Some References


Matilde Marcolli
Menger Sponge

- start with unit cube $I^3$
- divide into 27 cubes of side $1/3$
- remove central cube on each face and central cube in the middle
- repeat construction on each of the 20 remaining cubes . . .
Menger Sponge

- $n$-th stage $M_n$ of the construction of the Menger sponge consists of $20^n$ cubes

$$M = \bigcap_{n \in \mathbb{N}} M_n$$

of side $3^{-n}$, so that $\text{Vol}(M_n) = \left(\frac{20}{27}\right)^n$ and surface area $\Sigma(M_n) = 2\left(\frac{20}{9}\right)^n + 4\left(\frac{8}{9}\right)^n$

- volume goes to zero surface area to infinity: Hausdorff dimension is between 2 and 3

$$\dim_H(M) = \frac{\log 20}{\log 3} = 2.727\ldots$$

- each face is a Sierpinski carpet
- each intersection with a diagonal of the cube or a midline of the faces is a Cantor set
Topological dimension

- with the previous construction seen that the Menger sponge has Haudorff dimension $2 < \dim_H(M) < 3$
- so one would expect topological dimension is 2 but... topological dimension one $\dim_{\text{top}}(M) = 1$ (Menger curve)
- to see this use the following equivalent description of the topological dimension (for subsets of an ambient space $\mathbb{R}^N$): a space $M \subset \mathbb{R}^N$ has topological dimension $n$ if each point $x \in M$ has arbitrarily small neighborhoods $U$ such that $U \cap M$ is a set of topological dimension $n - 1$, and $n$ is the smallest non-negative integer with this property
Example: the Sierpinski Gasket has topological dimension 1
Example: Sierpinski Tetrahedron

Hausdorff dimension 2 (4 pieces, scaling 1/2) and topological dimension 1 (similar neighborhoods balls as for Sierpinski Gasket)
Example: the Koch Snowflake has topological dimension 1
Example: Sierpinski Carpet also has topological dimension 1 (like Sierpinski Gasket) and Menger Sponge also in a similar way more difficult to draw the right choice of neighborhoods here that make topological dimension 1 immediately visible.
Universality of the Menger Curve


- **universal property** of the Menger curve
  - universal space for the class of all compact metric spaces of topological dimension $\leq 1$
  - every such space embeds inside the Menger curve
- the Cantor set is similarly universal for all compact metric spaces of topological dimension 0 (and the Sierpinski carpet for Jordan curves)

- on embedding and universality properties
• A **continuum** is a connected compact metric (metrizable) topological space

• A **Peano continuum** is a locally-connected compact metrizable space

- **Menger curve** $M$ topologically characterized as a one-dimensional Peano continuum without locally separating points (for every connected neighbourhood $U$ of any point $x$ the set $U \setminus \{x\}$ is connected) and also without non-empty open subsets embeddable in the plane. Every one-dimensional Peano continuum can be embedded in $M$
$n$-dimensional Menger universal spaces


- Menger universal $M^m_n$-continuum
  - first step unit cube $I^m$
  - suppose at the $k$-th step of the construction have produced a configuration $\mathcal{F}_k$ of smaller $m$-cubes
  - at the $(k+1)$st step subdivide each cube $D$ in $\mathcal{F}_k$ into $3^{m(k+1)}$ subcubes with edges $3^{-m(k+1)}$
  - for each $D \in \mathcal{F}_k$ let $\mathcal{F}_{k+1}(D)$ be those smaller cubes that intersect the $n$-faces of $D$
  - take $\mathcal{F}_{k+1} = \bigcup_{D \in \mathcal{F}_k} \mathcal{F}_{k+1}(D)$
let $M^m_n(k) = \bigcup_{D \in \mathcal{F}_k} D \subset I^m$ union of the subcubes

$$M^m_n = \cap_{k=0}^{\infty} M^m_n(k)$$

- Menger curve is $M^3_1$
- Sierpinski carpet is $M^2_1$

**Universality of $M^m_n$**

- the Menger $M^m_n$-continuum is universal for all compact metric spaces (compacta) of topological dimension $\leq n$ that embed in $\mathbb{R}^m$ (Štanko, 1971)

- a continuum $X$ is homemorphic to $M^m_n$ iff it can be embedded in the sphere $S^{m+1}$ so that $S^{m+1} \setminus X$ has infinitely many connected components $C_i$ with $\text{diam}(C_i) \to 0$ and $\partial C_i \cap \partial C_j = \emptyset$ for $i \neq j$, the boundaries $\partial C_i$ are $m$-cells for each $i$ and $\bigcup_{i=1}^{\infty} \partial C_i$ is dense in $X$ (Cannon, 1973)
Universal mapping of Menger $M_n = M_n^{2n+1}$-continua


- (Bestvina, 1984): for $m \geq 2n + 1$ all the Menger compacta $M_n^m$ are homeomorphic

- $\exists$ continuous maps $f_n : M_n \to M_n$ universal in the class of maps between $n$-dimensional compacta

- $\forall f : X \to Y$ continuous map between $n$-dimensional compacta there are embeddings $\iota_X : X \hookrightarrow M_n$ and $\iota_Y : Y \hookrightarrow M_n$ such that commuting diagram up to homeomorphism

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\iota_X} & & \downarrow{\iota_Y} \\
M_n & \xrightarrow{f_n} & M_n
\end{array}
$$

- references added to the webpage
All Cantor sets are homeomorphic

- **Brouwer’s theorem**: a topological space is homeomorphic to the Cantor set if and only if it is non-empty, perfect, compact, totally disconnected, and metrizable


Cantor sets are projective limits of finite sets:

- projective system \( \{X_n\} \) of finite sets (discrete topology) with surjective maps \( \phi_{n,m} : X_n \to X_m \) for \( n > m \)
- projective limit \( X = \lim \leftarrow_n X_n \) is subspace of the product \( \prod_n X_n \) (with product topology)

\[
X = \{ x = (x_n) \in \prod X_n \mid x_m = \phi_{n,m}(x_m), \forall n \leq m \}
\]

- either use characterization above or construct a coding by strings on an alphabet
Categorical view of the Menger curve $M = M^3_1$


- **Menger prespace** $\mathbb{M}$ generic inverse limit in the category of finite connected graphs with surjective graph homomorphisms

- **Edge relation**: equivalence relation $\mathcal{R}$ on $\mathbb{M}$

- **Menger curve**: quotient by this equivalence $M = \mathbb{M}/\mathcal{R}$

- **Topological realization** $M = |\mathbb{M}|$ of combinatorial object $\mathbb{M}$
Category of graphs

- A graph $G$ is a pair $(V, R_V)$ where $V$ is a set (vertices) and $R_V \subseteq V \times V$ is a relation that is reflexive ($(v, v) \in R_V$) and symmetric ($(v, w) \in R_V \iff (w, v) \in R_V$) defining edges.

- Note nonconventional assumption that $(v, v) \in R_V$ (like presence of a “trivial” looping edge at each vertex).

- Homomorphism of graphs: function $f : V \rightarrow V'$ preserving edge relations (if $(v, w) \in R_V$ then $(f(v), f(w)) \in R_{V'}$); epimorphism if surjective on vertices and edges.

- Only consider induced subgraphs: subset of vertices $V$ and all edges of $R_V$ between them.

- Category: $C$ objects finite connected graphs morphisms between them that are connected (preimage of each connected subset of target is a connected subset of source graph).

- Epimorphism between connected graphs is connected iff preimages of vertices are connected.
• Projective limits of finite graphs
  
  - topological graph \((K, \mathcal{R}_K)\) with \(K\) a zero-dimensional compact metrizable topological space and \(\mathcal{R}_K \subset K \times K\) closed subset, continuous morphisms
  - for finite graph discrete topology
  - inverse system \(f^n_m : V_n \to V_m\) with \(f_{n,n} = \text{id}\) and \(f_{n,m} \cdot f_{m,k} := f_{m,k} \circ f_{n,m} = f_{n,k}\) for \(n \geq m \geq k\)
  - inverse limit is a topological graph
    
    \[(K, \mathcal{R}_K) = \lim_{\leftarrow n} (V_n, \mathcal{R}_{V_n})\]

  - no longer a finite graph in general: set of vertices \(K\) is Cantor-like
  - viewing projective limit as subset of product, \(x = (x_0, x_1, x_2, \ldots) \in K\) with \(x_i \in V_i\) and with projections \(f_i : K \to V_i\) satisfying \(f_{i,j} \circ f_i = f_j\)
  - connectedness: point \(x = (x_0, x_1, x_2, \ldots)\) and \(y = (y_0, y_1, y_2, \ldots)\) connected in \(K\) iff \(x_i\) connected to \(y_i\) in \(V_i\) for all coordinates
• Category of projective limits of finite graphs

  objects are projective limits \( K = \varprojlim_n (V_n, R_{V_n}, f_{n,m}) \) and morphisms are connected epimorphisms between these topological graphs

  \( K \) connected and locally-connected, coordinatewise in \( \prod_n K_n \) each \( f_{m,n}^{-1}(v) \) connected

  morphism of projective limits \( h : K \to L \) with \( K = \varprojlim_n K_n \) and \( L = \varprojlim_n L_n \) then for all \( m \) there is \( n \) and \( h_{n,m} : K_n \to L_m \) such that \( h_{n,m} \circ f_n = \ell_m \circ h \) for \( f_n : K \to K_n \) and \( \ell_m : L \to L_m \) projections, with \( h_{n,m} \) connected epimorphism of finite graphs so \( h : K \to L \) is connected epimorphism of topological graphs
• Conversely all connected and locally-connected topological graphs with connected epimorphisms are obtained as projective limits and morphisms of projective limits of finite graphs.

• $K$ has topology with a basis of connected clopen sets; can extract from this a sequence $U_n$ of finite coverings with $U_n$ a refinement of $U_{n-1}$ such that different $U, V \in U_n$ have $U \cap V = \emptyset$ and $\cup_n U_n$ separates vertices of $K$.

• Give to $U_n$ a graph structure by putting an edge between $U$ and $V$ iff $\exists x, y$ with $x \in U$ and $y \in V$ such that $(x, y) \in R_K$.

• Then have projection maps between these graphs $f_{n,m} : U_n \rightarrow U_m$ that are connected epimorphisms and $K = \lim_{\rightarrow n} U_n \text{ proj limit of graphs}$.
connected epimorphism $h : K \to L$ of connected and locally-connected topological graphs: know $K = \lim_{n} K_n$ and $L = \lim_{m} L_m$ with projections $f_n : K \to K_n$ and $\ell_m : L \to L_m$, so need to show for all $m$ there is $n$ and $h_{n,m} : K_n \to L_m$ with $\ell_m \circ h = h_{n,m} \circ f_n$

for given $m$ pick $n$ large enough that $f_n^{-1}(K_n)$ is a refinement of $(\ell_m \circ h)^{-1}(L_m)$, then there is a map $h_{n,m} : K_n \to L_m$ that is defined through this inclusion so that $\ell_m \circ h = h_{n,m} \circ f_n$

because $\ell_m$, $h$, $f_n$ are connected epimorphisms $h_{n,m}$ also is

- category of projective limits of finite graphs is same as category of connected and locally-connected topological graphs with connected epimorphisms
• Topological graphs and Peano continua

  • Peano continuum: locally-connected compact metrizable space

  • prespace: connected and locally-connected topological graph $K$ where the edge relation $\mathcal{R}_K$ is transitive (hence an equivalence relation)

  • any equivalence relation on a finite set gives a graph on that set of vertices that consists of a disjoint union of cliques (complete graphs) so for finite connected graphs just cliques

  • realization $|K|$ of a prespace $K$: topological space given by quotient $K/\mathcal{R}_K$

• Claim: $X$ Peano continuum iff $X = |K|$ for some prespace $K$

• A. Panagiotopoulos, S. Solecki, A combinatorial model for the Menger curve, arXiv:1803.02516
Projective Fraïssé class

- any sub-collection of pairwise non-isomorphic objects is countable
- identity maps in the class and maps in the class closed under composition (ok if morphisms of a category)
- for any objects $B, C$ in the class there is an object $D$ with morphisms $f : D \rightarrow B$ and $g : D \rightarrow C$
- for every morphisms $f' : B \rightarrow A$ and $g' : C \rightarrow A$ there are morphisms $f : D \rightarrow B$ and $g : D \rightarrow C$ with $f' \circ f = g' \circ g$ (projective amalgamation property)

- finite connected graphs with connected epimorphisms are a projective Fraïssé class
Menger Prespace

- given a projective Fraïssé class (here the one of finite graphs) there is an object $M$ (projective limit of a “generic sequence” of objects in the class) such that
  - for each object $A$ in the class there is a morphism $f : M \to A$ (morphism in the category of projective limits)
  - for any $A, B$ in the class and morphisms $f : M \to A$ and $g : B \to A$ there is a morphism $f : M \to B$ with $f = g \circ h$ (projective extension property)

- this $M$ is the Menger prespace
by first property of Fraïssé class have countable $A_n$ and $e_n : C_n \to B_n$ containing all isomorphism types of objects and morphisms

- inductive construction of projective system: $L_0 = A_0$ and assume have $L_n$ with maps $t_{n,i} : L_n \to L_i$ for $i < n$

by third property of Fraïssé class find $H$ with maps $f : H \to L_n$ and $g : H \to A_{n+1}$ and with a finite number $s_1, \ldots, s_k$ of morphisms (up to isoms) $s_i : H \to B_{n+1}$

- use $k$ times projective amalgamation to obtain $H'$ with maps $f' : H' \to H$ and $d_j : H' \to C_{n+1}$ with $s_j \circ f' = e_{n+1} \circ d_j$ for all $j \leq k$

- take $L_{n+1} = H'$ with $t_{n+1,i} = t_{n,i} \circ f \circ f'$

the way $(L_n, t_{n,i})$ constructed gives the two properties above of $\mathbb{M}$

- **Menger Prespace and Menger Curve**: realization $|\mathbb{M}|$ is a topologically one-dimensional Peano continuum without locally separating points, hence it is the Menger curve
Statements of some properties of Menger prespace and curve (Panagiotopoulos, Solecki)

- **Homogeneity**
  - $K$ closed subgraph of $\mathbb{M}$ “locally non-separating” if for each clopen connected $W$ in $\mathbb{M}$ the complement $W \setminus K$ is connected
  - $K, L$ locally non-separating subgraphs of $\mathbb{M}$: any isomorphism $f : K \rightarrow L$ extends to an automorphism of $\mathbb{M}$

- **Lifting property**
  - $K$ locally non-separating subgraph of $\mathbb{M}$: given finite graphs $A, B$ and connected epimorphisms $g : B \rightarrow A$ and $f : \mathbb{M} \rightarrow A$, for any morphism $p : K \rightarrow B$ with $g \circ p = f|_K$ there is $h : \mathbb{M} \rightarrow B$ with $g \circ h = f$ and $h|_K = p$

- **Universality**
  - for any Peano continuum $X$ there is a continuous connected surjective map $f : |\mathbb{M}| \rightarrow X$
Higher dimensional analogs

- a more general higher-dimensional theory of inverse limits of $n$-dimensional polyhedra with simplicial finite-to-one projections


- Menger compacta and inverse limits categories

  - simplicial sets/simplicial complexes (more about them later)
    - $k$-connected if homotopy groups $\pi_i$ vanish for $i \leq k$; simplicial map $k$-connected if preimage of every $k$-connected subcomplex is $k$-connected
  - category $C_n$ of all $n$-dimensional and $(n - 1)$-connected simplicial complexes with $(n - 1)$-connected simplicial maps
  - generic sequences and projective limit gives $n$-dimensional prespaces $\mathbb{M}_n$ with realization (with respect to faces relation) is Menger space $M_n = M_n^{2n+1}$
Can use Menger compacta as model spaces for fractals? just like simplicial sets or cubical sets?

- Simplicial sets in topology
  - J. Peter May, *Simplicial objects in algebraic topology*, University of Chicago Press, 1992

- Simplicial set: sequence of sets $X = \{X_n\}_{n \geq 0}$ with maps (faces and degeneracies) $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$

  \[
  \begin{align*}
  d_id_j &= d_{j-1}d_i, & \text{if } i < j, \\
  d_is_j &= s_{j-1}d_i, & \text{if } i < j, \\
  d_js_j &= d_{j+1}s_j = \text{id}, \\
  d_is_j &= s_{j+1}d_i, & \text{if } i > j + 1, \\
  s_is_j &= s_{j+1}s_i, & \text{if } i \leq j.
  \end{align*}
  \]
Categorical version of simplicial sets

- **Δ category**: objects finite ordered sets $[n] := \{1, 2, \ldots, n\}$ and morphisms $f : [m] \to [n]$ order-preserving functions: $f(i) \leq f(j)$ for $i \leq j$
- morphisms are generated by maps $D_i : [n] \to [n + 1]$ and $S_i : [n + 1] \to [n]$
  
  $D_i[0, \ldots, n] = [0, \ldots, \hat{i}, \ldots, n], \quad S_i[0, \ldots, n] = [0, \ldots, i, i, \ldots, n]$

- in $\Delta^{op}$ the $D_i$ become face maps $d_i : [n + 1] \to [n]$ and $S_i$ the degeneracy maps $s_i : [n] \to [n + 1]$

![Diagram](image.png)

image of degeneracy $s_1$ degenerate 2-simplex image of collapse map $S_1$
- **Simplicial set**: functor \( X : \Delta^{op} \to S \) to the category of sets (contravariant functor from \( \Delta \))

- **Realization**: \( |\Delta^n| \) the geometric simplex realization of combinatorial \( \Delta^n = [n] \)
  \[
  |X| := \sqcup_n (X_n \times |\Delta^n|) / \sim
  \]

  modulo equivalence relation \((x, S_i(t)) \sim (s_i(x), t)\) and \((x, D_i(t)) \sim (d_i(x), t)\)

- Interpret as recipe for gluing the geometric simplexes \(|\Delta^n|\) together according to the combinatorial scheme prescribed by the \( X_n \) so that faces and degeneracies match
• **Nerve**: simplicial sets from categories
  
  category $\text{Cat}$ of small categories with functors as morphisms, nerve functor $\mathcal{N} : \text{Cat} \to \Delta S$ to the category of simplicial sets $\Delta S = \text{Func}(\Delta^{op}, S)$
  
  for a small category $\mathcal{C}$ the nerve $\mathcal{N}(\mathcal{C})$ has a 0-simplex (vertex) for each object of $\mathcal{C}$, a 1-simplex (edge) for each morphism, a 2-simplex for each composition of two morphisms, a $k$-simplex for every chain of $k$ composable morphisms
  
  face maps: composition of two adjacent morphisms at the $i$-th place of a $k$-chain $d_i : \mathcal{N}_k(\mathcal{C}) \to \mathcal{N}_{k-1}(\mathcal{C})$ and degeneracies are insertions of the identity morphism at an object in the chain
• **Products**: product of simplexes is not a simplex but can be decomposed as a union of simplexes

Cubes behave better than simplexes with respect to products
Simplicial and cubical complexes
- Cubical sets in topology
  - $\mathcal{I}$ unit interval as combinatorial structure consisting of two vertices and an edge connecting them
  - $|\mathcal{I}| = [0, 1]$ geometric realization: unit interval as topological space (subspace of $\mathbb{R}$)
  - $\mathcal{I}^n$ for the $n$-cube as combinatorial structure and $|\mathcal{I}^n| = [0, 1]^n$ its geometric realization
  - $\mathcal{I}^0$ a single point
  - face maps $\delta^a_i : \mathcal{I}^n \to \mathcal{I}^{n+1}$, for $a \in \{0, 1\}$ and $i = 1, \ldots, n$

$$\delta^a_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, a, t_i, \ldots, t_n)$$

- degeneracy maps $s_i : \mathcal{I}^n \to \mathcal{I}^{n-1}$

$$s_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$$
- Cubical relations for $i < j$

\[ \delta_j^b \circ \delta_i^a = \delta_i^a \circ \delta_{j-1}^b \quad \text{and} \quad s_i \circ s_j = s_{j-1} \circ s_i \]

and relations

\[ \delta_i^a \circ s_{j-1} = s_j \circ \delta_i^a \quad i < j \]

\[ s_j \circ \delta_i^a = 1 \quad i = j \]

\[ \delta_{i-1}^a \circ s_j = s_j \circ \delta_i^a \quad i > j \]

- Cube category: $\mathcal{C}$ has objects $I^n$ for $n \geq 0$ and morphisms generated by the face and degeneracy maps $\delta_i^a$ and $s_i$

- Cubical set: functor $C : \mathcal{C}^{op} \to \mathcal{S}$ to the category of sets.

- Notation: $C_n := C(I^n)$
variant of the cube category $\mathcal{C}_c$ with additional degeneracy maps $\gamma_i : I^n \to I^{n-1}$ called connections

$$\gamma_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \max\{t_i, t_{i+1}\}, t_{i+2}, \ldots, t_n)$$

satisfying relations

$$\gamma_i \gamma_j = \gamma_j \gamma_{i+1}, \quad i \leq j; \quad s_j \gamma_i = \begin{cases} \gamma_i s_j + 1 & i < j \\ s_i^2 = s_i s_{i+1} & i = j \\ \gamma_{i-1} s_j & i > j \end{cases}$$

$$\gamma_j \delta_i^a = \begin{cases} \delta_i^a \gamma_{j-1} & i < j \\ 1 & i = j, j + 1, \ a = 0 \\ \delta_j^a s_j & i = j, j + 1, \ a = 1 \\ \delta_i^{a-1} \gamma_j & i > j + 1. \end{cases}$$

role of degeneracy maps: maps $s_i$ identify opposite faces of a cube, additional degeneracies $\gamma_i$ identify adjacent faces
cubical set with connection: functor $C : C^{op} \to \mathcal{S}$ to the category of sets

category of cubical sets has these functors as objects and natural transformations as morphisms

so morphisms given by collection $\alpha = (\alpha_n)$ of morphisms $\alpha_n : C_n \to C'_n$ satisfying compatibilities $\alpha \circ \delta^a_i = \delta^a_i \circ \alpha$ and $\alpha \circ s_i = s_i \circ \alpha$ (and in the case with connection $\alpha \circ \gamma_i = \gamma_i \circ \alpha$)

cubical nerve $\mathcal{N}_c C$ of a category $C$ is the cubical set with

$$(\mathcal{N}_c C)_n = \text{Fun}(I^n, C)$$

with $I^n$ the $n$-cube seen as a category with objects the vertices and morphisms generated by the 1-faces (edges), and $\text{Fun}(I^n, C)$ is the set of functors from $I^n$ to $C$

when working with cubical sets with connection homotopy equivalent to simplicial nerve

Building an analog of cubical sets using Menger spaces

- Menger spaces $M^m_n$ are modelled on cubes, so want the same faces and degeneracy maps (and connections) as in the cube category
- additional important data: the self-similarity structure
- the iterated function system $\{f_1, \ldots, f_N\}$ given by the affine contraction maps that take the cube $I^m$ to the $N$ smaller cubes, scaled by a factor $3^{-m}$, where $N = N(m, n)$ is the number of those subcubes that intersect the $n$-faces of $I^m$
- Menger category $\mathcal{M}$ with objects the $M^m_n$ (or better their corresponding combinatorial spaces $\mathcal{M}^m_n$) and morphisms generated by the $\delta_i^a$, $s_i$, $\gamma_i$, and the IFS maps $f_k$
- Menger sets: functors $M : \mathcal{M}^{op} \to \mathcal{S}$ to the category of sets
there is a good homology theory for cubical sets (introduced by Serre to study (co)homology of fibrations) see

it is also known that cubical sets with connections have the same homology theory as ordinary cubical sets (connections do not add any nontrivial cycles in the homology groups)

What is the effect on homology of introducing the contraction maps of the IFS for the Menger spaces? directed system of homology groups of cubical sets?

Similar approach with simplicial sets? (Sierpinski $n$-simplex instead of Menger space?)

What does this approach lead to?