Measure Theory
Introduction to Fractal Geometry and Chaos

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Measure Spaces rigorous theory of "volumes" (and of probabilities)

- measure space (X, Σ, μ)
  - σ-algebra Σ on a set X
    - Σ collection of subsets of X such that
    - $\emptyset$ and X belong to Σ
    - *complements*: $A^c = X \setminus A$ is in Σ whenever A is in Σ
    - *countable unions*: $\bigcup_{k=1}^{\infty} A_k$ is in Σ if $A_k \in \Sigma$ for all $k \in \mathbb{N}$
  - measure $\mu : \Sigma \to \mathbb{R}_+ \cup \{\infty\}$
    - $\mu(\emptyset) = 0$
    - *monotonicity*: $\mu(A_1) \leq \mu(A_2)$ if $A_1 \subseteq A_2$
    - *disjoint additivity*: $\mu(\sqcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$ additive on disjoint unions

- probability distribution: a measure space where $\mu(X) = 1$
  - $A \in \Sigma$ events
  - $\mu(A)$ their probabilities

Examples

1. $\Sigma = 2^X$ all subsets of X and $\mu = \delta_x$ (Dirac delta measure supported at a chosen point $x \in X$) with

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

2. $\Sigma = 2^X$ and counting (cardinality) measure

$$\mu(A) = \begin{cases} \#A & A \text{ finite} \\ \infty & A \text{ infinite} \end{cases}$$

3. If X finite, counting measure can be normalized to a probability (uniform probability)

$$\mu(A) = \frac{\#A}{\#X}$$

Support of a measure

- measure space (X, Σ, μ)
- assume Borel measure: $(X, \mathcal{T})$ topological space and open sets $U \in \mathcal{T}$ of the topology are all measurable sets $U \in \Sigma$
- support of μ

$$\mathrm{supp}(\mu) = \{x \in X \,|\, \mu(U) > 0 \,\forall U \in \mathcal{T} \text{ with } x \in U\}$$

- an open set $U \in \mathcal{T}$ has $\mu(U) > 0$ iff $U \cap \mathrm{supp}(\mu) \neq \emptyset$ and $\mu(U) = 0$ otherwise
- Dirac delta measure δ_x has $\mathrm{supp}(\delta_x) = \{x\}$

Continuity of measures

- limits of sets

$$\liminf_{n\to\infty} A_n = \cup_{k=1}^{\infty} \cap_{j=k}^{\infty} A_j \quad \limsup_{n\to\infty} A_n = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} A_j$$

same for increasing or decreasing sequence

- (X, Σ, μ) measure space
  - increasing sequence $A_1 \subset A_2 \subset \cdots A_n \subset \cdots$

$$\mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

  - decreasing sequence $A_1 \supset A_2 \supset \cdots A_n \supset \cdots$

$$\mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

  - arbitrary sequences

$$\mu(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mu(A_n)$$

- increasing sequence: limit is union, use $\cup_{n=1}^{\infty} A_n = A_1 \sqcup_{k=2}^{\infty} (A_k \smallsetminus A_{k-1})$ and additivity of measure

$$\mu(\cup_k A_k) = \mu(A_1) + \sum_{k=2}^{\infty} \mu(A_k \smallsetminus A_{k-1})$$

$$= \lim_k \mu(A_1 \sqcup_{k=2}^{\infty} (A_k \smallsetminus A_{k-1})) = \lim_k \mu(A_k)$$

- decreasing sequence: limit is intersection, use $B_k = A_1 \smallsetminus A_k$ increasing sequence

$$\mu(\cap_k A_k) = \mu(A_1) - \mu(\cup_j B_j) = \mu(A_1) - \lim_j \mu(B_j) = \lim_j \mu(A_j)$$

- arbitrary sequence: use $B_k = \cap_{j=k}^{\infty} A_j$ increasing sequence

$$\mu(\liminf_{n\to\infty} A_n) = \mu(\cup_{k=1}^{\infty} B_k) = \lim_k \mu(B_k) \leq \liminf_k \mu(A_k)$$

Almost everywhere properties

• (X, Σ, μ) measure space, a property of X holds *almost everywhere* if it fails to hold on a set of μ-measure zero

• Examples:

- X with δ_x Dirac measure: two functions $f_1, f_2 : X \to \mathbb{R}$ agree almost everywhere iff $f_1(x) = f_2(x)$ at the point x where δ_x is supported (sets of measure zero = those not containing x)
- X with counting measure: two functions $f_1, f_2 : X \to \mathbb{R}$ agree almost everywhere iff $f_1(x) = f_2(x)$ at all points $x \in X$ (no non-empty sets of measure zero)

• Note: in the case of probability spaces probability zero does not mean impossibility

• How to construct more interesting measures than the Dirac and counting examples?

Outer measures and measures

- $\mu^* : 2^X \to \mathbb{R}_+ \cup \{\infty\}$ defined on all subsets with
  - $\mu^*(\emptyset) = 0$
  - $\mu(A_1) \leq \mu(A_2)$ if $A_1 \subseteq A_2$
  - *countable subadditivity*:

$$\mu^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

  even when disjoint union require only $\leq$ not additivity

- use an outer measure μ^* to construct a σ-algebra Σ and a measure space (X, Σ, μ)
  - σ-algebra determined by μ^*

$$\Sigma := \{E \subseteq X \;:\; \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap (X \smallsetminus E)), \;\; \forall A \subseteq X\}$$

  - $\mu = \mu^*|_\Sigma$ is a measure (the definition of Σ forces additivity on disjoint unions)

- σ-algebra property of

$$\Sigma := \{E \subseteq X \,:\, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap (X \smallsetminus E)), \;\; \forall A \subseteq X\}$$

- $\emptyset$ and X are in Σ
- condition defining Σ is symmetric in E and $X \smallsetminus E$ so if a set is in Σ its complement also is
- given a sequence of sets E_k in Σ

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \smallsetminus E_1) =$$

$$\mu^*(A \cap E_1) + \mu^*((A \smallsetminus E_1) \cap E_2) + \mu^*((A \smallsetminus E_1) \smallsetminus E_2) =$$

$$\sum_{j=1}^{k} \mu^*((A \smallsetminus \cup_{i=1}^{j-1} E_i) \cap E_j) + \mu^*(A \smallsetminus \cup_{j=1}^{k} E_j)$$

$$\geq \sum_{j=1}^{k} \mu^*((A \smallsetminus \cup_{i=1}^{j-1} E_i) \cap E_j) + \mu^*(A \smallsetminus \cup_{j=1}^{\infty} E_j)$$

- estimate for all k so in the limit

$$\mu^*(A) \geq \sum_{j=1}^{\infty} \mu^*((A \setminus \cup_{i=1}^{j-1} E_i) \cap E_j) + \mu^*(A \setminus \cup_{j=1}^{\infty} E_j)$$

- then use $A \cap \cup_{j=1}^{\infty} E_j = \cup_{j=1}^{\infty}((A \setminus \cup_{i=1}^{j-1} E_i) \cap E_j)$

$$\mu^*(A) \leq \mu^*(A \cap \cup_{j=1}^{\infty} E_j) + \mu^*(A \setminus \cup_{j=1}^{\infty} E_j)$$

  and by previous also

$$\mu^*(A \cap \cup_{j=1}^{\infty} E_j) + \mu^*(A \setminus \cup_{j=1}^{\infty} E_j) \leq \mu^*(A)$$

- so $\cup_{j=1}^{\infty} E_j \in \Sigma$ closed under countable unions

- restriction $\mu = \mu^*|_\Sigma$ is a measure
  - disjoint union of sets E_i in Σ, use estimate

$$\mu^*(A) \geq \sum_{j=1}^{\infty} \mu^*((A \setminus \sqcup_{i=1}^{j-1} E_i) \cap E_j) + \mu^*(A \setminus \sqcup_{j=1}^{\infty} E_j)$$

  - applied to $A = \sqcup_j E_j$ gives

$$\mu^*(\sqcup_{j=1}^{\infty} E_j) \geq \sum_{j=1}^{\infty} \mu^*(E_j)$$

  - together with subadditivity of the outer measure gives additivity on disjoint unions in Σ

Carathéodory construction

- General idea:
  1. start with a class of sets $\mathcal{S}$ for which have a good definition of what an extension ℓ (length, area, volume) should be (example: intervals on the real line and their length)
  2. define an outer measure on all subsets of the ambient X by setting
  $$\mu^*(A) := \inf_{A \subset \cup_k S_k} \sum_k \ell(S_k)$$
  infimum over all sequences of sets in the sample class whose union covers A
  3. then pass to σ-algebra and measure (X, Σ, μ) defined by this outer measure
  4. this measure μ will still have the property that it agrees with the original ℓ on sets $S \in \mathcal{S}$ belonging to the chosen sample class, $\mu(S) = \ell(S)$

Existence of non-measurable sets

• distinction between the σ-algebra Σ defined by an outer measure and the full set 2^X of all subsets of X lies in the possible existence of non-measurable sets

• for the case of $\mathcal{S}$ = intervals in the real line and ℓ = length, with μ the Lebesgue measure on $\mathbb{R}$ (translation invariant measure) existence of non-measurable sets not constructive: use of the axiom of choice

• existence of non-measurable sets on the sphere S^2 with respect to the rotation invariant measure related to Banach-Tarski paradox of duplication of the sphere

Non-measurable sets: Banach-Tarski paradox

- F_2 free group on two generators a, b
- all words of arbitrary length in the letters a, b, a^{-1}, b^{-1} with no cancellations (no a preceded or followed by a^{-1} and no b preceded or followed by b^{-1})
- additional "empty word $\emptyset$ (identity element of the group)
- product = concatenation of words (with elimination of cancellations)

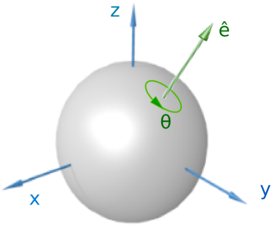
$$F_2 = \{\emptyset\} \cup \mathcal{W}(a) \cup \mathcal{W}(b) \cup \mathcal{W}(a^{-1}) \cup \mathcal{W}(b^{-1})$$

$\mathcal{W}(x)$ = words starting with letter x
can also write as

$$F_2 = \mathcal{W}(a) \cup a\,\mathcal{W}(a^{-1}) \quad \text{or} \quad F_2 = \mathcal{W}(b) \cup b\mathcal{W}(b^{-1})$$

piece $a\,\mathcal{W}(a^{-1})$ contains all other possibilities because of cancellation of aa^{-1}, same for $b\mathcal{W}(b^{-1})$

- realize F_2 as a subgroup of rigid motions of the sphere
  - choose two different axes of rotation e_1, e_2 (straight lines through the center of the sphere)
  - choose two angles θ_1, θ_2 that are not rationally related (θ_i/π lin independent over $\mathbb{Q}$)
  - group generated by $a = R_{e_1,\theta_1}$ and $b = R_{e_2,\theta_2}$



- choose a point P on the sphere not on the axes e_1, e_2
- take orbits of P under transformations in each set $\mathcal{W}(a)$, $\mathcal{W}(b)$, $\mathcal{W}(a^{-1})$, $\mathcal{W}(b^{-1})$
- these orbits do not cover the whole sphere S^2
- keep choosing new points outside of the orbits already obtained and take their orbits under the same sets of transformations until the whole S^2 is covered
- Note: this requires infinitely many choices (axiom of choice is implicit here)
- Then choose a point x_α from each of the orbits constructed (Note: again infinitely many choices)
- obtain 4 disjoint sets covering the sphere S^2

$$A = \cup_\alpha \mathcal{W}(a) x_\alpha, \quad B = \cup_\alpha \mathcal{W}(b) x_\alpha,$$

$$C = \cup_\alpha \mathcal{W}(a^{-1}) x_\alpha, \quad D = \cup_\alpha \mathcal{W}(b^{-1}) x_\alpha$$

- duplication of the sphere (Banach-Tarski paradox)
  - break S^2 into the pieces A, B, C, D
  - transform $C = \cup_\alpha \mathcal{W}(a^{-1})x_\alpha$ by the rigid motion $a = R_{e_1,\theta_1}$ into $C' = \cup_\alpha a\mathcal{W}(a^{-1})x_\alpha$ and transform $D = \cup_\alpha \mathcal{W}(b^{-1})x_\alpha$ by rigid motion $b = R_{e_2,\theta_2}$ into $D' = \cup_\alpha b\mathcal{W}(b^{-1})x_\alpha$
  - by $F_2 = \mathcal{W}(a) \cup a\mathcal{W}(a^{-1}) = \mathcal{W}(b) \cup b\mathcal{W}(b^{-1})$ and $\cup_\alpha F_2 x_\alpha = S^2$ obtain two copies of S^2 by reassembling these pieces as

$$S^2 = A \cup C' = \cup_\alpha \mathcal{W}(a)x_\alpha \bigcup \cup_\alpha a\mathcal{W}(a^{-1})x_\alpha$$

$$S^2 = B \cup D' = \cup_\alpha \mathcal{W}(b)x_\alpha \bigcup \cup_\alpha b\mathcal{W}(b^{-1})x_\alpha$$

The decomposition into sets A, B, C, D necessarily involves non-measurable sets!

Hausdorff measures $\mu_{H,s}$

• ambient space $X = \mathbb{R}^N$ some fixed N with Euclidean distance (or more generally a metric space (X, d))

• Hausdorff outer measure for a given $s \in \mathbb{R}_+^*$

$$\mu^*_{H,s}(A) := \lim_{\epsilon \to 0} \inf_{\mathcal{U}} \sum_\alpha (\operatorname{diam} U_\alpha)^s$$

where $\mathcal{U} = \{U_\alpha\}$ an open covering of A with diameters at most ϵ

• Hausdorff measure $\mu_{H,s}$ obtained from the Hausdorff outer measure via the Carathéodory construction

Behavior of the Hausdorff measures

• for fixed $\epsilon > 0$ define $\mu_{H,\epsilon,s}(E) := \inf_{\mathcal{U}} \sum_\alpha (\mathrm{diam}\, U_\alpha)^s$ for $E \subset \mathbb{R}^N$, infimum over all countable coverings $\mathcal{U} = \{U_\alpha\}$ with $0 < \mathrm{diam}\, U_\alpha \leq \epsilon$

- For a given $E \subset \mathbb{R}^N$ in the fixed ambient space, the function $s \mapsto \mu_{H,s}(E)$ is non-increasing as s grows from 0 to ∞
- If $s < t$ then $\mu_{H,\epsilon,s}(E) \geq \epsilon^{s-t} \mu_{H,\epsilon,t}(E)$
- this implies that if for E at some t have $\mu_{H,t}(E) > 0$ then for all $s < t$ have $\mu_{H,s}(E) = \infty$
- also if for some s have $\mu_{H,s}(E) < \infty$ then for all $t > s$ have $\mu_{H,t}(E) = 0$
- on a fixed set E values of Hausdorff measure as a function of s starts at ∞, somewhere jumps to 0 then remains 0; at the unique s where it jumps can have an intermediate value $0 \leq \mu_{H,s}(E) \leq \infty$

Hausdorff dimension   rigorous definition

• the Hausdorff dimension $\dim_H(E)$ is the value of s where the Hausdorff measure $\mu_{H,s}(E)$ jumps from ∞ to 0

$$\mu_{H,s}(E) = \begin{cases} \infty & s < \dim_H(E) \\ 0 & s > \dim_H(E) \end{cases}$$

• s-sets and Hausdorff measure

- s-set $E \subset \mathbb{R}^N$: if the value of the Hausdorff measure at $s = \dim_H(E)$ is neither zero nor infinity

$$0 < \mu_{H,\dim_H(E)}(E) < \infty$$

especially "well behaved" sets with respect to the Hausdorff measure: the measure $\mu_{H,s}$ defines a good s-dimensional volume on E

properties of Hausdorff measures and dimension

- affine transformations and the Hausdorff measure
  - $E \subset \mathbb{R}^N$, given any $v \in \mathbb{R}^N$, $\lambda \in \mathbb{R}_+^*$ and $s \in \mathbb{R}_+$
$$\mu_{H,s}(E+v) = \mu_{H,s}(E) \quad \text{and} \quad \mu_{H,s}(\lambda E) = \lambda^s \mu_{H,s}(E)$$
  - these properties follow directly from the definition of the Hausdorff measure
  - it follows (Haar measure for the group of translations) that $\mu_{H,n}$ on $\mathbb{R}^n$ agrees (up to a constant scaling factor $C_n > 0$) with the Lebesgue measure of dimension n

- properties of the Hausdorff dimension
  - *monotonicity*: for $A \leq B$ have $\dim_H(A) \leq \dim_H(B)$
  - *countably stability*: $\dim_H(A) = \sup_k \dim_H(A_k)$ for $A = \cup_k A_k$ inside ambient $\mathbb{R}^N$
  - *vanishing*: $\dim_H(A) = 0$ for A countable set
  - *smooth manifolds*: $\dim_H(M) = m$ if $M \subset \mathbb{R}^N$ is a smooth manifold of dimension $m \leq N$

Lipschitz maps and Hausdorff measures/dimension

- Lipschitz map $f : (X, d_X) \to (Y, d_Y)$

$$d_Y(f(x), f(y)) \leq C\, d_X(x, y), \quad \forall x, y \in X$$

for some $C > 0$; contraction if $C < 1$

- homeomorphism $f : X \to Y$ bi-Lipschitz if

$$C^{-1} d_X(x, y) \leq d_Y(f(x), f(y)) \leq C\, d_X(x, y) \;\; \forall x, y \in X$$

- Hausdorff measure and dimension
  - Lipschitz maps satisfy, $\forall$ measurable A and $s \in \mathbb{R}_+$

$$\mu_{H,s}(f(A)) \leq C^s\, \mu_{H,s}(A)$$

$$\dim_H(f(A)) \leq \dim_H(A)$$

  - bi-Lipschitz maps satisfy, $\forall$ measurable A and $s \in \mathbb{R}_+$

$$\dim_H(f(A)) = \dim_H(A)$$

  - if $d_Y(f(x), f(y)) = C\, d_X(x, y)$ then

$$\mu_{H,s}(f(A)) = C^s\, \mu_{H,s}(A)$$

Hausdorff measure and Hausdorff dimension of Cantor sets

- $s = \dim_H C = \frac{\log 2}{\log 3}$ and $\mu_{H,s}(C) = 1$
  - first show $\mu_{H,\frac{\log 2}{\log 3}}(C) \leq 1$ hence $\dim_H C \leq \frac{\log 2}{\log 3}$
  - for this enough to cover C with 2^j intervals of length 3^{-j} as in the iterative construction of C, then $\mu_{H,s}(C) \leq 2^j 3^{-sj} = 1$ for $s = \frac{\log 2}{\log 3}$
  - then show opposite estimate $\mu_{H,\frac{\log 2}{\log 3}}(C) \geq 1$ so that $\dim_H C \geq \frac{\log 2}{\log 3}$
  - for this show that for any collection $\mathcal{J} = \{J_\alpha\}$ of intervals covering C have $\sum_\alpha \ell(J_\alpha)^s \geq 1$ for $s = \frac{\log 2}{\log 3}$

- by compactness can use finite coverings, can also take smallest intervals that cover a pair of ternary intervals of the construction of C (J contains one such interval I_1, followed by an interval K in the complement of C followed by another such interval I_2)
- by construction of C these have $\ell(I_1), \ell(I_2) \leq \ell(K)$
- concavity of $x \mapsto x^s$ and $s = \frac{\log 2}{\log 3}$ give

$$\ell(J)^s = (\ell(I_1) + \ell(K) + \ell(I_2))^s \geq (\frac{3}{2}(\ell(I_1) + \ell(I_2)))^s$$

$$= 2(\frac{1}{2}\ell(I_1)^s + \frac{1}{2}\ell(I_2)^s) \geq \ell(I_1)^s + \ell(I_2)^s$$

- so replacing J with two subintervals I_j of the construction of C does not increase the sum of s-powers of lengths
- keep doing this operation until reduce to a covering of C by intervals of equal length 3^{-j} that include all the 2^j intervals in the construction of C so that $\sum_J \ell(J)^s \geq 1$ for $s = \frac{\log 2}{\log 3}$ for this covering (hence for the arbitrary initial one as well)

Hausdorff measures and self-similarity

- affine self-similar fractals $K = f_1(K) \cup \cdots f_N(K)$ fixed point with f_i contractions given by affine maps
  - if K is an s-set: $0 < \mu_{H,\dim_H(K)}(K) < \infty$ then

$$\mu_{H,\dim_H(K)}(K) = \mu_{H,\dim_H(K)}(\cup_i f_i(K))$$

$$\leq \sum_i \mu_{H,\dim_H(K)}(f_i(K)) = \sum_i \lambda_i^{\dim_H(K)} \mu_{H,\dim_H(K)}(K)$$

  - self-similarity dimension $\dim_{\text{self-sim}}(K)$ is unique $s > 0$ such that $1 = \sum_{i=1}^N \lambda_i^s$
  - $j(s) = \sum_{i=1}^N \lambda_i^s$ decreasing function with $j(0) = N > 1$ and $j(s) \to 0$ for $s \to \infty$
  - so above inequality gives $\sum_i \lambda_i^{\dim_H(K)} \geq 1$
  - obtain upper bound on Hausdorff dimension:

$$\dim_{\text{self-sim}}(K) \geq \dim_H(K)$$

  - not sharp in general due to overlaps between the images $f_i(K)$

Moran condition

Theorem (P. Moran, 1945)

Suppose that $A \subset R^d$ is a compact attractor of an IFS $\mathcal{F} = \{f_1, \ldots, f_k\}$ of similarity transformations with $0 < \lambda_i < 1$. Assume that either $f_j(A)$ are disjoint for $j = 1, \ldots, k$ or that A obtained in the following way: Suppose Ω_1 is an open bounded set and $\Omega_2^j = f_j(\Omega_1)$ be disjoint open sets for $j = 1, \ldots, k$ contained in Ω_1. Similarly let $\Omega_2^{j\ell} = f_\ell(\Omega_1^j)$ for $\ell = 1, \ldots, k$ be disjont in all j and so on. Suppose A is the intersection of

$$\overline{\Omega_1}, \quad \overline{\cup_j \Omega_2^j}, \quad \overline{\cup_{j\ell} \Omega_3^{j\ell}}, \quad \ldots$$

Then $\dim_H(A)$ is the similarity dimension, namely, the unique $s > 0$ solving

$$1 = \lambda_1^s + \cdots + \lambda_k^s.$$

For many affine self-similar examples this condition holds and the Hausdorff dimension can be computed directly as the self-similarity dimension

Vitali covering theorem

• given a sufficiently large collection of sets that covers E it is possible to select a subcollection of *disjoint* sets that *almost* covers E in a measure sense

- *semidisjoint* collection $\mathcal{C} = \{C_\alpha\}$ of sets if no C_α contained inside another C_β
- collection $\mathcal{C} = \{C_\alpha\}$ *Vitali class* for E if for all $x \in E$ and all $\delta > 0$ there is a $C_\alpha \in \mathcal{C}$ such that $x \in C_\alpha$ and $0 < \mathrm{diam}(C_\alpha) \leq \delta$

• Vitali covering result: $E \subset \mathbb{R}^N$ Hausdorff $\mu_{H,s}$-measurable and $\mathcal{C}$ Vitali class for E of closed sets: possible to select finite or countable subcollection $\mathcal{C}' = \{C_k\} \subset \mathcal{C}$ of disjoint sets such that either $\sum_k \mathrm{diam}(C_k)^s = \infty$ or $\mu_{H,s}(E \smallsetminus \cup_k C_k) = 0$;
if $\mu_{H,s}(E) < \infty$ can also obtain for a chosen $\epsilon > 0$ that $\mu_{H,s}(E) \leq \sum_k \mathrm{diam}(C_k)^s + \epsilon$

Probability measures on Bernoulli shifts

• alphabet $\mathfrak{A}$ finite set with $\#\mathfrak{A} = k$, shift space

$$\Sigma_k^+ := \{a_1 a_2 \cdots a_n \cdots \mid a_k \in \mathfrak{A}, \forall k \in \mathbb{N}\}$$

infinite sequences of letters in the alphabet $\mathfrak{A}$

• cylinder sets $\mathcal{C}_{w_1,\ldots,w_n} \in \mathcal{C}$

$$\mathcal{C}_{w_1,\ldots,w_n} := \{a = a_1 a_2 \cdots a_n \cdots \in \Sigma_k^+ \mid a_k = w_k,\ k = 1,\ldots,n\}$$

• choose a probability distribution on the finite set $\mathfrak{A}$, that is, a point $P = (p_a)_{a\in\mathfrak{A}}$ in the k-simplex: $p_a \geq 0$ and $\sum_{a\in\mathfrak{A}} p_a = 1$

• use cylinder sets as sample class for the Carathéodory construction by assigning probability

$$\mathbb{P}(\mathcal{C}_{w_1,\ldots,w_n}) := p_{w_1} \cdots p_{w_n}$$

• binary case $\mathfrak{A} = \{0,1\}$ and shift space Σ_2^+ gives $P = (p, 1-p)$ and

$$\mathbb{P}(\mathcal{C}_{w_1,\ldots,w_n}) = p^{a_n(w)}(1-p)^{b_n(w)}$$

with $a_n(w) = \#\{w_i = 0\}$ and $b_n(w) = \#\{w_i = 1\}$ in the string $w = w_1 \ldots w_n$

• $\mathbb{P}$ is additive on disjoint unions of cylinders

$$\mathcal{C}_{w_1,\ldots,w_n} = \mathcal{C}_{w_1,\ldots,w_n,1} \sqcup \cdots \sqcup \mathcal{C}_{w_1,\ldots,w_n,k}$$

decomposition of an n-cylinder into a disjoint union of $(n+1)$-cylinders and

$$\mathbb{P}(\mathcal{C}_{w_1,\ldots,w_n}) = p_{w_1} \cdots p_{w_n} = p_{w_1} \cdots p_{w_n} \sum_{a \in \mathfrak{A}} p_a =$$

$$= \sum_{a \in \mathfrak{A}} \mathbb{P}(\mathcal{C}_{w_1,\ldots,w_n,a})$$

• Bernoulli probability measure on Σ_k^+ obtained by applying the Carathéodory construction (outer measure, σ-algebra, and measure) to the assignment of cylinder sets $\mathcal{C}$ with the probabilities $\mathbb{P}(\mathcal{C}_{w_1,\dots,w_n}) := p_{w_1} \cdots p_{w_n}$

• Bernoulli measures on the Cantor set C using the coding of the Cantor set by Σ_2^+ with coding map $h : \Sigma_2^+ \to C$, define Bernoulli measure on C by $\mu_P(A) = \mathbb{P}(h^{-1}(A))$ for assigned $P = (p, 1-p)$.

• Hasdorff measure $\mu_{H,s}$ special case of Bernoulli measure where $p_i = \lambda_i^s$ contraction ratios $\sum_i \lambda_i^s = 1$ for $s =$ self-similarity dimension

Markov measures

• probabilities with memory of past events (one step), unlike Bernoulli measures (no memory)

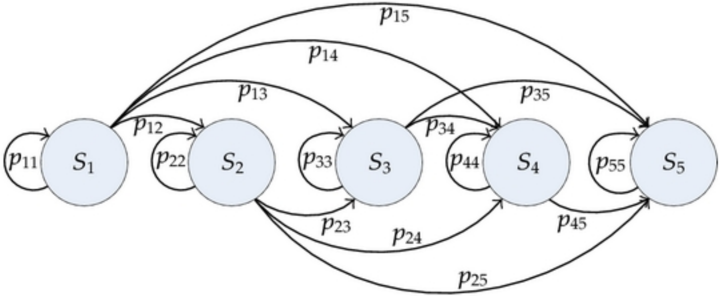
• shift space Σ_k^+ (one-side) infinite words on k letter

• data of a Markov chain

- Root probabilities: of letters in the alphabet $\pi = (\pi_1, \ldots, \pi_k)$ with $\pi_i \geq 0$ and $\sum_{i=1}^k \pi_i = 1$
- Stochastic matrix of transitions: $P = (p_{ij})_{i,j=1,\ldots,k}$ with $p_{ij} \geq 0$ and $\sum_{j=1}^k p_{ij} = 1$
- Stationary vector: $\pi P = \pi$ left Perron–Frobenius eigenvector of P

$$\sum_{i=1}^k \pi_i p_{ij} = \pi_i$$

Markov chain



$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ 0 & p_{22} & p_{23} & p_{24} & p_{25} \\ 0 & 0 & p_{33} & p_{34} & p_{35} \\ 0 & 0 & 0 & p_{44} & p_{45} \end{bmatrix} \text{ where } p_{ij} \geq 0 \text{ and } \sum_j p_{ij} = 1$$

Markov measure on the shift space Σ_k^+

- outer measure by assigning measure to *cylinder sets* $\mathcal{C}_{w_1,\ldots,w_n}$

$$\mu_{P,\pi}(\mathcal{C}_{w_1,\ldots,w_n}) := \pi_{w_1} p_{w_1 w_2} p_{w_2 w_3} \cdots p_{w_{n-1} w_n}$$

- seen as probability of starting in state w_1 and ending in state w_n with an oriented path in the graph of the Markov chain passing through intermediate vertices $w_2, \ldots, w_{n-1}$
- *additivity on cylinders* because of stochastic matrix property $\sum_{j=1}^k p_{ij} = 1$ for all $i = 1, \ldots, k$
- use Carathéodory construction to obtain Σ-algebra and Markov measure
- Shift invariant measure because $\pi P = \pi$

$$\mu_{P,\pi}(\sigma^{-1}(A)) = \mu_{P,\pi}(A)$$

$\sigma : \Sigma_k^+ \to \Sigma_k^+$ non-invertible one-sided shift
$w = w_1 w_2 \cdots w_N \cdots \mapsto \sigma(w) = w_2 w_2 \cdots w_{N+1} \cdots$

- Σ_k^+ parameterizes all possible infinite directed paths in the graph of the Markov chain
- the measure $\mu_{P,\pi}(\mathcal{C}_{w_1,\dots,w_n})$ gives the probability that a random path starts with the steps $w_1, \dots, w_n$
- if an entry $p_{ij} = 0$ can remove the corresponding edge in the graph: probability zero to all paths where the letter j follows the letter i (reaching state j from state i)
- Bernoulli measures: if $p_i > 0$ for $i = 1, \dots, k$ have $\mathrm{supp}(\mu_P) = \Sigma_k^+$
- Markov measures with all $\pi_i > 0$ and all $p_{ij} > 0$ have $\mathrm{supp}(\mu_{P,\pi}) = \Sigma_k^+$

- Markov measures with all $\pi_i > 0$ and all $p_{ij} > 0$ have $\mathrm{supp}(\mu_{P,\pi}) = \Sigma_k^+$
- otherwise transition matrix

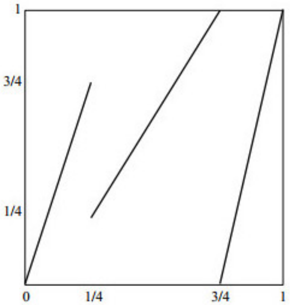
$$A_{ij} = \begin{cases} 0 & p_{ij} = 0 \\ 1 & p_{ij} > 1 \end{cases}$$

- transition matrix only measures if an edge is "accessible" to paths or not (without quantifying the probability)
- primitive A if $\exists n \in \mathbb{N}$ such that A^n positive $A^n_{ij} > 0$ all i, j
- primitive A implies given any two states i, j there is an oriented path in the Markov chain graph of length at least n given by an admissible word that goes from i to j
- admissible sequences $\Sigma_A^+ \subset \Sigma_k^+$

$$\Sigma_A^+ = \{ w = w_1 w_2 \cdots w_N \cdots \in \Sigma_k^+ \mid A_{w_i w_{i+1}} = 1, \forall i \geq 1 \}$$

- support of Markov measure $\mathrm{supp}(\mu_{P,\pi}) = \Sigma_A^+$

Dynamical systems: one-dimensional Markov maps



- collection $I_1, \dots, I_k$ of disjoint subintervals of $[0,1]$ and map $f : I_1 \cup \cdots \cup I_k \to [0,1]$ such that $f_i = f|_{I_i}$ is homeomorphism to image $f(I_i)$
- each image $f(I_i)$ contains each interval I_j it intersects: if $I_j \cap f(I_i) \neq \emptyset$ then $I_j \subseteq f(I_i)$

Cantor sets and coding of one-dimensional Markov maps

- Transition matrix for one-dimensional Markov maps

$$A_{ij} = \begin{cases} 1 & f(I_i) \cap I_j \neq \emptyset \\ 0 & f(I_i) \cap I_j = \emptyset \end{cases}$$

- Cantor set of one-dimensional Markov map $f : I_1 \cup \cdots \cup I_k \to [0,1]$

$$C_A = \{ x \in I_1 \cup \cdots \cup I_k \mid \text{ all iterates } f^{\circ n} \text{ are defined at } x \}$$

- dynamical system $f : C_A \to C_A$
- coding map for $f : C_A \to C_A$

$$\sigma : \Sigma_A^+ \to \Sigma_A^+ \quad \text{one-sided shift map}$$

- subshifts of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$ models of dynamics for many different kinds of maps and fractals

Hiroshi Kawano, *Design 3-1: Color Markov Chain Pattern*, 1964