

# Random Fractals

## Introduction to Fractal Geometry and Chaos

Matilde Marcolli

MAT1845HS Winter 2020, University of Toronto  
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## Some References

- Barry Simon, *Functional Integration and Quantum Physics*, Academic Press, 1979
- Farzad Fathizadeh, Yeorgia Kafkoulis, Matilde Marcolli, *Bell polynomials and Brownian bridge in Spectral Gravity models on multifractal Robertson-Walker cosmologies*, arXiv:1811.02972
- Peter Mörters, *Random Fractals*, in “New Perspectives in Stochastic Geometry”

## Gaussian Random Variables and Gaussian Processes

- on a probability space

$(\Omega, \mathcal{F}, \mathbb{P})$  has a **standard Gaussian** (or **standard normal**) distribution if

$$\mathbb{P}(X > x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-u^2/2} du.$$

- $n$ -dim vector-valued Gaussian random variable  $X = AY + b$  with  $Y$  with  $k$ -dim standard Gaussian distribution and  $n \times k$  matrix  $A$  and  $n$ -vector  $b$
- **Gaussian process**: stochastic process  $\{X_t\}_{t \in \mathcal{I}}$  such that for all  $k \in \mathbb{N}$  and  $t_1, \dots, t_k \in \mathcal{I}$  the vector  $X = (X_{t_1}, \dots, X_{t_k})$  is a vector-valued Gaussian random variable
- covariance matrix of a random vector

$$\text{Cov}(X) = \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)^t]$$

for Gaussian  $\text{Cov}(X) = AA^t$

- distribution of a Gaussian vector completely determined by its mean and covariance

## Random Walk

- $\nu$  prob measure on  $\{\pm 1\}$  with  $\nu(-1) = \nu(1) = 1/2$  and  $d\mu = \otimes_{n=1}^{\infty} d\nu_n$  on  $\prod_{n=1}^{\infty} \{\pm 1\}$ ; take  $X_n = \sum_{k=1}^n y_k$  with  $y_k$  the  $k$ -th coord; family of random variables  $\{X_n\}_{n=1}^{\infty}$  random walk
- expectation  $\mathbb{E}(y_n y_m) = \delta_{n,m}$  and  $\mathbb{E}(X_n^2) = n$ ; rescaled  $\tilde{X}_n = n^{-1/2} X_n$  approach a Gaussian random variable  $X_{\infty}$  (central limit theorem); for any continuous bounded function

$$\mathbb{E}(f(\tilde{X}_n)) \rightarrow (2\pi)^{-1/2} \int e^{-y^2/2} f(y) dy$$

## Brownian Motion

- $b(t) = \lim_{n \rightarrow \infty} n^{-1/2} X_{[nt]}$  integral part  $[nt]$ ; Gaussian of variance  $t$ ; since  $X_n - X_m$  and  $X_m$  independent for  $m < n$  get  $b(t) - b(s)$  and  $b(s)$  independent for  $s < t$  so

$$\mathbb{E}(b(s)(b(t) - b(s))) = 0 \text{ hence } \mathbb{E}(b(s)b(t)) = s \quad \forall s < t$$

- **Wiener Process** (Brownian motion) family  $\{b(t)\}_{t \geq 0}$  of random variables with covariance  $\mathbb{E}(b(s)b(t)) = \min\{s, t\}$
- characterization as unique invariant Gaussian Markov process up the changes of scale

## Brownian motion and the Heat Kernel

- semigroup  $e^{-tH_0}$  with  $H_0 = -\frac{1}{2}\frac{d^2}{dx^2}$  with integral kernel

$$P_t(x, y) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2t} |x - y|^2\right)$$

- joint probability distribution of  $b(s_1), \dots, b(s_n)$  is

$$P_{t_1}(0, x_1)P_{t_2}(x_1, x_2) \cdots P_{t_n}(x_{n-1}, x_n)$$

with  $t_1 = s_1, t_2 = s_2 - s_1, \dots, t_n = s_n - s_{n-1}$

- since  $b(s_1), b(s_2) - b(s_1), \dots, b(s_n) - b(s_{n-1})$  independent Gaussian random variables of variance  $t_k$ , joint distribution

$$P_{t_1}(0, y_1)P_{t_2}(0, y_2) \cdots P_{t_n}(0, y_n)$$

with  $y_1 = x_1, y_2 = x_2 - x_1, \dots, y_n = x_n - x_{n-1}$

## Brownian bridge

- Gaussian process  $\{\alpha(s)\}_{0 \leq s \leq 1}$  with covariance

$$\mathbb{E}(\alpha(s)\alpha(t)) = s(1-t) \text{ for } 0 \leq s \leq t \leq 1$$

- relation to Brownian motion:  $\alpha(s) = b(s) - sb(1)$
- **setting for Feynman-Kac formula**: operator  $H = H_0 + V$  with potential and heat kernel  $e^{-tH}$
- **Trotter product formula**:

$$\langle f, e^{-tH} g \rangle = \lim_{n \rightarrow \infty} \langle f, (e^{-tH_0/n} e^{-iV/n})^n g \rangle$$

- consequence of relation between Brownian motion and heat kernel of  $H_0$ :

$$\langle f_0, e^{-t_1 H_0} f_1 \cdots e^{-t_n H_0} f_n \rangle = \int f_0(\omega(s_0)) \cdots f_n(\omega(s_n)) D[\omega]$$

with  $D[\omega]$  Wiener measure;  $t_k = s_k - s_{k-1}$  and  $0 \leq s_0 < s_1 < \cdots < s_n$ , with  $L^\infty$  functions, and  $\omega(t) = x + b(t)$

- use previous two to write

$$\langle f, e^{-tH} g \rangle = \lim_{n \rightarrow \infty} \int \overline{f(\omega(0))} g(\omega(t)) \exp\left(-\frac{t}{n} \sum_{j=0}^{n-1} V(\omega(tj/n))\right) D[\omega]$$

$$\frac{t}{n} \sum_{j=0}^{n-1} V(\omega(tj/n)) \rightarrow \int_0^t V(\omega(s)) ds$$

- resulting **Feynman-Kac formula**:

$$\langle f, e^{-tH} g \rangle = \int \overline{f(\omega(0))} g(\omega(t)) \exp\left(-\int_0^t V(\omega(s)) ds\right) D[\omega]$$

- this gives  $(e^{-tH} f)(0) = \int \exp\left(-\int_0^t V(b(s)) ds\right) f(b(t)) D[b]$
- **Brownian bridge formulation**:

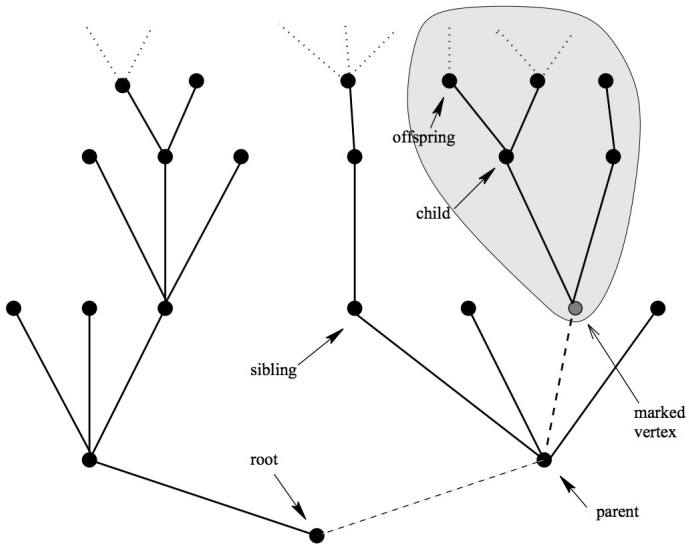
$$e^{-tH} = \frac{1}{2\sqrt{\pi t}} \int \exp\left(-t \int_0^1 V_n(t + \sqrt{2t}\alpha(u)) du\right) D[\alpha]$$

## Fractals from Weighted Trees

- $T = (E, V)$  an infinite rooted tree root  $\rho$
- each vertex  $v \in V \Rightarrow$  marking by a nonempty compact set  $I_v \subset \mathbb{R}^D$  (closure of its interior)
  - if  $v$  is a child of  $u$ , then  $I_v \subset I_u$ ;
  - if  $u$  and  $v$  are siblings, then  $\text{int } I_u \cap \text{int } I_v = \emptyset$ ;
  - for all rays  $\xi = (v_0, v_1, \dots)$  we have  $\lim_{n \rightarrow \infty} \text{diam}(I_{v_n}) = 0$ .
- set associated to the tree

$$\mathcal{I}(T) := \bigcup_{\xi \in \partial T} \bigcap_{v \in \xi} I_v$$

- this realization of a set by a tree non-unique (similar to realization of a self-similar set by a iterated system of contraction maps)



## Hausdorff Dimension

- **Capacities** (weights on edges)  $C : E \rightarrow \mathbb{R}_+$
- **Flow of strength  $c > 0$** : map  $\theta : E \rightarrow [0, c]$  such that
  - we have  $\sum_{\bar{w}=\rho} \theta((\rho, w)) = c$ ,
  - for every vertex  $v \neq \rho$  we have  $\theta((\bar{v}, v)) = \sum_{\bar{w}=v} \theta((v, w))$ ,
  - for every  $e \in E$  we have  $\theta(e) \leq C(e)$ ,
- **Computation of Hausdorff Dimension**:  $A = \mathcal{I}(T)$  for some tree  $T$  with markings  $\{\mathcal{I}_v\}_{v \in V}$
- assume that

$$\inf_{v \neq \rho} \frac{\text{diam}(\mathcal{I}_v)}{\text{diam}(\mathcal{I}_{\bar{v}})} > 0 \quad \text{and} \quad \inf_v \frac{\text{vol}(\text{interior}(\mathcal{I}_v))}{\text{diam}(\mathcal{I}_v)^D} > 0$$

with  $\bar{v}$  parent vertex of  $v$

- take as capacities  $C_s(e) := \text{diam}(\mathcal{I}_v)^s$  for  $e = (\bar{v}, v)$  then

$$\begin{aligned} \dim A &= \inf \left\{ s : \inf_{\Pi \text{ cutset}} \sum_{e \in \Pi} C_s(e) = 0 \right\} \\ &= \sup \left\{ s : \text{there is a flow with capacities } C_s \right\} \end{aligned}$$

## Idea of Proof

- first equality from definition of Hausdorff dimension
- second equality **max-flow min-cut** theorem in graph theory: case of trees, maximal strength of flow with capacities  $C$  equals minimal sum of capacities over the edges in a cut-set (every ray from root to leaves includes an edge in the cut-set)
- **Example: Middle Third Cantor Set**
  - binary tree, markings  $\mathcal{I}_v$  interval of length  $3^{-|v|}$  with  $|v| =$  number of edges in path from root to  $v$ , capacities  $C_s = 3^{-sn}$  for all edges with endpoint  $n$ -th generation ( $|v| = n$ ); condition for a flow to exist  $3^s \leq 2$ , then Hausdorff dim computation gives  $\dim_H(A) = \frac{\log 2}{\log 3}$

## Galton–Watson Trees

- $X = (N, A_1, \dots, A_N)$  random variable weights  $0 < A_i \leq 1$
- tree  $T$ : assign to each vertex a copy of  $X$  and  $N$  children vertices with weights  $A_i$
- diameter of set  $\mathcal{I}_v$  taken equal to product of weights along path from root to  $v$
- When is resulting Galton–Watson tree infinite? Probability

$$\mathbb{P}(T \text{ infinite}) > 0 \Leftrightarrow \mathbb{E}N > 1$$

- *inherited property of a rooted tree*: every finite tree has it and if a tree  $T$  has it and  $T(v)$  subtree stemming from a vertex then  $T(v)$  also has it
- *zero-one-law* for Galton–Watson trees: the probability that a tree in the class satisfying a given inherited property is infinite is either zero or one

## Galton–Watson Fractals

- tree  $T$  with markings  $\mathcal{I}_v$  with property

$$\inf_v \frac{\text{vol}(\text{interior}(\mathcal{I}_v))}{\text{diam}(\mathcal{I}_v)^D} > 0$$

and with weights

$$\frac{\text{diam}(\mathcal{I}_v)}{\text{diam}(\mathcal{I}_\rho)} = \prod_{i=1}^n A(v_i)$$

product of weights along path from root  $\rho$  to  $v$

- resulting set  $\mathcal{I}(T)$  is a Galton–Watson Fractal

## Dimension of Galton–Watson Fractals

- compute Hausdorff dimension as above using capacities

$$C_s(e) = \prod_{i=1}^n A(v_i)^s$$

for  $e = (\bar{v}, v)$

- for this need to know the existence of flows: result of Falconer

$$\gamma = \mathbf{E} \left[ \sum_{i=1}^N A_i^s \right].$$

(a) If  $\gamma \leq 1$  then almost surely no flow is possible.

(b) If  $\gamma > 1$  then flow is possible almost surely given that the tree is infinite.

- then Hausdorff dimension

$$\dim_H \mathcal{I}(T) = \min \left\{ s \in \mathbb{R}_+ \mid \mathbf{E} \left[ \sum_{i=1}^N A_i^s \right] \leq 1 \right\}$$

- other result: if  $\dim_H \mathcal{I}(T) = s$  and  $\sum_{i=1}^N A_i^s \neq 1$  with positive probability then measure  $\mu_H(\mathcal{I}(T)) = 0$  with probability one

## Zero Set of the Brownian Bridge

- Brownian bridge  $B(t)$  from Brownian Motion  $W(t)$  (symmetrizes the Brownian motion)
  - The process  $B(t) = W(t) - tW(1)$ , for  $t \in [0, 1]$ , is a Brownian bridge.
  - Let  $T = \sup\{t < 1: W(t) = 0\}$ , then the process  $C(t) = \sqrt{1/T}W(tT)$ , for  $t \in [0, 1]$ , is also a Brownian bridge.
- zero set of the Brownian Bridge

$$T_1 := \sup\{t \leq 1/2 \mid B(t) = 0\}, \quad T_2 := \inf\{t \geq 1/2 \mid B(t) = 0\}$$

- zero set of Brownian bridge is a Galton–Watson fractal
- remove interval  $(T_1, T_2)$  where no zeros happen, to the left and right have independent Brownian bridges

$$\left\{ \sqrt{\frac{1}{T_1}} B(tT_1) \right\}_{0 \leq t \leq 1} \quad \left\{ \sqrt{\frac{1}{1 - T_2}} B(1 - t(1 - T_2)) \right\}_{0 \leq t \leq 1}$$

- apply same procedure to remaining bridges and iteratively construct the zero set

## Dimension of Zero Set of the Brownian Bridge

- Galton–Watson fractal with generating random variable  $X = (2, T_1, 1 - T_2)$
- use previous computation of dimension: get  $\dim_H Z(B) =$  unique solution of  $\mathbb{E}[T_1^s + (1 - T_2)^s] = 1$  where  $Z(B) = \{t \in [0, 1] \mid B(t) = 0\}$
- compute this dimension: first note that  $\mathbb{E}[\sqrt{T_1} + \sqrt{1 - T_2}] = 1$ . By symmetry this is equivalent to  $\mathbb{E}[\sqrt{1 - T_2}] = 1/2$

- to show  $\mathbb{E}[\sqrt{1 - T_2}] = 1/2$

$$\begin{aligned}
 T_2 &= \inf \{1/2 \leq t \leq 1 : B(t) = 0\} \\
 &= \inf \{1/2 \leq t \leq 1 : W(t) - tW(1) = 0\} \\
 &\stackrel{d}{=} \inf \{1/2 \leq t \leq 1 : tW(1/t) - tW(1) = 0\} \\
 &= \inf \{1/2 \leq t \leq 1 : W(1/t) - W(1) = 0\} \\
 &= 1/\sup \{1 \leq s \leq 2 : W(s) - W(1) = 0\}.
 \end{aligned}$$

As  $\{W(s) - W(1) : s \geq 1\}$  has the same law as  $\{W(s-1) : s \geq 1\}$ , we have

$$T_2 \stackrel{d}{=} \frac{1}{1 + \sup \{0 \leq t \leq 1 : W(t) = 0\}}$$

and, in particular,

$$\mathbb{E}\sqrt{1 - T_2} = \int_0^1 \sqrt{\frac{x}{1+x}} f(x) dx,$$

with  $f(x)$  density of the random variable

$$L = \sup\{0 \leq t \leq 1 \mid W(t) = 0\}$$

- **Reflection principle of Brownian motion:** the random variable  $L$  has arcsin distribution

$$\begin{aligned}\mathbf{E}\sqrt{1 - T_2} &= \frac{1}{\pi} \int_0^1 \sqrt{\frac{x}{1+x}} \frac{dx}{\sqrt{x(1-x)}} \\ &= \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi} \arcsin(1) = \frac{1}{2}\end{aligned}$$

- obtain that almost surely the dimension of the set  $Z(B)$  of zeros of the Brownian bridge is  $1/2$

## Random Cantor Sets

- zero-one valued random variables  $Y(j_1, \dots, j_n)$  with  $j_k \in \{0, 1\}$
- take  $\mathcal{S}_0 = [0, 1]$
- given  $\mathcal{S}_n$  collection of compact intervals of length  $2^{-n}$  construct  $\mathcal{S}_{n+1}$  by
  - *splitting each interval in  $\mathcal{S}_{n-1}$  into two nonoverlapping intervals of half the length,*
  - *adding any interval thus constructed to the collection  $\mathcal{S}_n$  if*

$$Y(j_1, \dots, j_n) = 1,$$

where  $\sum_{i=1}^n j_i 2^{-i}$  is the left endpoint of the interval.

- random Cantor set is

$$A = \bigcap_{n=1}^{\infty} \bigcup_{\mathcal{I} \in \mathcal{S}_n} \mathcal{I}$$