Schottky Groups and Limit Sets
Introduction to Fractal Geometry and Chaos

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Some References

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Kleinian Groups and Schottky Groups

- Kleinian Group: discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$
- $\mathrm{PSL}_2(\mathbb{C})$ acts by isometries of real hyperbolic 3-space $\mathbb{H}^3 = \mathrm{PGL}(2,\mathbb{C})/\mathrm{SU}(2)$ with $ds^2 = \frac{dz d\bar{z} + dy^2}{y^2}$
- in upper half space model $\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^+$

  $$\gamma : (z,y) \mapsto \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}y^2}{|cz+d|^2 + |c|^2 y^2}, \frac{y\,|ad-bc|}{|cz+d|^2 + |c|^2 y^2} \right)$$

  for $(z,y) \in \mathbb{C} \times \mathbb{R}^+$ and

  $$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

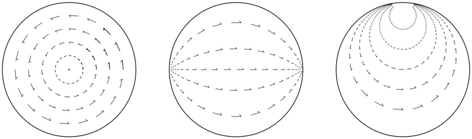
- $\mathrm{PSL}_2(\mathbb{C})$ also acts by fractional linear transformations on $\mathbb{P}^1(\mathbb{C})$

  $$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

- these two actions are compatible when viewing $\mathbb{P}^1(\mathbb{C}) = \partial \mathbb{H}^3$ as the boundary at infinity of hyperbolic 3-space

Elliptic, parabolic, loxodromic elements of $\mathrm{PSL}_2(\mathbb{C})$

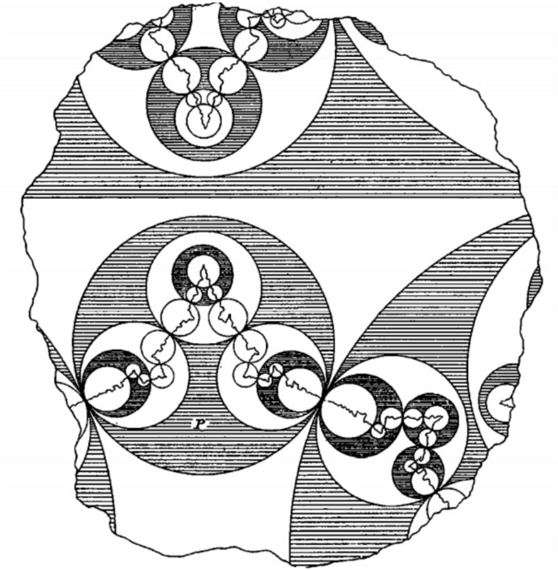
- $\gamma \neq \mathrm{Id}$ in $\mathrm{PSL}_2(\mathbb{C})$:
  - elliptic: $|\mathrm{Tr}(\gamma)| < 2$
  - parabolic: $|\mathrm{Tr}(\gamma)| = 2$
  - loxodromic: $|\mathrm{Tr}(\gamma)| > 2$
- Fixed points:
  - elliptic: has a fixed point inside $\mathbb{H}^3$
  - parabolic: has no fixed point inside $\mathbb{H}^3$ and one fixed point on $\mathbb{P}^1(\mathbb{C}) = \partial\mathbb{H}^3$
  - loxodromic: has no fixed point inside $\mathbb{H}^3$ and two fixed points in $\mathbb{P}^1(\mathbb{C})$
- picture for $\mathbb{H}^2$ with boundary $\mathbb{P}^1(\mathbb{R})$ (real case): elliptic, hyperbolic, parabolic



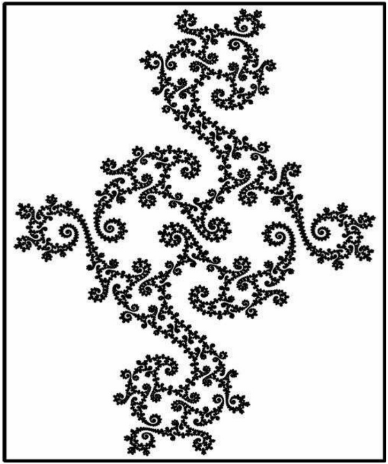
Schottky groups

- purely loxodromic subgroup of $\mathrm{PSL}_2(\mathbb{C})$: every non-identity element $\gamma \neq \mathrm{Id}$ in the subgroup is loxodromic
- Schottky group: a purely loxodromic discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ isomorphic to a free group on $g \geq 1$ generators, $\Gamma \simeq \mathbb{Z}^{\star g} = F^g$
- symmetric set of generators $\{\gamma_1, \dots, \gamma_g, \gamma_1^{-1}, \dots, \gamma_g^{-1}\}$
- $2g$ disjoint Jordan curves $A_1, B_1, \dots, A_g, B_g$ in $\mathbb{P}^1(\mathbb{C})$ with disjoint interiors: if there are $\gamma_i \in \mathrm{PSL}_2(\mathbb{C})$ taking the outside of A_i onto the inside of B_i, then the group generated by these transformations is a Schottky group
- all Schottky groups can be constructed in this way
- Classical Schottky Group: when all the Jordan curves in this construction can be taken to be circles

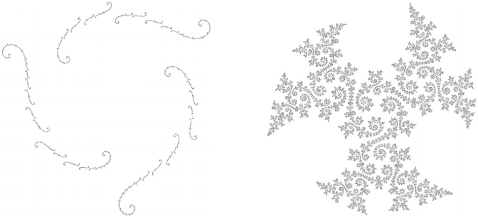
- Limit set Λ_Γ of a Schottky group: smallest non–empty closed Γ–invariant subset of $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$
- Γ acts freely and properly discontinuously on $\mathbb{H}^3$, the set Λ_Γ is contained in $\mathbb{P}^1(\mathbb{C})$
- limit set is the closure of the set of the attractive and repelling fixed points $z^{\pm}(g)$ of the loxodromic elements $g \in \Gamma$
- limit set consists of limits of orbits of points of $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ under the action of Γ
- domain of discontinuity: $\Omega_\Gamma = \mathbb{P}^1(\mathbb{C}) \smallsetminus \Lambda_\Gamma$
- free and properly discontinuous action of Γ on $\mathbb{H}^3 \cup \Omega_\Gamma$



Limit sets of Kleinian groups as originally drawn by Fricke and Klein in 1897



Limit set of a Schottky group (McMullen)



(a) $q = .8 + .3i$, $\delta \approx 1.09$.

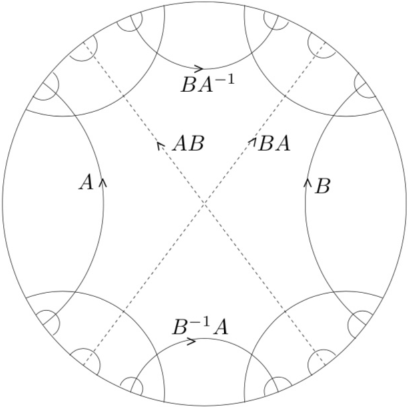
(b) $q = .8 + .44i$, $\delta \approx 1.29$.

Two limit sets of Schottky groups with $\mathbb{Z}/3\mathbb{Z}$ symmetry, with estimate of Hausdorff dimension (Shaun Maguire)

- fundamental domain $\mathcal{F}_\Gamma$ for the action of a Schottky group Γ on $\mathbb{P}^1(\mathbb{C})$ is given by the exterior of the Jordan curves

$$\Omega_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma \cdot \mathcal{F}_\Gamma$$

- Fuchsian Schottky group: Schottky group in $\mathrm{SL}_2(\mathbb{R})$ (preserves upper and lower hemispheres of $\mathbb{P}^1(\mathbb{C})$ and equator circle $\mathbb{P}^1(\mathbb{R})$)
- Fuchsian Schottky group G produces a quotient $G \backslash \mathbb{H}^2$ which is topologically a Riemann surface with boundary



Generators, fixed points and geodesics for a Fuchsian Schottky group acting on $\mathbb{P}^1(\mathbb{R})$ and on $\mathbb{H}^2$

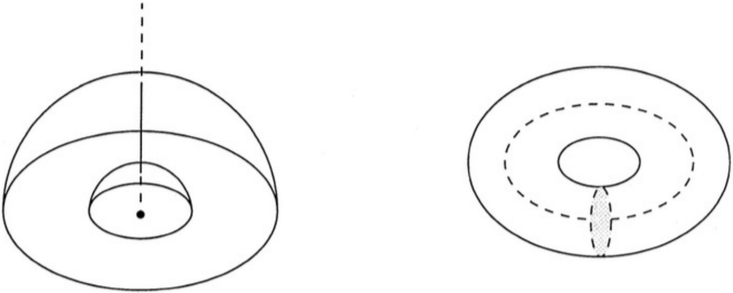
genus one case

- generator γ conjugate to $q \in \mathbb{C}$ with $0 < |q| < 1$
- action on $\mathbb{H}^3$ by

$$\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} (z, y) = (qz, |q| y)$$

- limit set just two points $\Lambda_\Gamma = \{0, \infty\}$
- quotient $\mathfrak{X} = \mathbb{H}^3/(q^{\mathbb{Z}})$ is a solid torus with the elliptic curve $X_{/\mathbb{C}} = \mathbb{C}^*/(q^{\mathbb{Z}})$ as its boundary at infinity
- length of closed geodesic in $\Gamma\backslash\mathbb{H}^3$ is ℓ_γ with $q = e^{-\frac{1}{2}(\ell_\gamma + i\theta)}$

$$\ell_\gamma = \cosh^{-1}(|\frac{\mathrm{Tr}\gamma}{2}|^2 + |(\frac{\mathrm{Tr}\gamma}{2})^2 - 1|)$$



Euclidean BTZ Black Hole

- Bañados–Teitelboim–Zanelli black holes: asymptotically AdS space–times global identifications of AdS_{2+1} by a discrete group of isometries generated by a single loxodromic element
- group of isometries of AdS_{2+1} is $SO(2,2)$
- hyperboloid model of anti de Sitter space $-t^2 - u^2 + x^2 + y^2 = -1$ in $\mathbf{R}^{2,2}$
- non–rotating case: group Γ in a diagonal $SO(2,1) \cong PSL(2,\mathbf{R})$ in $SO(2,2)$
- Euclidean analog of the BTZ black hole $\mathbb{H}^3$ replacing AdS_{2+1}
- not purely real q: quotient space $\mathfrak{X}_q(\mathbb{C})$ represents a spinning black hole

- physical meaning of q is clarified by the following expression:

$$q = \exp\left(\frac{2\pi(i|r_-| - r_+)}{\ell}\right)$$

- parameters $r_\pm$ depend on mass M and angular momentum J of the black hole

$$r_\pm^2 = \frac{1}{2}\left(M\ell \pm \sqrt{M^2\ell^2 + J}\right)$$

- ℓ determines the cosmological constant $\Lambda = -1/\ell^2$ and normalizes the metric as

$$ds^2 = \frac{\ell^2}{y^2}(|dz|^2 + dy^2)$$

- coordinates in the upper half space model of $\mathbb{H}^3$ in terms of Schwarzschild coordinates (r, τ, ϕ) with Euclidean time τ

$$z = \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \exp\left(\left(\frac{r_+}{\ell}\phi - \frac{|r_-|}{\ell^2}\tau\right) + i\left(\frac{r_+}{\ell^2}\tau + \frac{|r_-|}{\ell}\phi\right)\right),$$

$$y = \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{1/2} \exp\left(\frac{r_+}{\ell}\phi - \frac{|r_-|}{\ell^2}\tau\right).$$

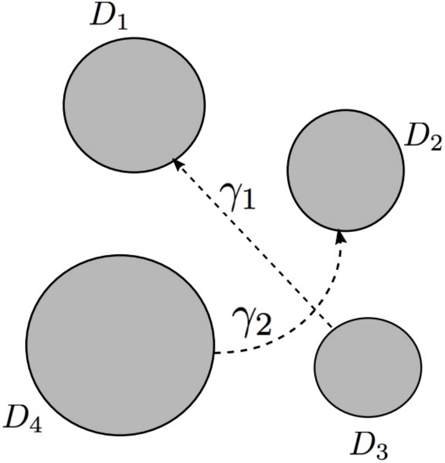
- then action

$$(z, y) \mapsto (e^{2\pi(i|r_-|-r_+)/\ell}\, z,\; e^{-2\pi r_+/\ell}\, y)$$

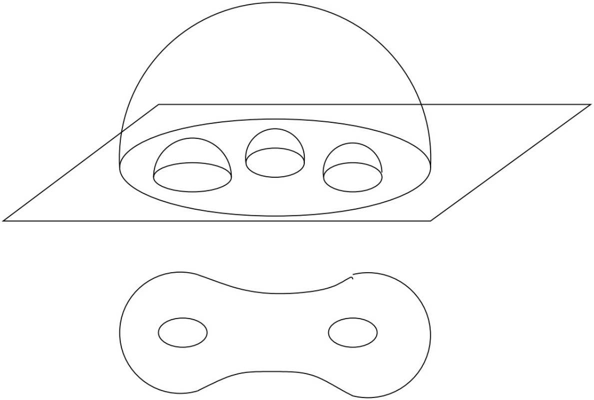
- $r_- \neq 0$ (non purely real q) spinning black hole

Higher genus

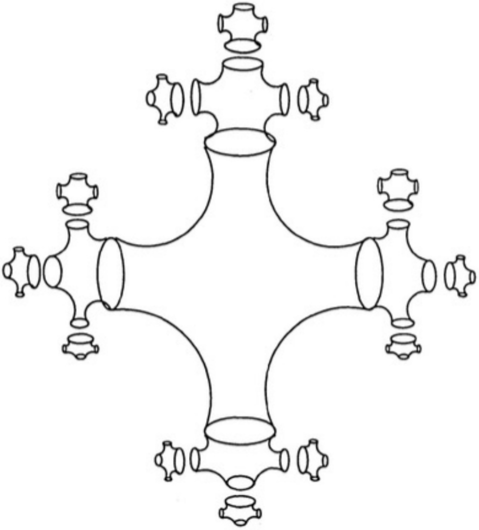
- $g \geq 1$ quotient space is topologically a handlebody of genus g
$$\mathfrak{X}_\Gamma := \Gamma\backslash\mathbb{H}^3$$
- $g \geq 2$ limit set Λ_Γ is uncountable and quotient
$$X_\mathbb{C} = \Gamma\backslash\Omega_\Gamma$$
is a Riemann surface of genus g
- covering $\Omega_\Gamma \to X_\mathbb{C}$ a *Schottky uniformization* of $X_\mathbb{C}$
- every complex Riemann surface $X_\mathbb{C}$ of genus $g \geq 1$ admits a Schottky uniformization
- Schottky uniformization determines a set of generators a_i, $i = 1, \dots g$, for $Ker(I_*)$, where $I_* : H_1(X_\mathbb{C}, \mathbb{Z}) \to H_1(\mathfrak{X}_\Gamma, \mathbb{Z})$ induced by the inclusion of $X_\mathbb{C}$ as the conformal boundary at infinity of $\mathfrak{X}_\Gamma$
- generators a_i are the images under the quotient map $\Omega_\Gamma \to X_\mathbb{C}$ of Jordan curves A_i



Circle pairing of the action of a classical rank two Schottky group on $\mathbb{P}^1(\mathbb{C})$



Fundamental domains of the action of a rank two Schottky group on $\mathbb{H}^3$



Fundamental domains of the action of a rank two Schottky group on $\mathbb{H}^3$

Limit Sets of Schottky groups and Shift Operator

- choice of a set of generators $\{g_i\}_{i=1}^{g}$ for the Schottky group Γ gives bijection between the elements of Γ and the set of all *reduced words* in the $\{g_i\}_{i=1}^{2g}$, with notation $g_{i+g} := g_i^{-1}$, for $i = 1, \dots, g$
- reduced words: all finite sequences $w = a_0 \dots a_\ell$ in the g_i, for any $\ell \in \mathbb{N}$, satisfying $a_{i+1} \neq a_i^{-1}$ for all $i = 0 \dots, \ell - 1$
- set $\mathcal{S}^+$ of all right–infinite *reduced sequences* in the $\{g_i\}_{i=1}^{2g}$,

$$\mathcal{S}^+ = \{a_0 a_1 \dots a_\ell \dots \ |a_i \in \{g_i\}_{i=1}^{2g}, \ a_{i+1} \neq a_i^{-1}, \forall i \in \mathbb{N}\}$$

- set $\mathcal{S}$ of *doubly infinite reduced sequences* in the $\{g_i\}_{i=1}^{2g}$,

$$\mathcal{S} = \{\dots a_{-m} \dots a_{-1} a_0 a_1 \dots a_\ell \dots \ |a_i \in \{g_i\}_{i=1}^{2g}, \ a_{i+1} \neq a_i^{-1}, \forall i \in \mathbb{Z}\}$$

- $\mathcal{S}$ with topology generated by the cylinder sets $W^s(x,\ell) = \{y \in \mathcal{S} | x_k = y_k, k \geq \ell\}$, and the $W^u(x,\ell) = \{y \in \mathcal{S} | x_k = y_k, k \leq \ell\}$ for $x \in \mathcal{S}$ and $\ell \in \mathbb{Z}$
- topology on $\mathcal{S}^+$ also generated by cylinder sets
- one-sided shift operator T on $\mathcal{S}^+$

$$T(a_0 a_1 a_2 \ldots a_\ell \ldots) = a_1 a_2 \ldots a_\ell \ldots$$

- two-sided invertible shift operator T on $\mathcal{S}$ as the map

$$T\left(\begin{array}{ccccccccc} \ldots & a_{-m} & \ldots & a_{-1} & a_0 & a_1 & \ldots & a_\ell & \ldots \end{array} \right) = \begin{array}{ccccccccc} \ldots & a_{-m+1} & \ldots & a_0 & a_1 & a_2 & \ldots & a_{\ell+1} & \ldots \end{array}$$

- choice of base point $x_0 \in \mathbb{H}^3 \cup \Omega_\Gamma$ defines a map $Z : \mathcal{S}^+ \to \Lambda_\Gamma$
- for eventually periodic sequence $w\overline{a_0 \dots a_N} \in \mathcal{S}^+$ with initial word w, take

$$Z(w\overline{a_0 \dots a_N}) = wz^+(a_0 \dots a_N)$$

- identify finite reduced word w with an element in Γ so $wz^+(a_0 \dots a_N)$ is image under $w \in \Gamma$ of attractive fixed point of element $a_0 \dots a_N$ of Γ
- for sequences $a_0 \dots a_\ell \dots$ not eventually periodic take

$$Z(a_0 \dots a_\ell \dots) = \lim_{\ell \to \infty} (a_0 \dots a_\ell) x_0$$

- cylinder sets $\mathcal{S}^+(w) \subset \mathcal{S}^+$ right-infinite reduced sequences that start with word $w = a_0 \dots a_\ell$
- $g \in \Gamma$ as reduced word w in the generators g_i, write $\Lambda_\Gamma(g) := Z(\mathcal{S}^+(w))$
- $\Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$ quotient by diagonal action of Γ of *complement of the diagonal* $(\Lambda_\Gamma \times \Lambda_\Gamma)^0$
- Γ acts on $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations, hence also on Λ_Γ since Γ-invariant subset of $\mathbb{P}^1(\mathbb{C})$
- Γ also acts on $\mathcal{S}^+$: identify with a reduced word $\gamma = c_0 \dots c_k$ in the g_i, so it maps a sequence $a_0 \dots a_\ell \dots$ to the sequence obtained from $c_0 \dots c_k a_0 \dots a_\ell \dots$ by necessary cancellations to get reduced sequence

- spaces $\mathcal{S}^+$ and $\mathcal{S}$ are topologically Cantor sets. One-sided shift T is continuous surjective map on $\mathcal{S}^+$, and two-sided shift T homeomorphism of $\mathcal{S}$
- limit set Λ_Γ with topology induced by embedding in $\mathbb{P}^1(\mathbb{C})$ is also Cantor set, and map Z is homeomorphism
- map Z is Γ-equivariant
- shift operator T on $\mathcal{S}^+$ induces map $ZTZ^{-1} : \Lambda_\Gamma \to \Lambda_\Gamma$

$$ZTZ^{-1}|_{\Lambda_{\Gamma(g_i)}}(z) = g_i^{-1}(z)$$

- $A = (A_{ij})$ elementary matrix (entries in $\{0, 1\}$)
- subshift of finite type with transition matrix A set of all (doubly) infinite sequences in alphabet $\{1, \dots, N\}$

$$\mathcal{S}_A := \{ \dots i_{-m} \dots i_{-1} i_0 i_1 \dots i_\ell \dots \, | 1 \leq i_k \leq N, \, A_{i_k i_{k+1}} = 1, \forall k \in \mathbb{Z} \}$$

- double sided shift operator T on subshift of finite type space $\mathcal{S}_A$
- consider the case where the elementary matrix A is the symmetric $2g \times 2g$ matrix with $A_{ij} = 0$ for $|i - j| = g$ and $A_{ij} = 1$ otherwise
- reduced words condition on the generators of the Schottky group
- space $\mathcal{S}$ identified with subshift of finite type $\mathcal{S}_A$ with symmetric $2g \times 2g$ matrix $A = (A_{ij})$ with $A_{ij} = 0$ for $|i - j| = g$ and $A_{ij} = 1$ otherwise

Transfer Operator

- Perron–Frobenius theory for the shift T with Ruelle transfer operator
$$(\mathcal{R}_{-sf}\, g)(z) := \sum_{y:\, Ty=z} e^{-sf(y)} g(y),$$
depending on a parameter $s \in \mathbb{C}$ and with the function $f(z) = \log|T'(z)|$
- analog of the Gauss–Kuzmin operator for continued fractions
- for $s = \delta_H$, Hausdorff dimension of limit set, the operator $\mathcal{R} := \mathcal{R}_{-\delta f}$, with $f(z) = \log|T'(z)|$ is the Perron–Frobenius operator of T, adjoint of composition with T,
$$\int_{\Lambda_\Gamma} h(x) \mathcal{R} f(x)\, d\mu(x) = \int_{\Lambda_\Gamma} h(Tx)\, f(x)\, d\mu(x),$$
for all $h, f \in L^2(\Lambda_\Gamma, \mu)$ and δ_H-Hausdorff measure

- this gives

$$\int_{\Lambda_\Gamma} h(Tx)\, f(x)\, d\mu(x) = \sum_i \int_{\Lambda_\Gamma(g_i^{-1})} h(g_i x) f(x)\, d\mu(x)$$

$$= \sum_i \int_{\Lambda_\Gamma} h(g_i x)\, \chi_{g_i^{-1}}(x)\, f(x)\, d\mu(x),$$

- $\chi_\gamma(x)$ characteristic function of cylinder set $\Lambda_\Gamma(\gamma)$
- then get

$$= \int_{\Lambda_\Gamma} h(x) \sum_i A_{ij} \chi_{g_i^{-1}}(x)\, f(g_i^{-1}(x))\, |g_i'(g_i^{-1}(x))|^{-\delta_H}\, d\mu(x),$$

sum is over all admissible i's, with $g_i \neq g_j$, where $x = g_j a_1 \dots a_n \dots$.

• M. Pollicott, *Kleinian groups, Laplacian on forms and currents at infinity*, Proc. Amer. Math. Soc. 110 (1990) 269–279.

- Banach space $(\mathbb{V}, \| \cdot \|_{\mathbb{V}})$ of functions on Λ_Γ with $\|f\|_\infty \leq \|f\|_{\mathbb{V}}$ for all $f \in \mathbb{V}$, and $\mathbb{V} \cap C(\Lambda_\Gamma)$ dense in $C(\Lambda_\Gamma)$
- on this space $\mathcal{R}$ bounded with spectral radius $r(\mathcal{R}) = 1$
- point $\lambda = 1$ simple eigenvalue, with (normalized) eigenfunction density of the unique T-invariant measure on Λ_Γ

- $\mathcal{B}_q$ a function space of q–forms on $\mathcal{U} \subset \mathbb{P}^1(\mathbb{C})$, with $\mathcal{U} = \cup_i D_i^{\pm}$, with analytic coefficients and uniformly bounded
- $\mathcal{R}_s^{(q)}$ the Ruelle transfer operator acting on $\mathcal{B}_q$
- case $q = 0$, for $s = \delta_H$ the Hausdorff dimension of Λ_Γ, the operator $\mathcal{R}_{\delta_H}^{(0)}$ has top eigenvalue 1
- simple eigenvalue, with a non–negative eigenfunction which is the density of the invariant measure
- for $\mathrm{Re}(s) >> 0$ the operators $\mathcal{R}_s^{(q)}$ are nuclear
- $\{\lambda_{i,q}(s)\}_i$ eigenvalues of $\mathcal{R}_s^{(q)}$

Ruelle Zeta Function

- $\{\lambda_{i,q}(s)\}_i$ the eigenvalues of $\mathcal{R}_s^{(q)}$:

$$P_q(s) = \det(1 - \mathcal{R}_s^{(q)}) = \prod_i (1 - \lambda_{i,q}(s))$$

- Ruelle Zeta Function

$$Z_\Gamma(s) := \frac{P_1(s)}{P_0(s) P_2(s)}$$

- (Pollicott) relation to Selberg zeta function of hyperbolic handlebody
- Selberg zeta for handlebody:

$$Z(s) = \prod_{\gamma \in \Gamma} (1 - N(\gamma)^{-s})$$

for $N(\gamma) = \exp(\ell_\gamma)$ length of the corresponding primitive closed geodesic (equivalently $N(\gamma)$ norm of the derivative of γ at the repelling fixed point $z^+(\gamma) \in \mathbb{P}^1(\mathbb{C})$)

Patterson–Sullivan measure on the limit set

- Poincaré series of a Schottky group

$$P_\Gamma(x_0, s) := \sum_{\gamma\in\Gamma} e^{-s\, d(x_0,\gamma(x_0))}$$

  base point $x_0 \in \mathbb{H}^3$, hyperbolic distance
- Poincaré series is convergent for $s > \delta_H$ Hausdorff dimension of limit set
- delta measures on Γ orbits

$$\mu_x^s := \frac{1}{P_\Gamma(x_0, s)} e^{-s\, d(x,\gamma(x_0))} \delta_{\gamma(x_0)}$$

  with δ_x Dirac delta measure, then take weak limit for $s \to \delta_H$
- transformation property $\gamma^*\mu_x = \mu_{\gamma^{-1}x}$

- Busemann cocycle: $\xi \in \mathbb{P}^1(\mathbb{C})$

$$B_\xi(x, x') := \lim_{u \to \xi} (d(x, u) - d(u, x'))$$

a smooth function expressible in terms of Poisson kernel

- main property of Busemann cocycle

$$e^{-B_\xi(\gamma^{-1}x_0, x_0)} = |\gamma'(\xi)|_{S^2}$$

norm in the spherical metric on $S^2 = \mathbb{P}^1(\mathbb{C})$ (Fubini-Study metric)

- get induced Patterson–Sullivan measure on limit set Λ_Γ with

$$\int_{\Lambda_\Gamma} f \, d\mu_{PS} = \int_{\Lambda_\Gamma} (f \circ \gamma) \, |\gamma'(z)|_{S^2}^{\delta_H} d\mu$$

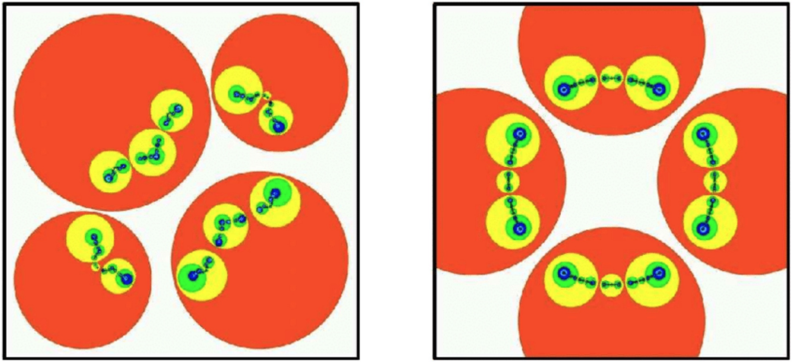
Quasi-circles for Schottky groups

- Γ be a Kleinian group acting on $\mathbb{P}^1(\mathbb{C})$
- $\Omega \subset \mathbb{P}^1(\mathbb{C})$ a Γ-invariant domain
- subset $\Omega_0 \subset \Omega$ is Γ-stable if, for every $\gamma \in \Gamma$, either $\gamma(\Omega_0) = \Omega_0$ or $\gamma(\Omega_0) \cap \Omega_0 = \emptyset$
- Γ-stabilizer of Ω_0 is the subgroup Γ_0 of those $\gamma \in \Gamma$ such that $\gamma(\Omega_0) = \Omega_0$
- quotient map $\pi_\Gamma : \Omega \to \Gamma \backslash \Omega$
- Γ Kleinian group, a quasi-circle for Γ is a Jordan curve C in $\mathbb{P}^1(\mathbb{C})$ invariant under the action of Γ
- a quasi-circle for Γ contains the limit set Λ_Γ

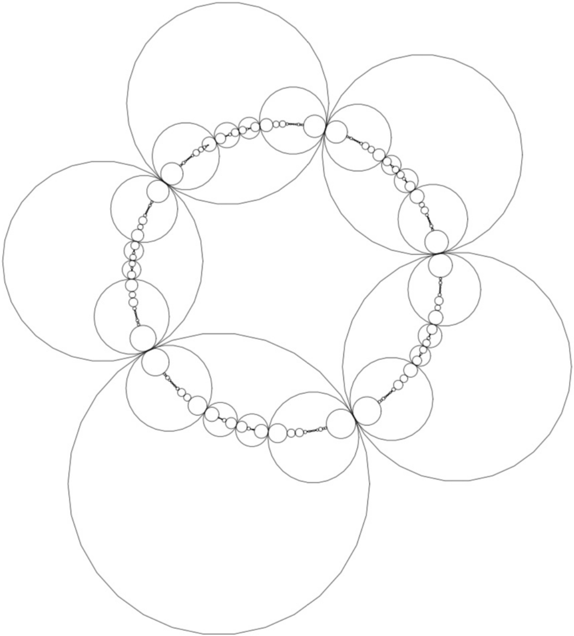
Bowen construction of quasi-circles for Schottky groups

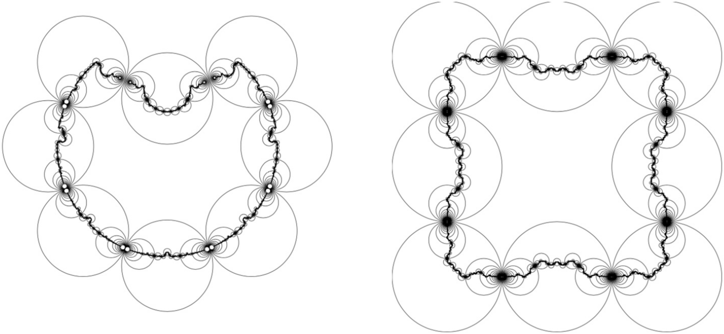
- $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ be a Schottky group of rank $g \geq 2$
- choice of set of generators $\{g_i\}_{i=1}^{g}$ for Γ determines $2g$ Jordan curves C_i, $i = 1 \ldots 2g$ in $\mathbb{P}^1(\mathbb{C})$ with pairwise disjoint interiors D_i such that, if we write $g_{i+g} = g_i^{-1}$ for $i = 1 \ldots g$, the fractional linear transformation g_i maps the interior of C_i to the exterior of $C_{i+g \mod 2g}$
- fix a choice of $2g$ pairs of points $\rho_i^{\pm}$ on the curves C_i so that g_i maps the two points $\rho_i^{\pm}$ to the two points $\rho_{i+g \mod 2g}^{\mp}$
- choose a collection $\mathcal{C}_0$ of pairwise disjoint oriented arcs in $\mathbb{P}^1(\mathbb{C})$ that do not intersect interior of the D_i
- can choose $\mathcal{C}_0$ so that oriented boundary of $\mathcal{C}_0$ as a 1-chain is given by $\partial \mathcal{C}_0 = \sum_i \rho_i^+ - \sum_i \rho_i^-$
- then quasi-circle for Γ given by curve

$$C := \Lambda_\Gamma \cup \bigcup_{\gamma \in \Gamma} \gamma \mathcal{C}_0$$



Limit sets and quasi-circles (Mumford)





Circles and quasi-circles (Seade)

- image under quotient map of collection of points $\{\gamma\rho_i^{\pm}\}_{\gamma\in\Gamma, i=1\dots 2g}$ gives two points on each curve a_i
- image of $C\cap\Omega_\Gamma$ consists of a collection $\hat{C}$ of pairwise disjoint arcs on $X_{\mathbb{C}}$ connecting these $2g$ points
- cutting the surface $X_{\mathbb{C}}$ along $\hat{C}$ obtain two surfaces X_i, $i=1,2$, with boundary $\partial X_i = \hat{C}$
- C is Γ-invariant, so the two connected components Ω_i, $i=1,2$, of $\mathbb{P}^1(\mathbb{C})\backslash C$ are Γ-stable
- Γ_i denote the Γ-stabilizer of Ω_i
- have $\Gamma_1=\Gamma_2$: suppose there is $\gamma\in\Gamma_1$ but $\gamma\notin\Gamma_2$, then $\gamma(\mathbb{P}^1(\mathbb{C}))\subset\Omega_1\cup C$, so image $\gamma(\mathbb{P}^1(\mathbb{C}))$ contractible in $\mathbb{P}^1(\mathbb{C})$, thus γ has topological degree zero, but an orientation preserving fractional linear transformation has topological degree one
- $\tilde{\Gamma}$ the Γ-stabilizer: $\tilde{\Gamma}=\Gamma_1=\Gamma_2$

Bers simultaneous uniformization

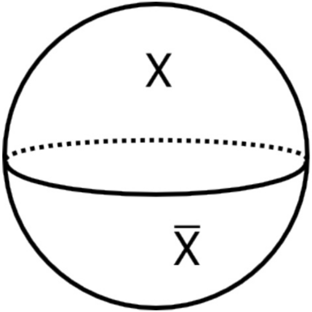
- quasi-circle C is a Jordan curve in $\mathbb{P}^1(\mathbb{C})$: by Riemann mapping theorem there exist conformal maps α_i of the two connected components Ω_i to the two hemispheres U_i of $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$

$$\alpha_i : \Omega_i \xrightarrow{\simeq} U_i \qquad U_1 \cup U_2 = \mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$$

- consider the two groups

$$G_i := \{\alpha_i \gamma \alpha_i^{-1} : \gamma \in \tilde{\Gamma}\}$$

- isomorphic as groups to $\tilde{\Gamma}$, $G_i \simeq \tilde{\Gamma}$
- the G_i preserve the upper/lower hemisphere U_i, hence they are Fuchsian groups, $G_i \subset \mathrm{PSL}(2,\mathbb{R})$
- group $\tilde{\Gamma} \subset \Gamma$ is itself discrete purely loxodromic subgroup of $\mathrm{PSL}(2,\mathbb{C})$ isomorphic to free group, hence a Schottky group, so the G_i are Fuchsian Schottky groups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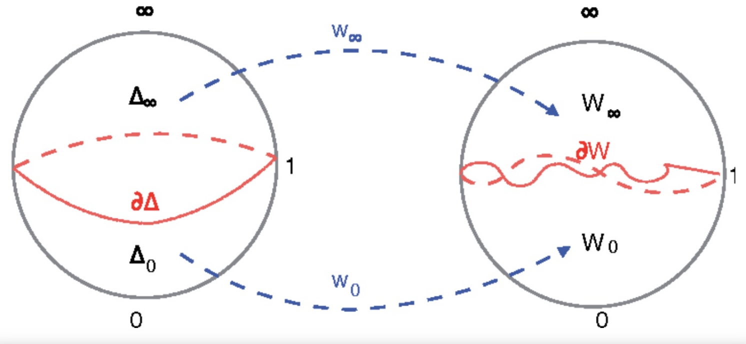
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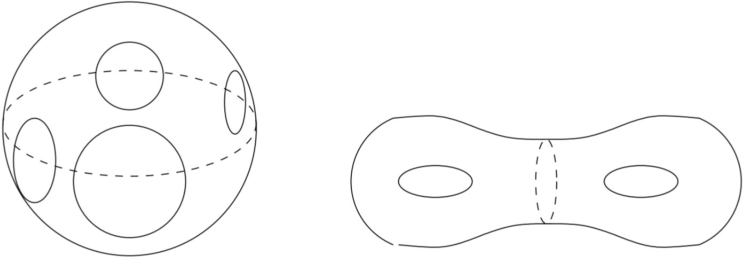
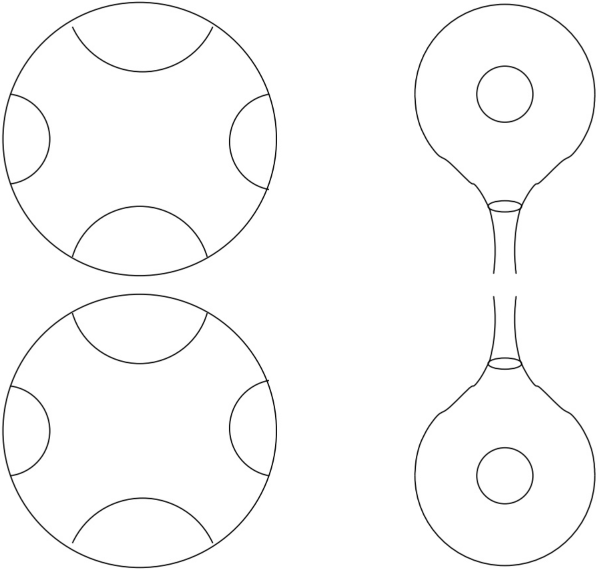




Geodesics in the handlebody

- Geodesics in $\mathfrak{X}_\Gamma$ can be lifted to geodesics in $\mathbb{H}^3$ with ends on $\mathbb{P}^1(\mathbb{C})$
- geodesics with one or both ends on $\Omega_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ correspond to geodesics in $\mathfrak{X}_\Gamma$ that reach the boundary at infinity $X_{\mathbb{C}} = \Gamma\backslash\Omega_\Gamma$ in infinite time
- geodesics in $\mathbb{H}^3$ with both ends on $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ project in the quotient to geodesics contained in the convex core of $\mathfrak{X}_\Gamma$

$$\mathfrak{C}_\Gamma = \Gamma\backslash \text{ Hull } (\Lambda_\Gamma)$$

- trapped region inside the handlebody $\mathfrak{X}_\Gamma$
- higher genus Euclidean BTZ black holes (Krasnov)
- Schottky group Γ is geometrically finite: $\mathfrak{C}_\Gamma$ bounded region inside $\mathfrak{X}_\Gamma$
- geodesics in $\mathfrak{X}_\Gamma$ that lift to geodesics in $\mathbb{H}^3$ with both ends on Λ_Γ are called bounded geodesics

- $\Xi \subset \mathfrak{X}_\Gamma$ image under quotient map of all geodesics in $\mathbb{H}^3$ with endpoints on $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$, with induced topology
- $\Xi_c \subset \Xi$ the image in $\mathfrak{X}_\Gamma$ of all geodesics in $\mathbb{H}^3$ with endpoints of the form $\{z^-(h), z^+(h)\}$ for some primitive $h \in \Gamma$ with $z^\pm(h)$ are the attractive and repelling fixed points of h
- h is primitive in Γ if it is not a power of some other element of Γ
- infinite tangle of bounded geodesics: $\tilde{\Xi}$ orientation double cover of Ξ
- tangle of primitive closed geodesics $\tilde{\Xi}_c$ orientation double cover of Ξ_c .
- Ξ is orientable so $\tilde{\Xi} \cong \Xi \times \mathbb{Z}/2$, second coordinate is choice of an orientation on each geodesic

- $L_{\{a,b\}}$ denote the geodesic in $\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}^+$ with endpoints $\{a, b\}$
- parameterization for the geodesic $L_{\{a,b\}}$:

$$\tilde{L}_{\{a,b\}}(s) = \left(\frac{ae^s + be^{-s}}{e^s + e^{-s}}, \frac{|a-b|}{e^s + e^{-s}} \right) \qquad s \in \mathbb{R}$$

- parameter s parameterization by arc length in

$$\tilde{\Xi} = \{\pi_\Gamma(\tilde{L}_{\{a,b\}}(s)) : \ s \in \mathbb{R}, \ (a,b) \in (\Lambda_\Gamma \times \Lambda_\Gamma)^0\},$$

$$(\Lambda_\Gamma \times \Lambda_\Gamma)^0 := (\Lambda_\Gamma \times \Lambda_\Gamma) \smallsetminus \Delta$$

  denotes the complement of the diagonal in $\Lambda_\Gamma \times \Lambda_\Gamma$

- $\mathbb{Z}/2$ involution on $\tilde{\Xi}$ with Ξ quotient corresponds to involution exchanging the two factors in $\Lambda_\Gamma \times \Lambda_\Gamma$

Coding of bounded geodesics in the handlebody

- because Γ free group of loxodromic elements, coding of geodesics in $\mathfrak{X}_\Gamma$ in terms of the dynamical system $(\mathcal{S}, T)$ particularly simple
- denote by $\mathcal{S}^p \subset \mathcal{S}$ set of periodic reduced sequences in the g_i, set of periodic points of the shift T
- define

$$\hat{\Xi} := \left\{ \pi_\Gamma(L_{\{a,b\}}) : (a,b) \in (\Lambda_\Gamma \times \Lambda_\Gamma)^0 \right\}$$

$$\hat{\Xi}_c := \left\{ \pi_\Gamma(L_{\{z^+(h),z^-(h)\}}) : h \in \Gamma \setminus id \right\},$$

  with $L_{\{a,b\}}$ geodesic in $\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}^+$ with endpoints $\{a, b\}$
- correspondence

$$\mathcal{L}_c : w\overline{a_0 \dots a_N} \mapsto \pi_\Gamma(L_{\{w\, z^+(a_0 \dots a_N), w\, z^-(a_0 \dots a_N)\}})$$

  induces a bijection between $\mathcal{S}^p/T$ and $\hat{\Xi}_c$

- because every closed geodesic in $\tilde{\Xi}_c$ is of the form

$$\pi_\Gamma(L_{\{z^+(a_0 \ldots a_N), z^-(a_0 \ldots a_N)\}})$$

for some reduced sequence $a_0 \ldots a_N$ with $a_N \neq a_0^{-1}$

- two elements $\overline{a_0, a_2, \ldots, a_N}$ and $\overline{b_0, b_1, \ldots, b_M}$ of $\mathcal{S}^p$ represent the same primitive closed geodesic if $h_a = a_0 a_2 \cdots a_N$ and $h_b = b_0 b_1 \cdots b_M$ are conjugate in Γ, $h_a = g h_b g^{-1}$, by an element $g = c_1 c_2 \cdots c_k$
- this implies $c_k = b_0^{-1}$, $c_{k-1} = b_1^{-1}$, etc so same orbit of action of T

$$\overline{b_0, b_1, \ldots, b_M} = T^{N_0}(\overline{a_0, a_1, \ldots, a_N}),$$

Smale Space

- pair $(\mathcal{S}, T)$ space and homeomorphism is a Smale space if locally $\mathcal{S}$ can be decomposed as the product of expanding and contracting directions for T
  1. For every point $x \in \mathcal{S}$ there exist subsets $W^s(x)$ and $W^u(x)$ of $\mathcal{S}$, such that $W^s(x) \times W^u(x)$ is homeomorphic to a neighborhood of x.
  2. The map T is contracting on $W^s(x)$ and expanding on $W^u(x)$, and $W^s(Tx)$ and $T(W^s(x))$ agree in some neighborhood of x, and so do $W^u(Tx)$ and $T(W^u(x))$

- map $Q : \mathcal{S} \to \Lambda_\Gamma \times \Lambda_\Gamma$

$$Q(\ldots a_{-m} \ldots a_{-1} a_0 a_1 \ldots a_\ell \ldots) =$$

$$\left(Z(a_{-1}^{-1} a_{-2}^{-1} \ldots a_{-m}^{-1} \ldots), \, Z(a_0 a_1 a_2 \ldots a_\ell \ldots) \right)$$

  with $Z(a_0 \ldots a_\ell \ldots) = \lim_{\ell \to \infty} (a_0 \ldots a_\ell) x_0$ is embedding of $\mathcal{S}$ in Cartesian product $\Lambda_\Gamma \times \Lambda_\Gamma$
- map $Q : \mathcal{S} \to \Lambda_\Gamma \times \Lambda_\Gamma$ descends to a homeomorphism of the quotients

$$\bar{Q} : \mathcal{S}/T \stackrel{\simeq}{\to} \Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$$

- two-sided shift operator T on $\mathcal{S}$ decomposes $\mathcal{S}$ in a product of expanding and contracting directions, so $(\mathcal{S}, T)$ is a Smale space

- sets

$$W^u(x) = \cup_{\ell \in \mathbb{Z}} W^u(x, \ell),$$

$$W^u(x, \ell) := \{ y \in \mathcal{S} | x_k = y_k, k \leq \ell \}$$

$$W^s(x) = \cup_{\ell \in \mathbb{Z}} W^s(x, \ell).$$

$$W^s(x, \ell) := \{ y \in \mathcal{S} | x_k = y_k, k \geq \ell \}$$

give expanding and contracting directions

Suspension flow

- mapping torus (suspension flow) of dynamical system $(\mathcal{S}, T)$ defined as

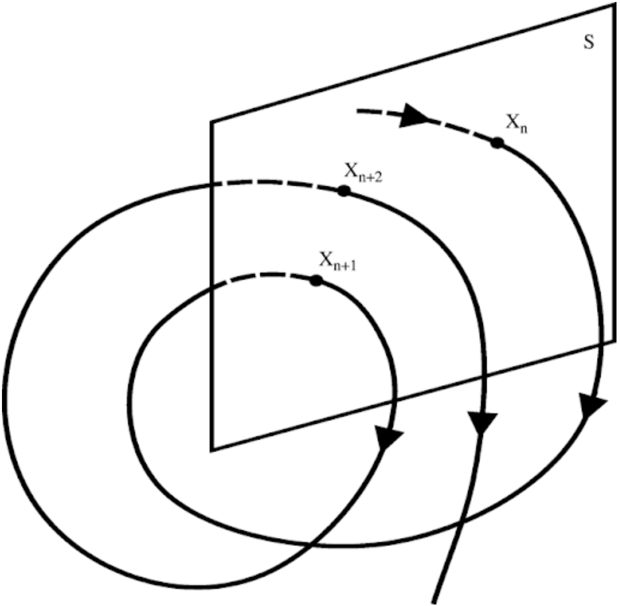
$$\mathcal{S}_T := \mathcal{S} \times [0,1]/(x,0) \sim (Tx, 1)$$

- $\tilde{Q} : \mathcal{S}_T \to \tilde{\Xi}$

$$\tilde{Q}([x,t]) = \pi_\Gamma \tilde{L}_{\{a,b\}}(s(x,t)),$$

  with $(a,b) = Q(x)$ in $(\Lambda_\Gamma \times \Lambda_\Gamma)^0$
- map $\tilde{Q}$ gives geodesic in handlebody, but parameterization by the time coordinate t on mapping torus, in general, not geodesic arc length s
- $s(x,t)$ such that geodesic $\tilde{L}_{\{a,b\}}(s(x,t))$ in $\mathbb{H}^3$ crosses fundamental domain for the action of Γ in time $t \in [0,1]$
- map $\tilde{Q} : \mathcal{S}_T \to \tilde{\Xi}$ continuous surjection, bijection away from the intersection points of different geodesics in $\tilde{\Xi}$

Poincaré Return Map



Cohomology and homology of $\mathcal{S}_T$

- cohomology $H^1(\mathcal{S}_T, \mathbb{Z})$ satisfies:
  1. identification $H^1(\mathcal{S}_T, \mathbb{Z})$ with co–invariants of the action of T
  2. $H^1(\mathcal{S}_T, \mathbb{Z})$ has a filtration by free abelian groups $F_0 \hookrightarrow F_1 \hookrightarrow \cdots F_n \hookrightarrow \cdots$, with $\mathrm{rank} F_0 = 2g$ and $\mathrm{rank} F_n = 2g(2g-1)^{n-1}(2g-2)+1$, for $n \geq 1$, so that

$$H^1(\mathcal{S}_T, \mathbb{Z}) = \varinjlim_n F_n$$

- invariants and coinvariants
- $\mathrm{C}(\mathcal{S}, \mathbb{Z})$ continuous functions from $\mathcal{S}$ to the integers, abelian group generated by characteristic functions of clopen sets of $\mathcal{S}$
- invariants: $\mathrm{C}(\mathcal{S}, \mathbb{Z})^T := \{ f \in \mathrm{C}(\mathcal{S}, \mathbb{Z}) \,|\, f - f \circ T = 0 \}$
- coinvariants: $\mathrm{C}(\mathcal{S}, \mathbb{Z})_T := \mathrm{C}(\mathcal{S}, \mathbb{Z}) / \mathrm{B}(\mathcal{S}, \mathbb{Z})$, with $\mathrm{B}(\mathcal{S}, \mathbb{Z}) := \{ f - f \circ T \,|\, f \in \mathrm{C}(\mathcal{S}, \mathbb{Z}) \}$

- can identify $H^1(\mathcal{S}_T, \mathbb{Z})$ with the group of homotopy classes of continuous maps of $\mathcal{S}_T$ to the circle $S^1 = B(1, \mathbb{Z})$
- isomorphism with coinvariants

  $$\mathrm{C}(\mathcal{S}, \mathbb{Z})_T \cong H^1(\mathcal{S}_T, \mathbb{Z})$$

  seen explicitly as $f \mapsto [\exp(2\pi i t f(x))]$ for $f \in \mathrm{C}(\mathcal{S}, \mathbb{Z})$ and $[\cdot]$ homotopy class
- map is well defined on equivalence class of f mod $B(\mathcal{S}, \mathbb{Z})$ since, for $f - h + h \circ T$ function

  $$\exp(2\pi i t(f - h + h \circ T)(x)) =$$

  $$\exp(2\pi i t f(x)) \exp(2\pi i((1-t)h(x) + t h(T(x))))$$

  since h integer valued, but $\exp(2\pi i((1-t)h(x) + t h(T(x))))$ homotopic to constant function equal to 1

Filtration of $H^1(\mathcal{S}_T, \mathbb{Z})$

- identification

$$\mathrm{C}(\mathcal{S},\mathbb{Z})_T = \mathrm{C}(\mathcal{S},\mathbb{Z})/B(\mathcal{S},\mathbb{Z}) \cong \mathcal{P}/\delta\mathcal{P}$$

  with $\mathcal{P} \subset \mathrm{C}(\mathcal{S},\mathbb{Z})$ set of functions that depend only on future coordinates and $\delta(f) = f - f \circ T$

- because characteristic functions of clopen sets in $\mathcal{S}$ depend only on finitely many coordinates, so any function in $\mathrm{C}(\mathcal{S},\mathbb{Z})$, when composed with a sufficiently high power of T becomes a function only of the future coordinates
- $\mathcal{P}$ can be identified with $\mathrm{C}(\mathcal{S}^+,\mathbb{Z})$ viewed as submodule of $\mathrm{C}(\mathcal{S},\mathbb{Z})$
- basis of clopen sets for $\mathcal{S}^+$ cylinder sets $\mathcal{S}^+(w) \subset \mathcal{S}^+$, with $w = a_0 \dots a_N$ reduced words in the g_i

- so $\mathcal{P}$ has filtration $\mathcal{P} = \cup_{n=0}^{\infty} \mathcal{P}_n$, with $\mathcal{P}_n$ generated by the characteristic functions of $\mathcal{S}^+(w)$ with w of length at most $n+1$
- because of relations between these, $\mathcal{P}_n$ free abelian group generated by the characteristic functions of $\mathcal{S}^+(w)$ with w of length exactly $n+1$
- number of these reduced words in words is $\mathrm{rank} \mathcal{P}_n = 2g(2g-1)^n$
- map δ satisfies $\delta : \mathcal{P}_n \to \mathcal{P}_{n+1}$, with a 1-dimensional kernel given by the constant functions

$$(\delta f)(a_0 \ldots a_n a_{n+1}) = f(a_0 \ldots a_n) - f(a_1 \ldots a_{n+1})$$

- resulting quotients

$$F_n = \mathcal{P}_n / \delta \mathcal{P}_{n-1}$$

$$\mathrm{rank} F_n = 2g(2g-1)^{n-1}(2g-2) + 1$$

with $F_0 \cong \mathcal{P}_0$ of rank $2g$

- $F_n \hookrightarrow F_{n+1}$ induced by the inclusion $\mathcal{P}_n \subset \mathcal{P}_{n+1}$
- $\mathcal{P}/\delta\mathcal{P}$ is the direct limit of the F_n under inclusions

$$H^1(\mathcal{S}_T, \mathbb{Z}) = \varinjlim_n F_n$$

- description as coinvariants also gives identification of $H^1(\mathcal{S}_T, \mathbb{Z})$ with the K_0-group of the crossed product C^*-algebra for the action of T on $\mathcal{S}$,

$$H^1(\mathcal{S}_T, \mathbb{Z}) \cong K_0(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z})$$

Multipliers of Schottky groups

- each $\gamma \neq \mathrm{Id}$ has two fixed points, one repelling $r(\gamma)$ one attracting $a(\gamma)$ for orbits $\gamma^{\mathbb{Z}} z_0$ with $z_0 \in \Omega_\Gamma$
- from trace (trace/det) classification see γ conjugate to a transformation $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \neq 1$ (can take $|\lambda| > 1$ for generators γ_i and $|\lambda| < 1$ for inverses γ_i^{-1}
- $\lambda = \lambda(\gamma)$ multiplier of $\gamma \in \Gamma$

Schottky Space

- marked Schottky group a Schottky group Γ together with a set of generators $\{\gamma_1, \dots, \gamma_g\}$
- equivalent marked Schottky groups $(\Gamma, \gamma_1, \dots, \gamma_g)$ and $(\Gamma', \gamma'_1, \dots, \gamma'_g)$ if $\exists \delta \in \mathrm{PSL}_2(\mathbb{C})$ such that $\delta \gamma_i \delta^{-1} = \gamma'_i$
- Schottky space $\mathcal{S}_g$: set of equivalence classes of marked Schottky groups of rank g
- genus one: $\mathcal{S}_1 = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$ by mapping generator γ to $1/\lambda(\gamma)$
- genus $g \geq 2$: assign to set $\{\gamma_1, \dots, \gamma_g\}$ the $3g$-tuple $(r(\gamma_1), a(\gamma_1), \lambda(\gamma_1), \dots, r(\gamma_g), a(\gamma_g), \lambda(\gamma_g)) \in \mathbb{P}^1(\mathbb{C})^{3g}$: these uniquely determine the marked Schottky group
- up to conjugacy by an element $\delta \in \mathrm{PSL}_2(\mathbb{C})$ can always set $r(\gamma_1), a(\gamma_1), r(\gamma_2)$ at $0, \infty, 1$ respectively, so left with

$$(\lambda_1, a_2, \lambda_2, r_3, a_3, \lambda_3, \dots, r_g, a_g, \lambda_g) \in \mathbb{P}^1(\mathbb{C})^{3g-3}$$

- Schottky space $\mathcal{S}_g$ embeds as open subset of $\mathbb{C}^{3g-3}$
- because representing Schottky groups in terms of $2g$ Jordan curves pairwise mapped by the γ_i, the curves depend continuously on the Schottky parameters $(r(\gamma_i), a(\gamma_i), \lambda(\gamma_i))$ (can be seen directly when discs and general non-classical case is continous deformation of classical case)
- group $\mathrm{Aut}(F^g)$ automorphisms of free group on g generators acts on $\mathcal{S}_g$
- think of $\mathcal{S}_g$ equivalently as set of all possible embeddings $\sigma : F^g \to \mathrm{PSL}_2(\mathbb{C})$ such that the image is a Schottky group, up to conjugation in $\mathrm{PSL}_2(\mathbb{C})$
- then action of $\mathrm{Aut}(F^g)$ by precomposition and inner automorphisms of F^g act trivially so action of $\mathrm{Out}(F^g) = \mathrm{Aut}(F^g)/\mathrm{Inn}(F^g)$

Schottky Space and Teichmüller Space

- $\mathcal{T}_g$ Teichmüller space classifies marked Riemann surfaces of genus g
- marking given by an assigned group isomorphism $\pi_g \to \pi_1(X) = G$
- for $g \geq 2$ uniformization $X = G\backslash\mathbb{H}^2$ with $G \subset \mathrm{PSL}_2(\mathbb{R})$ Fuchsian group $G \simeq \pi_g = \langle a_i, b_i \mid \prod_{i=1}^{g}[a_i, b_i] = 1\rangle$ fundamental group
- action of $\mathcal{M}_g = \mathrm{Out}(\pi_g)$
- map from Teichmüller to Schottky space: fix generators $\{a_i, b_i\}_{i=1}^{g}$ of π_g, take normal subgroup $\mathcal{N} \subset \pi_g$ generated by $\{a_i\}_{i=1}^{g}$, isomorphism $F^g \to \pi_g/\mathcal{N}$ free group rank g

- quotient map $\pi_g \to F^g$
- diagram

$$\begin{tikzcd}
\pi_g \arrow[r, "\tau"] \arrow[d, "\alpha"'] & \pi_1(X) \arrow[r, "\sim"] \arrow[d, "\alpha_\tau"] & \mathrm{Deck}(\tilde{X}/X) \arrow[r, hook] \arrow[d] & \mathrm{PSL}_2(\mathbb{R}) \\
F_g \arrow[r, "\sigma"'] & \pi_1(X)/N_\tau \arrow[r, "\sim"] & \Gamma \arrow[r, hook] & \mathrm{PSL}_2(\mathbb{C})
\end{tikzcd}$$

- obtain map $\mathcal{T}_g \to \mathcal{S}_g$ by $(X, \tau) \mapsto (X, \sigma)$
- Reference: Frank Herrlich, *Schottky space and Teichmüller disks*, Handbook of group actions. Vol. I, 289–308, Adv. Lect. Math. (ALM), 31, Int. Press, 2015