Fractals and Spectral Triples
Introduction to Fractal Geometry and Chaos

Matilde Marcolli

MAT1845HS Winter 2020, University of Toronto
M 5-6 and T 10-12 BA6180

Some References

- Christensen, Erik; Ivan, Cristina; Lapidus, Michel L., *Dirac operators and spectral triples for some fractal sets built on curves*. Adv. Math. 217 (2008), no. 1, 42–78.
- Chamseddine, Ali H.; Connes, Alain, *The spectral action principle*. Comm. Math. Phys. 186 (1997), no. 3, 731–750.
- A. Ball, M. Marcolli, *Spectral Action Models of Gravity on Packed Swiss Cheese Cosmology*, Classical and Quantum Gravity, 33 (2016), no. 11, 115018, 39 pp.
- Farzad Fathizadeh, Yeorgia Kafkoulis, Matilde Marcolli, *Bell polynomials and Brownian bridge in Spectral Gravity models on multifractal Robertson-Walker cosmologies*, arXiv:1811.02972

Von Neumann Algebras

• Hilbert space $\mathcal{H}$ (infinite dimensional, separable, over $\mathbb{C}$) algebra of bounded operators $\mathcal{B}(\mathcal{H})$ with operator norm

• Commutant of $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$:

$$\mathcal{M}' := \{T \in \mathcal{B}(\mathcal{H}) : \; TS = ST, \forall S \in \mathcal{M}\}.$$

• von Neumann algebra: $\mathcal{M} = \mathcal{M}''$ (double commutant)
$\Leftrightarrow$ weakly closed

• Center: $Z(\mathcal{M}) = L^\infty(X, \mu)$.

• If $Z(\mathcal{M}) = \mathbb{C}$: factor

C^*-algebras

• involutive ($*$ anti-isomorphism) Banach algebra (complete in norm, $\|ab\| \leq \|a\| \cdot \|b\|$, $\|a^* a\| = \|a\|^2$)

• Gel'fand–Naimark correspondence: locally compact Hausdorff topological space $\Leftrightarrow$ commutative C^*-algebra:

$$X \Leftrightarrow C_0(X)$$

• representation of a C^*-algebra $\mathcal{A}$

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

C^*-algebra homomorphism

• state: continuous linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with positivity $\varphi(a^* a) \geq 0$ for all $a \in \mathcal{A}$ and $\varphi(1) = 1$

GNS representation

- cyclic vector ξ for a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C^*-algebra if set $\{\pi(a)\xi \ : \ a \in \mathcal{A}\}$ dense in $\mathcal{H}$
- state from unit norm cyclic vector $\varphi(a) = \langle \pi(a)\xi, \xi \rangle$
- given a state $\varphi : \mathcal{A} \to \mathbb{C}$ construct a representation (GNS) where state is as above
- define $\langle a, b \rangle = \varphi(a^* b)$ for $a, b \in \mathcal{A}$
- $\mathcal{N} = \{a \in \mathcal{A} : \varphi(a^* a) = 0\}$ linear subspace but for C^*-algebras also a *left* ideal in $\mathcal{A}$
- $\mathcal{H} = \mathcal{A}/\mathcal{N}$ with $\langle a, b \rangle = \varphi(a^* b)$ Hilbert space
- the representation $\pi(a)b + \mathcal{N} = ab + \mathcal{N}$
- cyclic vector $\xi = 1 + \mathcal{N}$ unit of $\mathcal{A}$

Spectral triple $(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{A} = \mathrm{C}^*$-algebra
- $\mathcal{H}$ Hilbert space: $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
- D unbounded self–adjoint operator on $\mathcal{H}$
- $(D - \lambda)^{-1}$ compact operator, $\forall \lambda \notin \mathbb{R}$
- $[D, a]$ bounded operator, $\forall a \in \mathcal{A}_0 \subset \mathcal{A}$, dense involutive subalgebra of $\mathcal{A}$.

Riemannian spin manifold X: $\mathcal{A} = C(X)$, $\mathcal{H} = L^2$-spinors, $D =$ Dirac operator, $\mathcal{A}_0 = C^\infty(X)$

Zeta functions

• spectral triple $(\mathcal{A},\mathcal{H},D) \Rightarrow$ family of zeta functions: for $a \in \mathcal{A}_0 \cup [D,\mathcal{A}_0]$

$$\zeta_{a,D}(z) := \mathrm{Tr}(a|D|^{-z}) = \sum_{\lambda} \mathrm{Tr}(a\,\Pi(\lambda,|D|))\lambda^{-z}$$

Dimension of a spectral triple $(\mathcal{A},\mathcal{H},D)$

- Simpler definition: dimension n (*n–summable*) if $|D|^{-n}$ infinitesimal of order one: $\lambda_k(|D|^{-n}) = O(k^{-1})$
- Refined definition: **dimension spectrum** $\Sigma \subset \mathbb{C}$: set of poles of the zeta functions $\zeta_{a,D}(z)$. (all zetas extend holomorphically to $\mathbb{C} \setminus \Sigma$)
- in sufficiently nice cases (almost commutative geometries) poles of $\zeta_D(s) = \zeta_{1,D}(s)$ suffice

Example: Fractal string

Ω bounded open in $\mathbb{R}$ (e.g. complement of Cantor set Λ in $[0,1]$)
$\mathcal{L} = \{\ell_k\}_{k\geq 1}$ lengths of connected components of Ω with

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k \cdots > 0.$$

Geometric zeta function (Lapidus and van Frankenhuysen)

$$\zeta_{\mathcal{L}}(s) := \sum_k \ell_k^s$$

Cantor set: spectral triple

$\Lambda =$ middle-third Cantor set: $\zeta_L(s) = \frac{3^{-s}}{1-2\cdot 3^{-s}}$

algebra commutative C^*-algebra $C(\Lambda)$.

Hilbert space: $E = \{x_{k,\pm}\}$ endpoints of intervals $J_k \subset \Omega = [0,1] \smallsetminus \Lambda$, with $x_{k,+} > x_{k,-}$

$$\mathcal{H} := \ell^2(E)$$

action $\mathrm{C}(\Lambda)$ acts on $\mathcal{H}$

$$f \cdot \xi(x) = f(x)\xi(x), \quad \forall f \in \mathrm{C}(\Lambda), \ \forall \xi \in \mathcal{H}, \ \forall x \in E.$$

sign operator subspace $\mathcal{H}_k$ of coordinates $\xi(x_{k,+})$ and $\xi(x_{k,-})$,

$$F|_{\mathcal{H}_k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Dirac operator

$$D|_{\mathcal{H}_k} \begin{pmatrix} \xi(x_{k,+}) \\ \xi(x_{k,-}) \end{pmatrix} = \ell_k^{-1} \cdot \begin{pmatrix} \xi(x_{k,-}) \\ \xi(x_{k,+}) \end{pmatrix}$$

- verify $[D, a]$ bounded for $a \in \mathcal{A}_0$:

$$[D, f]|_{\mathcal{H}_k} = \frac{(f(x_{k,+}) - f(x_{k,-}))}{\ell_k} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

for f Lipschitz: $\|[D, f]\| \leq C(f)$
take dense $\mathcal{A}_0 \subset C(\Lambda)$ to be locally constant or more generally Lipschitz functions

- same for any self-similar set in $\mathbb{R}$ (Cantor-like)

Zeta function

$$\mathrm{Tr}(|D|^{-s}) = 2\zeta_L(s) = \sum_{k \geq 1} 2^k 3^{-sk} = \frac{2 \cdot 3^{-s}}{1 - 2 \cdot 3^{-s}}$$

Dimension spectrum

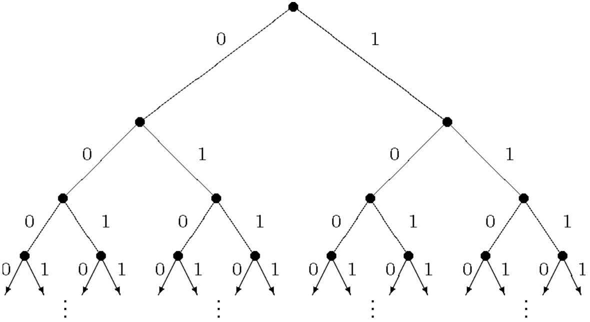
$$\Sigma = \left\{ \frac{\log 2}{\log 3} + \frac{2\pi i n}{\log 3} \right\}_{n \in \mathbb{Z}}$$

Example: AF algebras (noncommutative Cantor sets)

C^*-algebras approximated by finite dimensional algebras (direct limits of a direct system of finite dimensional algebras)

determined by Bratteli diagram: $\mathcal{F}_k$ = fin dim algebras $\phi_{k,k+1}$ embeddings with specified *multiplicities*

$C(\Lambda)$ = commutative AF algebra corresponding to the diagram

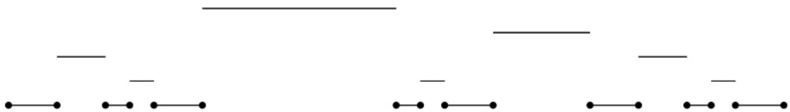


Example: Fibonacci spectral triple

Fibonacci AF algebra $\mathcal{F}_n = \mathcal{M}_{F_{n+1}} \oplus \mathcal{M}_{F_n}$, embeddings from partial embedding

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Fibonacci Cantor set from the interval $I = [0, 4]$ remove F_{n+1} open intervals $J_{n,j}$ of lengths $\ell_n = 1/2^n$, according to the rule:



Hilbert space E of endpoints $x_{n,j,\pm}$ of the intervals $J_{n,j}$:
$\mathcal{H} = \ell^2(E)$, completion of

$$\begin{array}{ccccccccccc} \mathbb{C} & \oplus & \mathbb{C} & \oplus & \mathbb{C}^2 & \oplus & \mathbb{C}^3 & \oplus & \mathbb{C}^5 & \cdots \\ & \oplus & \mathbb{C} & \oplus & \mathbb{C} & \oplus & \mathbb{C}^2 & \oplus & \mathbb{C}^3 & \cdots \end{array}$$

Action of $\mathcal{M}_{F_{n+1}} \oplus \mathcal{M}_{F_n} \Rightarrow$ of AF algebra

Sign on subspace $\mathcal{H}_{n,j}$ spanned by $\xi(x_{n,j,\pm})$

$$F|_{\mathcal{H}_{n,j}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Dirac operator

$$D|_{\mathcal{H}_{n,j}} \begin{pmatrix} \xi(x_{n,j,+}) \\ (x_{n,j,-}) \end{pmatrix} = \ell_n^{-1} \begin{pmatrix} \xi(x_{n,j,-}) \\ \xi(x_{n,j,+}) \end{pmatrix}$$

$\Rightarrow$ spectral triple with zeta function

$$Tr(|D|^{-s}) = 2\zeta_F(s) = \frac{2}{1 - 2^{-s} - 4^{-s}}$$

geometric zeta function $\zeta_F(s) = \sum_n F_{n+1} 2^{-ns}$

- bounded commutators condition: $[D, a]$ bounded for $a \in \mathcal{A}_0$:

$$[D, U]|_{\mathcal{H}_{n,j}} \begin{pmatrix} \xi(x_{n,j,+}) \\ \xi(x_{n,j,-}) \end{pmatrix} = \ell_n^{-1} \begin{pmatrix} (A_{n,+} - A_{n,-})\xi(x_{n,j,-}) \\ (A_{n,-} - A_{n,+})\xi(x_{n,j,+}) \end{pmatrix}.$$

$\Rightarrow$ for $U \in \cup_k \mathcal{F}_k$ (dense subalgebra)

Dimension spectrum with $\phi = \frac{1+\sqrt{5}}{2}$

$$\left\{ \frac{\log \phi}{\log 2} + \frac{2\pi i n}{\log 2} \right\}_{n \in \mathbb{Z}} \cup \left\{ -\frac{\log \phi}{\log 2} + \frac{2\pi i (n + 1/2)}{\log 2} \right\}_{n \in \mathbb{Z}}$$

The spectral action functional

- Ali Chamseddine, Alain Connes, *The spectral action principle*, Comm. Math. Phys. 186 (1997), no. 3, 731–750.

A good action functional for noncommutative geometries

$$\mathrm{Tr}(f(D/\Lambda))$$

D Dirac, Λ mass scale, $f > 0$ even smooth function (cutoff approx)
Simple dimension spectrum $\Rightarrow$ expansion for $\Lambda \to \infty$

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_k f_k \Lambda^k \fint |D|^{-k} + f(0)\,\zeta_D(0) + o(1),$$

with $f_k = \int_0^\infty f(v)\, v^{k-1}\, dv$ momenta of f
where $\mathrm{DimSp}(\mathcal{A}, \mathcal{H}, D) =$ poles of $\zeta_{b,D}(s) = \mathrm{Tr}(b|D|^{-s})$

Asymptotic expansion of the spectral action

$$\mathrm{Tr}(e^{-t\Delta}) \sim \sum a_\alpha \, t^\alpha \qquad (t \to 0)$$

and the ζ function

$$\zeta_D(s) = \mathrm{Tr}(\Delta^{-s/2})$$

- Non-zero term a_α with $\alpha < 0 \Rightarrow$ *pole* of ζ_D at -2α with

$$\mathrm{Res}_{s=-2\alpha}\, \zeta_D(s) = \frac{2\, a_\alpha}{\Gamma(-\alpha)}$$

- No log t terms $\Rightarrow$ regularity at 0 for ζ_D with $\zeta_D(0) = a_0$

- Get first statement from

$$|D|^{-s} = \Delta^{-s/2} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{-t\Delta}\, t^{s/2-1}\, dt$$

with $\int_0^1 t^{\alpha+s/2-1}\, dt = (\alpha + s/2)^{-1}$.

- Second statement from

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)} \sim \frac{s}{2} \qquad \text{as} \quad s \to 0$$

contrib to $\zeta_D(0)$ from pole part at $s=0$ of

$$\int_0^\infty \mathrm{Tr}(e^{-t\Delta})\, t^{s/2-1}\, dt$$

given by $a_0 \int_0^1 t^{s/2-1}\, dt = a_0 \frac{2}{s}$

Zeta function and heat kernel (manifolds)

- Mellin transform

$$|D|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tD^2} t^{\frac{s}{2}-1} dt$$

- heat kernel expansion

$$\mathrm{Tr}(e^{-tD^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for} \quad t \to 0$$

- zeta function expansion

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) = \sum_\alpha \frac{c_\alpha}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}$$

- taking residues

$$\mathrm{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

Spectral Action as a model of Euclidean (Modified) Gravity

$$\mathcal{S}_\Lambda = \mathrm{Tr}(f(\frac{D}{\Lambda})) = \sum_{\lambda \in \mathrm{Spec}(D)} f(\frac{\lambda}{\Lambda})$$

- D Dirac operator
- $\Lambda \in \mathbb{R}_+^*$ energy scale
- $f(x)$ test function (smooth approximation to cutoff function)

Why a model of (Euclidean) Gravity?

- M compact Riemannian 4-manifold

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

$$= \frac{48 f_4 \Lambda^4}{\pi^2} \int \sqrt{g}\, d^4x + \frac{96\, f_2 \Lambda^2}{24\pi^2} \int R \sqrt{g}\, d^4x$$

$$+ \frac{f_0}{10\pi^2} \int (\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}) \sqrt{g}\, d^4x$$

coefficients a_0, a_2 and a_4 cosmological, Einstein–Hilbert, and Weyl curvature $C^{\mu\nu\rho\sigma}$ and Gauss–Bonnet $R^* R^*$ gravity terms

Example spectral action of the round 3-sphere S^3

$$\mathcal{S}_{S^3}(\Lambda) = \mathrm{Tr}(f(D_{S^3}/\Lambda)) = \sum_{n\in\mathbb{Z}} n(n+1) f((n+\frac{1}{2})/\Lambda)$$

- zeta function

$$\zeta_{D_{S^3}}(s) = 2\zeta(s-2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2})$$

$\zeta(s,q) =$ Hurwitz zeta function

- by asymptotic expansion

$$\mathcal{S}_{S^3}(\Lambda) \sim \Lambda^3 f_3 - \frac{1}{4}\Lambda f_1$$

- can also compute using Poisson summation formula (Chamseddine–Connes): estimate error term $O(\Lambda^{-\infty})$

Example: round 3-sphere S_a^3 radius a

$$\zeta_{D_{S_a^3}}(s) = a^s (2\zeta(s-2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2}))$$

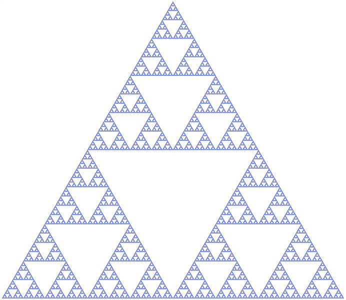
$$\mathcal{S}_{S_a^3}(\Lambda) \sim (\Lambda a)^3 f_3 - \frac{1}{4}(\Lambda a) f_1$$

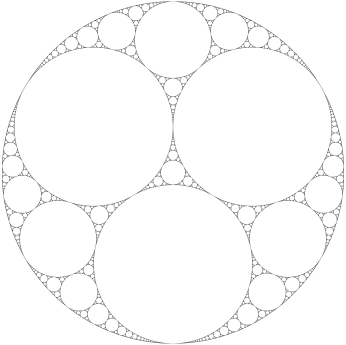
Example: spherical space form $Y = S_a^3/\Gamma$ (Ćaćić, Marcolli, Teh)

$$\mathcal{S}_Y(\Lambda) \sim \frac{1}{\#\Gamma}\, \mathcal{S}_{S_a^3}(\Lambda)$$

Spectral Triples on Fractals obtained by gluing smooth spaces

- Sierpinski gasket (treat each triangle like a smooth circle)
- Apollonian packings of circles
- higher dimensional analogs



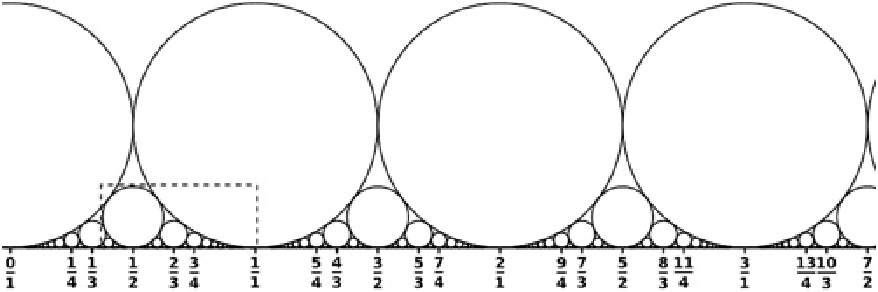


Spectral triple construction (Christensen, Ivan, Lapidus)

- Spectral triple for the model manifold (circle, sphere, etc) (A, H, D)
- a copy (H_i, D_i) of Hilbert space and Dirac operator of (A, H, D) for each copy of the model manifold in the fractal
- direct sum $(\mathcal{H}, \mathcal{D}) := \oplus_i (H_i, D_i)$
- algebra is subalgebra $\mathcal{A} \subset \oplus_i A_i$ of functions that agree at the points where the model manifolds are joined together to form the fractal
- the resulting $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies all the properties of a spectral triple

Example: Lower Dimensional Apollonian Ford Circles

- Ford circles: tangent to the real line at points $(k/n, 0)$ with centers at points $(k/n, 1/(2n^2))$



- number of circles of radius $r_n = (2n^2)^{-1}$ is number of integers $1 \leq k \leq n$ coprime to n: multiplicity $m(r_n)$ given by Euler totient function

$$m(r_n) = \varphi(n),$$

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

product over the distinct prime numbers dividing n

- Dirichlet series generating function of the Euler totient function

$$\mathcal{D}_\varphi(s) = \sum_{n \geq 1} \frac{\varphi(n)}{n^s}$$

- fractal string zeta function

$$\zeta_{\mathcal{L}}(s) = \sum_{n\geq 1} \varphi(n)\,(2n^2)^{-s} = 2^{-s} \sum_{n\geq 1} \varphi(n)\, n^{-2s} = 2^{-s} \mathcal{D}_\varphi(2s)$$

- using $\varphi(p^k) = p^k - p^{k-1}$

$$1 + \sum_k \varphi(p^k) p^{-sk} = \frac{1 - p^{-s}}{1 - p^{1-s}}$$

- using Euler product formula

$$\mathcal{D}_\varphi(s) = \frac{\zeta(s-1)}{\zeta(s)}$$

- so fractal string zeta function of Ford circles

$$\zeta_{\mathcal{L}}(s) = 2^{-s}\,\frac{\zeta(2s-1)}{\zeta(2s)}$$

Zeta function of the Spectral Triple: Ford Circles

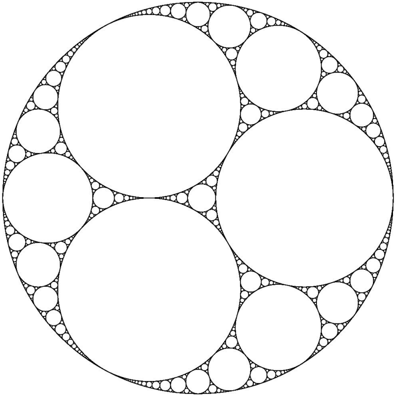
- Zeta function for an individual circle $D = -id/dx$ trivial spin structure has spectrum n and eigenfunctions e^{inx}
- for the other non-trivial spin structure spectrum $(n + 1/2)$ and eigenfunctions $e^{i(n+1/2)x}$
- zeta function (non-trivial spin structure) $\mathrm{Tr}(|D|^{-s}) = \zeta(s, 1/2)$ Hurwitz zeta $\zeta(s, 1/2) = \sum (n + 1/2)^{-s}$
- trivial spin structure with Riemann zeta (on complement of kernel)
- if scale circle by radius $a > 0$ scale zeta function by factor of a^s
- zeta function for the Ford Circles packing Dirac operator

$$\mathrm{Tr}(|\mathcal{D}|^{-s}) = \mathrm{Tr}(\oplus_k |D_{a_k}|^{-s}) = \sum_k a_k^s \mathrm{Tr}(|D|^{-s}) = \zeta_{\mathcal{L}}(s)\zeta_D(s)$$

$$= \zeta_{\mathcal{L}}(s)\zeta(s, 1/2) = 2^{-s} \, \frac{\zeta(2s-1)\zeta(s, 1/2)}{\zeta(2s)}$$

Apollonian sphere packings

• best known and understood case: Apollonian circle packing



Configurations of mutually tanget circles in the plane, iterated on smaller scales filling a full volume region in the unit $2D$ ball: residual set volume zero fractal of Hausdorff dimension 1.30568...

• Many results (geometric, arithmetic, analytic) known about Apollonian circle packings: see for example

- R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H.Yan, *Apollonian circle packings: number theory*, J. Number Theory 100 (2003) 1–45
- A. Kontorovich, H. Oh, *Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds*, Journal of AMS, Vol 24 (2011) 603–648.

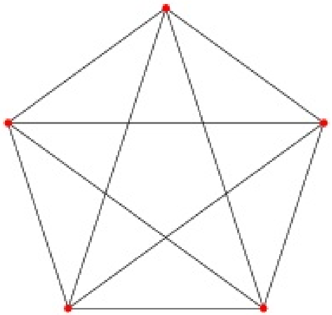
• Higher dimensional analogs of Apollonian packings: much more delicate and complicated geometry

- R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H.Yan, *Apollonian Circle Packings: Geometry and Group Theory III. Higher Dimensions*, Discrete Comput. Geom. 35 (2006) 37–72.

Some known facts on Apollonian sphere packings

• Descartes configuration in D dimensions: $D+2$ mutually tangent $(D-1)$-dimensional spheres

• Example: start with $D+1$ equal size mutually tangent S^{D-1} centered at the vertices of D-simplex and one more smaller sphere in the center tangent to all



4-dimensional simplex

• Quadratic Soddy–Gosset relation between radii a_k

$$\left(\sum_{k=1}^{D+2} \frac{1}{a_k}\right)^2 = D \sum_{k=1}^{D+2} \left(\frac{1}{a_k}\right)^2$$

• curvature-center coordinates: $(D+2)$-vector

$$w = (\frac{\|x\|^2 - a^2}{a}, \frac{1}{a}, \frac{1}{a}x_1, \ldots, \frac{1}{a}x_D)$$

(first coordinate curvature after inversion in the unit sphere)

• Configuration space $\mathcal{M}_D$ of all Descartes configuration in D dimensions = all solutions $\mathcal{W}$ to equation

$$\mathcal{W}^t \, Q_D \, \mathcal{W} = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2\, I_D \end{pmatrix}$$

with left and a right action of Lorentz group $O(D+1,1)$

• Dual Apollonian group $\mathcal{G}_D^\perp$ generated by reflections: inversion with respect to the j-th sphere

$$S_j^\perp = I_{D+2} + 2\,1_{D+2} e_j^t - 4\, e_j e_j^t$$

$e_j = j$-th unit coordinate vector

- $D \neq 3$: only relations in $\mathcal{G}_D^\perp$ are $(S_j^\perp)^2 = 1$
- $\mathcal{G}_D^\perp$ discrete subgroup of $\mathrm{GL}(D+2, \mathbb{R})$
- Apollonian packing $\mathcal{P}_D =$ an orbit of $\mathcal{G}_D^\perp$ on $\mathcal{M}_D$

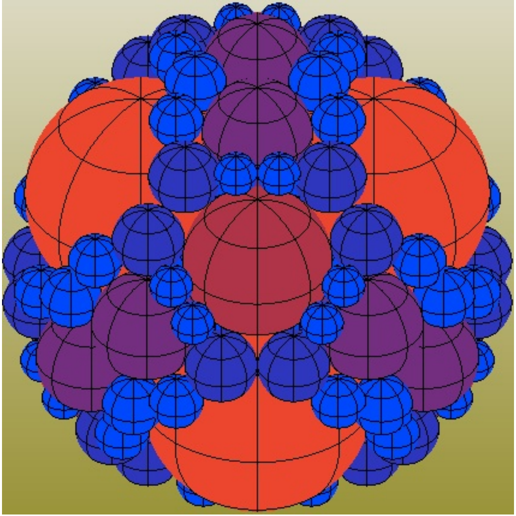
$\Rightarrow$ iterative construction: at n-th step add spheres obtained from initial Descartes configuration via all possible

$$S_{j_1}^\perp S_{j_2}^\perp \cdots S_{j_n}^\perp, \quad j_k \neq j_{k+1}, \forall k$$

there are N_n spheres in the n-th level

$$N_n = (D+2)(D+1)^{n-1}$$

iterative construction of sphere packings



- Length spectrum: radii of spheres in packing $\mathcal{P}_D$

$$\mathcal{L} = \mathcal{L}(\mathcal{P}_D) = \{a_{n,k} \,:\, n \in \mathbb{N}, 1 \leq k \leq (D+2)(D+1)^{n-1}\}$$

radii of spheres $S^{D-1}_{a_{n,k}}$

- Melzak's packing constant $\sigma_D(\mathcal{P}_D)$ exponent of convergence of series

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s$$

- Residual set: $\mathcal{R}(\mathcal{P}_D) = B^D \smallsetminus \cup_{n,k} B^D_{a_{n,k}}$ with $\partial B^D_{a_{n,k}} = S^{D-1}_{a_{n,k}} \in \mathcal{P}_D$
- Packing $\Rightarrow \mathrm{Vol}_D(\mathcal{R}(\mathcal{P}_D)) = 0 \Rightarrow \sum_{\mathcal{L}} a_{n,k}^D < \infty \Rightarrow \sigma_D(\mathcal{P}_D) \leq D$
- packing constant and Hausdorff dimension:

$$\dim_H(\mathcal{R}(\mathcal{P}_D)) \leq \sigma_D(\mathcal{P}_D)$$

for Apollonian circles known to be same

- Sphere counting function: spheres with given curvature bound

$$\mathcal{N}_\alpha(\mathcal{P}_D) = \#\{S^{D-1}_{a_{n,k}} \in \mathcal{P}_D \,:\, a_{n,k} \geq \alpha\}$$

curvatures $c_{n,k} = a_{n,k}^{-1} \leq \alpha^{-1}$

- for Apollonian circles power law (Kontorovich–Oh)

$$\mathcal{N}_\alpha(\mathcal{P}_2) \sim_{\alpha\to 0} \alpha^{-\dim_H(\mathcal{R}(\mathcal{P}_2))}$$

- for higher dimensions (Boyd): packing constant

$$\limsup_{\alpha\to 0} -\frac{\log \mathcal{N}_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$$

if limit exists $\mathcal{N}_\alpha(\mathcal{P}_D) \sim_{\alpha\to 0} \alpha^{-(\sigma_D(\mathcal{P}_D)+o(1))}$

Screens and Windows

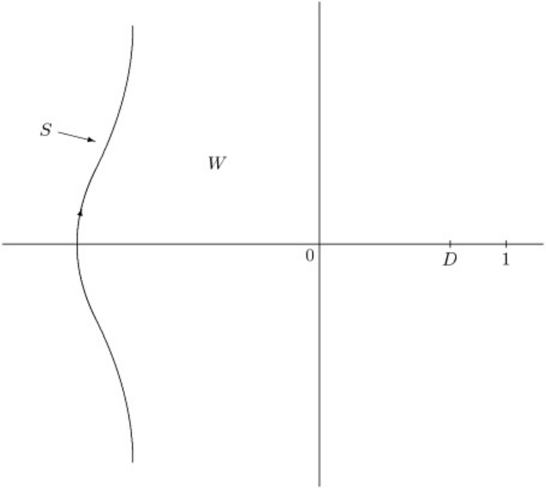
• in general $\zeta_{\mathcal{L}_D}(s)$ need *not* have analytic continuation to meromorphic on whole $\mathbb{C}$

• $\exists$ *screen* $\mathcal{S}$: curve $S(t) + it$ with $S : \mathbb{R} \to (-\infty, \sigma_D(\mathcal{P}_D)]$

• *window* $\mathcal{W}$ = region to the right of screen $\mathcal{S}$ where analytic continuation

- M.L. Lapidus, M. van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions. Geometry and spectra of fractal strings*, Second edition. Springer Monographs in Mathematics. Springer, 2013.

Screens and windows



Some additional assumptions

- Definition:

Apollonian packing $\mathcal{P}_D$ of $(D-1)$-spheres is *analytic* if

1. $\zeta_{\mathcal{L}}(s)$ has analytic to meromorphic function on a region $\mathcal{W}$ containing $\mathbb{R}_+$
2. $\zeta_{\mathcal{L}}(s)$ has only one pole on $\mathbb{R}_+$ at $s = \sigma_D(\mathcal{P}_D)$.
3. pole at $s = \sigma_D(\mathcal{P}_D)$ is simple

- Also assume: $\exists \lim_{\alpha \to 0} -\frac{\log \mathcal{N}_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$
- Question: in general when are these satisfied for packings $\mathcal{P}_D$?
- focus on $D = 4$ cases with these conditions

Rough estimate of the packing constant

- $\mathcal{P} = \mathcal{P}_4$ Apollonian packing of 3-spheres $S^3_{a_{n,k}}$
- at level n: average curvature

$$\frac{\gamma_n}{N_n} = \frac{1}{6 \cdot 5^{n-1}} \sum_{k=1}^{6 \cdot 5^{n-1}} \frac{1}{a_{n,k}}$$

- estimate $\sigma_4(\mathcal{P}_4)$ with averaged version: $\sum_n N_n (\frac{\gamma_n}{N_n})^{-s}$

$$\sigma_{4,av}(\mathcal{P}) = \lim_{n\to\infty} \frac{\log(6 \cdot 5^{n-1})}{\log\left(\frac{\gamma_n}{6 \cdot 5^{n-1}}\right)}$$

- generating function of the γ_n known (Mallows)

$$G_{D=4} = \sum_{n=1}^{\infty} \gamma_n x^n = \frac{(1-x)(1-4x)u}{1 - \frac{22}{3}x - 5x^2}$$

$u =$ sum of the curvatures of initial Descartes configuration

• obtain explicitly ($u = 1$ case)

$$\gamma_n = \frac{(11+\sqrt{166})^n(-64+9\sqrt{166})+(11-\sqrt{166})^n(64+9\sqrt{166})}{3^n \cdot 10 \cdot \sqrt{166}}$$

• this gives a value

$$\sigma_{4,av}(\mathcal{P}) = 3.85193\ldots$$

• in Apollonian circle case where $\sigma(\mathcal{P})$ known this method gives larger value, so expect $\sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P})$

• constraints on the packing constant:

$$3 < \dim_H(\mathcal{R}(\mathcal{P})) \leq \sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P}) = 3.85193\ldots$$

Packed Swiss Cheese Cosmology Physics Motivation: a cosmological model based on Apollonian packings of spheres

• Iterate construction removing more and more balls ⇒ Apollonian sphere packing of 3-dimensional spheres

• Residual set of sphere packing is fractal

• Proposed as explanation for possible fractal distribution of matter in galaxies, clusters, and superclusters

- F. Sylos Labini, M. Montuori, L. Pietroneo, *Scale-invariance of galaxy clustering*, Phys. Rep. Vol. 293 (1998) N. 2-4, 61–226.
- J.R. Mureika, C.C. Dyer, *Multifractal analysis of Packed Swiss Cheese Cosmologies*, General Relativity and Gravitation, Vol.36 (2004) N.1, 151–184.

Homogeneity versus Isotropy in Cosmology

• Homogeneous and isotropic: Friedmann universe $\mathbb{R} \times S^3$

$$\pm dt^2 + a(t)^2 \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)$$

with round metric on S^3 with $SU(2)$–invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$

• Homogeneous but not isotropic:
Bianchi IX mixmaster models $\mathbb{R} \times S^3$

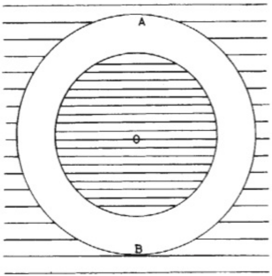
$$F(t)\,(\pm dt^2 + \frac{\sigma_1^2}{W_1^2(t)} + \frac{\sigma_2^2}{W_2^2(t)} + \frac{\sigma_3^2}{W_3^3(t)})$$

with a conformal factor $F(t) \sim W_1(t)W_2(t)W_3(t)$

• Isotropic but not homogeneous?
$\Rightarrow$ Swiss Cheese Models

Main Idea:

• M.J. Rees, D.W. Sciama, *Large-scale density inhomogeneities in the universe*, Nature, Vol.217 (1968) 511–516.



Cut off 4-balls from a FRW spacetime and replace with different density smaller region outside/inside patched across boundary with vanishing Weyl curvature tensor (isotropy preserved)

Different models of Fractal (Euclidean) Spacetimes

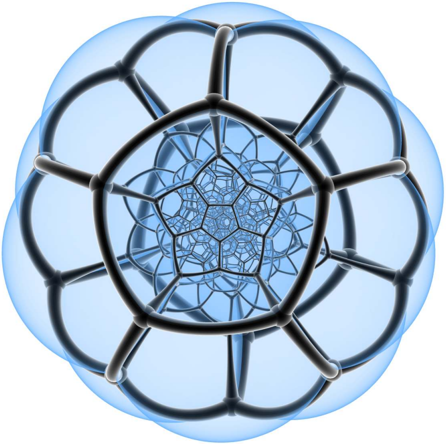
- Simplified model: stationary, $S^1 \times S^3$ replaced by $S^1 \times \mathcal{P}$ with S^1 compactified time direction and $\mathcal{P}$ Apollonian packing of 3-spheres; metric
- Same model with non-trivial cosmic topology: another spherical space form instead of S^3 (eg Poincare' dodecahedral space, homology 3-sphere) and fractal packing
- Better model: Robertson–Walker metrics on $\mathbb{R} \times S^3$ and on $\mathbb{R} \times \mathcal{P}$ with Apollonian packing $\mathcal{P}$; expanding/contracting universe

Models of (Euclidean, compactified) spacetimes

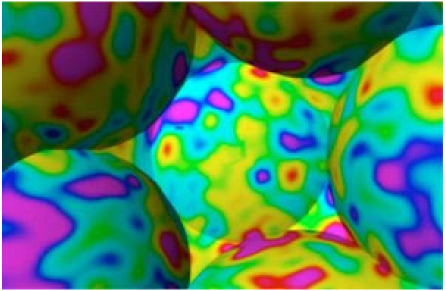
1. Homogeneous Isotropic cases: $S^1_\beta \times S^3_a$
2. Cosmic Topology cases: $S^1_\beta \times Y$ with Y a spherical space form S^3/Γ or a flat Bieberbach manifold T^3/Γ (modulo finite groups of isometries)
3. Packed Swiss Cheese: $S^1_\beta \times \mathcal{P}$ with Apollonian packing of 3-spheres $S^3_{a_{n,k}}$
4. Fractal arrangements with cosmic topology

Fractal arrangements with cosmic topology

• Example: Poincaré homology sphere, dodecahedral space $S^3/\mathcal{I}_{120}$, fundamental domain dodecahedron

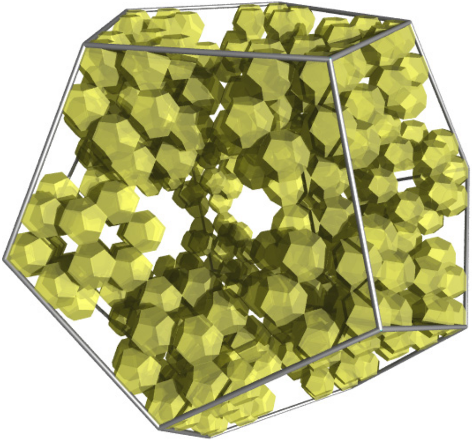


- considered a likely candidate for cosmic topology
  - S. Caillerie, M. Lachièze-Rey, J.P. Luminet, R. Lehoucq, A. Riazuelo, J. Weeks, *A new analysis of the Poincaré dodecahedral space model*, Astron. and Astrophys. 476 (2007) N.2, 691–696



- build a fractal model based on dodecahedral space

Fractal configurations of dodecahedra (Sierpinski dodecahedra)



• spherical dodecahedron has $Vol(Y) = Vol(S^3_a/\mathcal{I}_{120}) = \frac{\pi^2}{60} a^3$

• simpler than sphere packings because uniform scaling at each step: 20^n new dodecahedra, each scaled by a factor of $(2+\phi)^{-n}$

$$\dim_H(\mathcal{P}_{\mathcal{I}_{120}}) = \frac{\log(20)}{\log(2+\phi)} = 2.32958...$$

• close up all dodecahedra in the fractal identifying edges with $\mathcal{I}_{120}$: get fractal arrangement of Poincaré spheres $Y_{a(2+\phi)^{-n}}$

• zeta function has analytic continuation to all $\mathbb{C}$

$$\zeta_{\mathcal{L}}(s) = \sum_n 20^n \, (2+\phi)^{-ns} = \frac{1}{1 - 20(2+\phi)^{-s}}$$

exponent of convergence $\sigma = \dim_H(\mathcal{P}_{\mathcal{I}_{120}}) = \frac{\log(20)}{\log(2+\phi)}$ and poles

$$\sigma + \frac{2\pi i m}{\log(2+\phi)}, \quad m \in \mathbb{Z}$$

Spectral action on a fractal spacetime:

- $S^1_\beta \times \mathcal{P}$: Apollonian packing
- $S^1_\beta \times \mathcal{P}_Y$: fractal dodecahedral space

1. Construct a spectral triple for the geometries $\mathcal{P}$ and $\mathcal{P}_Y$
2. Compute the zeta function
3. Compute the asymptotic form of the spectral action
4. Effect of product with S^1_β

$\Rightarrow$ look for new terms in the spectral action (in additional to usual gravitational terms) that detect presence of fractality

The spectral triple of a fractal geometry

- case of Sierpinski gasket: Christensen, Ivan, Lapidus
- similar case for $\mathcal{P}$ and $\mathcal{P}_Y$
- for D-dim packing

$$\mathcal{P}_D = \{ S^{D-1}_{a_{n,k}} \, : \, n \in \mathbb{N}, \, 1 \leq k \leq (D+2)(D+1)^{n-1} \}$$

$$(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{\mathcal{P}_D}, \mathcal{D}_{\mathcal{P}_D}) = \oplus_{n,k} (\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{S^{D-1}_{a_{n,k}}}, \mathcal{D}_{S^{D-1}_{a_{n,k}}})$$

- for $\mathcal{P}_Y$ with $Y_a = S^3/\mathcal{I}_{120}$:

$$(\mathcal{A}_{\mathcal{P}_Y}, \mathcal{H}_{\mathcal{P}_Y}, \mathcal{D}_{\mathcal{P}_Y}) = (\mathcal{A}_{\mathcal{P}_Y}, \oplus_n \mathcal{H}_{Y_{a_n}}, \oplus_n D_{Y_{a_n}})$$

with $a_n = a(2+\phi)^{-n}$

Zeta functions for Apollonian packing of 3-spheres:

- Lengths zeta function (fractal string)

$$\zeta_{\mathcal{L}}(s) := \sum_{n\in\mathbb{N}} \sum_{k=1}^{6\cdot 5^{n-1}} a_{n,k}^s$$

with $\mathcal{L} = \mathcal{L}_4 = \{a_{n,k} \,|\, n \in \mathbb{N},\ k \in \{1,\ldots,6\cdot 5^{n-1}\}\}$

- zeta function of Dirac operator of the spectral triple

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n=1}^{\infty} \sum_{k=1}^{6\cdot 5^{n-1}} \mathrm{Tr}(|D_{S^3_{a_{n,k}}}|^{-s})$$

each term $\mathrm{Tr}(|D_{S^3_{a_{n,k}}}|^{-s}) = a_{n,k}^s(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2}))$ gives

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \left(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2})\right)\sum_{n,k} a_{n,k}^s$$

$$= \left(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2})\right)\zeta_{\mathcal{L}}(s)$$

Spectral action for Apollonian packing of 3-spheres:
(under good conditions on $\zeta_{\mathcal{L}}(s)$)

- Positive Dimension Spectrum: $\Sigma^+_{ST_{PSC}} = \{1, 3, \sigma_4(\mathcal{P})\}$

- asymptotic spectral action

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim \Lambda^3 \zeta_{\mathcal{L}}(3) f_3 - \Lambda \frac{1}{4} \zeta_{\mathcal{L}}(1) f_1$$

$$+\Lambda^{\sigma} (\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2})) \mathcal{R}_{\sigma} f_{\sigma} + \mathcal{S}^{osc}_{\Lambda}$$

$\sigma = \sigma_4(\mathcal{P})$ packing constant; residue $\mathcal{R}_{\sigma} = \mathrm{Res}_{s=\sigma} \zeta_{\mathcal{L}}(s)$, and momenta $f_{\beta} = \int_0^{\infty} v^{\beta-1} f(v) dv$

- additional term $\mathcal{S}^{osc}_{\Lambda}$ coming from series of contributions of poles of zeta function off the real line: oscillatory terms

Oscillatory terms (fractals)

• zeta function $\zeta_{\mathcal{L}}(s)$ on fractals in general has additional poles off the real line (position depends on Hausdorff and spectral dimension: depending on how homogeneous the fractal)

• best case exact self-similarity: $s = \sigma + \frac{2\pi i m}{\log \ell}$, $m \in \mathbb{Z}$

• heat kernel on fractals has additional log-oscillatory terms in expansion

$$\frac{C}{t^{\sigma}}(1 + A\cos(\frac{2\pi}{\log \ell}\log t + \phi)) + \cdots$$

for constants C, A, ϕ: series of terms for each complex pole

Log-oscillatory terms in expansion of the spectral action:

• G.V. Dunne, *Heat kernels and zeta functions on fractals*, J. Phys. A 45 (2012) 374016 [22p]

• M. Eckstein, B. Iochum, A. Sitarz, *Heat kernel and spectral action on the standard Podlés sphere*, Comm. Math. Phys. 332 (2014) 627–668

• M. Eckstein, A. Zajaç, *Asymptotic and exact expansion of heat traces*, arXiv:1412.5100 effect of product with S^1_β (leading term without oscillations)

- case of $S^1_\beta \times S^3_a$ (Chamseddine–Connes)

$$D_{S^1_\beta \times S^3_a} = \begin{pmatrix} 0 & D_{S^3_a} \otimes 1 + i \otimes D_{S^1_\beta} \\ D_{S^3_a} \otimes 1 - i \otimes D_{S^1_\beta} & 0 \end{pmatrix}$$

Spectral action

$$\mathrm{Tr}(h(D^2_{S^1_\beta \times S^3_a}/\Lambda)) \sim 2\beta\Lambda \mathrm{Tr}(\kappa(D^2_{S^3_a}/\Lambda)),$$

test function $h(x)$, and test function

$$\kappa(x^2) = \int_{\mathbb{R}} h(x^2 + y^2) dy$$

- Case of $S^1_\beta \times \mathcal{P}$:

$$\mathcal{S}_{S^1_\beta \times \mathcal{P}}(\Lambda) \sim 2\beta \left(\Lambda^4 \, \zeta_{\mathcal{L}}(3) \, \mathfrak{h}_3 - \Lambda^2 \, \frac{1}{4} \, \zeta_{\mathcal{L}}(1) \, \mathfrak{h}_1 \right)$$

$$+ 2\beta \, \Lambda^{\sigma+1} \, \left(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2}) \right) \, \mathcal{R}_\sigma \, \mathfrak{h}_\sigma$$

with momenta

$$\mathfrak{h}_3 := \pi \int_0^\infty h(\rho^2) \rho^3 d\rho, \quad \mathfrak{h}_1 := 2\pi \int_0^\infty h(\rho^2) \rho d\rho$$

$$\mathfrak{h}_\sigma = 2 \int_0^\infty h(\rho^2) \rho^\sigma d\rho$$

Interpretation:

- Term $2\Lambda^4 \beta a^3 \mathfrak{h}_3 - \frac{1}{2}\Lambda^2 \beta a \mathfrak{h}_1$, cosmological and Einstein–Hilbert terms, replaced by

$$2\Lambda^4 \beta \zeta_{\mathcal{L}}(3) \mathfrak{h}_3 - \frac{1}{2}\Lambda^2 \beta \zeta_{\mathcal{L}}(1) \mathfrak{h}_1$$

  zeta regularization of divergent series of spectral actions of 3-spheres of packing

- Additional term in gravity action functional: corrections to gravity from fractality

$$2\beta\,\Lambda^{\sigma+1} \left(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}) \right) \mathcal{R}_\sigma \mathfrak{h}_\sigma$$

Case of fractal dodecahedral space $\mathcal{P}_Y$

- Zeta functions

$$\zeta_{\mathcal{L}(\mathcal{P}_Y)}(s) = \sum_{n\geq 0} 20^n (2+\phi)^{-ns}$$

$$\zeta_{\mathcal{D}_{\mathcal{P}_Y}}(s) = \frac{a^s}{120}\left(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2})\right)\zeta_{\mathcal{L}(\mathcal{P}_Y)}(s)$$

- Spectral action:

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}_Y}/\Lambda)) \sim (\Lambda a)^3 \frac{\zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} f_3 - \Lambda a \frac{\zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} f_1$$

$$+(\Lambda a)^\sigma \frac{\zeta(\sigma-2,\frac{3}{2}) - \frac{1}{4}\zeta(\sigma,\frac{3}{2})}{120\log(2+\phi)} f_\sigma + \mathcal{S}^{osc}_{Y,\Lambda}$$

$\sigma = \dim_H(\mathcal{P}_Y) = \frac{\log(20)}{\log(2+\phi)} = 2.3296\ldots$

- on product geometry $S^1_\beta \times \mathcal{P}_Y$

$$\mathcal{S}_{S^1_\beta \times \mathcal{P}_Y}(\Lambda) \sim 2\beta \left(\Lambda^4 \frac{a^3 \zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} \mathfrak{h}_3 - \Lambda^2 \frac{a \zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} \mathfrak{h}_1 \right)$$

$$+2\beta\, \Lambda^{\sigma+1} \frac{a^\sigma (\zeta(\sigma-2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}))}{120 \log(2+\phi)} \mathfrak{h}_\sigma + \mathcal{S}^{osc}_{S^1_\beta \times Y, \Lambda}$$

- Note: correction term now at different σ than Apollonian $\mathcal{P}$
- oscillatory terms $\mathcal{S}^{osc}_{Y,\Lambda}$ more explicit than in the Apollonian case

Oscillatory terms: dodecahedral case

• zeros of zeta function $\zeta_{\mathcal{L}}(s)$

$$s_m = \sigma + \frac{2\pi i m}{\log(2+\phi)}, \qquad m \in \mathbb{Z}$$

with $\sigma = \log(20)/\log(2+\phi)$

• contribution to heat kernel expansion of non-real zeros:

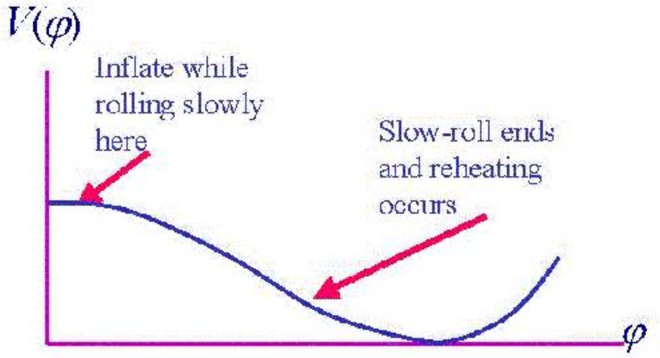
$$\frac{C}{t^{\sigma}}(a_0 + 2\Re(a_1 t^{-2\pi i/\log(2+\phi)}) + \cdots)$$

with coefficients a_m proportional to $\Gamma(s_m)$: for fixed real part σ decays exponentially fast along vertical line

• oscillatory terms are small

Slow-roll inflation potential from the spectral action

• perturb the Dirac operator by a scalar field $D^2 + \phi^2 \Rightarrow$ spectral action gives potential $V(\phi)$



• shape of $V(\phi)$ distinguishes most cosmic topologies: spherical forms and Bieberbach manifolds (Marcolli, Pierpaoli, Teh)

Fractality corrections to potential $V(\phi)$

- additional term in potential

$$\mathcal{U}_\sigma(x) = \int_0^\infty u^{(\sigma-1)/2}(h(u+x) - h(u))du$$

depends on σ fractal dimension

- size of correction depends on (leading term)

$$(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}))\mathcal{R}_\sigma$$

- further corrections to $\mathcal{U}_\sigma$ come from the oscillatory terms

$\Rightarrow$ presence of fractality (in this spectral action model of gravity) can be read off the slow-roll potential (hence the slow-roll coefficients, which depend on V, V', V'')

Robertson–Walker spacetime

- Topologically $S^3 \times \mathbb{R}$
- Metric (Euclidean)

$$ds^2 = dt^2 + a(t)^2 d\sigma^2$$

scaling factor $a(t)$, round metric $d\sigma^2$ on S^3

- A.H. Chamseddine, A. Connes, *Spectral action for Robertson-Walker metrics*, J. High Energy Phys. (2012) N.10, 101

- form of Dirac-Laplacian D^2 for Robertson–Walker metric

$$D^2 = -(\frac{\partial}{\partial t} + \frac{3a'(t)}{2a(t)})^2 + \frac{1}{a(t)^2}(\gamma^0 D_3)^2 - \frac{a'(t)}{a^2(t)}\gamma^0 D_3$$

- $\gamma^0 D_3 = D_{S^3} \oplus -D_{S^3}$, Dirac operator on S^3
- Dirac spectrum on S^3

$$\mathrm{Spec}(D_{S^3}) = \{k + \frac{3}{2}\} \text{ multiplicities } \mu(k + \frac{3}{2}) = (k+1)(k+2)$$

• use basis of eigenfunctions of the Dirac operator on S^3 to decompose D^2 as direct sum of operators

$$H_n = -(\frac{d^2}{dt^2} - \frac{(n+\frac{3}{2})^2}{a^2} + \frac{(n+\frac{3}{2})a'}{a^2})$$

multiplicity $4(n+1)(n+2)$

• spectral action for test function $f(u) = e^{-su}$

$$\mathrm{Tr}(f(D^2)) \sim \sum_{n\geq 0} \mu(n)\, \mathrm{Tr}(f(H_n))$$

multiplicities $\mu(n) = 4(n+1)(n+2)$ and operator H_n

$$H_n = -\frac{d^2}{dt^2} + V_n(t),$$

$$V_n(t) = \frac{(n+\frac{3}{2})}{a(t)^2}((n+\frac{3}{2}) - a'(t))$$

Result of this approach (Chamseddine–Connes)

• to compute the spectral action for the Robertson–Walker metric need to evaluate the trace $\mathrm{Tr}(e^{-sH_n})$ which requires computing $e^{-sH_n}(t,t)$ (for coeffs prior to time integration)

Feynman–Kac formula

$$e^{-sH_n}(t,t) = \frac{1}{2\sqrt{\pi s}} \int \exp\left(-s \int_0^1 V_n(t + \sqrt{2s}\alpha(u))du\right) D[\alpha]$$

$D[\alpha]$ Brownian bridge integrals

Brownian bridge: Gaussian stochastic process characterized by the covariance

$$\mathbb{E}(\alpha(v_1)\alpha(v_2)) = v_1(1 - v_2), \qquad 0 \leq v_1 \leq v_2 \leq 1$$

Background reference for Brownian bridge and Feynman–Kac:

- Barry Simon, *Functional Integration and Quantum Physics*, Academic Press, 1979

Problem: technique used on Chamseddine–Connes for computing the Brownian bridge integrals becomes computationally intractable after the 10th or 12th term

New Method for computing the Brownian bridge integrals more efficiently and obtain the full expansion of the spectral action

Quick summary of results in our work:

- use this Brownian bridge computation to obtain explicit formula for all the coefficients a_{2n} of the heat kernel expansion in terms of Bell polynomials
- consider isotropic non-homogeneous versions of Robertson–Walker spacetimes based on Apollonian packings of spheres (multifractal cosmologies)
- extend computation of the spectral action to these multifractal cases
- identify correction terms that *detect fractality*

Brownian bridge integrals and expansion

- notation $A(t) = 1/a(t)$ and $B(t) = A(t)^2$ so potential V_n

$V_n(t) = x^2 A(t)^2 + xA'(t) = x^2 B(t) + xA'(t), \qquad$ with $x = n + 3/2$

Integral in Feynman–Kac formula becomes

$$-s \int_0^1 V_n(t + \sqrt{2s}\,\alpha(v))\, dv = -x^2 U - xV$$

where

$$U = s \int_0^1 A^2 \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv = s \int_0^1 B \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv$$

$$V = s \int_0^1 A' \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv$$

- in heat kernel spectral multiplicities (Dirac eigenvalues on S^3)

$$\sum_n \mu(n) \mathrm{Tr}(e^{-s H_n}) \qquad \mu(n) = 4(n+1)(n+2) \qquad H_n = -\frac{d^2}{dt^2} + V_n(t)$$

replace sum over multiplicities by an integration of a continuous variable (Poisson summation) $x = n + 3/2$

- including multiplicities: $f_s(x) := \left(x^2 - \frac{1}{4}\right) e^{-x^2 U - xV}$

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \frac{\sqrt{\pi}\, e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}}$$

Generating function for the full expansion of the spectral action

$$\frac{1}{\sqrt{\pi s}} \frac{\sqrt{\pi}\, e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}} = \frac{1}{\sqrt{s}} \frac{e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}}$$

then consider Laurent series expansion in the variable $\tau = s^{1/2}$

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!}\, \tau^n \quad \text{and} \quad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!}\, \tau^n$$

$$u_n = B^{(n)}(t)\, 2^{n/2} x_n(\alpha) = \left(\sum_{k=0}^{n} \binom{n}{k} A^{(k)}(t) A^{(n-k)}(t) \right) 2^{n/2}\, x_n(\alpha)$$

$$v_n = A^{(n+1)}(t) 2^{n/2}\, x_n(\alpha)$$

$$x_k(\alpha) = \int_0^1 \alpha(v)^k\, dv$$

resulting expansion

$$\mathrm{Tr}(\exp(-\tau^2 D^2)) \sim \sum_{M=0}^{\infty} \tau^{2M-4} \int a_{2M}(t)\, dt,$$

$$a_{2M}(t) = \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} \left(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)} \right) \right) D[\alpha]$$

coefficients and Bell polynomials

$$C_{2M}^{(r,m)} = \sum_{\substack{0 \le k,p \le 2M \\ 0 \le n \le M \\ 0 \le \beta \le 2M-2n}} \left(\frac{\binom{-n+r}{k} \binom{2n+m}{p} \binom{2M-2n}{\beta}\, k!\, p!}{4^n\, n!\, (2M-2n)!} u_0^{-n+r-k} v_0^{2n+m-p} \times \right.$$

$$\left. B_{\beta,k}\,(u_1, \ldots, u_{\beta-k+1})\, B_{2M-2n-\beta,\, p}\,(v_1, \ldots, v_{2M-2n-\beta-p+1}) \right)$$

Bell polynomials: Faà di Bruno derivatives of composite functions

$$\frac{d^n}{dt^n} f(g(t)) = \sum_{m=1}^{n} f^{(m)}(g(t))\, B_{n,m}(g'(t), g''(t), \ldots, g^{(n-m+1)}(t))$$

Structure of Brownian Bridge Integrals

Step 1: integrals of monomials on the standard simplex

$$\Delta^n = \{(v_1, v_2, \dots, v_n) \in \mathbb{R}^n : 0 \le v_1 \le v_2 \le \cdots \le v_n \le 1\}.$$

monomial $v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$

$$\int_{\Delta^n} v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} \, dv_1 \, dv_2 \cdots dv_n =$$

$$\frac{1}{(k_1+1)(k_1+k_2+2)\cdots(k_1+k_2+\cdots+k_n+n)}$$

Similarly for $1 \le j_1 < j_2 < \cdots < j_k \le n$

$$\int_{\Delta^n} v_{j_1} v_{j_2} \cdots v_{j_k} \, dv_1 \, dv_2 \cdots dv_n = \frac{j_1(j_2+1)(j_3+2)\cdots(j_k+k-1)}{(n+k)!}$$

Step 2: Brownian Bridge and integration on the simplex

- Using variance property of Brownian Bridge: $(v_1, v_2, \dots, v_{2n}) \in \Delta^{2n}$

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\, D[\alpha] = \sum v_{i_1}(1-v_{j_1})v_{i_2}(1-v_{j_2})\cdots v_{i_n}(1-v_{j_n})$$

summation over indices with $i_1 < j_1$, $i_2 < j_2$, $\dots$, $i_n < j_n$, and $\{i_1, j_1, i_2, j_2, \dots, i_n, j_n\} = \{1, 2, \dots, 2n\}$

- equivalently for $(v_1, v_2, \dots, v_{2n}) \in \Delta^{2n}$

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\, D[\alpha] =$$

$$\sum_{\sigma\in S^*_{2n}} v_{\sigma(1)}(1 - v_{\sigma(2)})v_{\sigma(3)}(1 - v_{\sigma(4)})\cdots v_{\sigma(2n-1)}(1 - v_{\sigma(2n)})$$

S^*_{2n} set of all permutations σ in symmetric group S_{2n} with $\sigma(1) < \sigma(2)$, $\sigma(3) < \sigma(4)$, $\dots$, $\sigma(2n-1) < \sigma(2n)$

Brownian Bridge Integrals

• Notation: $\mathcal{J}_{k,n}$ = set of all k-tuples of integers $J = (j_1, j_2, \ldots, j_k)$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq n$; for $J \in \mathcal{J}_{k,n}$ and $\sigma \in S^*_{2n}$ define $\sigma_J(1)$, $\sigma_J(2)$, ..., $\sigma_J(n+k)$ by property that

$$\sigma_J(1) < \sigma_J(2) < \cdots < \sigma_J(n+k)$$

and that the set of such σ_J's is given by
$\{\sigma_J(1) < \sigma_J(2) < \cdots < \sigma_J(n+k)\}$

$$= \{\sigma(1), \sigma(3), \ldots, \sigma(2n-1), \sigma(2j_1), \ldots, \sigma(2j_k)\}$$

$$x_k(\alpha) = \int_0^1 \alpha(v)^k \, dv$$

• Brownian Bridge Integrals

$$\int x_1(\alpha)^{2n} \, D[\alpha] = \int \Big(\int_0^1 \alpha(v) \, dv \Big)^{2n} \, D[\alpha] =$$

$$(2n)! \sum_{\sigma \in S^*_{2n}} \sum_{k=0}^{n} \sum_{J \in \mathcal{J}_{k,n}} (-1)^k \frac{\sigma_J(1) \, (\sigma_J(2)+1) \cdots (\sigma_J(n+k)+n+k-1)}{(3n+k)!}$$

Monomial Brownian Bridge Integrals

- for $(v_1, v_2, \dots, v_n) \in \Delta^n$ and for $i_1, i_2, \dots, i_n \in \mathbb{Z}_{\geq 0}$ such that $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$

$$\int \alpha(v_1)^{i_1}\alpha(v_2)^{i_2}\cdots\alpha(v_n)^{i_n}\, D[\alpha] =$$

$$\binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!} \left(\sum \binom{|I|/2}{k_{m,j}} \sum_{r_1=0}^{K_1} \sum_{r_2=0}^{K_2} \cdots \sum_{r_n=0}^{K_n} \prod_{p=1}^{n} (-1)^{r_p} v_p^{i_p - r_p} \right),$$

with $I = (i_1, i_2, \dots, i_n)$, first summation over non-negative integers $k_{j,m}$, $j, m = 1, 2, \dots, n$ such that

$$\sum_{j,m=1}^{n} k_{j,m} = \frac{|I|}{2}, \qquad \sum_{m=1}^{n} (k_{j,m} + k_{m,j}) = i_j \;\text{ for all }\; j = 1, 2, \dots, n$$

and for each $m = 1, 2, \dots, n$,

$$K_m := k_{m,m} + \sum_{j=1}^{m-1} (k_{j,m} + k_{m,j})$$

Sketch of proof

$$\int \exp\left(\sqrt{-1}\sum_{j=1}^{n} u_j \alpha(v_j)\right) D[\alpha] = \exp\left(-\frac{1}{2}\sum_{j,m=1}^{n} c_{j,m} u_j u_m\right)$$

where the terms $c_{j,m}$ are given by

$$c_{j,m} = v_j(1-v_m) \quad \text{if} \quad j \leq m, \quad \text{and} \quad c_{j,m} = v_m(1-v_j) \quad \text{if} \quad m \leq j$$

Expanding gives

$$\frac{(\sqrt{-1})^{i_1+i_2+\cdots+i_n}}{(i_1+i_2+\cdots+i_n)!}\binom{i_1+i_2+\cdots+i_n}{i_1, i_2, \ldots, i_n}\int \alpha(v_1)^{i_1}\alpha(v_2)^{i_2}\cdots\alpha(v_n)^{i_n}\, D[\alpha] =$$

$$\frac{(-1/2)^{(i_1+i_2+\cdots+i_n)/2}}{((i_1+i_2+\cdots+i_n)/2)!}\left(\text{Coefficient of } u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n} \text{ in } \Big(\sum_{j,m=1}^{n} c_{j,m}u_j u_m\Big)^{(i_1+i_2+\cdots+i_n)/2}\right)$$

$$= \frac{(-1/2)^{(i_1+i_2+\cdots+i_n)/2}}{((i_1+i_2+\cdots+i_n)/2)!}\sum\binom{(i_1+i_2+\cdots+i_n)/2}{k_{1,1}, k_{1,2}, \ldots, k_{1,n}, k_{2,1}, \ldots, k_{n,n}}\prod_{j,m=1}^{n} c_{j,m}^{k_{j,m}}$$

from which then can group terms as stated

Shuffle Product

• for $(v_1, v_2, \ldots, v_n) \in \Delta^n$ and $i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\geq 0}$ with $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$

$$V^b(i_1, i_2, \ldots, i_n) := \int \alpha(v_1)^{i_1}\alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n}\, D[\alpha]$$

• extend V^b linearly to vector space generated by all words $(i_1, i_2, \ldots, i_n)$ in the letters $i_1, i_2, \ldots, i_n$

• Shuffle product $\alpha \sqcup\!\sqcup \beta$ of two words $\alpha = (i_1, i_2, \ldots, i_p)$ and $\beta = (j_1, j_2, \ldots, j_q)$ sum of $\binom{p+q}{p}$ words obtained by interlacing letters of these two words so that in each term the order of the letters of each word is preserved

• $2n = m_1 i_1 + m_2 i_2 + \cdots + m_r i_r$ even positive integer with $i_1, i_2, \ldots, i_r$ distinct positive integers and $m_1, m_2, \ldots, m_r$ positive integers

$$\int x_{i_1}(\alpha)^{m_1} x_{i_2}(\alpha)^{m_2} \cdots x_{i_r}(\alpha)^{m_r} \, D[\alpha] =$$

$$m! \int_{\Delta^{|m|}} V^b(\underbrace{(i_1, \ldots, i_1)}_{m_1} \sqcup \underbrace{(i_2, \ldots, i_2)}_{m_2} \sqcup \cdots \sqcup \underbrace{(i_r, \ldots, i_r)}_{m_r}) dv_1 dv_2 \cdots dv_{|m|}$$

$$m! = (m_1!)(m_2!) \cdots (m_r!), \qquad |m| = m_1 + m_2 + \cdots + m_r \,.$$

follows directly from writing

$$\int x_{i_1}(\alpha)^{m_1} x_{i_2}(\alpha)^{m_2} \cdots x_{i_r}(\alpha)^{m_r} \, D[\alpha]$$

$$= \int \Big(\int_0^1 \alpha(v_1)^{i_1} \, dv_1 \Big)^{m_1} \Big(\int_0^1 \alpha(v_2)^{i_2} \, dv_2 \Big)^{m_2} \cdots \Big(\int_0^1 \alpha(v_r)^{i_r} \, dv_r \Big)^{m_r} D[\alpha],$$

Brownian Bridge Integrals in the Coefficients of the Spectral Action

$$\int C_{2M}^{(r,m)}\, D[\alpha] =$$

$$\sum \left(\frac{\binom{-n+r}{k}\binom{2n+m}{p} k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b\big(\underbrace{(1,\dots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\dots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \big)\, dv_1 \cdots dv_{k+p} \right.$$

$$\left. \times B(t)^{-n+r-k}\, (A'(t))^{2n+m-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \right)$$

summation is over integers $0 \leq k, p \leq 2M, 0 \leq n \leq M$, $0 \leq \beta \leq 2M - 2n$, and over sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ of non-negative integers for each choice of k, p, n, β, such that $|\lambda|' = \beta, |\lambda| = k,\ |\mu|' = 2M - 2n - \beta, |\mu| = p$

coefficients of the expansion of the spectral action of Robertson–Walker metric

$$a_{2M}(t) =$$

$$\frac{1}{2}\sum{}' \Big(\frac{\binom{-n-3/2}{k}\binom{2n}{p} k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b\big(\underbrace{(1,\ldots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\ldots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \big)\, dv_1 \cdots dv_{k+p} \times$$

$$B(t)^{-n-(3/2)-k}\ \big(A'(t)\big)^{2n-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \Big)$$

$$+\frac{1}{4}\sum{}'' \Big(\Big(\binom{-n-5/2}{k}\binom{2n+2}{p} B(t)^{-5/2}\big(A'(t)\big)^2 - \binom{-n-1/2}{k}\binom{2n}{p} B(t)^{-1/2} \Big) \times$$

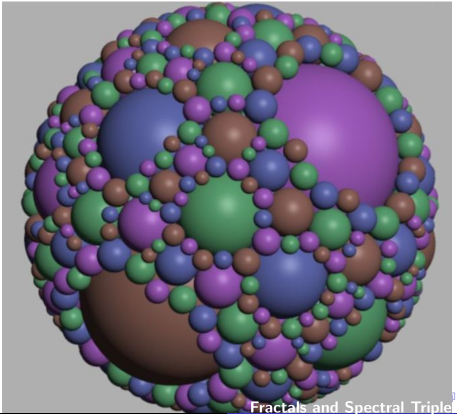
$$\frac{k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b\big(\underbrace{(1,\ldots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\ldots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \big)\, dv_1 \cdots dv_{k+p} \times$$

$$B(t)^{-n-k}\ \big(A'(t)\big)^{2n-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \Big)$$

summation $\sum{}'$ is over all integers $0 \le k, p \le 2M, 0 \le n \le M$, $0 \le \beta \le 2M - 2n$, and sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ of non-negative integers (for each choice of k, p, n, β) such that $|\lambda|' = \beta, |\lambda| = k$, $|\mu|' = 2M - 2n - \beta, |\mu| = p$; second summation $\sum{}''$ is over all integers $0 \le k, p \le 2M - 2, 0 \le n \le M - 1$, $0 \le \beta \le 2M - 2 - 2n$, over all sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$ of non-negative integers such that $|\lambda|' = \beta, |\lambda| = k$, $|\mu|' = 2M - 2 - 2n - \beta, |\mu| = p$

Packed Swiss Cheese Cosmology: more refined model based on Robertson–Walker metrics

- $\mathcal{P}$ Apollonian packing of 3-spheres radii $\{a_{n,k} : n \in \mathbb{N}, k = 1, \ldots, 6 \cdot 5^{n-1}\}$
- iterative construction of packing: at n-th step $6 \cdot 5^{n-1}$ spheres $S^3_{a_{n,k}}$ are added
- spacetime that are isotropic but not homogeneous



- two possible choices of associated Robertson–Walker metrics
  1. round scaling (of full 4-dim spacetime)

$$ds^2_{n,k} = a^2_{n,k}\,(dt^2 + a(t)^2\, d\sigma^2), \qquad n \in \mathbb{N}, k = 1, \dots, 6 \cdot 5^{n-1}$$

  2. non-round scaling (of spatial sections only)

$$ds^2_{n,k} = dt^2 + a(t)^2\, a^2_{n,k}\, d\sigma^2, \qquad n \in \mathbb{N}, k = 1, \dots, 6 \cdot 5^{n-1}$$

- $D_{n,k}$ resulting Dirac operators on $\mathbb{R} \times S^3_{a_{n,k}}$
- entire (multifractal) spacetime $\mathbb{R} \times \mathcal{P}$
- spectral triple for $\mathbb{R} \times \mathcal{P}$: $\mathcal{A}$ subalgebra of $C_0(\mathbb{R} \times \mathcal{P})$, Hilbert space $\mathcal{H} = \oplus_{n,k} \mathcal{H}_{n,k}$ with $\mathcal{H}_{n,k} = L^2(S_{a_{n,k}}, \mathbb{S})$ and Dirac

$$D = D_{\mathbb{R}\times\mathcal{P}} := \bigoplus_{n\in\mathbb{N}} \bigoplus_{k=1}^{6\cdot 5^{n-1}} D_{n,k}$$

Mellin Transform and Zeta Functions

- meromorphic function $\phi(z)$ with poles at $\mathcal{S} \subset \mathbb{C}$, Laurent series expansion at a pole $z_0 \in \mathcal{S}$

$$\phi(z) = \sum_{-N \le k} c_k (z - z_0)^k$$

- singular element at $z_0 \in \mathcal{S}$

$$S(\phi, z_0) := \sum_{-N \le k \le 0} c_k (z - z_0)^k$$

- singular expansion of ϕ

$$S_\phi(z) := \sum_{z \in \mathcal{S}} S(\phi, z)$$

- Example: for the Gamma function

$$\Gamma(z) \asymp \sum_{k \ge 0} \frac{(-1)^k}{k!} \frac{1}{z + k}$$

- Mellin transform

$$\phi(z) = \mathcal{M}(f)(z) = \int_0^\infty f(\tau)\tau^{z-1} d\tau$$

- relation between asymptotic expansion at $u \to 0$ of a function $f(u)$ and singular expansion of its Mellin transform $\phi(z) = \mathcal{M}(f)(z)$
- small time asymptotic expansion

$$f(u) \sim_{u\to 0^+} \sum_{\alpha\in\mathcal{S}, k_\alpha} c_{\alpha,k_\alpha} u^\alpha \log(u)^{k_\alpha}$$

- coefficients c_{α,k_α} determined by singular expansion of Mellin transform

$$\mathcal{M}(f)(z) \asymp S_{\mathcal{M}(f)}(z) = \sum_{\alpha\in\mathcal{S}, k_\alpha} c_{\alpha,k_\alpha} \frac{(-1)^{k_\alpha} k_\alpha!}{(s+\alpha)^{k_\alpha+1}}$$

- index k_α ranges over terms in singular element of $\phi(z) = \mathcal{M}(f)(z)$ at $z = \alpha$, up to order of pole at α

Example: Packing of 4-Spheres

- round S^4 is a Robertson–Walker metric $dt^2 + a(t)^2 d\sigma^2$ with $a(t) = \sin t$ $(0 \leq t \leq \pi)$ and $d\sigma^2$ round metric on S^3
- spectrum of Dirac operator on S_r^{D-1} radius $r > 0$

$$\mathrm{Spec}(D_{S_r^{D-1}}) = \left\{ \lambda_{\ell,\pm} = \pm r^{-1}\left(\frac{D-1}{2} + \ell\right) \mid l \in \mathbb{Z}_+ \right\}$$

multiplicities

$$\mathrm{m}_{\ell,\pm} = 2^{[\frac{D-1}{2}]}\binom{\ell + D}{\ell}.$$

- zeta function of Dirac operator

$$\zeta_D(s) = \mathrm{Tr}(|D_{S_r^4}|^{-s}) = \sum_{\ell,\pm} \mathrm{m}_{\ell,\pm} |\lambda_{\ell,\pm}|^{-s} = \frac{4}{3}\, r^s\, (\zeta(s-3) - \zeta(s-1))$$

$\zeta(s)$ Riemann zeta function

- fractal string zeta function $\zeta_{\mathcal{L}}(s) = \sum_{n,k} a_{n,k}^s$ of Apollonian packing $\mathcal{P}$ of $S^3_{a_{n,k}}$ with radii sequence $\mathcal{L} = \{a_{n,k}\}$
- resulting Dirac operator $\mathcal{D}_{\mathcal{P}}$ on associated packing of 4-spheres (each 3-sphere equator in a fixed hyperplane of a corresponding 4-sphere)
- zeta function of Dirac $\mathcal{D}_{\mathcal{P}}$ factors as product of zetas

$$\zeta_{\mathcal{D}_{\mathcal{P}}}(s) = \mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n,k} \frac{4}{3} a_{n,k}^s (\zeta(s-3) - \zeta(s-1))$$

$$= \zeta_{\mathcal{L}}(s)\zeta_{D_{S^4}}(s)$$

- Mellin transform relation between the zeta function of the Dirac operator and the heat-kernel of the Dirac Laplacian

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \frac{1}{\Gamma(s/2)} \int_0^\infty \mathrm{Tr}(e^{-t\mathcal{D}_{\mathcal{P}}^2})\, t^{s/2-1} dt$$

- use to compute spectral action leading terms from zeta function: $\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim$

$$f(0)\zeta_{\mathcal{D}_{\mathcal{P}}}(0) + f_2\Lambda^2 \frac{\zeta_{\mathcal{L}}(2)}{2} + f_4\Lambda^4 \frac{\zeta_{\mathcal{L}}(4)}{2} + \sum_{\sigma\in\mathcal{S}(\mathcal{L})} f_\sigma \Lambda^\sigma \frac{\zeta_{D_{S^4}}(\sigma)}{2} \mathcal{R}_\sigma$$

- $\mathcal{S}(\mathcal{L})$ set of poles of fractal string zeta $\zeta_{\mathcal{L}}(s)$ residues

$$\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)$$

- $\zeta_{\mathcal{L}}(2)$ and $\zeta_{\mathcal{L}}(4)$ replace radii r^2 and r^4 for a single sphere S^4_r: zeta regularization of $\sum_{n,k} a^2_{n,k}$ and $\sum_{n,k} a^4_{n,k}$

Round scaling case: $\mathbb{R} \times \mathcal{P}$ with metric $a_{n,k}^2(dt^2 + a(t)^2 d\sigma^2)$

- $R = \{r_n\}$ sequence of $r_n \in \mathbb{R}_+^*$ so that $\zeta_R(z) = \sum_n r_n^{-z}$ converges for $\Re(z) > C$ for some $C > 0$
- function $f(\tau)$ with small time asymptotics

$$f(\tau) \sim \sum_N c_N \tau^N$$

- associated series

$$g_R(\tau) = \sum_n f(r_n \tau)$$

- then small time asymptotic expansion of $g_R(\tau)$

$$g_R(\tau) \sim_{\tau \to 0^+} \sum_N c_N \, \zeta_R(-N) \, \tau^N + \sum_{\sigma \in \mathcal{S}(\zeta_R)} \mathcal{R}_{R,\sigma} \, \mathcal{M}(f)(\sigma) \, \tau^{-\sigma}$$

with $\mathcal{S}(\zeta_R)$ poles of $\zeta_R(z)$

$$\mathcal{R}_{R,\sigma} := \mathrm{Res}_{z=\sigma} \zeta_R(z)$$

Sketch of proof:

- write associated series as

$$g_R(\tau) \sim \sum_{N,n} c_N r_n^N \tau^N = \sum_N \zeta_R(-N)\tau^N$$

- Mellin transform $\mathcal{M}(g)(z) = \int_0^\infty g(\tau)\tau^{z-1} d\tau$ gives

$$\mathcal{M}(g)(z) = (\sum_n r_n^{-z}) \int_0^\infty \sum_N c_N\, u^{N+z-1} du = \zeta_R(z) \cdot \mathcal{M}(f)(z)$$

- asymptotic expansion of $g_R(\tau)$ from Mellin transform $\mathcal{M}(g_R)(z)$ singular expansion

$$S_{\mathcal{M}(g_R)}(z) = \sum_{\sigma \in \mathcal{S}(\zeta_R)} \frac{\mathcal{R}_{R,\sigma}\, \mathcal{M}(f)(\sigma)}{z-\sigma} + \sum_{\sigma \in \mathcal{S}(\mathcal{M}(f))} \frac{\zeta_R(\sigma) c_\sigma}{z-\sigma}$$

- and from small time asymptotics of $f(\tau)$ know

$$S_{\mathcal{M}(f)}(z) = \sum_N \frac{c_N}{z+N}$$

Feynman–Kac formula on $\mathbb{R} \times \mathcal{P}$

- on each $\mathbb{R} \times S^3_{a_{n,k}}$ decompose Dirac $D_{a_{n,k}}$ using operators

$$H_{m,n,k} = -a_{n,k}^{-2} \frac{d^2}{dt^2} + V_{m,n,k}(t)$$

$$V_{m,n,k} = \frac{(m+\frac{3}{2})}{a_{n,k}^2 \cdot a(t)^2}((m+\frac{3}{2}) - a_{n,k} \cdot a'(t)$$

  as in Chamseddine–Connes

- Feynman–Kac formula

$$e^{-\tau^2 H_{m,n,k}}(t,t) = e^{-\frac{\tau^2}{a_{n,k}^2}(\frac{d^2}{dt^2} + a_{n,k}^2 V_{m,n,k})}(t,t)$$

$$= \frac{a_{n,k}}{2\sqrt{\pi}\tau} \int \exp(-\tau^2 \int_0^1 V_{m,n,k}(t + \sqrt{2}\frac{\tau}{a_{n,k}} \alpha(u))du) D[\alpha]$$

- Poisson summation to replace sum

$$\sum_m \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t)$$

with multiplicities $\mu(m)$ with the integral

$$\int_{-\infty}^{\infty} f_{\tau,\,n,k}(x)\,dx$$

$$f_{\tau,\,n,k}(x) = \left(x^2 - \frac{1}{4}\right) e^{-x^2 a_{n,k}^{-2} U - x a_{n,k}^{-1} V}$$

with U and V as in single sphere case

$$\sum_m \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t) =$$

$$\int \frac{a_{n,k}}{\tau}\left(\frac{e^{\frac{V^2}{4U}}}{4}(-a_{n,k}U^{-1/2} + 2a_{n,k}^3 U^{-3/2} + a_{n,k}^3 V^2 U^{-5/2})\right) D[\alpha]$$

$$= \int \frac{1}{\tau}\left(\frac{e^{\frac{V^2}{4U}}}{4}(-a_{n,k}^2 U^{-1/2} + a_{n,k}^4(2U^{-3/2} + V^2 U^{-5/2}))\right) D[\alpha]$$

- same Taylor expansion method

$$e^{\frac{V^2}{4U}} U^r V^\ell = \tau^{2(r+\ell)} \sum_{M=0}^{\infty} a_{n,k}^{-M-2(r+\ell)} \, C_M^{(r,\ell)} \tau^M$$

with $C_M^{(r,\ell)}$ as in single sphere case $dt^2 + a(t)^2 d\sigma^2$

- resulting expansion

$$\frac{1}{\tau}\left(\frac{e^{\frac{V^2}{4U}}}{4}(-a_{n,k}^2 U^{-1/2} + a_{n,k}^4(2U^{-3/2} + V^2 U^{-5/2}))\right) =$$
$$\frac{1}{4}\sum_{M=0}^{\infty}\left(C_M^{(-5/2,2)} - C_M^{(-1/2,0)}\right)\zeta_{\mathcal{L}}(-M+2)\tau^{M-2}$$
$$+\frac{1}{2}\sum_{M=0}^{\infty} C_M^{(-3/2,0)}\zeta_{\mathcal{L}}(-M+4)\tau^{M-4}.$$

- Feynman–Kac formula for the whole $\mathbb{R} \times \mathcal{P}$

$$\sum_{n,k} \sum_{m} \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t) =$$

$$\sum_{M=0}^{\infty} \tau^{2M-4} \zeta_{\mathcal{L}}(-2M+4) \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} (C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}) \right) D[\alpha]$$

with only the term $\frac{1}{2} C_0^{(-3/2,0)}$ when $M = 0$

- obtained as a series

$$g_{\mathcal{L}}(\tau) = \sum_{n,k} f(a_{n,k}^{-1} \tau)$$

$$f(\tau) \sim \sum_{M} \tau^{2M-4} \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} (C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}) \right) D[\alpha]$$

Result: Spectral Action on Multifractal Robertson–Walker $\mathbb{R} \times \mathcal{P}$

$$\mathrm{Tr}(f(\mathcal{D}/\Lambda)) \sim$$

$$\sum_{M=0}^{\infty} \Lambda^{n_M} f_{n_M} \zeta_{\mathcal{L}}(n_M) \int \left(\frac{1}{2} C_{4-n_M}^{(-3/2,0)} + \frac{1}{4} (C_{2-n_M}^{(-5/2,2)} - C_{2-n_M}^{(-1/2,0)}) \right) D[\alpha]$$
$$+ \sum_{\sigma \in \mathcal{S}_{\mathcal{L}}} \tilde{f}(\sigma) \cdot f_\sigma \cdot \mathrm{Res}_{z=\sigma} \zeta_{\mathcal{L}} \cdot \Lambda^\sigma$$

$n_M = 4 - 2M$, set of poles $\mathcal{S}_{\mathcal{L}}$ of $\zeta_{\mathcal{L}}$ Mellin transform $\tilde{f}(z) = \mathcal{M}(f)(z)$ of $f(\tau) = \mathrm{Tr}(\exp(-\tau^2 D^2))$ with Dirac on $\mathbb{R} \times S^3$ with $dt^2 + a(t)^2 d\sigma^2$

Conclusion: presence of fractality detected by two types of effects

1. zeta regularization of coefficients $\zeta_{\mathcal{L}}(4-2M)$ in terms Λ^{4-2M} (including effective gravitational and cosmological constant in top terms)
2. additional terms from non-real poles of order $\Lambda^{\Re\sigma}$ (and log periodic) with $3 < \Re\sigma = \dim_H \mathcal{P} < 4$ between cosmological and Einstein–Hilbert term

Multifractal Robertson–Walker with non-round scaling

$dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

- rescaling $ds_a^2 = dt^2 + a^2 \cdot a(t)^2\, d\sigma^2$ with $a > 0$ gives $U \mapsto a^{-2}\, U$ and $V \mapsto a^{-1}\, V$
- this gives rescaling

$$\frac{1}{4}\sum_{M=0}^{\infty}(a^3\, C_M^{(-5/2,2)} - a\, C_M^{(-1/2,0)})\,\tau^{M-2} + \frac{1}{2}\sum_{M=0}^{\infty} a^3\, C_M^{(-3/2,0)}\,\tau^{M-4}$$

- expect presence of zeta regularized coefficients $\zeta_{\mathcal{L}}(3)$, $\zeta_{\mathcal{L}}(1)$
- to see this use a Mellin transform with respect to the "multiplicity variable" x in $f_s(x)$

Kummer confluent hypergeometric function

- notation: $a^{(n)} := a(a+1)\cdots(a+n-1)$ and $a^{(0)} := 1$
- Kummer confluent hypergeometric function defined by series

$$ {}_1F_1(a,b,t) = \sum_{n=0}^{\infty} \frac{a^{(n)} t^n}{b^{(n)} n!} $$

- solution of the Kummer equation

$$ t\frac{d^2 f}{dt^2} + (b-t)\frac{df}{dt} - af = 0. $$

Mellin transform and hypergeometric function

- Mellin transform in the x-variable of the function

$$f_{s,-}(x) := f_s(x) = (x^2 - \frac{1}{4})e^{-x^2U - xV}$$

given by

$$\mathcal{M}((x^2 - \frac{1}{4})e^{-x^2U-xV})(z) = \frac{1}{8}U^{-(z+3)/2}\times$$

$$\Big(U^{1/2}\,\Gamma(\frac{z}{2})(-U\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}) + 2z\,{}_1F_1(\frac{z+2}{2},\frac{1}{2},\frac{V^2}{4U}))$$

$$+V\,\Gamma(\frac{z+1}{2})(U\,{}_1F_1(\frac{z+1}{2},\frac{3}{2},\frac{V^2}{4U}))-2(z+1)\,{}_1F_1(\frac{z+3}{2},\frac{3}{2},\frac{V^2}{4U}))\Big)$$

- similar expression for transform of $f_{s,+}(x) := (x^2 - \frac{1}{4})e^{-x^2U+xV}$

- multiplicity integral

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \int_0^{\infty} f_{s,-}(x)\, dx + \int_0^{\infty} f_{s,+}(x)\, dx$$

$$f_{s,\pm}(x) = (x^2 - \frac{1}{4})e^{-x^2 U \pm xV}$$

- multiplicity integral as special value at $z = 1$ of Mellin

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \mathcal{M}(f_{s,-})(z)|_{z=1} + \mathcal{M}(f_{s,+})(z)|_{z=1}$$

- Mellin transform $\mathcal{M}(f_{s,-})(z) + \mathcal{M}(f_{s,+})(z)$

$$= -\frac{1}{4}\, U^{-1-\frac{z}{2}}\, \Gamma(\frac{z}{2})(U\, {}_1F_1(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}) - 2z\, {}_1F_1(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}))$$

- value at $z = 1$

$$\left(-\frac{1}{4}\, U^{-(1+\frac{z}{2})}\, \Gamma(\frac{z}{2})(U\, {}_1F_1(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}) - 2z\, {}_1F_1(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}))\right)|_{z=1} =$$

$$e^{\frac{V^2}{4U}} \frac{\sqrt{\pi}}{4}(-U^{-1/2} + 2U^{-3/2} + V^2 U^{-5/2})$$

Effect of scaling

- notation:

$$H_\lambda(\tau,z) := \quad U^{-z/2}\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\lambda,\frac{V^2}{4U})$$

$$H(\tau,z) := \quad H_{1/2}(\tau,z) = U^{-z/2}\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U})$$

$$H_{\mathcal{L}}(\tau,z) := \quad U^{-z/2}\,\zeta_{\mathcal{L}}(z)\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}) = \zeta_{\mathcal{L}}(z)\,H(\tau,z)$$

- multiplicity integral with scaling $dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

$$\mathcal{M}(f_{s,n,k,-})(z) + \mathcal{M}(f_{s,n,k,+})(z) =$$

$$-\frac{1}{4}\,a_{n,k}^z\,U^{-\frac{z}{2}}\,\Gamma(\frac{z}{2})_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U})$$

$$+a_{n,k}^{z+2}U^{1-\frac{z}{2}}\,\Gamma(1+\frac{z}{2})\,{}_1F_1(1+\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}))$$

$$-\frac{1}{4}a_{n,k}^z\,H(\tau,z) + a_{n,k}^{z+2}\,H(\tau,z+2).$$

- multiplicity integral over the full sphere packing $\mathbb{R} \times \mathcal{P}$

$$f_{\mathcal{P},s}(x) = \left(x^2 - \frac{1}{4}\right) \sum_{n,k} e^{-x^2 a_{n,k}^{-2} U - x a_{n,k}^{-1} V}$$

- as value of Mellin transform

$$\int_{-\infty}^{\infty} f_{\mathcal{P},s}(x)\, dx = \mathcal{M}(f_{\mathcal{P},s,-})(z)|_{z=1} + \mathcal{M}(f_{\mathcal{P},s,+})(z)|_{z=1}$$

$f_{\mathcal{P},s,\pm} = (x^2 - 1/4) \sum_{n,k} \exp(-x^2 a_{n,k}^{-2} U \pm x a_{n,k}^{-1} V)$

- Mellin transforms

$$\mathcal{M}(f_{\mathcal{P},s,-})(z) + \mathcal{M}(f_{\mathcal{P},s,+})(z) = -\frac{1}{4} H_{\mathcal{L}}(\tau, z) + H_{\mathcal{L}}(\tau, z+2)$$

- this shows one gets zeta regularized $\zeta_{\mathcal{L}}(3)$ and $\zeta_{\mathcal{L}}(1)$

Sketch of how to see the log periodic terms for non-round scaling $dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

- τ expansion

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!}\, \tau^n, \quad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!}\, \tau^n$$

- gives expansion of confluent hypergeometric function

$$H_{\mathcal{L}}(\tau, z) = \zeta_{\mathcal{L}}(z)\Gamma(z/2)U^{-z/2} \sum_{n=0}^{\infty} \frac{(z/2)_n}{4^n\, n!\, (1/2)_n}\, V^{2n} U^{-n}$$

  with $(a)_n = a(a+1)\cdots(a+n-1)$ the rising factorial

- the term $U^{-z/2}$ contributes a term with τ^z times a power series in τ, while confluent hypergeometric function contributes a power series in τ

- to see why τ^z gives rise to log periodic terms in the spectral action consider simplified case
- product of the Mellin transforms $\mathcal{M}(f_1)(z)\cdot\mathcal{M}(f_2)(z)$ is Mellin transform of convolution

$$\mathcal{M}(f_1)(z)\cdot\mathcal{M}(f_2)(z)=\mathcal{M}(f_1\star f_2)(z),$$

$$(f_1\star f_2)(x)=\int_0^\infty f_1(\frac{x}{u})f_2(u)\,\frac{du}{u}$$

- Mellin transform of a delta distribution

$$\tau^{z-1}=\mathcal{M}(\delta(x-\tau))$$

- Mellin transform of distribution $\Lambda_{\mathcal{P},\tau}:=\sum_{n,k}\tau\, a_{n,k}\ \delta(x-\tau\cdot a_{n,k})$

$$\langle\sum_{n,k}\tau\, a_{n,k}\ \delta(x-\tau\cdot a_{n,k}),\phi(x)\rangle=\sum_{n,k}\tau\, a_{n,k}\ \phi(\tau\, a_{n,k})$$

given by

$$\tau^z\,\zeta_{\mathcal{L}}(z)=\mathcal{M}(\sum_{n,k}\tau\, a_{n,k}\ \delta(x-\tau\cdot a_{n,k}))$$

- given function $g(x)$
  will want $g_\gamma(x) := \mathcal{M}^{-1}(\Gamma(z/2)\,{}_1F_1(z/2, 1/2, \gamma))$

$$\mathcal{M}(\Lambda_{\mathcal{P},\tau})(z) \cdot \mathcal{M}(g)(z) = \mathcal{M}(\Lambda_{\mathcal{P},\tau} \star g)(z)$$

$$= \mathcal{M}(\sum_{n,k} \tau a_{n,k} \int_0^\infty \delta(u - \tau a_{n,k})\, g(\frac{x}{u})\, \frac{du}{u}) = \sum_{n,k} \mathcal{M}(g(\frac{x}{\tau \cdot a_{n,k}}))$$

- take $h_z(\tau) := \mathcal{M}(g(\frac{x}{\tau}))$

$$L_z(\tau) := \sum_{n,k} h_z(\tau \cdot a_{n,k})$$

- asymptotic expansion for this function through singular expansion of Mellin transform in τ

$$\mathcal{M}_\tau(L_z(\tau))(\beta) = \zeta_{\mathcal{L}}(\beta) \cdot \mathcal{M}(h_z(\tau))(\beta)$$

- contributions from poles of $\zeta_{\mathcal{L}}(\beta)$ and of $\mathcal{M}(h_z(\tau))(\beta)$: log-periodic and zeta regularized terms as expected