1-p)^{b_n(w)}$ with $a_n(w)$ number of 1's and $b_n(w) = n - a_n(w)$ the number of zeros
- for large $n$ neural code $C$ in Shannon Random Code Ensemble of $(\Sigma_2^+, \mu_P)$

$$\lim_{n\to\infty} \frac{a_n(w)}{n} \overset{a.e.}{=} p$$

- category $\mathcal{P}_f$ of finite probabilities with fiberwise measures (not normalized) as morphisms
- $\phi = (f, \Lambda) : (X, P_X) \to (Y, P_Y)$ with $f : X \to Y$ pointed map $f(\mathrm{supp}(P_X)) \subset \mathrm{supp}(P_Y)$ and $\Lambda = \{\lambda_y\}$ on fibers $f^{-1}(y) \subset X$, with $\lambda_{y_0}(x_0) > 0$ and $P_X(A) = \sum_{y \in Y} \lambda_y(A \cap f^{-1}(y))\, P_Y(y)$
- $\mathcal{P}_f$ has zero object and sum
- functor $P : \mathrm{Codes}_{n,*} \to \mathcal{P}_f$

$$P_C(c) = \begin{cases} \frac{b(c)}{n(\#C-1)} & c \neq c_0 \\ 1 - \sum_{c' \neq c_0} \frac{b(c')}{n(\#C-1)} & c = c_0 \end{cases}$$

- consistent assignments of codes to a network $\Rightarrow$ assignment of probabilities
- information: $\mathcal{P}_{f,s}$ subcategory with $f : X \to Y$ surjections and $\lambda_y(x)$ for $x \in f^{-1}(y)$ probability measures, then Shannon entropy is a functor $S : \mathcal{P}_{f,s} \to \mathbb{R}$ (with $(\mathbb{R}, \geq)$ thin category)

## Example of categories of resources: concurrent/distributed computing architectures

- category of transition systems
  - G. Winskel, M. Nielsen, *Categories in concurrency*, in "Semantics and logics of computation (Cambridge, 1995)", pp. 299–354, Publ. Newton Inst., 14, Cambridge Univ. Press, 1997.
- models of computation that involve parallel and distributed processing
- objects $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$ with $S$ set of possible states of the system, $\iota$ initial state, $\mathcal{L}$ set of labels, $\mathcal{T}$ set of possible transitions, $\mathcal{T} \subseteq S \times \mathcal{L} \times S$ (specified by initial state, label of transition, final state)
- directed graph with vertex set $S$ and with set of labelled directed edges $\mathcal{T}$

- $\mathrm{Mor}_{\mathcal{C}}(\tau, \tau')$ of transition systems pairs $(\sigma, \lambda)$, function $\sigma : S \to S'$ with $\sigma(\iota) = \iota'$ and (partially defined) function $\lambda : \mathcal{L} \to \mathcal{L}'$ of labeling sets such that, for any transition $s_{in} \xrightarrow{\ell} s_{out}$ in $\mathcal{T}$, if $\lambda(\ell) \in \mathcal{L}'$ is defined, then $\sigma(s_{in}) \xrightarrow{\lambda(\ell)} \sigma(s_{out})$ is a transition in $\mathcal{T}'$
- categorical sum

$$(S, \iota, \mathcal{L}, \mathcal{T}) \oplus (S', \iota', \mathcal{L}', \mathcal{T}') = (S \times \{\iota'\} \cup \{\iota\} \times S', (\iota, \iota'), \mathcal{L} \cup \mathcal{L}', \mathcal{T} \sqcup \mathcal{T}')$$

$$\mathcal{T} \sqcup \mathcal{T}' := \{(s_{in}, \ell, s_{out}) \in \mathcal{T}\} \cup \{(s'_{in}, \ell', s'_{out}) \in \mathcal{T}'\}$$

where both sets are seen as subsets of

$$(S \times \{\iota'\} \cup \{\iota\} \times S') \times (\mathcal{L} \cup \mathcal{L}') \times (S \times \{\iota'\} \cup \{\iota\} \times S')$$

- zero object is given by the stationary single state system $S = \{\iota\}$ with empty labels and transitions

## Grafting

- $\tau_i = (S_i, \iota_i, \mathcal{L}_i, \mathcal{T}_i)$ for $i = 1, 2$ objects in category $\mathcal{C}$ of transition systems
- a choice of two states $s \in S_1$ and $s' \in S_2$
- grafting $\tau_{s,s'} = (S, \iota, \mathcal{L}, \mathcal{T})$ in $\mathcal{C}$ with $S = S_1 \cup S_2$, $\iota = \iota_1$, $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{e\}$ and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \{(s, e, s')\}$
- $\mathcal{C}' \subset \mathcal{C}$ subcategory of transition systems $\tau$ with a single final state $q \in S$
- then grafting $\tau_1 \star \tau_2$ given by $\tau_{q_1,\iota_2}$ with final state of $\tau_1$ grafted to initial state of $\tau_2$
- $G$ finite acyclic directed graph and $\omega$ a topological ordering vertex set $V_G$ then given $\{\tau_v\}_{v \in V}$ objects of $\mathcal{C}'$ there is a well defined grafting $\tau_{G,\omega}$ of the $\tau_v$ that is also an object in $\mathcal{C}'$

## Strongly connected components, condensation graph, and computational architecture functor

- finite directed graph $G$: subset $V' \subset V_G$ is a strongly connected component if each vertex in $V'$ reachable through oriented path in $G$ from any other
- condensation graph $\bar{G}$ is a directed acyclic graph: obtained from $G$ by contracting each subgraph of a strongly connected component to a single vertex
- $\mathcal{G} := \mathrm{Func}(\mathbf{2}, \mathcal{F})$ category of finite directed graphs
- $\Delta_{\mathcal{G}}$ category with objects pairs $(G, \Phi)$ with $G \in \mathrm{Obj}(\mathcal{G})$ and $\Phi \in \Sigma_{\mathcal{C}}(V_G)$, morphisms $(\alpha, \alpha_*)$ with $\alpha : G \to G'$ and $\alpha_*(\Phi)(A) = \Phi(\alpha_V^{-1}(A))$
- $\Delta'_{\mathcal{G}}$ subcategory of $\Delta_{\mathcal{G}}$ with objects $(G, \Phi)$ where summing functor $\Phi$ takes values in $\mathcal{C}'$
- functor $\Xi_0 : \Delta'_{\mathcal{G}} \to \mathcal{C}'$ assigning to an object $(G, \Phi)$ the grafting $\tau_{\bar{G},\bar{\omega}}$ along the condensation graph $\bar{G}$ of the $\Phi(V_{G_i})$ with $G_i$ the strongly connected components of $G$

## Modeling computational architectures of neuronal networks

- local automata model (discretized) individual neurons with pre-synaptic and post-synaptic activity
- grafting of these automata where their inputs and outputs are connected model connectivity of the network
- can adapt this setting to model non-local neuromodulation effects (distributed computing models of neuromodulation: Potjans–Morrison–Diesmann)

## Example: distributed systems for neuromodulation

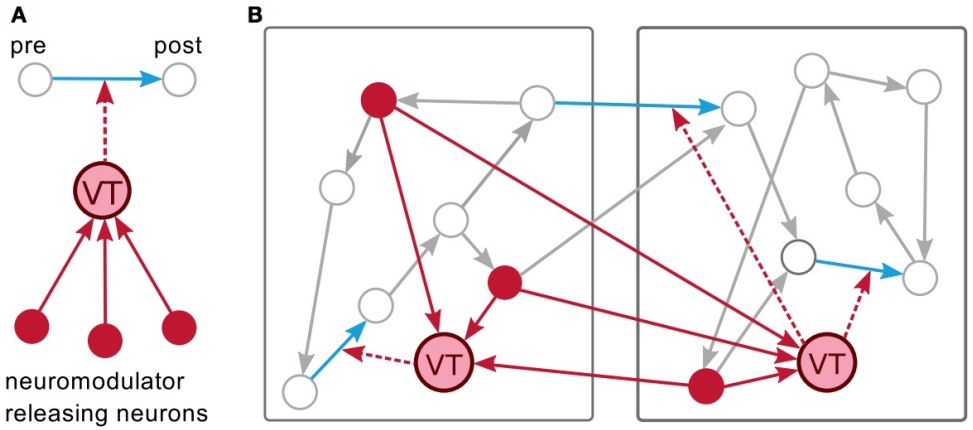



**FIGURE 4 | A distributed volume transmitter object. (A)** The volume transmitter (VT) collects all spikes from the neurons releasing the neuromodulator into a given volume and delivers them to any associated synapses. **(B)** An example network is distributed over two machines. The volume transmitter (VT) is duplicated on each machine. It collects globally the spikes of the neuromodulator releasing neurons (red) and delivers them locally to the neuromodulated synapses (blue).

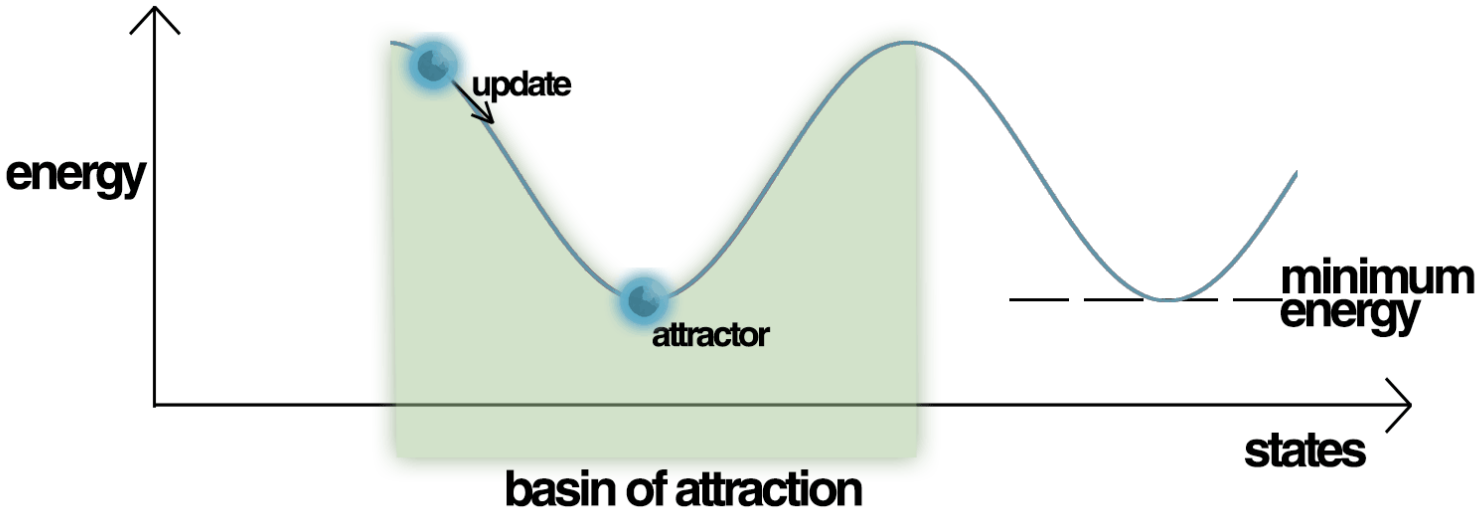
- W.Potjans, A.Morrison, M.Diesmann, *Enabling Functional Neural Circuit Simulations with Distributed Computing of Neuromodulated Plasticity*, Front. Comput. Neurosci. Vol.4 (2010) 141

## Hopfield Network

- historical connection between statistical physics of spin glass models and neural networks
- nodes variables $s_i = \pm 1$, update

$$s_i = \begin{cases} +1 & \sum_j w_{ij} s_j \geq \theta_i \\ -1 & \text{otherwise} \end{cases}$$

$$E = -\frac{1}{2} \sum_{i,j} w_{ij} s_i s_j - \sum_i \theta_i s_i$$




Energy landscape of the Hopfield network

## Discrete and continuous Hopfield dynamics

- discrete version (binary neurons)

$$\nu_j(n+1) = \begin{cases} 1 & \text{if } \sum_k T_{jk}\nu_k(n) + \eta_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

- continuous version (neuron firing rates as variables and threshold-linear dynamics)

$$\frac{dx_j}{dt} = -x_j + \left( \sum_k W_{jk} x_j + \theta_j \right)_+$$

$W_{jk}$ real-valued connection strengths, $\theta_j$ constant external inputs, and $(\cdot)_+ = \max\{0, \cdot\}$ threshold function

- finite difference version

$$\frac{x_j(t+\Delta t) - x_j(t)}{\Delta t} = -x_j + (\sum_k W_{jk} x_k(t) + \theta_j)_+$$

(versions with or without "leak term" $-x_j$ on r.h.s.)

## Categorical Hopfield dynamics: Step 1

- as above $\Sigma^{eq}_{\mathcal{C}}(G)$ for a network $G$ and category of resources $\mathcal{C}$
- this category of summing functors is like a physical configuration space: kinematic space describing all possible assignments of resources of type $\mathcal{C}$ to network $G$
- $\rho : \mathcal{C} \to \mathcal{R}$ functor to another category of resources (maybe same) with respect to which dynamics is measured
- $(R, +, \succeq)$ preordered semigroup of category $\mathcal{R}$
- will use relation $r_C \succeq 0$ for class of $\rho(C)$ for threshold-dynamics
- $\mathcal{E}(\mathcal{C}) = \mathrm{Func}(\mathcal{C}, \mathcal{C})$ category of monoidal endofunctors of $\mathcal{C}$, morphisms natural transformations
- sum of endofunctors defined pointwise $(F \oplus F')(C) = F(C) \oplus F'(C)$ for all $C \in \mathrm{Obj}(\mathcal{C})$.

## Categorical Hopfield dynamics: Step 2

- *bisumming functors* $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ summing in both arguments
- *coordinates*: $T_{ee'}$ with $T_{A,B} = \oplus_{e \in A, e' \in B} T_{ee'}$
- $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(E)$ category of bisumming functors with invertible natural transformations
- $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(G)$ equalizer of functors

$$s, t : \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(E) \rightrightarrows \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(V)$$

## Categorical Hopfield dynamics: Step 3

- initial condition $\Phi_0 \in \Sigma^{eq}_{\mathcal{C}}(G)$: set $X_A(0) := \Phi_0(A)$ (or just $X_e(0) := \Phi_0(e)$)
- fixed summing functor $\Psi \in \Sigma^{eq}_{\mathcal{C}}(G)$: set $\Theta_e = \Psi(e)$
- take $Y_e(n) := \oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e$
- $r_{Y_e(n)}$ the class in $(R, +, \succeq)$ of the object $\rho(Y_e(n))$ in $\mathcal{R}$
- threshold $(\cdot)_+$: $(Y_e(n))_+ = \oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e$ if $r_{Y_e(n)} \succeq 0$ and zero-object of $\mathcal{C}$ otherwise
- equation

$$X_e(n+1) = X_e(n) \oplus (\oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e)_+$$

or variant $X_e(n+1) = (\oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e)_+$ (leaking term or not)

## Some properties of the dynamics

- $X_A(n) =: \Phi_n(A)$ defines a sequence of summing functors in $\Sigma^{eq}_{\mathcal{C}}(G)$
- assignment $\mathcal{T} : \Phi_n \mapsto \Phi_{n+1}$ defined by solution defines endofunctor $\mathcal{T} : \Sigma^{eq}_{\mathcal{C}}(G) \to \Sigma^{eq}_{\mathcal{C}}(G)$
- induced discrete topological dynamical system $\tau$ on realization $|\mathcal{N}(\Sigma^{eq}_{\mathcal{C}}(G))| = B\Sigma^{eq}_{\mathcal{C}}(G)$
- for $\mathcal{C} = \mathcal{W}\mathrm{Codes}_{n,*}$ with a measuring semigroup, categorical Hopfield dynamics induces usual (finite difference) Hopfield dynamics on the weights
- Question: general results in categorical setting about existence of solutions and behavior?