Notions of Dimension
Introduction to Fractal Geometry and Chaos

Matilde Marcolli

MAT1845HS Winter 2020, University of Toronto
M 1-2 and T 10-12 BA6180

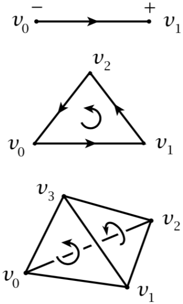
References

- Yuri I. Manin, *The notion of dimension in geometry and algebra*, Bull. Amer. Math. Soc. (N.S.) 43 (2006), no. 2, 139–161.

Euclid and dimension

- Euclid's elements:
  - The extremities of a line are points (Book I)
  - An extremity of a solid is a surface (Book XI)
- Modern concepts:
  - The boundary of an N-dimensional space (simplicial set, ...) is an $(N-1)$-dimensional space
  - The boundary of a boundary is empty (crucial idea of *homology*)
- guiding principle for our investigation of fractal geometry in this class:
  - Question: what happens to the idea of boundary and homology when the notion of dimension changes and becomes more general?

Boundary and dimension

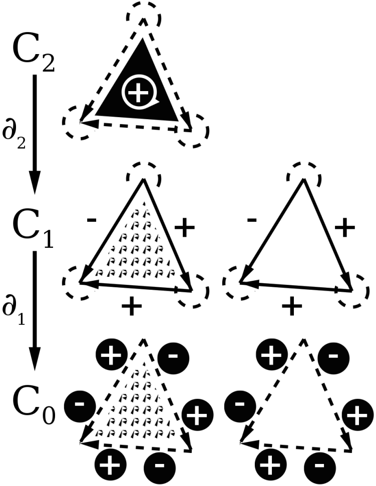


$$\partial[v_0, v_1] = [v_1] - [v_0]$$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

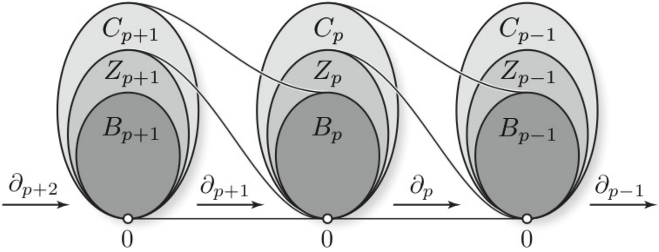
$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

Boundary operators in dimension one, two, and three



Boundary of a boundary is zero, $\partial^2 = 0$

Chain complexes and homology



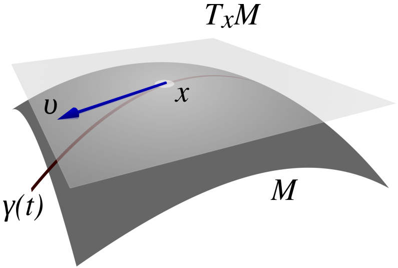
C_n abelian group generated by n-dim simplexes; $Z_n = \mathrm{Ker}(\partial_n)$; $B_n = \mathrm{Image}(\partial_{n+1})$; homology $H_n = Z_n/B_n$

Linear dimension: dimension of a vector space

• *linear independence*: $v_1, \dots, v_N \in V$ linearly independent if $a_1 v_1 + \cdots + a_N v_N = 0$ (with a_i in underlying field K) implies $a_i = 0$ for all $i = 1, \dots, N$

• *linear dimension*: maximal cardinality of a set of linearly independent vectors in V (or infinite if no such max)

• dimension of a smooth manifold: locally modelled by a vector space (tangent space), so linear dimension of the tangent space ... well defined under changes of coordinate charts (invertible linear transformations)



Dimension for Topological Spaces

• topological space: $(X, \mathcal{T})$ set with collection of subsets

- $\emptyset$ and X are in $\mathcal{T}$
- arbitrary unions $\bigcup_{\alpha\in\mathcal{I}} U_\alpha$ of sets $U_\alpha \in \mathcal{T}$ are also in $\mathcal{T}$
- finite intersections $\bigcap_{k=1}^{N} U_k$ of sets $U_k \in \mathcal{T}$ are also in $\mathcal{T}$

The collection $\mathcal{T}$ is the family of open sets of the topology

• open cover of a topological space X: family of open sets $U \in \mathcal{T}$ whose union contains X

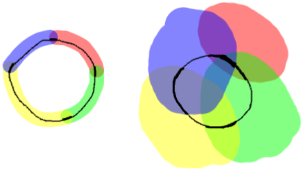
• construct a notion of dimension based on the overlapping of sets in open coverings

• open covering $\mathcal{C}$ of a topological space X

• refinement of an open covering $\mathcal{C}$: open covering $\mathcal{C}'$ such that all $U' \in \mathcal{C}'$ satisfies $U' \subset U$ for some $U \in \mathcal{C}$

• order of covering $\mathcal{C}$: smallest $n \in \mathbb{N}$ (or ∞ if no such) such that each point $x \in X$ belongs to at most n sets in $\mathcal{C}$

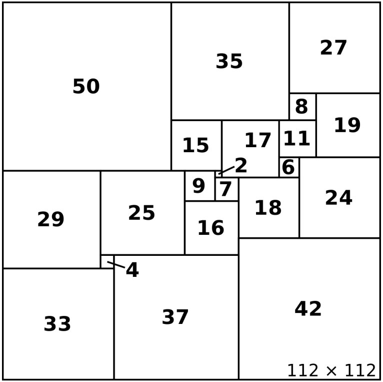
• dimension of X is smallest $D \in \mathbb{N}$ such that every open covering $\mathcal{C}$ of X has a refinement $\mathcal{C}'$ with order at most $D+1$ (infinite dimension if no such D)



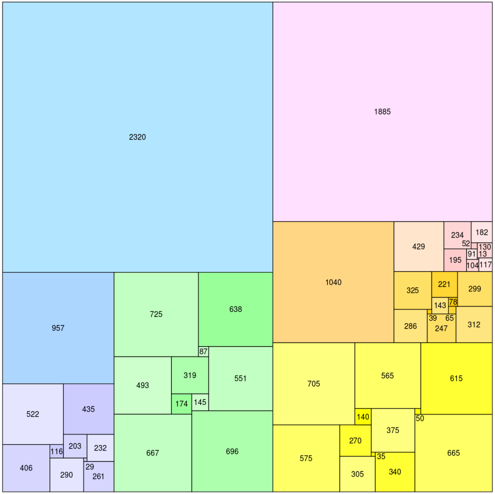
covering (right) and refinement (left) with $D = 1$ for the circle

Squares of squares

- perfect squared square: square decomposed into smaller squares all of different sizes
- 1978, A. J. W. Duijvestijn (1978) simplest perfect squared square, side 112 tiled with 21 squares



• a more elaborate example of perfect squared square: side 4205 with 55 tiles



• *dimensionality*: sides ℓ_i of the subsquares and side ℓ of the big square $\sum_i \ell_i^2 = \ell^2$

• obviously *not sufficient*: it is known that there are no solutions to the perfectly cubed cube problem and higher dimensional analogs; condition $\sum_i \ell_i^D = \ell^D$ not sufficient

• the whole story is here:

- R. L. Brooks, C. A. B. Smith, A. H. Stone, W. T. Tutte, *The dissection of rectangles into squares*, Duke Math. J. 7 (1940) N.1, 312–340

• Simple observation: scaled replicas of the same figure satisfy a *self-similarity* equation (if $\ell = 1$)

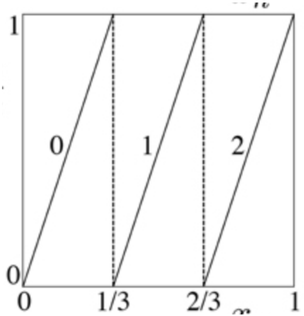
$$\sum_i \ell_i^D = 1$$

This leads to a notion of self-similarity dimension
self-similarity dimensions need not be integers

Cantor set and dynamics

- Piecewise linear map with chaotic behavior
  - $f : \mathcal{I}_1 \cup \mathcal{I}_2 \to \mathcal{I}$ with $\mathcal{I} = [0,1]$, $\mathcal{I}_1 = [0,1/3]$ and $\mathcal{I}_2 = [2/3,1]$ and

$$f(x) = \begin{cases} 3x & x \in \mathcal{I}_1 \\ 3x - 2 & x \in \mathcal{I}_2 \end{cases}$$



- cannot iterate f because range not contained in domain
- $f^2 = f \circ f$ defined where both x and $f(x)$ are in $\mathrm{Dom}(f) = \mathcal{I}_1 \cup \mathcal{I}_2$: intervals $\mathrm{Dom}(f^2) = \mathcal{I}_{11} \cup \mathcal{I}_{21} \cup \mathcal{I}_{12} \cup \mathcal{I}_{22}$

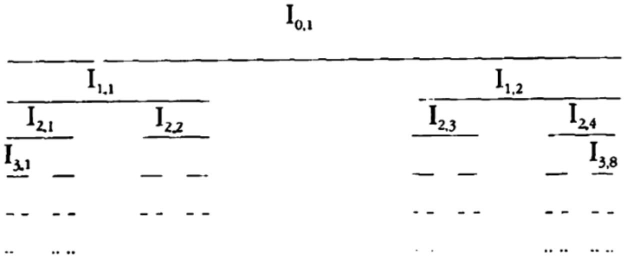
$$\mathcal{I}_{11} = [0, \frac{1}{9}], \;\; \mathcal{I}_{21} = [\frac{2}{3}, \frac{7}{9}], \;\; \mathcal{I}_{12} = [\frac{2}{9}, \frac{1}{3}], \;\; \mathcal{I}_{22} = [\frac{8}{9}, 1]$$

$$f(\mathcal{I}_{11}) = f(\mathcal{I}_{21}) = [0, 1/3], \quad f(\mathcal{I}_{12}) = f(\mathcal{I}_{22}) = [2/3, 1]$$

- domain of n-th iterate $f^n = f \circ \cdots \circ f$ subset of $\mathcal{I}$ given by union of intervals

$$\mathcal{I}_{v_1,\ldots,v_n} = \mathcal{I}_{v_1} \cap f^{-1}(\mathcal{I}_{v_2}) \cap \cdots \cap f^{-(n-1)}(\mathcal{I}_{v_n})$$

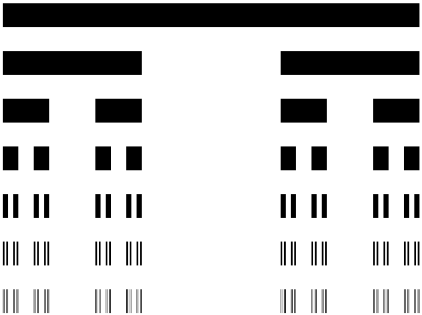
with $v_i \in \{1,2\}$ total of 2^n intervals of length 3^{-n}



- domain in $\mathcal{I} = [0,1]$ where all iterates are defined is Cantor set

$$C := \bigcap_{n\geq 1} \Big(\bigcup_{v_1,\ldots,v_n \in \{1,2\}} \mathcal{I}_{v_1,\ldots,v_n} \Big)$$

- *middle third Cantor set*: C obtained removing middle third interval from $[0,1]$ (leaving $\mathcal{I}_1 \cup \mathcal{I}_2$) then repeatedly remove middle third of each remaining intervals (leaving at the n-th step the 2^n intervals $\mathcal{I}_{v_1,\ldots,v_n}$



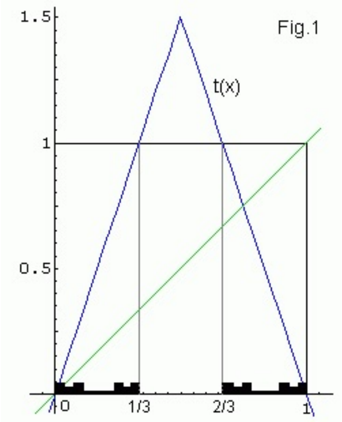
- dynamical system $f : C \to C$ with all iterates
- C largest invariant set in $\mathcal{I}$ under f (if A invariant then all iterates defined so $A \subseteq C$ and $C = \cap_n \mathrm{Dom}(f^n)$ with $f(\mathrm{Dom}(f^n)) = \mathrm{Dom}(f^{n-1})$ so $f(C) \subseteq C$)

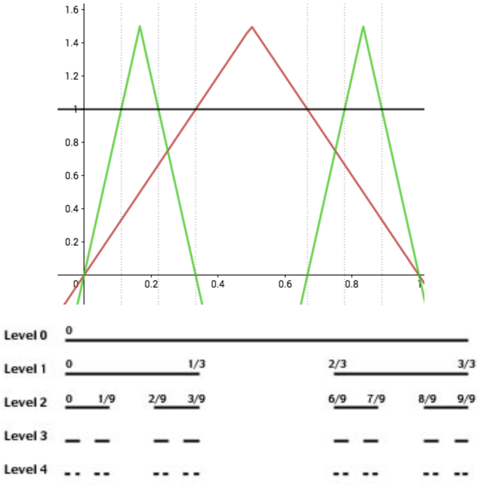
- Variant of this dynamical system construction: Tent map

$$T(x) = \begin{cases} 3x & 0 \leq x < 1/2 \\ 3(1-x) & 1/2 \leq x \leq 1 \end{cases}$$

- points $x \in \mathcal{I} = [0,1]$ that never leave $\mathcal{I}$ under iteration of T: nonwandering set
- points that leave $\mathcal{I}$ after applying T: middle third $[1/3, 2/3]$ interval; after second iteration middle thirds of $\mathcal{I}_1$ and $\mathcal{I}_2$ etc.
- nonwandering set of the tent map is Cantor set C

Tent map and nonwandering set




1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
Level 0
0
Level 1
0
1/3
2/3
3/3
Level 2
0
1/9
2/9
3/9
6/9
7/9
8/9
9/9
Level 3
Level 4

Geometry of the Cantor set

- write points $x \in \mathcal{I} = [0,1]$ in ternary instead of decimal digits $x = 0.a_1 a_2 a_3 \cdots = \sum_{k\geq 1} a_k 3^{-k}$ with $a_k \in \{0,1,2\}$
- up to ambiguity $0.1 \simeq 0.0222\ldots$ at endpoint $\mathcal{I}_1 \cup \mathcal{I}_2$ is subset of $\mathcal{I}$ with $a_1 \in \{0,2\}$
- $\mathcal{I}_{v_1,\ldots,v_n}$ is subset with $a_k = v_k$ for $k = 1,\ldots,n$
- Cantor set is all points of $\mathcal{I}$ with no 1's in ternary digits (up to ambiguities at endpoints of intervals)
- the Cantor set is uncountable (change alphabet $\{0,2\}$ to $\{0,1\}$ get all binary expansions of $[0,1]$)
- $x \in \mathcal{I}_{v_1} \cap \mathcal{I}_{v_1,v_2} \cap \cdots \cap \mathcal{I}_{v_1,\ldots,v_n} \cap \cdots$ gives the address of a point x in the Cantor set x by choosing at each step the left or right interval

Properties of Topological spaces $(X, \mathcal{T})$

• Hausdorff: $\forall x, y \in X \, \exists U, V$ open in X with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$

• connectedness: topological space X connected if *not* a union $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$ and X_1, X_2 closed non-empty; equivalent: $Y \subseteq X$ is both open and closed iff $Y = X$ or $Y = \emptyset$

• connected components: all subsets of X partially ordered by inclusions, maximal connected subsets

• totally disconnected: all connected components are points

• totally separated: $\forall x, y \in X \, \exists U, V$ open in X with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$ with $U \cup V = X$

• totally separated $\Rightarrow$ totally disconnected; if Hausdorff other implication too

• accumulation point (of $A \subset X$): $x \in X$ such that every open U in X with $x \in U$ contains at least another point $y \neq x$ with $y \in A$

• perfect: all points are accumulation points

• compactness: from every open covering $\{U_\alpha\}$ of X can extract a finite open subcovering $X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}$

• compactness in $\mathbb{R}$ with the standard metric topology: $X \subset \mathbb{R}$ compact iff both closed and bounded

Topology of the Cantor set

- infinite intersection of closed sets so C is closed
- also contained inside $\mathcal{I} = [0,1]$ so bounded $\Rightarrow$ compact
- all points are accumulation points: use open intervals in $\mathcal{I}$ around a point $x \in C$ and previous description in ternary expansion to see these intervals always contain other points of C: perfect
- also C is totally disconnected: use address of points $x, y \in C$ by intervals $\mathcal{I}_{v_1,\ldots,v_n}$ to place in two different intervals, use these to obtain U, V for totally separated property

Metric geometry

- Metric Space:
  - distance function (metric) $d : X \times X \to \mathbb{R}_+$
  - symmetry: $d(x, y) = d(y, x)$
  - positivity: $d(x, y) = 0$ iff $x = y$
  - triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$
- Euclidean distance in $\mathbb{R}$ by $d(x, y) = |x - y|$
- Metric topology: topology generated by the open balls

$$B_d(x, r) := \{y \in X \mid d(x, y) < r\}$$

topology generated by taking all finite intersections and arbitrary unions of $\{B_d(x, r)\}_{x \in X, r \in \mathbb{R}_+^*}$

- $U \subset X$ open in the metric topology iff $\forall x \in U$ there is a ball $B_d(x, r)$ for some $r > 0$ with $B_d(x, r) \subset U$

Metric on the Cantor set

- Use a coding of points $x \in C$ in Cantor set by sequences $a_1 a_2 a_3 \cdots$ with digits $a_k \in \{1, 2\}$ (as in ternary expansion but symbols $1, 2$ instead of $0, 2$ bijection anyway)
- $\Sigma_2^+ = \{a_1 a_2 \cdots a_n \cdots \mid a_k \in \{1, 2\}\}$ coding space of sequences
- choose $\alpha > 2$ a scale
- define distance

$$d(x, y) := \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{\alpha^k}$$

  where $x = a_1 a_2 \cdots a_n \cdots$ and $y = b_1 b_2 \cdots b_n \cdots$ in Σ_2^+
- metric ball $B_d(x, r)$ with $r = \alpha^{-1}$ is *first cylinder set*

$$B_d(x, \frac{1}{\alpha}) = \mathcal{C}(a_1) := \{y = b_1 b_2 \cdots \in \Sigma_2^+ \mid b_1 = a_1\}$$

- Cylinder sets
  $\mathcal{C}(a_1, \ldots, a_n) := \{y = b_1 b_2 \cdots \in \Sigma_2^+ \mid b_1 = a_1, \cdots b_n = a_n\}$
  first n digits are fixed

Cylinder sets and coding map

- cylinder sets $\mathcal{C}(a_1, \dots, a_n)$ both open and closed
  - closed: $x_k = a_{k,1} a_{k,2} \cdots a_{k,m} \cdots$ sequence in Σ_2^+ with $x_k \in \mathcal{C}(a_1, \dots, a_n)$ for all k, convergent to $y = b_1 b_2 \cdots \in \Sigma_2^+$, then $d(x_k, y) \to 0$ hence $|a_{k,m} - b_m| \to 0$ for $k \to \infty$ for all m, but for $m = 1, \dots, n$ $a_{k,m} = a_m$ for all k so also $b_m = a_m$ so $y \in \mathcal{C}(a_1, \dots, a_n)$
  - open: show for all $x \in \mathcal{C}(a_1, \dots, a_n)$ there is a ball $B_d(x, r) \subset \Sigma_2^+$ for some $r > 0$ with $B_d(x, r) \subset \mathcal{C}(a_1, \dots, a_n)$. If $b_m \neq a_m$ for some $m \in \{1, \dots, n\}$ then $d(x, y) \geq \alpha^{-n}$ hence necessarily $y \in B_d(x, r)$ is also $y \in \mathcal{C}(a_1, \dots, a_n)$ whenever $r < \alpha^{-n}$

- coding map $h : \Sigma_2^+ \to C$ is homeomorphism
    - $h(a_1 a_2 \cdots a_n \cdots)$ is ternary expansion $x = 0.c_1 c_2 \cdots c_n \cdots$ with $c_k = 0$ if $a_k = 1$ and $c_k = 2$ if $a_k = 2 \Rightarrow$ bijective
    - coding maps cylinder sets to intersections of C with intervals $C \cap \mathcal{I}_{v_1 \ldots, v_n}$

$$h(\mathcal{C}(v_1, \ldots, v_n)) = C \cap \mathcal{I}_{v_1 \ldots, v_n}$$

    - continuous bijection from compact to Hausdorff space is homeomorphism (continuous inverse): closed sets A inside compact are compact, images of compact by continuous map are compact and compact sets in Hausdorff space are closed, so preimages of closed under inverse function are closed (continuous inverse)

Self-similarity of the Cantor set

- consider again piecewise linear map $f : \mathcal{I}_1 \cup \mathcal{I}_2 \to \mathcal{I}$ with $f(x) = 3x$ for $x \in \mathcal{I}_1$ and $f(x) = 3x - 2$ for $x \in \mathcal{I}_2$
- inverse maps $g_1 : \mathcal{I} \to \mathcal{I}_1$ and $g_2 : \mathcal{I} \to \mathcal{I}_2$
- restrictions of these maps to $C \subset \mathcal{I}$ send bijectively $g_i|_C : C \to C \cap \mathcal{I}_i$
- in terms of coding these maps take sequence $x = a_1 a_2 \cdots a_n \cdots$ in Σ_2^+ and map it to sequence $a_0 a_1 a_2 \cdots a_n \cdots$ with a_0 either 1 or 2
- obtain $C = g_1(C) \cup g_2(C)$ with g_i linear maps with scaling factor $\lambda_1 = \lambda_2 = 3$
- self-similarity equation $2 \cdot 3^{-s} = 1$ gives

$$\dim(C) = s = \frac{\log 2}{\log 3}$$

self-similarity dimension of the Cantor set

• Hausdorff dimension (discussed later) agrees for Cantor set with self-similarity dimension: not true in general for all fractals

Hausdorff dimension: first heuristic encounter

- volume of the ball B_ρ of radius $\rho > 0$ in Euclidean space $\mathbb{R}^D$

$$V_D(B_\rho) = \frac{\Gamma(1/2)^D}{\Gamma(1 + D/2)} \rho^D$$

- Note: this expression continues to make sense for $D \in \mathbb{R}_+^*$ not an integer (in fact even for complex D)
- covering $\mathcal{U}$ of a set S (inside an ambient Euclidean space) with balls of varying radii ρ_k and measure volume *as if* they'd be α-dimensional ($\alpha \in \mathbb{R}_+^*$); compute

$$V_\alpha(S) := \lim_{\rho \to 0} \inf_{\mathcal{U} : \rho_k < \rho} \sum_k V_\alpha(B_{\rho_k})$$

- there is a unique value $D \in \mathbb{R}_+^*$ such that $V_\alpha(S) = 0$ for all $\alpha > D$ and $V_\alpha(S) = \infty$ for all $\alpha < D$
- that boundary value $D = \dim_H(S)$

Rigorous definition later: heuristic idea of analytic continuation of volumes of balls here connected to idea of "dimensional regularization" in physics

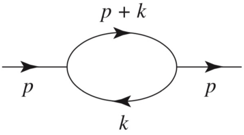
Dimensional regularization: a strange idea in physics

- Gaussian integral

$$\int e^{-\lambda k^2} d^D k = \left(\frac{\pi}{\lambda}\right)^{D/2}$$

- the right-hand-side continues to make sense for D real or complex
- can *define* the left-hand-side in dimension $D \in \mathbb{C}$ to be what the right-hand-side gives
- no "geometric space" of dimension $D \in \mathbb{C}$ over which the integration is taken but the integration is defined by analytic continuation from the integer case
- WHY? ... because of divergent Feynman integrals

Dimensional regularization of Feynman integrals



$$\to \int \frac{1}{k^2+m^2} \frac{1}{((p+k)^2+m^2)} d^D k$$

- Schwinger parameters

$$(k^2+m^2)^{-1}((p+k)^2+m^2)^{-1} = \int_{\mathbb{R}_+ \times \mathbb{R}_+} e^{-s(k^2+m^2} e^{-t((p+k)^2+m^2)} \, ds \, dt$$

- after changing variables $s = (1-x)\lambda$, $t = x\lambda$ and use dimensional regularization of Gaussian

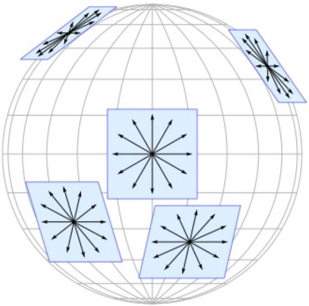
$$\pi^{D/2} \Gamma(2 - D/2) \int_0^1 ((x - x^2)p^2 + m^2)^{D/2-2} \, dx$$

- pole at $D = 4$ but regular at $D = 4 + \epsilon$ with $\epsilon \in \mathbb{C}$ small

Differential forms and cohomology

• smooth manifolds have duals to chains, cycles, boundaries and homology defined in terms of differential forms

- M has a tangent bundle TM



Tangent bundle on a 2-sphere

- dual cotangent bundle T^*M, fiber $T_x^*M = \mathrm{Hom}(T_xM, \mathbb{R})$ dual vector space

Vector bundles and sections

- $E \to M$ vector bundle rank N over manifold M of dim n
  - projection $\pi : E \to M$
  - $M = \cup_i U_i$ with $\phi_i : U_i \times \mathbb{R}^N \stackrel{\simeq}{\to} \pi^{-1}(U_i)$ homeomorphisms with $\pi \circ \phi_i(x, v) = x$
  - transition functions: $\phi_j^{-1} \circ \phi_i : (U_i \cap U_j) \times \mathbb{R}^N \to (U_i \cap U_j) \times \mathbb{R}^N$ with

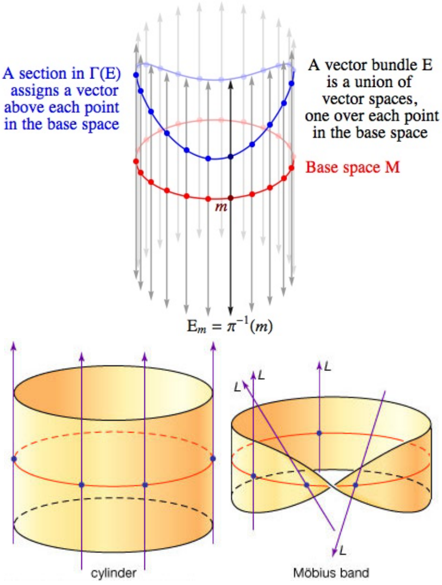
$$\phi_j^{-1} \circ \phi_i(x, v) = (x, \phi_{ij}(x)v)$$

$$\phi_{ij} : U_i \cap U_j \to \mathrm{GL}_N(\mathbb{C})$$

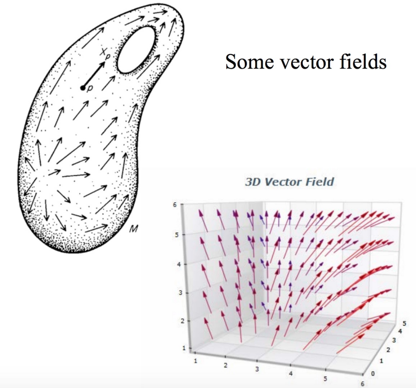
satisfy cocyle property: $\phi_{ii}(x) = id$ and $\phi_{ij}(x)\phi_{jk}(x)\phi_{ki}(x) = id$

  - Sections: $s \in \Gamma(U, E)$ open $U \subseteq M$ maps $s : U \to E$ with $\pi \circ s(x) = x$

$$s(x) = \phi_i(x, s_i(x)) \text{ and } s_i(x) = \phi_{ij}(x)s_j(x)$$


A section in Γ(E) assigns a vector above each point in the base space
A vector bundle E is a union of vector spaces, one over each point in the base space
Base space M
m
$E_m = \pi^{-1}(m)$
L
L
L
L
cylinder
Möbius band

- sections of tangent bundle *TM* are *vector fields*



- sections of the cotangent bundle are 1-forms (dual to vector fields)

Tensors and differential forms as sections of vector bundles

- tangent bundle TM (tangent vectors), cotangent bundle T^*M (1-forms)
- vector field sections $V = (v^\mu) \in \Gamma(M, TM)$
- metric tensor $g_{\mu\nu}$ symmetric tensor section of $T^*M \otimes T^*M$
- (p,q)-tensors: $T = (T^{i_1\dots i_p}_{j_1\dots j_q})$ section in $\Gamma(M, TM^{\otimes p} \otimes T^*M^{\otimes q})$
- 1-form: section $\alpha = (\alpha_\mu) \in \Gamma(M, T^*M)$
- k-form $\omega \in \Omega^k(M) := \Gamma(M, \bigwedge^k(T^*M))$
- exterior algebra $\bigwedge^k(V)$ of vector space: quotient of tensor algebra, spanned by $v_1 \wedge \cdots \wedge v_k$ with relations $v_i \wedge v_j = -v_j \wedge v_i$
- dimension $\dim \bigwedge^k(T^*_x M) = \binom{n}{k}$ with $\dim M = n$
- in local coordinates $x = (x^i)$ on M differential forms $\omega \in \Omega^k(M)$, smooth functions $f_{i_1,\dots,i_k}$ as coefficients

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1,\dots,i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Differential forms and de Rham cochain complex

- differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$

$$d(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

  where terms are zero if $i \in \{i_1, \ldots, i_k\}$ and reordered with appropriate sign otherwise
- coboundary identity: $d^2 = 0$
- cohomology:

$$H^k(M) := \mathrm{Ker}(d : \Omega^k(M) \to \Omega^{k+1}(M))/\mathrm{Image}(d : \Omega^{k-1}(M) \to \Omega^k(M))$$

- duality pairing of homology and cohomology: integration of a differential form on a cycle

$$([\Sigma] \in H_k(M), [\omega] \in H^k(M)) \mapsto \int_\Sigma \omega \in \mathbb{R}$$

Observations

- this construction of cohomology depends essentially on M being a smooth manifold (tangent and cotangent bundle)
- no direct analog for spaces (like fractals) that are not smooth
- any way to make sense of an analog of this de Rham cochain complex of differential forms and cohomology for non-smooth spaces?
- sections of smooth vector bundles have an algebraic definition: finitely generated projective modules over the algebra of smooth functions (Serre–Swan theorem)
- can use an algebra definition for differential forms then?
- yes, through Hochschild and cyclic homology (will be discussed later)
- ... but, these still provide only integer-graded cohomologies even for spaces of non-integer dimension
- ... ways to account for non-integer dimension in integration pairing?

Densities of non-integer dimension

- M smooth diffentiable manifold, $s \in \mathbb{C}$
- complex line bundle $\mathcal{L}_s$ of s-densities on M: transition functions for $y = y_j(x_i)$

$$\left| \det \left(\frac{\partial x_i}{\partial y_j} \right) \right|^s$$

- sections W_s of $\mathcal{L}_s$ written locally as $h(x)\,|dx|^s$
- integration pairing

$$W_s \times W_{1-s} \to \mathbb{C}, \quad (f|dx|^s, g|dx|^{1-s}) \mapsto \int_M \bar{f} g\,|dx|$$

where 1-density $|dx|$ is a volume element (a measure)
- s-dimensional currents dual to W_{1-s}: think of as geometric subspaces of M of dimension s that densities in W_s can be integrated on

More rigorous discussion of these ideas later

Eigenvalues growth: Weyl's law

- $\Omega \subset \mathbb{R}^D$ bounded domain, Dirichlet problem (drum)

$$\Delta u + \lambda u = 0 \quad \text{with} \quad u|_{\partial\Omega} = 0$$

number $N(\lambda)$ of eigenvalues of the Laplacian less than or equal to λ

$$\lim_{\lambda\to\infty} N(\lambda)\,\lambda^{-D/2} = (2\pi)^{-D}\, V_D(B_1)\, V_D(\Omega)$$

so $N(\lambda) \sim \lambda^{D/2}$ satisfies power law depending on dimension

- M compact Riemannian manifolds, $g = g_{ij}$ metric with Laplace–Bertrami operator

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_i \partial_i \Big(\sqrt{|g|} \sum_j g^{ij} \partial_j f\Big)$$

Weyl law with $D = \dim M$

$$N(\lambda) \sim \lambda^{D/2} \frac{V_D(B_1)}{(2\pi)^D} \mathrm{Vol}_g(M)$$

Weyl's law and heat kernel

- heat kernel expansion

$$\mathrm{Tr}(e^{-t\Delta}) = \sum_{\lambda} e^{-t\lambda} = \int_M k(t,x,x) dVol_g(x) \sim_{t\to 0} (4\pi t)^{-\frac{\dim(M)}{2}} \sum_{k=0}^{\infty} a_k t^k$$

  with $a_k = \int_M a_k(x,\Delta)\, dVol_g(x)$ with $a_k(x,\Delta)$ universal polynomials in the curvature tensor of g and its derivatives

- when convergence holds have (abelian Tauberian theorem)

$$\lim_{t\to 0^+} t^r \sum_{k=1}^{\infty} e^{-t\lambda_k} = \alpha \quad \text{iff} \quad \lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^r} = \frac{\alpha}{\Gamma(r+1)}$$

- read Weyl law in limit of $(4\pi t)^{\frac{\dim(M)}{2}} \mathrm{Tr}(e^{-t\Delta})$
- same with Laplacian Δ replaced by Dirac square $\not{D}^2$

Spin Geometry: introducing Dirac operators

- H. Blaine Lawson, Marie-Louise Michelsohn, *Spin Geometry*, Princeton 1989
- John Roe, *Elliptic Operators, Topology, and Asymptotic Methods*, CRC Press, 1999

- Spin manifold
  - Smooth n-dim manifold M has tangent bundle TM
  - Riemannian manifold (orientable): orthonormal frame bundle FM on each fiber E_x inner product space with oriented orthonormal basis
  - FM is a principal $SO(n)$-bundle
  - Principal G-bundle: $\pi : P \to M$ with G-action $P \times G \to P$ preserving fibers $\pi^{-1}(x)$ on which free transitive (so each fiber $\pi^{-1}(x) \simeq G$ and base $M \simeq P/G$)

- Fundamental group $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$ so double cover universal cover:
$$Spin(n) \to SO(n)$$
- Manifold M is *spin* if orthonormal frame bundle FM lifts to a principal $Spin(n)$-bundle PM
- *Warning*: not all compact Riemannian manifolds are spin: there are topological obstruction
- In dimension $n = 4$ not all spin, but all at least $spin^{\mathbb{C}}$
- $spin^{\mathbb{C}}$ weaker form than spin: lift exists after tensoring TM with a line bundle (or square root of a line bundle)

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Spin^{\mathbb{C}}(n) \to SO(n) \times U(1) \to 1$$

- Spinor bundle
  - Spin group $Spin(n)$ and Clifford algebra: vector space V with quadratic form q

$$Cl(V,q) = T(V)/I(V,q)$$

    tensor algebra mod ideal gen by $uv + vu = 2\langle u, v\rangle$ with $\langle u, v\rangle = (q(u+v) - q(u) - q(v))/2$
  - Spin group is subgroup of group of units

$$Spin(V,q) \hookrightarrow \mathrm{GL}_1(Cl(V,q))$$

    elements $v_1 \cdots v_{2k}$ prod of even number of $v_i \in V$ with $q(v_i) = 1$
  - $Cl^{\mathbb{C}}(\mathbb{R}^n)$ complexification of Clifford alg of $\mathbb{R}^n$ with standard inn prod: unique min dim representation dim $\Delta_n = 2^{[n/2]} \Rightarrow$ rep of $Spin(n)$ on Δ_n not factor through $SO(n)$

- Associated vector bundle of a principal G-bundle: V linear representation $\rho : G \to GL(V)$ get vector bundle $E = P \times_G V$ (diagonal action of G)
- *Spinor bundle* $\mathbb{S} = P \times_\rho \Delta_n$ on spin manifold M
- *Spinors* sections $\psi \in \mathcal{C}^\infty(M, \mathbb{S})$
- Module over $\mathcal{C}^\infty(M)$ and also action by forms (Clifford multiplication) $c(\omega)$
- as vector space $Cl(V, q)$ same as $\Lambda^\bullet(V)$ not as algebra: under this vector space identification Clifford multiplication by a diff form

Dirac operator as "square root" of geometry

- connection on a vector bundle E on sections $\mathcal{E} = \Gamma(M, E)$ by $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ with $\mathcal{A} = \mathcal{C}^\infty(M)$ with Leibniz rule

$$\nabla(\eta a) = \nabla(\eta)a + \eta \otimes da$$

  for $a \in \mathcal{A}$ and $\eta \in \mathcal{E}$
- Dirac operator: first order linear differential operator (elliptic on M compact): "square root of Laplacian"
- $\gamma_a = c(e_a)$ Clifford action o.n.basis of (V, q)
- even dimension $n = 2m$: $\gamma = (-i)^m \gamma_1 \cdots \gamma_n$ with $\gamma^* = \gamma$ and $\gamma^2 = 1$ sign

$$\frac{1+\gamma}{2} \quad \text{and} \quad \frac{1-\gamma}{2}$$

  orthogonal projections: $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$
- Spin connection $\nabla^S : \mathbb{S} \to \mathbb{S} \otimes \Omega^1(M)$

$$\nabla^S(c(\omega)\psi) = c(\nabla\omega)\psi + c(\omega)\nabla^S\psi$$

  for $\omega \in \Omega^1(M)$ and $\psi \in \mathcal{C}^\infty(M, \mathbb{S})$ and $\nabla =$ Levi-Civita connection

- Dirac operator $\not{D} = -ic \circ \nabla^S$

$$\not{D} : \mathbb{S} \xrightarrow{\nabla^S} \mathbb{S} \otimes_{\mathcal{C}^\infty(M)} \Omega^1(M) \xrightarrow{-ic} \mathbb{S}$$

- $\not{D}\psi = -ic(dx^\mu)\nabla^S_{\partial_\mu}\psi = -i\gamma^\mu \nabla^S_\mu \psi$
- Hilbert space $\mathcal{H} = L^2(M, \mathbb{S})$ square integrable spinors

$$\langle \psi, \xi \rangle = \int_M \langle \psi(x), \xi(x) \rangle_x \sqrt{g}\, d^n x$$

- $\mathcal{C}^\infty(M)$ acting as bounded operators on $\mathcal{H}$ (Note: M compact)
- Commutator: $[\not{D}, f]\psi = -ic(\nabla^S(f\psi)) + ifc(\nabla^S \psi)$

$$= -ic(\nabla^s(f\psi) - f\nabla^S\psi) = -ic(df \otimes \psi) = -ic(df)\psi$$

$[\not{D}, f] = -ic(df)$ bounded operator on $\mathcal{H}$ (M compact)

- main advantage of Dirac operator over Laplace-Beltrami: as "square root" it has a *sign* (positive and negative spectrum) and this sign contains additional topological information of M not seen by Laplacian

Analytic properties of Dirac on $\mathcal{H} = L^2(M, \mathbb{S})$ on a compact Riemannian M

- Unbounded operator
- Self adjoint: $\not{D}^* = \not{D}$ with dense domain
- Compact resolvent: $(1 + \not{D}^2)^{-1/2}$ is a compact operator (if no kernel $\not{D}^{-1}$ compact)
- Lichnerowicz formula: $\not{D}^2 = \Delta^S + \frac{1}{4}R$ with R scalar curvature and Laplacian

$$\Delta^S = -g^{\mu\nu}(\nabla^S_\mu \nabla^S_\nu - \Gamma^\lambda_{\mu\nu} \nabla^S_\lambda)$$

Main Idea: abstract these properties into an algebraic definition of Dirac on spaces that are *not* smooth Riemannian spin manifolds

Zeta function and heat kernel

- Mellin transform

$$|\mathcal{D}|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-t\mathcal{D}^2} t^{\frac{s}{2}-1} dt$$

- heat kernel expansion

$$\mathrm{Tr}(e^{-t\mathcal{D}^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for} \quad t \to 0$$

- zeta function expansion

$$\zeta_{\mathcal{D}}(s) = \mathrm{Tr}(|\mathcal{D}|^{-s}) = \sum_\alpha \frac{c_\alpha}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}$$

- taking residues

$$\mathrm{Res}_{s=-2\alpha} \zeta_{\mathcal{D}}(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

Spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$

- Alain Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. 34 (1995), no. 3, 203–238.

- $\mathcal{A} = \mathrm{C}^*$-algebra
- $\mathcal{H}$ Hilbert space: $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$
- $\mathcal{D}$ unbounded self–adjoint operator on $\mathcal{H}$
- $(\mathcal{D} - \lambda)^{-1}$ compact operator, $\forall \lambda \notin \mathbb{R}$
- $[\mathcal{D}, a]$ bounded operator, $\forall a \in \mathcal{A}_0 \subset \mathcal{A}$, dense involutive subalgebra of $\mathcal{A}$.

Riemannian spin manifold X: $\mathcal{A} = C(X)$, $\mathcal{H} = L^2$-spinors, $\mathcal{D} =$ Dirac operator, $\mathcal{A}_0 = C^\infty(X)$

Zeta functions

• spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}) \Rightarrow$ family of zeta functions: for $a \in \mathcal{A}_0 \cup [\mathcal{D}, \mathcal{A}_0]$

$$\zeta_{a,\mathcal{D}}(z) := \mathrm{Tr}(a|\mathcal{D}|^{-z}) = \sum_{\lambda} \mathrm{Tr}(a\,\Pi(\lambda, |\mathcal{D}|))\lambda^{-z}$$

Example: Fractal string

Ω bounded open in $\mathbb{R}$ (e.g. complement of Cantor set Λ in $[0,1]$)
$\mathcal{L} = \{\ell_k\}_{k\geq 1}$ lengths of connected components of Ω with

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k \cdots > 0.$$

Geometric zeta function (Lapidus and van Frankenhuysen)

$$\zeta_{\mathcal{L}}(s) := \sum_k \ell_k^s$$

Dimension from operators: spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$

- Simpler definition: dimension n (*n–summable*) if $|\mathcal{D}|^{-n}$ infinitesimal of order one: $\lambda_k(|\mathcal{D}|^{-n}) = O(k^{-1})$
- Refined definition: **dimension spectrum** $\Sigma \subset \mathbb{C}$: set of poles of the zeta functions $\zeta_{a,\mathcal{D}}(z)$. (all zetas extend holomorphically to $\mathbb{C} \setminus \Sigma$)
- in sufficiently nice cases (almost commutative geometries) poles of $\zeta_{\mathcal{D}}(s) = \zeta_{1,\mathcal{D}}(s)$ suffice

Cantor set: spectral triple

$\Lambda =$ middle-third Cantor set: $\zeta_L(s) = \frac{3^{-s}}{1-2\cdot 3^{-s}}$

algebra commutative C^*-algebra $C(\Lambda)$.

Hilbert space: $E = \{x_{k,\pm}\}$ endpoints of intervals $J_k \subset \Omega = [0,1] \smallsetminus \Lambda$, with $x_{k,+} > x_{k,-}$; ℓ_k length of k-th interval in the construction

$$\mathcal{H} := \ell^2(E)$$

action $\mathrm{C}(\Lambda)$ acts on $\mathcal{H}$

$$f \cdot \xi(x) = f(x)\xi(x), \quad \forall f \in \mathrm{C}(\Lambda),\ \forall \xi \in \mathcal{H},\ \forall x \in E.$$

sign operator subspace $\mathcal{H}_k$ of coordinates $\xi(x_{k,+})$ and $\xi(x_{k,-})$,

$$F|_{\mathcal{H}_k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Dirac operator

$$\mathcal{D}|_{\mathcal{H}_k} \begin{pmatrix} \xi(x_{k,+}) \\ \xi(x_{k,-}) \end{pmatrix} = \ell_k^{-1} \cdot \begin{pmatrix} \xi(x_{k,-}) \\ \xi(x_{k,+}) \end{pmatrix}$$

• verify $[\mathcal{D}, a]$ bounded for $a \in \mathcal{A}_0$:

$$[\mathcal{D}, f]|_{\mathcal{H}_k} = \frac{(f(x_{k,+}) - f(x_{k,-}))}{\ell_k} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

for f Lipschitz: $\|[\mathcal{D}, f]\| \leq C(f)$
take dense $\mathcal{A}_0 \subset C(\Lambda)$ to be locally constant or more generally Lipschitz functions

• same for any self-similar set in $\mathbb{R}$ (Cantor-like)

Zeta function

$$\mathrm{Tr}(|\mathcal{D}|^{-s}) = 2\zeta_L(s) = \sum_{k\geq 1} 2^k 3^{-sk} = \frac{2 \cdot 3^{-s}}{1 - 2 \cdot 3^{-s}}$$

Dimension spectrum of the Cantor set

$$\Sigma = \left\{ \frac{\log 2}{\log 3} + \frac{2\pi i n}{\log 3} \right\}_{n \in \mathbb{Z}}$$

Observations

- the only real point of the dimension spectrum of the Cantor set is $\dim_H(\Lambda) = \frac{\log 2}{\log 3}$ (Hausdorff dimension, self-similarity dimension)
- can one see this in terms of a Weyl law? in terms of heat kernel expansion?
- we will be discussing heat kernel expansions on fractals later
- we will also return to fractal strings and complex dimensions for more interesting examples of fractals with more complicated dimension spectra