Continued Fractions and Modular Symbols
Introduction to Fractal Geometry and Chaos

Matilde Marcolli

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Some References

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- Paul E. Gunnels, *Modular Symbols* `http://www.uncg.edu/mat/numbertheory/summerschool/2017.html`

Continued Fractions

- $k_1, \dots, k_n \in \mathbb{N}$ and $n \geq 1$

$$[k_1, \dots, k_n] := \frac{1}{k_1 + \frac{1}{k_2 + \dots \frac{1}{k_n}}} = \frac{P_n(k_1, \dots, k_n)}{Q_n(k_1, \dots, k_n)}$$

P_n, Q_n polynomials with integer coefficients calculated inductively from relations

$$Q_{n+1}(k_1, \dots, k_n, k_{n+1}) = k_{n+1} Q_n(k_1, \dots, k_n) + Q_{n-1}(k_1, \dots, k_{n-1}),$$

$$P_n(k_1, \dots, k_n) = Q_{n-1}(k_2, \dots, k_n)$$

put formally $Q_{-1} = 0$, $Q_0 = 1$

- also have the relation

$$[k_1, \ldots, k_{n-1}, k_n + x_n] =$$

$$\frac{P_{n-1}(k_1, \ldots, k_{n-1})\, x_n + P_n(k_1, \ldots, k_n)}{Q_{n-1}(k_1, \ldots, k_{n-1})\, x_n + Q_n(k_1, \ldots, k_n)} = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} (x_n)$$

where action of GL_2 by fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z)$$

- for $\alpha \in (0,1)$ irrational, unique sequence of integers $k_n(\alpha) \geq 1$ such that α is the limit of $[\, k_1(\alpha), \ldots, k_n(\alpha)\,]$ as $n \to \infty$
- unique sequence $x_n(\alpha) \in (0,1)$ such that

$$\alpha = [\, k_1(\alpha), \ldots, k_{n-1}(\alpha), k_n(\alpha) + x_n(\alpha)\,]$$

  for each $n \geq 1$
- rational numbers $\alpha \in \mathbb{Q}$ have a finite continued fraction expansion $\alpha = [\, k_1(\alpha), \ldots, k_n(\alpha)\,]$ for some $n \in \mathbb{N}$ and an ambiguity in the continued fraction representation $[k_1, \ldots, k_n] = [k_1, \ldots, k_n - 1, 1]$ (resolve ambiguity by taking shortest)

- convergents of the continued fractions expansion
  - irrational α

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & k_1(\alpha) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n(\alpha) \end{pmatrix} (x_n(\alpha)).$$

  - successive numerators and denominators

$$p_n(\alpha) := P_n(k_1(\alpha), \ldots, k_n(\alpha)), \; q_n(\alpha) := Q_n(k_1(\alpha), \ldots, k_n(\alpha))$$

  - so that $p_n(\alpha)/q_n(\alpha)$ is the sequence of convergents of α.
  - associated matrix

$$g_n(\alpha) := \begin{pmatrix} p_{n-1}(\alpha) & p_n(\alpha) \\ q_{n-1}(\alpha) & q_n(\alpha) \end{pmatrix}$$

Semigroup of reduced matrices

- describing the set of matrices $g_n(\alpha)$
- define set

$$\mathrm{Red}_n := \left\{ \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n \end{pmatrix} \,|\, k_1, \ldots, k_n \geq 1;\ k_i \in \mathbb{Z} \right\}$$

- semigroup of reduced matrices

$$\mathrm{Red} := \cup_{n \geq 1} \mathrm{Red}_n \subset \mathrm{GL}_2(\mathbb{Z})$$

- reduced matrices in $\mathrm{GL}_2(\mathbb{Z})$

$$\mathrm{Red} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \,|\, 0 \leq a \leq b,\ 0 \leq c \leq d \}$$

- reduced matrices in $\mathrm{GL}_2(\mathbb{Z})$
  - generators of $\mathrm{GL}_2(\mathbb{Z})$

$$\sigma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \rho = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

    relations $(\sigma^{-1}\rho)^2 = (\sigma^{-2}\rho^2)^6 = 1$
  - not free as group but free semigroup generated by σ and ρ
  - the semigroup of reducible matrices is subsemigroup generated by all words in σ, ρ that end in ρ
  - so elements are products of $\sigma^{n-1}\rho = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$
  - for $\gamma \in \mathrm{Red}$ take $\ell(\gamma) =$ number of ρ's in the word
  - every conjugacy class of hyperbolic matrices in $\mathrm{GL}_2(\mathbb{Z})$ contains reduced matrices γ all with the same value of $\ell(\gamma)$

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & n_{\ell(\gamma)} \end{pmatrix}$$

  - number of these representatives is $\ell(\gamma)/k(\gamma)$ with $k(\gamma) \in \mathbb{N}$ largest with $\gamma = \gamma_0^{k(\gamma)}$ for some (primitive) $\gamma_0 \in \mathrm{GL}_2(\mathbb{Z})$

Shift map on the continued fraction expansion

- take points in $[0,1]$ continued fraction $[k_1, \ldots, k_n, \ldots]$ (for rationals continue with zeros)
- map $T : [0,1] \to [0,1]$

$$T : \alpha \mapsto \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right] = \begin{pmatrix} -[1/\alpha] & 1 \\ 1 & 0 \end{pmatrix} (\alpha)$$

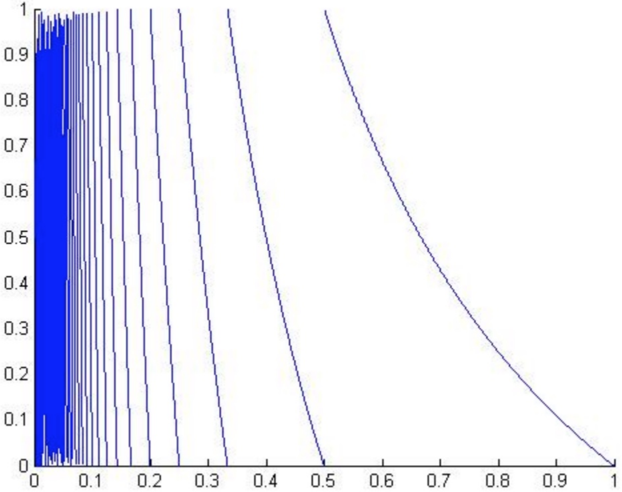
integer part $[1/\alpha]$

- effect of T on continued fraction expansion is one-sided shift

$$T : [k_1, \ldots, k_n, \ldots] \mapsto [k_2, \ldots, k_{n+1}, \ldots]$$

- so considering a shift space on infinitely many letters $k_i \in \mathbb{N}$ and with the dynamical system given by the shift map
- generalization of the usual (Σ_A^+, σ) shift spaces
- invariant measures? (Gauss–Kuzmin problem)

Shift map on the continued fraction expansion



Generalized Gauss–Kuzmin problem

- consider a finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$ so finite coset $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$
- Example: congruence subgroup
  $G = \Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \,|\, c = 0 \mod N \}$
- shift map $T : [0,1] \times \mathbb{P} \to [0,1] \times \mathbb{P}$

$$T : (\alpha, s) \mapsto \left(\frac{1}{\alpha} - \left[\frac{1}{\alpha} \right], \begin{pmatrix} -[1/\alpha] & 1 \\ 1 & 0 \end{pmatrix} s \right)$$

  action of $\mathrm{GL}_2(\mathbb{Z})$ on the left on $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$
- for any $x \in [0,1]$, $s \in \mathbb{P}$, $n \geq 0$ take

$$m_n(x, s) := \lambda(\{\alpha \in (0,1) \,|\, x_n(\alpha) \leq x,\, g_n(\alpha)^{-1}(s_0) = s\})$$

  λ prod of Lebesgue and counting measure, s_0 base point of $\mathbb{P}$

Invariant measure (Gauss-Kuzmin measure)

- transitivity condition: $\mathrm{Red} \cdot s = \mathbb{P}$ for all $s \in \mathbb{P}$ orbits under reduced matrices semigroup
- then limit exists

$$m(x, s) = \lim_{n \to \infty} m_n(x, s)$$

- and equal to

$$m(x, t) = \frac{1}{|\mathbb{P}| \log 2} \log(1 + x)$$

- method of proof: Ruelle transfer operators

Gauss–Kuzmin operator

- consider sets

$$M_n(y,s) := \{\beta \in (0,1) \,|\, x_n(\beta) \leq y,\; g_n(\beta)^{-1} s_0 = s\}$$

- inductive relation (neglecting the rationals that have measure zero)

$$M_{n+1}(x,s) = \coprod_{k=1}^{\infty} \left\{ M_n\left(\frac{1}{k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(s)\right) \setminus M_n\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(s)\right) \right\}.$$

- so measures recursively satisfy

$$m_{n+1}(x,s) = \sum_{k=1}^{\infty} \left\{ m_n\left(\frac{1}{k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(s)\right) - m_n\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(s)\right) \right\}$$

- differentiation in x variable for equation for densities

$$m'_{n+1}(x,s) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} m'_n\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(s)\right) =: (\mathcal{L} m'_n)(x,s)$$

- linear operator

$$\mathcal{L} f(x,s) := \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} (s) \right)$$

- $\beta = 1$ value of family of linear operators

$$\mathcal{L}_\beta f(x,s) := \sum_{k=1}^{\infty} \frac{1}{(x+k)^{2\beta}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} (s) \right)$$

- transfer operator because adjoint to shift T (with appropriate spaces of functions, discuss later)

$$\int_{[0,1]\times\mathbb{P}} f \cdot \mathcal{L} h \, d\lambda = \int_{[0,1]\times\mathbb{P}} (f \circ T) \cdot h \, d\lambda$$

- the iteration property of densities $m'_{n+1} = \mathcal{L} m'_n$ and transfer operator property show the problem of limit of the m_n densities becomes a fixed point problem for $\mathcal{L}$ and determined a T-invariant density

- Comment on transitivity hypothesis
  - we will be assuming $\mathrm{Red} \cdot s = \mathbb{P}$ for all $s \in \mathbb{P}$, how easy is this to satisfy?
  - Example: for G subgroup generated by lift of $\Gamma_0(N)$ (and sign change) already $\mathrm{Red}_3(t) = \mathbb{P}$
  - to see this: elements of $\mathbb{P}$ formal quotients of residues mod N represented by pairwise prime integers ("proj space" of $\mathbb{Z}/N\mathbb{Z}$)
  - three groups of such points: (i) $\{u/1 \,|\, (u, N) = 1\}$, (ii) $\{du/1 \,|\, d/N, d > 1, (u, N) = 1\}$ and (iii) $\{1/du\} \,|\, d/N, d > 1, (u, N) = 1$
  - "one step" means $s' \in \mathrm{Red}_1(s)$: from any element of I ∪ II obtain in one step all I ∪ III and from any element of III obtain in one step an element of II

Analysis of the Gauss–Kuzmin operator

- enlarging the domain: $\mathbb{D} \times \mathbb{P}$ with $\mathbb{D} := \{z \in \mathbb{C} \,|\, |z-1| < 3/2\}$ with sheets $\mathbb{D} \times \{s\}$
- map $z \mapsto (z+k)^{-1}$ transforms $\mathbb{D}$ strictly into itself
- Banach space $B_{\mathbb{C}} := V_{\mathbb{C}}(\mathbb{D} \times \mathbb{P})$ functions holomorphic on each sheet continuous on boundary, sup norm
- real Banach space of such functions that are real at real points of each sheet
- these stable with respect to $\mathcal{L}_\beta$ for real $\beta > 1/2$
- $B_{\mathbb{C}}$ also stable with respect to $\mathcal{L}_\beta$ with $\Re(\beta) > 1/2$

Positive Cone

- $K \subset B$ cone of functions taking non-negative values at real points of each sheet
- we have $K \cap -K = \{0\}$ (K is proper) because a nonzero analytic function cannot vanish on an interval
- also have $B = K - K$ (K is reproducing) because $f = (f + r) - r$, and if r is large and positive, $f + r, r \in K$
- functions positive at all real points of all sheets form the interior of K
- partial ordering from cone: $f \leq g$ if $f - g \in K$
- operator $\mathcal{L}_\beta$ is K-positive: i.e. $\mathcal{L}_\beta(K) \subset K$

Upper and Lower Bounds

- assume $\mathbb{P}$ contains no proper invariant subsets with respect Red
- then for each nonzero $f \in K$ there exist two real positive constants $a, b > 0$ and an integer $p \geq 1$ such that

$$a \leq \mathcal{L}_\beta^p f \leq b$$

- upper bound just from sup bound on functions in K and K-positivity of $\mathcal{L}_\beta$
- suppose lower bound is zero: for each $p \geq 1$, $\mathcal{L}_\beta^p f$ vanishes at some point (x_p, s_p) with x_p real in the closure of $\mathbb{D}$
- but all summands are non-negative at real points when $f \in K$, so f itself must vanish at all points in $\cup_{p\geq 1}\mathrm{Red}_p(x_p, s_p)$
- for any $p \in \mathbb{N}$, by transitivity assumption $\mathcal{L}_\beta^p f$ has real zeroes on all sheets
- for each $s \in \mathbb{P}$ there is a sequence of integers $q_n \to \infty$ and real points y_n in $\mathbb{D}$ such that $f(x, s)$ vanishes at all $x \in \cup_n \mathrm{Red}_{q_n}(y_n)$
- intersection with $[0, 1]$ is dense in $[0, 1]$ but f holomorphic!

Nuclear Operator

- nuclear operator: compact operator with a finite trace (on Hilbert space trace class ops)
- the operator $\mathcal{L}_\beta : B_{\mathbb{C}} \to B_{\mathbb{C}}$ is a nuclear operator for $\beta > 1/2$
- Note: operator is a series $\mathcal{L}_\beta = \sum_{k=1}^\infty \pi_{\beta,k}$ with

$$\pi_{\beta,k} f(x,s) = \frac{1}{(x+k)^{2\beta}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} (s) \right)$$

- show that each operator $\pi_{\beta,k}$ is nuclear and that the series of the norms converges $\sum_{k=1}^\infty \|\pi_{\beta,k}\| < \infty$
- in fact spectrum of $\pi_{\beta,k}$: take z_k unique fixed point of $\gamma_k := \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}$ in $\mathbb{D}$, and $\mu_i^{(k)}$ the spectrum of the permutation induced by this matrix on $\mathbb{P}$. Then the spectrum of $\pi_{\beta,k}$ is $\{(-1)^n (z_k + k)^{-2(\beta+n)} \mu_i^{(k)}\}$, $n \geq 0$

Existence of eigenfunctions in invariant cones

(analog of Perron–Frobenius)

- K a cone in a real Banach space B satisfying $\overline{K-K}=B$
- $\mathcal{L}$ a compact operator with $\mathcal{L}(K)\subseteq K$ and with positive spectral radius $r(L)$
- then $r(L)$ is an eigenvalue of L with a corresponding eigenfunction in K

• simple eigenvalue

- an operator $\mathcal{L}$ is u-bounded, with respect to a function $u\in K$, if for any $f\in K$ there exist some $n>0$ and $a,b>0$ with

$$au\leq \mathcal{L}^n f\leq bu$$

- $\mathcal{L}_\beta$ is u-bounded with respect to constant function $u(x,s)=1$
- lower bound (previous estimates) implies positivity of the spectral radius $r(\mathcal{L}_\beta)$
- if cone K is reproducing and the K–positive operator $\mathcal{L}$ is u-bounded then $\mathcal{L}$ has an eigenvalue $\lambda_0>0$ with an eigenvector $f\in K$. Also the eigenvalue λ_0 is simple

- with same hypothesis as previous, every eigenvalue λ of $\mathcal{L}$ different from λ_0 satisfies $|\lambda| < \lambda_0$
- because if the operator L is u–bounded then it is also f–bounded, where again f is the eigenfunction in K with eigenvalue λ_0
- then if h is an eigenfunction with eigenvalue λ have estimate

$$-\alpha(\lambda_0 - \epsilon)f \leq \lambda h \leq \alpha(\lambda_0 - \epsilon)f$$

  for some $\epsilon > 0$ with $\alpha > 0$ the smallest positive number such that $-\alpha f \leq h \leq \alpha f$ is satisfied
- this gives $|\lambda| \leq \lambda_0 - \epsilon$.

• all these results on eigenvalues and eigenfunctions in cones Perron–Frobenius type

- M. Krasnoselskij, Je. Lifshits, A. Sobolev *Positive linear systems*. Heldermann Verlag, 1989.

Operators on cones and adjoints

(also from Krasnoselskij-Lifshits-Sobolev)

- if cone K contains ball of positive radius, then adjoint $\mathcal{L}^*$ acting on dual Banach space B' has eigenfunctional f^* in adjoint cone K^* of linear K-positive functionals, with eigenvalue $\lambda \leq r(\mathcal{L}) = \lambda_0$
- in our case this holds and eigenfunction with topo eigenvalue is interior point of the cone
- if $\mathcal{L}^* f^* = \lambda f^*$ and $f^*(f) > 0$ then $\lambda = \lambda_0$ because

$$\lambda_0 \, f^*(f) = f^*(\lambda_0 f) = f^*(\mathcal{L} f) = (\mathcal{L}^* f^*)(f) = \lambda \, f^*(f)$$

- given an operator $\mathcal{L}$ with a simple eigenvalue equal to the spectral radius, $\lambda_0 = r(L)$, and rest of spectrum in disk $|\lambda| < q\, r(L)$ for some $q < 1$
- $f \in K$ eigenfunction of eigenvalue λ_0
- f^* eigenfunctional of $\mathcal{L}^*$ in K^*, with eigenvalue λ_0 satisfying $f^*(f) = 1$
- then sequence of iterates

$$f_{n+1} = \mathcal{L} f_n$$

converges to eigenfunction f in the following sense

- iterates converge as fast as a geometric progression with ratio arbitrarily close to the spectral margin $q = q(L)$

- more precise statement: define $Uh := f^*(h)f$, and $U^\perp h := h - f^*(h)f$
- them have

$$\lim_n \frac{\|U^\perp f_n\|}{\|Uf_n\|} = 0$$

- rate of convergence estimated by

$$\frac{\|U^\perp f_n\|}{\|Uf_n\|} \leq c(\epsilon)(q+\epsilon)^n \frac{\|U^\perp f_1\|}{\|Uf_1\|},$$

  for arbitrarily small $\epsilon > 0$
- for any $f \in B$ and $\epsilon > 0$

$$\mathcal{L}_\beta^n f = \kappa \, \lambda_{0,\beta}^n \, f_\beta + O(c(\epsilon)(q+\epsilon)^n \lambda_{0,\beta}^n)$$

  as $n \to \infty$, for some constant $\kappa > 0$ and where $q = q(\mathcal{L}_\beta) < 1$ is the spectral margin of $\mathcal{L}_\beta$
- convergence $\lambda_{0,\beta}^{-n} \mathcal{L}_\beta^n h \to f_\beta^*(h) f_\beta$

Shift invariant measure

- measure on $[0,1]\times\mathbb{P}$ and set $A=\cup_{s\in\mathbb{P}}A_s\times\{s\}$ with $A_s\subset[0,1]$

$$m(A)=\frac{1}{|\mathbb{P}|\log(2)}\sum_{s\in\mathbb{P}}\int_{A_s}\frac{1}{1+x}\,dx$$

- check on intervals that indeed $(1+x)^{-1}\,dx$ is T-invariant measure

$$\int_a^b\frac{1}{1+x}\,dx=\log(\frac{1+a}{1+b})$$

$$T^{-1}(a,b)=\bigcup_{n=1}^{\infty}(\frac{1}{n+b},\frac{1}{n+a})$$

$$m(T^{-1}(a,b))=\frac{1}{\log 2}\sum_n\int_{\frac{1}{n+b}}^{\frac{1}{n+a}}\frac{1}{1+x}\,dx$$

$$=\frac{1}{\log 2}\sum_n\log\left(\frac{1+\frac{1}{n+a}}{1+\frac{1}{n+b}}\right)$$

$$= \frac{1}{\log 2} \sum_{n=1}^{\infty} \log \left(\frac{n+a+1}{n+b+1} \cdot \frac{n+b}{n+a} \right) = \frac{1}{\log 2} \log(\frac{1+a}{1+b})$$

telescopic sum

- uniform counting measure in the finite set $\mathbb{P}$ direction because of transitivity hypothesis
- so we have $f \circ T = f$ for $f(x) = \frac{1}{|\mathbb{P}| \log(2)} \frac{1}{1+x}$
- for our case the spectral radius of $\mathcal{L}$ is one and convergence is

$$\mathcal{L}^n h \to (\int h d\lambda) f$$

where $d\lambda$ prod of Lebesgue and uniform on $[0,1] \times \mathbb{P}$ and $f(x) = \frac{1}{|\mathbb{P}| \log(2)} \frac{1}{1+x}$ invariant density

Ergodic theorem

- view this through an ergodic theorem

$$\frac{1}{n}\sum_{k=1}^{n} u\circ T^k(x,s) \to \frac{1}{|\mathbb{P}|\log(2)}\int_{[0,1]\times\mathbb{P}} u(x,s)\,\frac{1}{1+x}\,d\lambda$$

if invariant measure is also ergodic

- if so then

$$\int \frac{1}{n}\sum_{k=1}^{n}\mathcal{L}^k(h)\,u\,d\lambda = \int h\frac{1}{n}\sum_{k=1}^{n} u\circ T^k\,d\lambda$$

$$\to \int h(\int uf\,d\lambda)d\lambda = \int hd\lambda\cdot\int u\,f\,d\lambda$$

- holds for arbitrary u so almost everywhere

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\mathcal{L}^k(h) = (\int h\,d\lambda)\cdot f$$

- eigenfunctional $f^* : h \mapsto \int h \, d\lambda$ with eigenvalue one

$$\mathcal{L}^*(f^*)(h) = \sum_k \int_{I \times \mathbb{P}} h \circ g_k \; g_k' d\lambda = \sum_k \int_{I_k \times \mathbb{P}} h \, d\lambda = \int_{I \times \mathbb{P}} h \, d\lambda$$

- ergodicity of the invariant measure
- take cylinder sets $\mathcal{C}(a_1, \dots, a_k)$ of the continued fraction expansion
- for all Borel sets B and m the T-invariant measure

$$m(T^{-n}(B) \cap \mathcal{C}(a_1, \dots, a_k)) \sim m(T^{-n}(B)) m(\mathcal{C}(a_1, \dots, a_k))$$

- given a shift invariant measurable set $T^{-1}(A) = A$ this will then imply
$$m(A \cap B) \sim m(A) m(B)$$
- for $B = X \setminus A$ for $X = [0,1] \times \mathbb{P}$ have then $m(A) \in \{0, 1\}$

- to compute measure of $T^{-n}(B) \cap \mathcal{C}(a_1, \ldots, a_k)$ enough to check for $B = [d, e]$ interval
- then $T^{-n}(B) \cap \mathcal{C}(a_1, \ldots, a_n)$ is also an interval with endpoints

$$\frac{p_n + d\, p_{n-1}}{q_n + d\, q_{n-1}} \quad \text{and} \quad \frac{p_n + e\, p_{n-1}}{q_n + e\, q_{n-1}}$$

  and Lebesgue measure $(q_n + dq_{n-1})^{-1}(q_n + eq_{n-1})^{-1}$ (relation $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$)

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots\cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

- Lebesgue measure of $\mathcal{C}(a_1, \ldots, a_n)$ is $q_n^{-1}(q_n + q_{n+1})^{-1}$
- (hand waiving argument) then use an argument similar to the case of the subshifts of finite type: for large enough n the continued fraction digits involved in $\mathcal{C}(a_1, \ldots, a_n)$ and in $T^{-n}(B)$ will be independent variables

- direct computation of the property $m(T^{-n}(B) \cap \mathcal{C}(a_1, \dots, a_k)) \sim m(T^{-n}(B)) m(\mathcal{C}(a_1, \dots, a_k))$ works but is long
- another way of proving ergodicity based on Schweiger's "fibered systems"
  - Fritz Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford University Press, 1995.
- identify a list of conditions on the shift map T that ensure ergodicity (with the unique invariant measure)
- fibered system $T : X \to X$
  1. there is a finite or countable set of digits $\mathfrak{A}$
  2. there is a map $k : X \to \mathfrak{A}$ such that the sets $X(a) = k^{-1}(a)$ for $a \in \mathfrak{A}$ form a partition of X
  3. the restriction of the map T to any $X(a)$ is injective
- any shift space on a finite or infinite alphabet: take the cylinder sets with fixed first digit

Properties ensuring ergodicity for fibered systems (Schweiger)

• for $X = [0,1]$ if all the following properties are satisfied then $T : X \to X$ is ergodic for unique invariant measure μ satisfying $C^{-1}\lambda(E) \leq \mu(E) \leq C\lambda(E)$ Lebesgue measure λ

1. there are intervals (a_k, b_k) with $(a_k, b_k) \subseteq X(k) \subseteq [a_k, b_k]$
2. map $T : X(k) \to X = [0,1]$ extends to a $\mathcal{C}^2$-function on $[a_k, b_k]$
3. $T([a_k, b_k]) = [0,1]$ for all $k \in \mathfrak{A}$
4. there is a constant $\theta > 1$ such that $|T'(x)| \geq \theta$ for all $x \in X$
5. there is a constant $M > 0$ such that for all x

$$\left| \frac{T''(x)}{T'(x)^2} \right| \leq M,$$

The fourth condition can be replaced by a weaker form:

- there is an $N \in \mathbb{N}$ such that $|(T^N)'(x)| \geq \theta > 1$ for all x in the interval $I(k_1, \ldots, k_N)$ closure of $X(k_1, \ldots, k_N)$

Shift of continued fraction expansion satisfies these

- intervals $I_k = [\frac{1}{k+1}, \frac{1}{k}]$ satisfy first, second and third property (fixing first digit of continued fraction expansion)
- fourth property holds in the modified form for second iterate T^2: fixing first two digits I_{k_1,k_2} interval, on this interval

$$T(x) = \frac{1}{x} - k_1, \qquad T^2(x) = \frac{x}{1-k_1 x} - k_2$$

$$(T^2(x))' = \frac{1}{(1-k_1 x)^2} \geq 4$$

- fifth condition holds

$$\left| \frac{T''(x)}{(T'(x))^2} \right| = |(2/x^3)/(1/x^4)| = |2x| \leq 2$$

Application: mixmaster universe models in cosmology

- introduced in the 1970s by V. Belinskii, I. M. Khalatnikov, E. M. Lifshitz and I. M. Lifshitz as cosmological models that exhibit chaotic dynamics
- basic building blocks: Kasner metrics, non-isotropic solutions of Einstein equations

$$ds^2 = dt^2 - a(t)^2 dx^2 - b(t)^2 dy^2 - c(t)^2 dz^2$$

$$a(t) = t^{p_1}, \; b(t) = t^{p_2}, \; c(t) = t^{p_3}, \qquad \sum p_i = \sum p_i^2 = 1$$

coefficients $a(t)$, $b(t)$, $c(t)$ are non-isotropic scale factors

- the mixmaster universe model transitions between different Kasner metrics during different epochs of cosmic time

Building mixmaster dynamics with a discretized system

- local logarithmic time Ω

$$d\Omega := -\frac{dt}{abc}$$

- time evolution divided into Kasner eras $[\Omega_n, \Omega_{n+1}]$, $n \geq 1$
- within each era evolution of a, b, c described by Kasner metrics where variables p_i depend on a parameter u: written in increasing order

$$p_1(u) = -\frac{u}{1+u+u^2}, \; p_2(u) = \frac{1+u}{1+u+u^2}, \; p_3(u) = \frac{u(1+u)}{1+u+u^2}$$

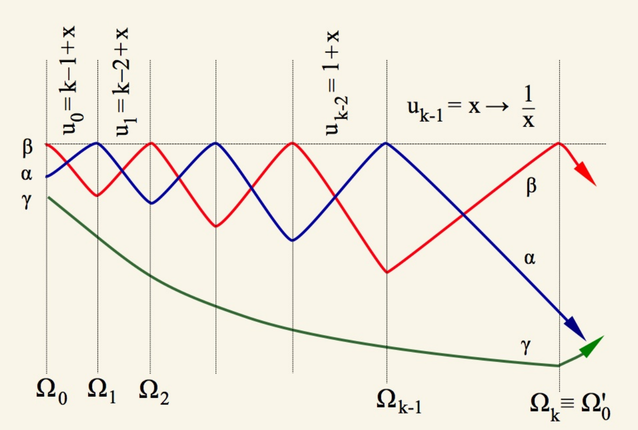
discretize the dependence on u

- evolution starts with a value $u_n > 1$ proceeds decreasing u with growing Ω until u becomes less than 1
- then transitional period to new Kasner era with transition formula for parameter u_{n+1}

$$u_{n+1} = \frac{1}{u_n - [u_n]}$$

- reordering of the spatial directions (of the exponents $p_i(u)$) to increasing order:
  - when u decreases by one permutation (12)(3)
  - when transition to next era when at $u < 1$ permutation (1)(23)
- shape of the dynamics: at each time one of the axis is driving an overall expansion or contraction of the universe, while the remaining two axes exhibit oscillating behavior, at the end of each cycle within each era and at the end of each era permutation of which axis drives the expansion/contraction and which oscillate

Kasner eras of the mixmaster universe



Mixmaster dynamics and geodesics on modular curves

- a geometric classification of the mixmaster solutions constructed as above
- the individual evolution of a typical trajectory is determined by a number $\alpha \in (0,1)$ whose continued fraction $[k_1, k_2, k_3, \dots]$ determines the number of oscillations in each successive Kasner era
- α is defined only up to a shift T^n, because the initial point of the backward evolution can be chosen arbitrarily
- also want to encode the permutations that determine the leading scaling factor
- take lift to $\mathrm{GL}_2(\mathbb{Z})$ of subgroup $\Gamma_0(2)$ of $\mathrm{SL}_2(\mathbb{Z})$, this has

$$\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/\Gamma_0(2) = \mathbb{P}^1(\mathbb{F}_2) = \{0, 1, \infty\}$$

- use these three points as labels for the spatial axes, then the $\mathrm{GL}_2(\mathbb{Z})$ matrices implementing the transformations $u \mapsto 1/u$ and $u \mapsto u - 1$ act on $\mathbb{P}$ as the permutations (1)(23) and (12)(3), respectively

Two sided shift and geodesics

- Modular curve $X_{\Gamma_0(2)} = \Gamma_0(2)\backslash\mathbb{H} \simeq \mathrm{PGL}_2(\mathbb{Z})\backslash(\mathbb{H}\times\mathbb{P})$ hyperbolic plane $\mathbb{H}$ and $\mathbb{P} = \mathbb{P}^1(\mathbb{F}_2)$
- correspondence as said between points $\{0,1,\infty\} = \mathbb{P}^1(\mathbb{F}_2)$ and spatial axes

$$\begin{array}{lcl} 0 = [0:1] & \mapsto & z \\ \infty = [1:0] & \mapsto & y \\ 1 = [1:1] & \mapsto & x \end{array}$$

- infinite geodesics for the hyperbolic metric in $\mathbb{H}$ are arcs circles with endpoints on the boundary and orthogonal to the boundary (or vertical straight lines if one end at infinity)
- an infinite geodesic on $X_{\Gamma_0(2)}$ is the image under the projection map $\pi : \mathbb{H}\times\mathbb{P} \to X_{\Gamma_0(2)}$ of an infinite geodesic on a sheet $\mathbb{H}\times\{s\}$

- to identify uniquely an infinite geodesic in $\mathbb{H}$ it suffices to specify its two endpoints (for instance by giving their continued fraction expansion):
  (ω^+, ω^-, s) endpoints at $t \to \pm\infty$ and sheet in $\mathbb{P}$
- up to action of $\mathrm{GL}_2(\mathbb{Z})$ sufficient to pick the two endpoints $(\omega^+, \omega^-) \in (-\infty, -1] \times [0, 1]$

$$\omega^+ = [k_0, k_1, k_2, \ldots], \qquad \omega^- = [k_{-1}; k_{-2}, k_{-3}, \ldots]$$

- two-sided invertible shift

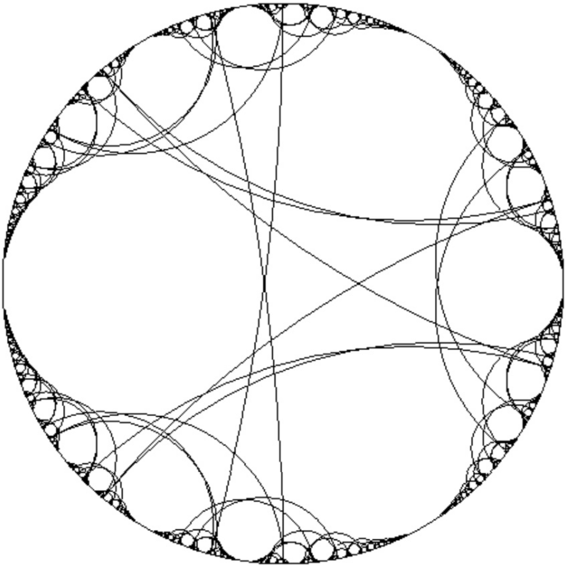
$$T(\omega^+, s) = (\frac{1}{\omega^+} - \left[\frac{1}{\omega^+}\right], \begin{pmatrix} -[1/\omega^+] & 1 \\ 1 & 0 \end{pmatrix} s)$$

$$T(\omega^-, s) = (\frac{1}{\omega^- + [1/\omega^+]}, \begin{pmatrix} -[1/\omega^+] & 1 \\ 1 & 0 \end{pmatrix} s)$$

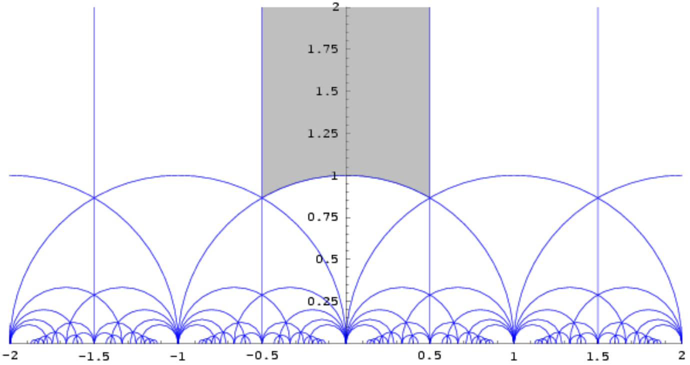
- it is the invertible shift that moves one step to the left the doubly infinite sequence

$$\ldots k_{-\ell}, \ldots, k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots, k_n, \ldots$$

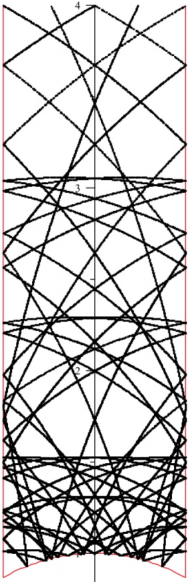
Geodesics in the hyperbolic plane (Pointcaré disc model)



Fundamental domain of the $SL_2(\mathbb{Z})$ action on the hyperbolic plane



Geodesics on the modular curve $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



- endpoints in the same $\mathrm{GL}_2(\mathbb{Z})$ orbit if differ by a power T^k of the shift action
- so data (ω^+, ω^-, s) up to the action of $T^{\mathbb{Z}}$ parameterize geodesics in $X_{\Gamma_0(2)}$
- but data (ω^+, ω^-, s) up to the action of $T^{\mathbb{Z}}$ also parameterize solutions of the mixmaster dynamics, where (ω^+, ω^-, s) and $T(\omega^+, \omega^-, s)$ determine the same solution up to a different choice of initial time

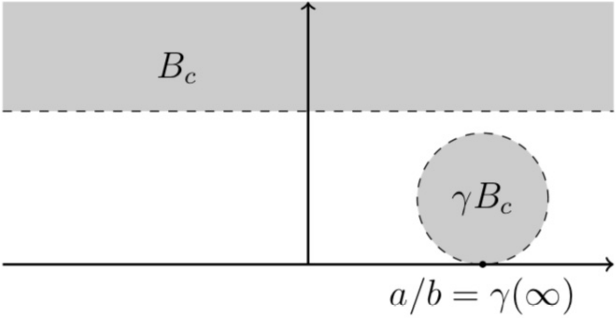
Modular curves

- for any finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$ there is a corresponding modular curve $X_G = \mathrm{GL}_2(\mathbb{Z})\backslash(\mathbb{H} \times \mathbb{P})$ with $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$
- often modular curves defined for finite index subgroups $G_0 \subset \mathrm{PSL}_2(\mathbb{Z})$ with $X_{G_0} = G_0\backslash\mathbb{H}$
- pass to other description by lifting G_0 to $\mathrm{GL}_2(\mathbb{Z})/G$ and taking the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ generated by this lift and by the sign $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- can identify $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$ with $\mathbb{P}_0 = \mathrm{PSL}_2(\mathbb{Z})/G_0$
- algebro-geometric compactification: $\overline{X}_G = \mathrm{GL}_2(\mathbb{Z})\backslash(\bar{\mathbb{H}} \times \mathbb{P})$ with

$$\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

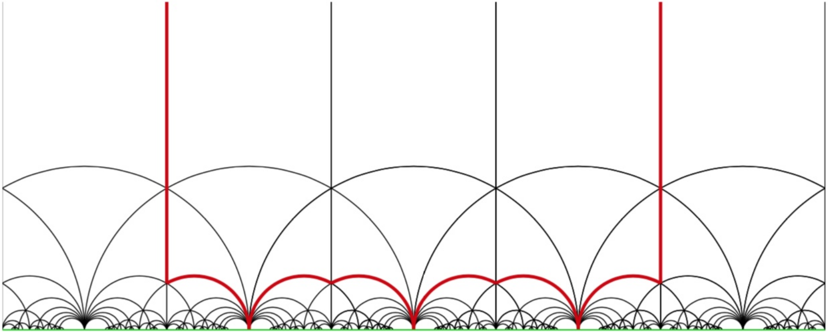
compactification by finitely many cusp points $G\backslash\mathbb{P}^1(\mathbb{Q}) \simeq \mathrm{GL}_2(\mathbb{Z})\backslash(\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P})$

Topology of compactification



Basis of open neighborhoods of the the cusps define topology on the compactification $\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ by cusp points

Example



Fundamental domain of $\Gamma(3) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \,|\, \gamma \equiv \mathrm{id} \mod 3\}$ with four cusps (three on the real line and one at ∞)

Modular Forms

- action of $\mathrm{SL}_2(\mathbb{Z})$ by fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$
- action on functions $f : \mathbb{H} \to \mathbb{C}$ slash operator of weight k

$$(f|_k\gamma)(z) := f(\frac{az+b}{cz+d})\cdot(cz+d)^{-k} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

- $f : \mathbb{H} \to \mathbb{C}$ modular form of weight k if
  1. f is holomorphic (Note: sometime people also consider meromorphic modular forms)
  2. $(f|_k\gamma) = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (but also consider modular forms for some finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$)
  3. f is holomorphic at infinity: when $\Im(z) \to \infty$ growth of $|f(z)|$ bounded above by a polynomial in $\max\{1, \Im(z)^{-1}\}$
- $\mathcal{M}_k$ the $\mathbb{C}$-vector space of modular forms of weight k; $\mathcal{M}_{G,k}$ for some finite index subgroup
- $\mathcal{S}_k$ and $\mathcal{S}_{G,k}$ subspaces of *cusp forms*: decay at infinity (vanishing at cusps in X_G) $|f(z)|$ bounded above by $\Im(z)^{-k/2}$ for $\Im(z) \to \infty$

Homology and cohomology of modular curves

- modular curve X_G is a one-dim complex manifold, so interesting (co)homology is H^1, H_1 and their pairing by integration of a one-form on a one-cycle
- holomorphic differentials (1-forms) on X_G can be viewed on $\mathbb{H}$ as modular forms of weight 2, among these cusp forms (vanishing at cusps): vector space $\mathcal{S}_{G,2}$
- f weight 2 cusp form: holomorphic differential form $f(z)\,dz$ transforms under $\gamma \in G_0 \subset \mathrm{SL}_2(\mathbb{Z})$ like

$$f(\frac{az+b}{cz+d})d(\frac{az+b}{cz+d}) = (cz+d)^2 f(z)\frac{ad-bc}{(cz+d)^2}\,dz = f(z)\,dz$$

  invariant so it descends to a holomorphic one-form on the quotient X_{G_0}
- in fact all holomorphic one-forms obtained in this way: known fact $\dim S_{G,2} = g(X_G)$

Modular symbols and homology of modular curves

- take two cusps $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$, same $\mathrm{SL}_2(\mathbb{Z})$ orbit, so geodesic in $\mathbb{H}$ with those endpoint becomes closed geodesic $\{\alpha, \beta\}$ in $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$: homology class in $H_1(X, \mathbb{Z})$
- pairing with cohomology

$$(f, \{\alpha, \beta\}) \mapsto \int_\alpha^\beta f(z)\, dz$$

  (for non-cusp forms the integral diverges)
- perfect pairing $\mathcal{S}_2 \times H_1(X, \mathbb{R}) \to \mathbb{C}$ so dual $\mathcal{S}_2^\vee \simeq H_1(X, \mathbb{R})$

Properties of Modular Symbols

- pair of cusps $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ geodesic with these endpoints and modular symbol $\{\alpha, \beta\}$ in $H_1(X, \mathbb{Z})$
- $\{\alpha, \beta\} = -\{\beta, \alpha\}$ change of orientation: two-term relation
- $\{\alpha, \beta\} = \{\alpha, \gamma\} + \{\gamma, \beta\}$ difference is a boundary: three-term relation
- $\{g\alpha, g\beta\} = \{\alpha, \beta\}$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$
- for finite index subgroup G similarly define modular symbols $\{\alpha, \beta\}_G \in H_1(X_G, \mathbb{Z})$ for cusps $\alpha, \beta \in G\backslash\mathbb{P}^1(\mathbb{Q})$ cusps of G, same relations but $\{g\alpha, g\beta\} = \{\alpha, \beta\}$ for all $g \in G$
- for subgroup G classes of α, β in $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$
- perfect pairing between cusp forms and modular symbols $\mathcal{S}_{G,2} \times H_1(X_G, \mathbb{R}) \to \mathbb{C}$

Continued fractions and the Modular Symbol algorithm (Manin)

- can always take just symbols $\{0, \alpha\}$ with $\alpha = p/q$ by relations
- continued fraction of rational $\alpha = [a_1, \ldots, a_r]$
- convergents $p_k/q_k = [a_1, \ldots, a_k]$

$$\{0,\alpha\} = \{0,\infty\} + \{\infty, \frac{p_1}{q_1}\} + \{\frac{p_1}{q_1}, \frac{p_2}{q_2}\} + \cdots + \{\frac{p_{r-1}}{q_{r-1}}, \frac{p_r}{q_r}\}$$

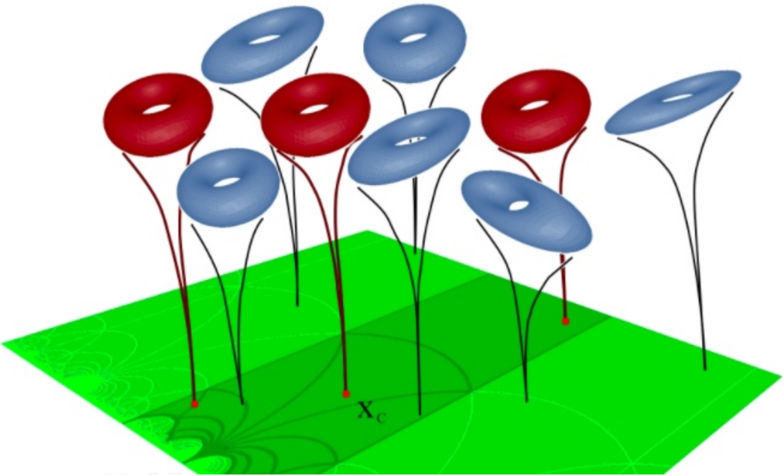
$$\{0,\alpha\} = \sum_{k=1}^{r} \{g_k(\alpha)\cdot 0, g_k(\alpha)\cdot\infty\}$$

$$g_k(\alpha) = \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

note that can also use $\begin{pmatrix} (-1)^{k-1} p_{k-1}(\alpha) & p_k(\alpha) \\ (-1)^{k-1} q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

What happens at irrational endpoints α, β?

- boundary $\mathbb{P}^1(\mathbb{R})$ of $\mathbb{H}$ not seen by usual compactification of modular curves, only cusps $\mathbb{P}^1(\mathbb{Q})$
- action of $\mathrm{GL}_2(\mathbb{Z})$ and finite index subgroups on $\mathbb{P}^1(\mathbb{R})$ is "bad" (dense orbits) so not a nice quotient $G\backslash\mathbb{P}^1(\mathbb{R})$ (non-Hausdorff), unlike finitely many points $G\backslash\mathbb{P}^1(\mathbb{Q})$ with nice topology
- do boundary points in $\mathbb{P}^1(\mathbb{R})$ *represent* something geometrically like cusps?
- the modular curves are moduli spaces of elliptic curves (with some level structure)
- going to a cusp: elliptic curve degenerates to multiplicative group $\mathbb{C}^*$
- going to an irrational point? elliptic curve degenerates to a *noncommutative torus*
- these objects do not exist algebro-geometrically so the irrational points of the boundary not seen in usual compactification: *what if want to see them?*



Modular curves as moduli spaces of elliptic curves

Limiting Modular Symbols

- extending the theory of modular symbols to the irrational boundary $\mathbb{P}^1(\mathbb{R})$ through the ergodic theorem for continued fractions shift
- take irrational $\beta \in \mathbb{P}^1(\mathbb{R}) \smallsetminus \mathbb{P}^1(\mathbb{Q})$ consider the limit

$$\{\{\star, \beta\}\}_G := \lim \frac{1}{T(x,y)} \{x, y\}_G \in H_1(X_G, \mathbb{R})$$

  with $T(x, y) =$ geodesic length of geodesic arc between x, y when $y \to \beta$
- if limit exists it does not depend on the other end α of the geodesic (nor on $x \in \mathbb{H}$ for fixed α)
- to see this compare behavior for geodesic Γ_2 coming from some real $\alpha < \beta$ and Γ_1 from ∞

Comparison of limits

- $p_n(\beta)/q_n(\beta) = p_n/q_n$ convergents to β
- for large enough $n \in \mathbb{N}$ (depending on parity)

$$\frac{\alpha+\beta}{2} < \frac{p_{n-1}}{q_{n-1}} < \beta < \frac{p_n}{q_n}$$

- also always have

$$\left|\frac{p_{n-1}}{q_{n-1}} - \beta\right| > \left|\frac{p_n}{q_n} - \beta\right|$$

- consider sequences of points (with $[a, b]$ geodesic from a to b)

$$z_n := \Gamma_1 \cap \left[\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}\right], \ \zeta_n := \Gamma_2 \cap \left[\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}\right]$$

- combining these get

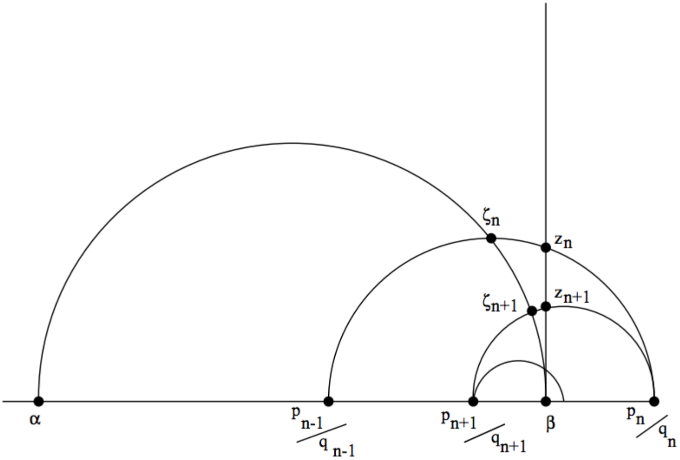
$$\frac{1}{2q_n q_{n+1}} < \Im(z_n) < \frac{1}{2q_{n-1}q_n},$$

and if moreover n is large enough

$$\frac{\theta}{2q_{n+2}q_{n+1}} < \theta\,\Im(z_{n+1}) < \Im(\zeta_n) \le \frac{1}{2q_{n-1}q_n}$$

where θ is some fixed constant between 0 and 1.

Comparison between geodesics



- geodesic distance from a fixed $x_1 \in \Gamma_1$ to $z \in \Gamma_1$ equals $-\log \Im(z) + O(1)$ as $z \to \beta$
- distance from a fixed $x_2 \in \Gamma_2$ to $\zeta \in \Gamma_2$ equals $-\log \Im(\zeta) + O(1)$ as $\zeta \to \beta$
- additivity of modular symbols:

$$\frac{1}{T(x_1, z_n)} \{x_1, z_n\} = \frac{1}{T(x_2, \zeta_n) + O(1)} (\{x_2, \zeta_n\} + O(1))$$

- both limits exist or otherwise simultaneously and have a common value if both exist
- Khintchin–Lévy theorem (instance of Lyapunov exponent)

$$\log q_n(\beta) \stackrel{a.e.\beta}{=} Cn(1+o(1)), \text{ as } n \to \infty, \quad \text{with} \quad C = \frac{\pi^2}{12 \log 2}$$

- so almost everywhere (in Lebesgue or abs.cont.T-inv.meas.) limit given by

$$\lim_{n\to\infty} \frac{1}{2Cn} \{i\infty, z_n\} = \lim_{n\to\infty} \frac{1}{2Cn} \sum_{i=1}^{n} \left\{ \frac{p_{i-1}(\beta)}{q_{i-1}(\beta)}, \frac{p_i(\beta)}{q_i(\beta)} \right\}$$

Birkhoff Average

- can interpret the limit as a Birkhoff average
- $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$ such that $\mathrm{Red}^{-1}(s) = \mathbb{P}$ for all $s \in \mathbb{P}$; base point $s_0 \in \mathbb{P}$
- take a function φ on $\mathbb{P}$ values in a real vector space
- obtain limit (almost everywhere in β)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(g_i(\beta)^{-1}\, s_0) = \frac{1}{|\mathbb{P}|} \sum_{s\in\mathbb{P}} \varphi(s)$$

- because can write this as a Birkhoff average over the iterates of the shift of the continued fraction expansion and use ergodicity

$$\lim_{n\to\infty} \frac{1}{Cn} \sum_{k=1}^{n} \varphi \circ T^k(s_0)$$

$$T(x,s) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} s \right)$$

Weak convergence via transfer operator

- can also obtain the limit via the transfer operator

$$\int_{[0,1]\times\mathbb{P}} f\cdot\mathcal{L}h\,d\lambda = \int_{[0,1]\times\mathbb{P}} (f\circ T)\,h\,d\lambda$$

- eigenfunctional of of $\mathcal{L}^*$

$$h\mapsto \int_{[0,1]\times\mathbb{P}} h(x,t)d\lambda(x,t)$$

- for any $h\in B_{\mathbb{C}}$ strong convergence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(\mathcal{L}^k h)(x,t) = \frac{1}{|\mathbb{P}|\,\log 2}\,\frac{1}{1+x}\int_{[0,1]\times\mathbb{P}} h\,d\lambda$$

- equivalent to the convergence

$$\int_{[0,1]\times\mathbb{P}}\frac{1}{n}\sum_{k=1}^{n} f(T^k(x,t))h(x,t)d\lambda(x,t)\to$$

$$\frac{1}{|\mathbb{P}|\,\log 2}\left(\int_{[0,1]\times\mathbb{P}}\frac{f(x,t)}{1+x}d\lambda(x,t)\right)\int_{[0,1]\times\mathbb{P}} h\,d\lambda$$

Vanishing result

- modular symbols are left G-invariant
- consider function φ and $s_0 \in \mathbb{P}$

$$\varphi(g_k(\beta)^{-1} s_0) = \{g_k(\beta)(0), g_k(\beta)(\infty)\} = \left\{ \frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)} \right\}$$

- then the limit:

$$\frac{1}{2C\,|\mathbb{P}|} \sum_k \{h_k(0), h_k(\infty)\}$$

with h_k running over complete set of representatives of $\mathbb{P}$

- this sum vanishes: take

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then $\{h_k \sigma\}$ also a complete system of representatives and

$$\{\sigma(0), \sigma(\infty)\} = -\{0, \infty\}.$$

Exceptional sets   non-vanishing limits

- over a set of full Lebesgue (or Gauss–Kuzmin) measure the limiting modular symbol vanishes
- however, many interesting sets on which it is non-vanishing
- in particular quadratic irrationalities (β in real quadratic field)
- these points β have stably periodic continued fraction expansion (some finite length then periodic)
- β quadratic irrationality and its Galois conjugate β' are the fixed points of a hyperbolic element g in G: geodesic in $\mathbb{H}$ between these fixed points becomes closed geodesic in X_G with length $\ell(\Gamma) = \lambda(g)$ eigenvalue of g
- then non-vanishing limit modular symbol

$$\{\{\star, \beta\}\} = \frac{\{0, g(0)\}}{\lambda(g)}$$

Lyapunov spectrum

- Lyapunov exponent of the shift map $T : [0,1] \to [0,1]$, $T\beta = 1/\beta - [1/\beta]$, is given by

$$\lambda(\beta) := \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(\beta)| = \lim_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |T'(T^k\beta)|$$

$$\lambda(\beta) = 2 \lim_{n\to\infty} \frac{1}{n} \log q_n(\beta)$$

  $q_n(\beta)$ the successive denominators of the continued fraction expansion

- the function $\lambda(\beta)$ is T–invariant
- Khintchin-Lévy theorem: almost everywhere (in Lebesgue or Gauss-Kuzmin measure) limit equal to $2C$ with

$$C = \frac{\pi^2}{12 \log 2}$$

- Lyapunov spectrum: decompose unit interval in level sets of Lyapunov exponent $L_c = \{\beta \in [0,1] \,|\, \lambda(\beta) = c \in \mathbb{R}\}$

$$[0,1] = \bigcup_{c\in\mathbb{R}} L_c \cup \{\beta \in [0,1] \,|\, \lambda(\beta) \text{ does not exist}\}$$

- these level sets are uncountable dense T–invariant subsets of $[0,1]$, of varying Hausdorff dimension
- measure how the Hausdorff dimension varies, as a function $h(c) = \dim_H(L_c)$
- consider function function $\varphi : \mathbb{P} \to H_1(\overline{X_G}, \text{cusps}, \mathbb{R})$

$$\varphi(s) = \{g(0), g(\infty)\}_G$$

for $g \in \mathrm{GL}_2(\mathbb{Z})$ representative of the coset $s \in \mathbb{P}$

- for $c \in \mathbb{R}$, and for all $\beta \in L_c$ limiting modular symbol computed by same previous argument as Birkhoff average

$$\lim_{n\to\infty} \frac{1}{cn} \sum_{k=1}^{n} \varphi \circ T^k(s_0)$$

$T : [0,1] \times \mathbb{P} \to [0,1] \times \mathbb{P}$ shift operator and $s_0 \in \mathbb{P}$ base point

Case of quadratic irrationalities

- another equivalent expression for the limiting modular symbol of quadratic irrationalities
- geodesic γ_β in $\mathbb{H}$ with an endpoint at β with eventually periodic continued fraction expansion
- shift $T^\ell(\beta, s) = (\beta, s)$, for a minimal non-negative integer ℓ
- for a quadratic irrationality β the limit defininig Lyapunov exponent $\lambda(\beta) = 2 \lim_{n\to\infty} \log(q_n(\beta))/n$ exists and belongs to the interval $[2\log((1+\sqrt{5})/2), \infty)$ (result by Pollicott–Weiss)
- limiting modular symbol is given by

$$\{\{*, \beta\}\}_G = \frac{\sum_{k=1}^{\ell} \{g_k^{-1}(\beta)\cdot g(0), g_k^{-1}(\beta)\cdot g(\infty)\}_G}{\lambda(\beta)\ell}$$

with $g \in \mathrm{GL}_2(\mathbb{Z})$ representative of $s \in \mathbb{P}$

General case: limiting modular symbol on the Lyapunov spectrum

- case of limit for given $c \in \mathbb{R}$ and $\beta \in L_c$

$$\lim_{n\to\infty} \frac{1}{cn} \sum_{k=1}^{n} \varphi \circ T^k(s_0)$$

- set L_c for $c \neq \pi^2/(12\log 2)$ is of Lebesgue measure zero
- need an appropriate transfer operator adapted to the Lyapunov spectrum and resulting invariant measure on L_c
- General setting: two constructions of operators used to analyze the dynamics of iterates of map T: Ruelle transfer operator (or Perron–Frobenius operator) and Gauss-Kuzmin operator
- can apply these constructions to any $E \subset [0,1] \times \mathbb{P}$ T-invariant subset

Ruelle (or Perron–Frobenius) Transfer Operator

- on a T-invariant subset $E \subset [0,1] \times \mathbb{P}$

$$(\mathcal{R}_h f)(\beta, s) = \sum_{(\alpha,u) \in T^{-1}(\beta,s)} \exp(h(\alpha,u)) f(\alpha,u)$$

- in particular consider case where the function h is of the form

$$h(\beta, s) = -\sigma \log |T'(\beta)|$$

for a parameter σ

Gauss–Kuzmin Operator

- same T-invariant subset $E \subset [0,1] \times \mathbb{P}$ with Hausdorff dimension $\delta_E = \dim_H(E)$
- Gauss–Kuzmin Operator defined as adjoint of the shift T

$$\int_E (\mathcal{L}f) \cdot h \, d\mathcal{H}^{\delta_E} = \int_E f \cdot (h \circ T) \, d\mathcal{H}^{\delta_E}$$

for all f, h in $L^2(E, d\mathcal{H}^{\delta_E})$, Hausdorff measure $d\mathcal{H}^s$ with $s = \delta_E$

- equivalently operator given by

$$(\mathcal{L}f)(\beta, s) = \sum_{k \in \mathcal{N}_E} \frac{1}{(\beta + k)^{2\delta_E}} f\left(\frac{1}{\beta + k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot s \right)$$

with sum over the set

$$\mathcal{N}_E := \left\{ k : \left(\left[\frac{1}{k+1}, \frac{1}{k} \right] \times \mathbb{P} \right) \cap E \neq \emptyset \right\}$$

Relation

- $\mathcal{L}$ Gauss–Kuzmin part of a one-parameter family $\mathcal{L}_\sigma$ for $\sigma = 2\delta_E$

$$(\mathcal{L}_{\sigma,E} f)(x,s) = \sum_{k\in\mathcal{N}_E} \frac{1}{(x+k)^\sigma} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot s\right)$$

- in general these operators are different: Ruelle transfer operator more information on dynamics of T, Gauss–Kuzmin operator better functional analytic properties
- If invariant set of the form $E = B \times \mathbb{P}$ with $B = \{\beta \in [0,1] : a_i \in \mathcal{N}\}$ for $\beta = [a_1, \dots, a_n, \dots]$, and $\mathcal{N} \subset \mathbb{N}$ a given subset, then

$$\mathcal{R}_h = \mathcal{L}_{2\sigma,E}$$

for $h = -\sigma \log |T'|$

Conditions for good spectral theory for Gauss–Kuzmin

- T–invariant set $E \subset [0,1] \times \mathbb{P}$ has good spectral theory (GST) if the following conditions are satisfied:
  1. have real Banach space $\mathbb{V}$ such that, for each $f \in \mathbb{V}$, restriction $f|_E$ lies in $L^2(E, d\mathcal{H}^{\delta_E})$, and restrictions satisfy
  $$\overline{\{f|_E \, | f \in \mathbb{V}\}} = L^2(E, d\mathcal{H}^{\delta_E})$$
  2. operator $\mathcal{L}_{\sigma,E}$ acting on $\mathbb{V}$ is compact
  3. exists a cone $K \subset \mathbb{V}$ of functions positive at points of E, and an element u in the interior of K, such that $\mathcal{L}_{\sigma,E}$ is u–positive: for all non–trivial $f \in K$ $\exists k > 0$ and real $a, b > 0$ such that
  $$au \leq \mathcal{L}^k_{\sigma,E} f \leq bu$$
  order defined by $f \leq g$ iff $g - f \in K$
  4. for $\sigma = 2\delta_E$ the spectral radius of $\mathcal{L}_{2\delta_E,E}$ is one,
  $$\rho(\mathcal{L}_{2\delta_E,E}) = 1$$

Perron–Frobenius Theorem for Gauss–Kuzmin Operator

- E a T–invariant set with GST
- *simple eigenvalue* $\lambda_\sigma = \rho(\mathcal{L}_{\sigma,E})$ and unique (normalized) eigenfunction in the cone K

$$\mathcal{L}_{\sigma,E} h_\sigma = \lambda_\sigma h_\sigma$$

- exists a unique T-invariant measure on E, absolutely continuous with respect to Hausdorff measure $d\mathcal{H}^{\delta_E}$, with density the normalized eigenfunction $h_{2\delta_E}$
- $\mathcal{L}^*_{\sigma,E}$ adjoint operator acting on dual Banach space
- unique eigenfunctional ℓ_σ with $\mathcal{L}^*_{\sigma,E}\ell_\sigma = \lambda_\sigma \ell_\sigma$: for $f \in \mathbb{V}$

$$\lambda_\sigma^{-n} \mathcal{L}^n_{\sigma,E} f = \ell_\sigma(f) h_\sigma$$

  with eigenfunctional $\ell_{2\delta_E}$ given by $f \mapsto \int_E f \, d\mathcal{H}^{\delta_E}$
- iterates $\mathcal{L}^k_{2\delta_E,E} f$, for $f \in \mathbb{V}$, converge to the invariant density $h_{2\delta_E}$
- rate of convergence order $O(q^n)$ spectral margin $|\lambda| < q\lambda_\sigma$ for all points $\lambda \neq \lambda_\sigma$ in the spectrum of $\mathcal{L}_{\sigma,E}$

Invariant measure from Gauss–Kuzmin operator

- given any F in $L^2(E, d\mathcal{H}^{\delta_E})$

$$\int_E \frac{1}{n}\sum_{k=1}^{n} F(T^k(\beta,s))\, f(\beta,s)\, d\mathcal{H}^{\delta_E}(\beta,s) \to$$

$$\left(\int_E F\, h_{2\delta_E}\, d\mathcal{H}^{\delta_E}\right)\cdot\left(\int_E f\, d\mathcal{H}^{\delta_E}\right)$$

for and any test function $f \in L^2(E, d\mathcal{H}^{\delta_E})$

- $E = B \times \mathbb{P}$ a T-invariant subset with GST, $s_0 \in \mathbb{P}$ base point
- sequence of functions

$$m_n(x,s) := \mathcal{H}^{\delta_B}\left(\left\{\alpha \in B \,\middle|\, x_n(\alpha) \le x \text{ and } g_n^{-1}(\alpha)\cdot s_0 = s\right\}\right)$$

- recursive relation

$$m_{n+1}(x,s) = \sum_{k=1}^{\infty} \left(m_n \left(\frac{1}{k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot s \right) - m_n \left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot s \right) \right)$$

- densities

$$m'_{n+1}(x,s) = (\mathcal{L}_{2\delta_E,E}\, m'_n)(x,s)$$

- measures $m_n(x,s)$ converge to the unique T–invariant measure

$$m(x,s) := \int_E h_{2\delta_E}(x,s)\, d\mathcal{H}^{\delta_E}(x,s),$$

at a rate $O(q^n)$ for some $0 < q < 1$
- $h_{2\delta_B}$ is the top eigenfunction for Gauss–Kuzmin operator $\mathcal{L}_{2\delta_B}$ on B

$$h_{2\delta_E}(x,s) = \frac{1}{|\mathbb{P}|} h_{2\delta_B}(x)$$

Lyapunov exponent and variation of Gauss–Kuzmin operator

- variation of the Gauss–Kuzmin operator along the 1–parameter family, $\mathcal{A}_{\sigma,E} := \frac{d}{d\sigma}\mathcal{L}_{\sigma,E}$
- variation operator explicitly

$$(\mathcal{A}_{\sigma,E} f)(\beta, s) = \sum_{k\in\mathcal{N}_E} \frac{\log(\beta+k)}{(\beta+k)^\sigma} f\left(\frac{1}{\beta+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot s\right)$$

- λ_σ be the top eigenvalue of the Gauss–Kuzmin operator $\mathcal{L}_{\sigma,E}$ acting on $\mathbb{V}$, and let h_σ be the corresponding unique (normalized) eigenfunction

$$\int_E (\mathcal{A}_{2\delta_E,E} h_{2\delta_E})\, d\mathcal{H}^{\delta_E} = \lambda'_\sigma|_{\sigma=2\delta_E}$$

$$\lambda'_\sigma = \frac{d}{d\sigma}\lambda_\sigma$$

- for $\mathcal{H}^{\delta_B}$–almost every $\beta \in B$, the Lyapunov exponent satisfies

$$\lambda(\beta) = 2\lambda'_{2\delta_B}$$

- limiting modular symbol (with same φ function as before)

$$\lim_n \frac{1}{2\lambda'_{2\delta_B}\, n} \sum_{k=1}^{n} \varphi(T^k s_0)$$

Vanishing Result: Weak Convergence

- $E = B \times \mathbb{P}$ a T–invariant set with GST
- for all test functions $f \in L^2(E, d\mathcal{H}^{\delta_E})$ we have

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_E \{x_0, y(\tau)\} \cdot f \, d\mathcal{H}^{\delta_E} = 0.$$

- in terms of Birkhoff averages $\lim_{n\to\infty} \frac{1}{2\lambda'_{2\delta}\, n} \sum_{k=1}^n \varphi(T^k s)$

$$\int_{B\times\mathbb{P}} \frac{1}{n} \sum_{k=1}^n \varphi \circ T^k \, f d\mathcal{H}^{\delta_E} = \int_{B\times\mathbb{P}} \frac{1}{n} \sum_{k=1}^n (\mathcal{L}^k f)\varphi \, d\mathcal{H}^{\delta_E}.$$

- this converges to

$$\left(\int_{B\times\mathbb{P}} \varphi h_{2\delta_E} \, d\mathcal{H}^{\delta_E}\right)\left(\int_{B\times\mathbb{P}} f \, d\mathcal{H}^{\delta_E}\right) = \left(\frac{1}{|\mathbb{P}|}\sum_{s\in\mathbb{P}} \varphi(s)\right)\left(\int_{B\times\mathbb{P}} f \, d\mathcal{H}^{\delta_E}\right)$$

  where $h_{2\delta_E} = h_{2\delta_B}/|\mathbb{P}|$
- vanishing because as before $\sum_{s\in\mathbb{P}} \varphi(s) = 0$

Vanishing Result: Strong Convergence

- $E = B \times \mathbb{P}$ a T–invariant set with GST
- for $\mathcal{H}^{\delta_E}$–almost every $\beta \in E$ we have $\lim_{n\to\infty} \frac{1}{2\lambda'_{2\delta}\, n} \sum_{k=1}^{n} \varphi(T^k s) = 0$
- treat $\varphi_k = \varphi(T^k s_0)$ as random variables: strong law of large numbers (again an ergodic theorem in fact)
- weak vanishing shows expectation $E = E(\varphi_k) = 0$
- evaluate deviations

$$D^2 = \int_E \left| \sum_{k=1}^{n} \varphi(T^k s_0) \right|^2 d\mathcal{H}^{2\delta_E}$$

- pairing $|\varphi_k|^2 = \langle \varphi_k, \varphi_k \rangle$

$$\langle \varphi(s), \varphi(t) \rangle := \{g(0), g(\infty)\}_G \bullet \{h(0), h(\infty)\}_G,$$

with $s = gG$, $t = hG$, and $\bullet$ the intersection product in the homology of mod curve

- want to evaluate the difference

$$\int \langle \varphi_k, \varphi_{k+j} \rangle - \langle \int \varphi_k, \int \varphi_{k+j} \rangle$$

- probabilities $P(g_k^{-1}(\beta) \cdot s_0 = s)$ that $g_k^{-1}(\beta) \cdot s_0 = s$

$$P(g_k^{-1}(\beta) \cdot s_0 = s) = \int_B dm_k(x, s) d\mathcal{H}^{2\delta}(x) = m_k(1, s)$$

- write difference above as

$$\sum_{s \in \mathbb{P}} \sum_{t \in \mathbb{P}} \langle \varphi(s), \varphi(t) \rangle \quad (P(g_k^{-1}(\beta) \cdot s_0 = s \text{ and } g_{k+j}^{-1}(\beta) \cdot s_0 = t)$$
$$- P(g_k^{-1}(\beta) \cdot s_0 = s) \cdot P(g_{k+j}^{-1}(\beta) \cdot s_0 = t)).$$

- want to show sufficiently well approximated by weakly dependent events (usual ergodicity argument)

$$P(g_k^{-1}(\beta) \cdot s_0 = s \text{ and } g_{k+j}^{-1}(\beta) \cdot s_0 = t)$$

$$= P(g_k^{-1}(\beta) \cdot t_0 = s) \cdot P(g_{k+j}^{-1}(\beta) \cdot s_0 = t)(1 + O(q^j))$$

- this weak dependence does not hold in general for the measures $m_n(x,s)$ but still true that over cylinder sets after applying a sufficient shift dependences on different sets of variables (same proof as for finite shift spaces)
- considering a truncation at some large N, with $x_k^* = [a_{k+1}, a_{k+2}, \dots, a_{k+N}]$, the probabilities

$$m_n^*(x,s) = \mathcal{H}^{2\delta}\left(\{\alpha \in [0,1] \,|x_k^*(\alpha) \leq x \text{ and } g_k^{-1}(\alpha)\cdot s_0 = s\}\right)$$

  satisfy the weak dependence condition
- so obtain the almost everywhere convergence as an ergodic theorem (take Hausdorff measure of the right dimension and then invariant measure that is absolutely continuous with respect to Hausdorff measure)

Henseley Cantor Sets

- explicit examples of T-invariant sets with these properties
- family of Cantor sets associated to the continued fraction expansion of numbers in $[0,1]$
- sets E_N given by numbers $\alpha \in [0,1]$ with $\alpha = [a_1, a_2, \ldots, a_\ell, \ldots]$, where all $a_i \leq N$
- Hausdorff dimensions which tend to one as $N \to \infty$ according to the asymptotic formula

$$\delta_N := \dim_H(E_N) = 1 - \frac{6}{\pi^2 N} - \frac{72 \log N}{\pi^4 N^2} + O(1/N^2)$$

- Gauss–Kuzmin operator

$$(\mathcal{L}_{\sigma,N} f)(\beta, t) = \sum_{k=1}^{N} \frac{1}{(\beta+k)^\sigma} f\left(\frac{1}{\beta+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \cdot t\right)$$

Sketch of argument for spectral properties on Henseley Cantor sets

- can consider these truncated Gauss–Kuzmin operators $\mathcal{L}_{\sigma,N}$ as perturbations of the Gauss–Kuzmin operator $\mathcal{L}_\sigma$ on $[0,1]$
- estimate error term $T_{\sigma,N} = \mathcal{L}_\sigma - \mathcal{L}_{\sigma,N}$

$$T_{\sigma,N} f(x,t) = \sum_{k=N+1}^{\infty} k^{-\sigma} \frac{1}{(x/k+1)^\sigma} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} s\right)$$

$$\| T_{\sigma,N} f \| \leq \sum_{k=N+1}^{\infty} k^{-\eta} \left\| \frac{1}{(1+x/k)^\sigma} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} s\right) \right\|$$

$$\leq \frac{C}{N^\eta} \| f \|,$$

for σ complex with real part $\Re(\sigma) = \eta > 1/2$

- then prove existence of a $\delta > 0$, such that, for $\| T_{\sigma,N} \| < \delta$, there is a unique $\rho_{\sigma,N} \in \mathbb{C}$, with $|\rho_{\sigma,N}| < \delta$, for which $L_\sigma - (\lambda_\sigma + \rho_{\sigma,N}) I$ is singular, and map $T_{\sigma,N} \mapsto \rho_{\sigma,N}$ is analytic

- use then to show exists a unique $f_{\sigma,N}$ satisfying

$$(\mathcal{L}_\sigma + T_{\sigma,N}) f_{\sigma,N} = (\lambda_\sigma + \rho_{\sigma,N}) f_{\sigma,N},$$

  and the map $T_{\sigma,N} \mapsto f_{\sigma,N}$ is also analytic
- eigenfunction for $\sigma = 2\delta_N$ is therefore of the form $f_N(\beta, t) = f_N(\beta)/|\mathbb{P}|$, with $f_N(\beta)$ the unique normalized eigenfunction of the truncated Gauss–Kuzmin operator of the Hensley Cantor set
- have corresponding vanishing results as stated above

• Note: all these vanishing results on average or almost everywhere, with non-vanishing at the quadratic irrationalities, shows the limiting modular symbols concentrates along the closed geodesics in X_G and averages out to zero elsewhere: an instance of the phenomenon of quantum chaos

Nontrivial cohomology classes?

- are there any nontrivial cohomology classes in the almost-everywhere sense associated to the boundary $\mathbb{P}^1(\mathbb{R})$?
- can construct something using another classical result on continued fractions
- Lévy identity: complex valued function f defined on pairs of coprime integers (q, q') with $q \geq q' \geq 1$ and with $f(q, q') = O(q^{-\epsilon})$ for some $\epsilon > 0$ then identity

$$\int_0^1 \ell(f, \beta) \, d\beta = \sum_{\substack{q \geq q' \geq 1 \\ (q, q') = 1}} \frac{f(q, q')}{q(q + q')}$$

$$\ell(f, \beta) := \sum_{k=1}^{\infty} f(q_k(\beta), q_{k-1}(\beta))$$

- use combination of Lévy identity and previous convergence results for Birkhoff averages to show that, for E a T–invariant subset of $[0,1] \times \mathbb{P}$ with GST, and with convergent Lyapunov exponent $0 < \lambda'_{2\delta_E} < \infty$, for $\mathcal{H}^{\delta_E}$-almost all $\beta \in E$, and for $\Re(t) > 0$ the limit

$$C(f,\beta) := \sum_{n=1}^{\infty} \frac{q_{n+1}(\beta) + q_n(\beta)}{q_{n+1}(\beta)^{1+t}} \left\{ 0, \frac{q_n(\beta)}{q_{n+1}(\beta)} \right\}_G$$

defines a class $C(f,\beta)$ in $H_1(X_G, \mathbb{R})$ that pairs with 1-forms to

$$\ell(f,\beta) = \int_{C(f,\beta)} \omega$$

- Note: specific to $G = \Gamma_0(N)$ but other similar constructions possible

Selberg's zeta function

- for subgroups of finite index $G \subset GL_2(\mathbb{Z})$ and $G_0 \subset SL_2(\mathbb{Z})$
- counting of geodesics on modular curves
- representation as Fredholm determinant
- g hyperbolic if $\mathrm{Tr}(g)$ and $D(g)$ are positive

$$D(g) := \mathrm{Tr}(g)^2 - 4\det(g), \quad N(g) := \left(\frac{\mathrm{Tr}(g) + D(g)^{1/2}}{2} \right)^2,$$

- hyperbolic matrix is primitive if not a nontrivial power of an element of $GL_2(\mathbb{Z})$

$$\chi_s(g) = \frac{N(g)^{-s}}{1 - \det(g)\, N(g)^{-1}}$$

- as before $\mathbb{P} := GL_2(\mathbb{Z})/G$ and $\rho_{\mathbb{P}}$ action of $GL_2(\mathbb{Z})$ on space of functions on $\mathbb{P}$

Selberg's zeta function

- Prim set of representatives of all $GL_2(\mathbb{Z})$-conjugacy classes of primitive hyperbolic elements of $GL_2(\mathbb{Z})$
- zeta function

$$Z_G(s) := \prod_{g\in\mathrm{Prim}} \prod_{m=0}^{\infty} \det\left(1 - \det(g)^m \, N(g)^{-s-m} \, \rho_{\mathbb{P}}(g)\right)$$

- Selberg zeta function and Gauss–Kuzmin operator
  - Selberg zeta function as determinant

$$Z_G(s) = \det(1 - L_s), \qquad Z_{G_0}(s) = \det(1 - L_s^2)$$

where L_s is the Gauss–Kuzmin operator of the shift of the continued fraction expansion as a nuclear operator acting on the space $B_{\mathbb{C}}$

  - from the identity $\det(1 - L_s) = Z_G(s)$ it follows that the zeroes of $Z_G(s)$ are exactly those values for which the deformed operator L_s has Perron–Frobenius eigenvalue $\lambda_s = 1$

Some identities

- operator $\pi_s(g)$ for a reduced matrix g as product of the π_{s,k_i}, and $\ell(g)$ length
- Hyp set of representatives of all conjugacy classes of hyperbolic matrices
- $k(g)$ maximal integer such that $g = h^{k(g)}$ and

$$\tau_g := \mathrm{Tr}(\rho_{\mathbb{P}}(g)) = \#\{s \in \mathbb{P} \,|\, g(s) = s\}$$

- then identities

$$-\log\det(1 - L_s) = \sum_{\ell=1}^{\infty} \frac{\mathrm{Tr} L_s^\ell}{\ell} = \mathrm{Tr}\left(\sum_{\ell=1}^{\infty} \frac{1}{\ell}\left(\sum_{n=1}^{\infty} \pi_{s,n}\right)^\ell\right) =$$

$$\mathrm{Tr}\left(\sum_{g\in\mathrm{Red}} \frac{1}{\ell(g)} \pi_s(g)\right) = \sum_{g\in\mathrm{Hyp}} \frac{1}{k(g)} \chi_s(g)\, \tau_g$$

- appearance of τ_g because $\pi_s(g)$ acts as tensor product of the $\pi_s(g)$ of $G = GL_2(\mathbb{Z})$ and $\rho_{\mathbb{P}}$ and trace is product of respective traces

- for semigroup Red

$$\sum_{g\in \mathrm{Hyp}} \frac{1}{k(g)} \chi_s(g)\, \tau_g = \sum_{g\in \mathrm{Prim}} \sum_{k=1}^{\infty} \frac{1}{k} \frac{N(g)^{-ks}\, \tau_{g^k}}{1-\det(g)^k\, N(g)^{-k}}$$

- then get

$$-\log Z_G(s) = \sum_{g\in \mathrm{Prim}} \sum_{m=0}^{\infty} \mathrm{Tr} \sum_{k=1}^{\infty} \frac{1}{k} \det(g)^{mk}\, N(g)^{-(s+m)k}\, \rho_{\mathbb{P}}(g^k)$$

$$= \sum_{g\in \mathrm{Prim}} \sum_{k=1}^{\infty} \frac{1}{k} \frac{N(g)^{-ks}\, \tau_{g^k}}{1-\det(g)^k\, N(g)^{-k}}$$

- to interpret in terms of closed geodesics on modular curves: if $G \subset GL_2(\mathbb{Z})$ lift of $G_0 \subset PSL_2(\mathbb{Z})$ then modular curve $X_{G_0} = G_0 \backslash \mathbb{H}$ identified with with $GL_2(\mathbb{Z})\backslash(\mathbb{H}\times\mathbb{P})$, any closed geodesic on X_{G_0} covered by geodesics $[\alpha_g^-, \alpha_g^+]$ on sheets that left invariant by hyperbolic matrix $g \in G$