Menger Universal Spaces

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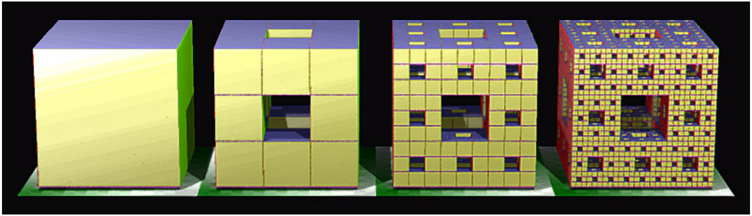
Some References

- Stephen Lipscomb, *Fractals and Universal Spaces in Dimension Theory*, Springer, 2008
- A. Panagiotopoulos, S. Solecki, *A combinatorial model for the Menger curve*, arXiv:1803.02516
- B.A. Pasynkov, *Partial topological products*, Trans. Moscow Math. Soc. 13 (1965), 153–271
- Greg Friedman, *An elementary illustrated introduction to simplicial sets*, Rocky Mountain Journal of Mathematics 42 (2012) 353–424

Menger Sponge



- start with unit cube $\mathcal{I}^3$
- divide into 27 cubes of side $1/3$
- remove central cube on each face and central cube in the middle
- repeat construction on each of the 20 remaining cubes ...

Menger Sponge

- n-th stage M_n of the construction of the Menger sponge consists of 20^n cubes

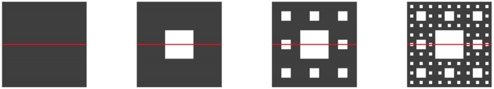
$$M = \bigcap_{n \in \mathbb{N}} M_n$$

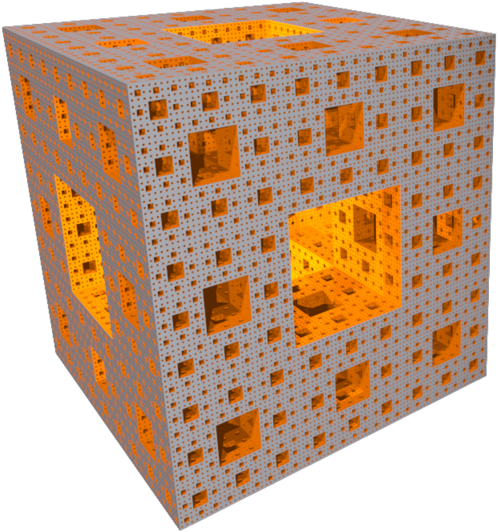
of side 3^{-n}, so that $\mathrm{Vol}(M_n) = (20/27)^n$ and surface area $\Sigma(M_n) = 2(20/9)^n + 4(8/9)^n$

- volume goes to zero surface area to infinity: Hausdorff dimension is between 2 and 3

$$\dim_H(M) = \frac{\log 20}{\log 3} = 2.727\ldots$$

- each face is a Sierpinski carpet
- each intersection with a diagonal of the cube or a midline of the faces is a Cantor set





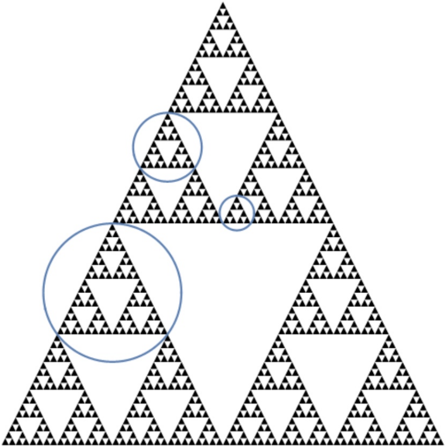
Inductive Dimension

- idea: n dimensional spaces have $n-1$ dimensional boundary
- small and large inductive dimension $ind(X)$ and $Ind(X)$
- both zero for $\emptyset$
- $ind(X)$ smallest n with: $\forall x \in X$ and $\forall U$ open $x \in U$ $\exists V \subset U$ with $x \in V$ and $\bar{V} \subset U$ with $ind(\partial V) < n$
- $Ind(X)$ smallest n with: $\forall F$ closed and U open $F \subset U$, $\exists V$ open $F \subset V \subset U$ with $Ind(\partial V) < n$
- X metrizable: $ind(X) \leq Ind(X) = \dim_{top}(X)$
- X compact Hausdorff: $\dim_{top}(X) \leq ind(X) \leq Ind(X)$
- so in good cases all agree: other way of computing topological dimension

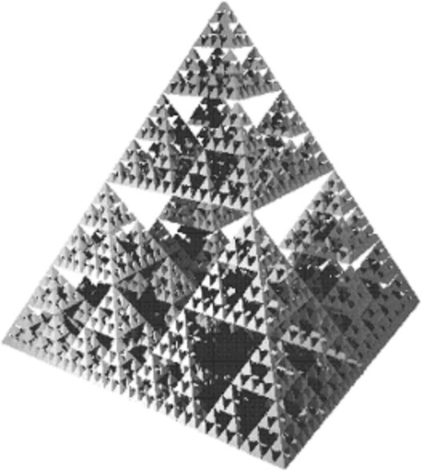
Topological Dimension and Inductive Dimension

- with the previous construction seen that the Menger sponge has Haudorff dimension $2 < \dim_H(M) < 3$
- so one would expect topological dimension is 2 but ... topological dimension one $\dim_{\rm top}(M) = 1$ (Menger curve)
- to see this use the following equivalent description of the topological dimension (for subsets of an ambient space $\mathbb{R}^N$): a space $M \subset \mathbb{R}^N$ has topological dimension n if each point $x \in M$ has arbitrarily small neighborhoods U such that $\partial U \cap M$ is a set of topological dimension $n-1$, and n is the smallest non-negative integer with this property

Example: the Sierpinski Gasket has topological dimension 1

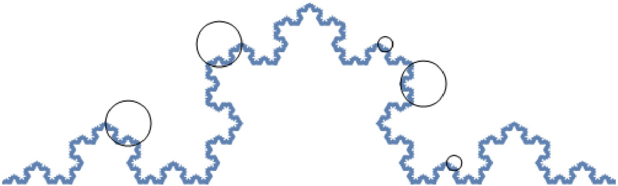


Example: Sierpinski Tetrahedron

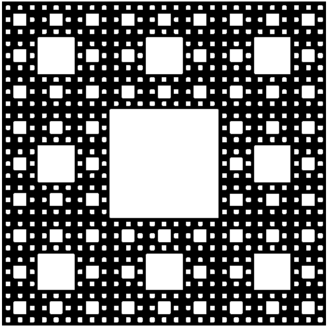


Hausdorff dimension 2 (4 pieces, scaling 1/2) and topological dimension 1 (similar neighborhoods balls as for Sierpinski Gasket)

Example: the Koch Snowflake has topological dimension 1



Example: Sierpinski Carpet also has topological dimension 1 (like Sierpinski Gasket) and Menger Sponge also in a similar way



more difficult to draw the right choice of neighborhoods here that make topological dimension 1 immediately visible

Universality of the Menger Curve

- K. Menger, *Kurventheorie* , Teubner, 1932.
- R. Anderson, *One-dimensional continuous curves and a homogeneity theorem*, Ann. of Math. 68 (1958) 1–16

• universal property of the Menger curve

- universal space for the class of all compact metric spaces of topological dimension ≤ 1
- every such space embeds inside the Menger curve

• the Cantor set is similarly universal for all compact metric spaces of topological dimension 0 (and the Sierpinski carpet for Jordan curves)

• on embedding and universality properties

- Stephen Lipscomb, *Fractals and Universal Spaces in Dimension Theory*, Springer, 2008.

- a continuum is a connected compact metric (metrizable) topological space
- a Peano continuum is a locally-connected compact metrizable space

• Menger curve M topologically characterized as a one-dimensional Peano continuum without locally separating points (for every connected neighbourhood U of any point x the set $U \setminus \{x\}$ is connected) and also without non-empty open subsets embeddable in the plane. Every one-dimensional Peano continuum can be embedded in M

n-dimensional Menger universal spaces

- A.N. Dranishnikov, *Universal Menger compacta and universal mappings*, Math. USSR-Sb. 57 (1987), no. 1, 131–149.
- B.A. Pasynkov, *Partial topological products*, Trans. Moscow Math. Soc. 13 (1965), 153–271
- M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Bull. AMS 11 (1984) 2, 369–370
- R. Engelking, Dimension theory, North Holland, 1978

- Menger universal M_n^m-continuum
  - first step unit cube $\mathcal{I}^m$
  - suppose at the k-th step of the construction have produced a configuration $\mathcal{F}_k$ of smaller m-cubes
  - at the $(k+1)$st step subdivide each cube D in $\mathcal{F}_k$ into $3^{m(k+1)}$ subcubes with edges $3^{-m(k+1)}$
  - for each $D \in \mathcal{F}_k$ let $\mathcal{F}_{k+1}(D)$ be those smaller cubes that intersect the n-faces of D
  - take $\mathcal{F}_{k+1} = \cup_{D\in\mathcal{F}_k} \mathcal{F}_{k+1}(D)$

- let $M_n^m(k) = \cup_{D \in \mathcal{F}_k} D \subset \mathcal{I}^m$ union of the subcubes

$$M_n^m = \cap_{k=0}^{\infty} M_n^m(k)$$

- Menger curve is M_1^3
- Sierpinski carpet is M_1^2

Universality of M_n^m

• the Menger M_n^m-continuum is universal for all compact metric spaces (compacta) of topological dimension $\leq n$ that embed in $\mathbb{R}^m$ (Štanko, 1971)

• a continuum X is homemorphic to M_n^{n-1} iff it can be ambedded in the sphere S^n so that $S^n \smallsetminus X$ has infinitely many connected components C_i with $\mathrm{diam}(C_i) \to 0$ and $\partial C_i \cap \partial C_j = \emptyset$ for $i \neq i$, the boundaries ∂C_i are m-cells for each i and $\cup_{i=1}^{\infty} \partial C_i$ is dense in X (Cannon, 1973)

Universal mapping of Menger $M_n = M_n^{2n+1}$-continua

• A.N. Dranishnikov, *Universal Menger compacta and universal mappings*, Math. USSR-Sb. 57 (1987), no. 1, 131–149

- (Bestvina, 1984): for $m \geq 2n+1$ all the Menger compacta M_n^m are homeomorphic
- $\exists$ continuous maps $f_n : M_n \to M_n$ universal in the class of maps between n-dimensional compacta
- $\forall f : X \to Y$ continuous map between n-dimensional compacta there are embeddings $\iota_X : X \hookrightarrow M_n$ and $\iota_Y : Y \hookrightarrow M_n$ such that commuting diagram up to homeomorphism

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\ M_n \arrow[r, "f_n"] & M_n \end{tikzcd}$$

- references added to the webpage

All Cantor sets are homeomorphic

• Brouwer's theorem: a topological space is homeomorphic to the Cantor set if and only if it is non-empty, perfect, compact, totally disconnected, and metrizable

- L.E.J. Brouwer, *On the structure of perfect sets of points*, Proc. Koninklijke Akademie van Wetenschappen, 12 (1910) 785–794.

Cantor sets are projective limits of finite sets:

- projective system $\{X_n\}$ of finite sets (discrete topology) with surjective maps $\phi_{n,m} : X_n \to X_m$ for $n > m$
- projective limit $X = \varprojlim_n X_n$ is subspace of the product $\prod_n X_n$ (with product topology)

$$X = \{x = (x_n) \in \prod_n X_n \mid x_m = \phi_{n,m}(x_m), \forall n \leq m\}$$

- either use characterization above or construct a coding by strings on an alphabet

Categorical view of the Menger curve $M = M_1^3$

- A. Panagiotopoulos, S. Solecki, *A combinatorial model for the Menger curve*, arXiv:1803.02516

- Menger prespace $\mathbb{M}$ generic inverse limit in the category of finite connected graphs with surjective graph homomorphisms
- Edge relation: equivalence relation $\mathcal{R}$ on $\mathbb{M}$
- Menger curve: quotient by this equivalence $M = \mathbb{M}/\mathcal{R}$
- topological realization $M = |\mathbb{M}|$ of combinatorial object $\mathbb{M}$

- Category of graphs
  - a graph G is a pair $(V, \mathcal{R}_V)$ where V is a set (vertices) and $\mathcal{R}_V \subset V \times V$ is a relation that is reflexive ($(v,v) \in \mathcal{R}_V$) and symmetric ($(v,w) \in \mathcal{R}_V \Leftrightarrow (w,v) \in \mathcal{R}_V$) defining edges
  - Note nonconventional assumption that $(v,v) \in \mathcal{R}_V$ (like presence of a "trivial" looping edge at each vertex)
  - homomorphism of graphs: function $f : V \to V'$ preserving edge relations (if $(v,w) \in \mathcal{R}_V$ then $(f(v), f(w)) \in \mathcal{R}_{V'}$); epimorphism if surjective on vertices and edges
  - only consider induced subgraphs: subset of vertices V and all edges of $\mathcal{R}_V$ between them
  - category: $\mathcal{C}$ objects finite connected graphs morphisms epimorphisms between them that are connected (preimage of each connected subset of target is a connected subset of source graph)
  - epimorphism between connected graphs is connected iff preimages of vertices are connected

- Projective limits of finite graphs
  - topological graph $(K, \mathcal{R}_K)$ with K a zero-dimensional compact metrizable topological space and $\mathcal{R}_K \subset K \times K$ closed subset, continuous morphisms
  - for finite graph discrete topology
  - inverse system $f^n_m : V_n \to V_m$ with $f_{n,n} = \mathrm{id}$ and $f_{n,m} \cdot f_{m,k} := f_{m,k} \circ f_{n,m} = f_{n,k}$ for $n \geq m \geq k$
  - inverse limit is a topological graph
$$(K, \mathcal{R}_K) = \varprojlim_n (V_n, \mathcal{R}_{V_n})$$
  - no longer a finite graph in general: set of vertices K is Cantor-like
  - viewing projective limit as subset of product, $x = (x_0, x_1, x_2, \ldots) \in K$ with $x_i \in V_i$ and with projections $f_i : K \to V_i$ satisfying $f_{i,j} \circ f_i = f_j$
  - connectedness: point $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2 \ldots)$ connected in K iff x_i connected to y_i in V_i for all coordinates

- Category of projective limits of finite graphs
  - objects are projective limits $K = \varprojlim_n (V_n, \mathcal{R}_{V_n}, f_{n,m})$ and morphisms are connected epimorphisms between these topological graphs
  - K connected and locally-connected, coordinatewise in $\prod_n K_n$ each $f_{m,n}^{-1}(v)$ connected
  - morphism of projective limits $h : K \to L$ with $K = \varprojlim_n K_n$ and $L = \varprojlim_n L_n$ then for all m there is n and $h_{n,m} : K_n \to L_m$ such that $h_{n,m} \circ f_n = \ell_m \circ h$ for $f_n : K \to K_n$ and $\ell_m : L \to L_m$ projections, with $h_{n,m}$ connected epimorphism of finite graphs so $h : K \to L$ is connected epimorphism of topological graphs

- conversely all connected and locally-connected topological graphs with connected epimorphisms are obtained as projective limits and morphisms of projective limits of finite graphs
- K has topology with a basis of connected clopen sets; can extract from this a sequence $\mathcal{U}_n$ of finite coverings with $\mathcal{U}_n$ a refinement of $\mathcal{U}_{n-1}$ such that different $U, V \in \mathcal{U}_n$ have $U \cap V = \emptyset$ and $\cup_n \mathcal{U}_n$ separates vertices of K
- give to $\mathcal{U}_n$ a graph structure by putting an edge between U and V iff $\exists x, y$ with $x \in U$ and $y \in V$ such that $(x, y) \in \mathcal{R}_K$
- then have projection maps between these graphs $f_{n,m} : \mathcal{U}_n \to \mathcal{U}_m$ that are connected epimorphisms and $K = \varinjlim_n \mathcal{U}_n$ proj limit ot graphs

- connected epimorphism $h : K \to L$ of connected and locally-connected topological graphs: know $K = \varprojlim_n K_n$ and $L = \varprojlim_m L_m$ with projections $f_n : K \to K_n$ and $\ell_m : L \to L_m$, so need to show for all m there is n and $h_{n,m} : K_n \to L_m$ with $\ell_m \circ h = h_{n,m} \circ f_n$
- for given m pick n large enough that $f_n^{-1}(K_n)$ is a refinement of $(\ell_m \circ h)^{-1}(L_m)$, then there is a map $h_{n,m} : K_n \to L_m$ that is defined through this inclusion so that $\ell_m \circ h = h_{n,m} \circ f_n$
- because ℓ_m, h, f_n are connected epimorphisms $h_{n,m}$ also is

• category of projective limits of finite graphs is same as category of connected and locally-connected topological graphs with connected epimorphisms

- Topological graphs and Peano continua
  - Peano continuum: locally-connected compact metrizable space
  - prespace: connected and locally-connected topological graph K where the edge relation $\mathcal{R}_K$ is transitive (hence an equivalence relation)
  - any equivalence relation on a finite set gives a graph on that set of vertices that consists of a disjoint union of cliques (complete graphs) so for finite connected graphs just cliques
  - realization $|K|$ of a prespace K: topological space given by quotient $K/\mathcal{R}_K$
  - Claim: X Peano continuum iff $X = |K|$ for some prespace K
- A. Panagiotopoulos, S. Solecki, *A combinatorial model for the Menger curve*, arXiv:1803.02516

Projective Fraïsse class

- any sub-collection of pairwise non-isomorphic objects is countable
- identity maps in the class and maps in the class closed under composition (ok if morphisms of a category)
- for any objects B, C in the class there is an object D with morphisms $f : D \to B$ and $g : D \to C$
- for every morphisms $f' : B \to A$ and $g' : C \to A$ there are morphisms $f : D \to B$ and $g : D \to C$ with $f' \circ f = g' \circ g$ (projective amalgamation property)

• finite connected graphs with connected epimorphisms are a projective Fraïsse class

- Menger Prespace

- given a projective Fraïsse class (here the one of finite graphs) there is an object $\mathbb{M}$ (projective limit of a "generic sequence" of objects in the class) such that
  - for each object A in the class there is a morphism $f : \mathbb{M} \to A$ (morphism in the category of projective limits)
  - for any A, B in the class and morphisms $f : \mathbb{M} \to A$ and $g : B \to A$ there is a morphism $f : \mathbb{M} \to B$ with $f = g \circ h$ (projective extension property)

- this $\mathbb{M}$ is the Menger prespace

- generic sequence
  - by first property of Fraïsse class have countable A_n and $e_n : C_n \to B_n$ containing all isomorphism types of objects and morphisms
  - inductive construction of projective system: $L_0 = A_0$ and assume have L_n with maps $t_{n,i} : L_n \to L_i$ for $i < n$
  - by third property of Fraïsse class find H with maps $f : H \to L_n$ and $g : H \to A_{n+1}$ and with a finite number $s_1, \dots, s_k$ of morphisms (up to isoms) $s_i : H \to B_{n+1}$
  - use k times projective amalgamation to obtain H' with maps $f' : H' \to H$ and $d_j : H' \to C_{n+1}$ with $s_j \circ f' = e_{n+1} \circ d_j$ for all $j \leq k$
  - take $L_{n+1} = H'$ with $t_{n+1,i} = t_{n,i} \circ f \circ f'$
  - the way $(L_n, t_{n,i})$ constructed gives the two properties above of $\mathbb{M}$

- Menger Prespace and Menger Curve: realization $|\mathbb{M}|$ is a topologically one-dimensional Peano continuum without locally separating points, hence it is the Menger curve

Statements of some properties of Menger prespace and curve (Panagiotopoulos, Solecki)

- Homogeneity
  - K closed subgraph of $\mathbb{M}$ "locally non-separating" if for each clopen connected W in $\mathbb{M}$ the complement $W \smallsetminus K$ is connected
  - K, L locally non-separating subgraphs of $\mathbb{M}$: any isomorphism $f : K \to L$ extends to an automorphism of $\mathbb{M}$

- Lifting property
  - K locally non-separating subgraph of $\mathbb{M}$: given finite graphs A, B and connected epimorphisms $g : B \to A$ and $f : \mathbb{M} \to A$, for any morphism $p : K \to B$ with $g \circ p = f|_K$ there is $h : \mathbb{M} \to B$ with $g \circ h = f$ and $h|_K = p$

- Universality
  - for any Peano continuum X there is a continuous connected surjective map $f : |\mathbb{M}| \to X$

Higher dimensional analogs

- a more general higher-dimensional theory of inverse limits of n-dimensional polyhedra with simplicial finite-to-one projections
  - B.A. Pasynkov, *Partial topological products*, Trans. Moscow Math. Soc. 13 (1965), 153–271
  - A. Panagiotopoulos, S. Solecki, *A combinatorial model for the Menger curve*, arXiv:1803.02516

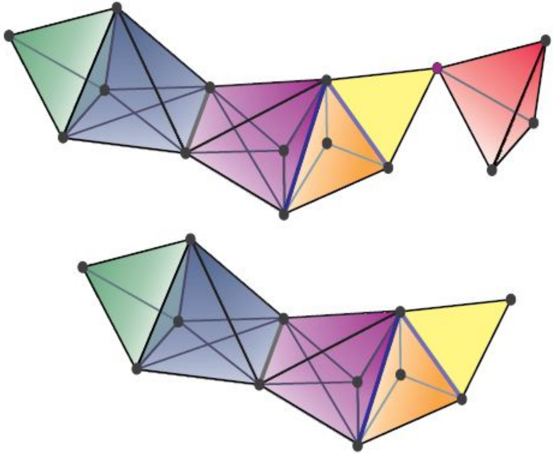
- Menger compacta and inverse limits categories
  - simplicial sets/simplicial complexes (more about them later) k-connected if homotopy groups π_i vanish for $i \leq k$; simplicial map k-connected if preimage of every k-connected subcomplex is k-connected
  - category $\mathcal{C}_n$ of all n-dimensional and $(n-1)$-connected simplicial complexes with $(n-1)$-connected simplicial maps
  - generic sequences and projective limit gives n-dimensional prespaces $\mathbb{M}_n$ with realization (with respect to faces relation) is Menger space $M_n = M_n^{2n+1}$

Can use Menger compacta as model spaces for fractals? just like simplicial sets or cubical sets?

- Simplicial sets in topology
  - Greg Friedman, *An elementary illustrated introduction to simplicial sets*, arXiv:0809.4221, Rocky Mountain Journal of Mathematics 42 (2012) 353–424
  - J. Peter May, *Simplicial objects in algebraic topology*, University of Chicago Press, 1992

- Simplicial set: sequence of sets $X = \{X_n\}_{n\geq 0}$ with maps (faces and degeneracies) $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i \quad \text{if } i < j, \\
d_i s_j &= s_{j-1} d_i \quad \text{if } i < j, \\
d_j s_j &= d_{j+1} s_j = \mathrm{id}, \\
d_i s_j &= s_j d_{i-1} \quad \text{if } i > j+1, \\
s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j.
\end{aligned}
$$



Categorical version of simplicial sets

- Δ category: objects finite ordered sets $[n] := \{1, 2, \dots, n\}$ and morphisms $f : [m] \to [n]$ order-preserving functions: $f(i) \leq f(j)$ for $i \leq j$
- morphisms are generated by maps $D_i : [n] \to [n+1]$ and $S_i : [n+1] \to [n]$

$$D_i[0, \dots, n] = [0, \dots, \hat{i}, \dots, n], \quad S_i[0, \dots, n] = [0, \dots, i, i, \dots, n]$$

- in Δ^{op} the D_i become face maps $d_i : [n+1] \to [n]$ and S_i the degeneracy maps $s_i : [n] \to [n+1]$

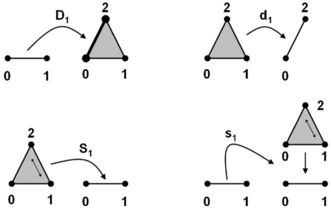


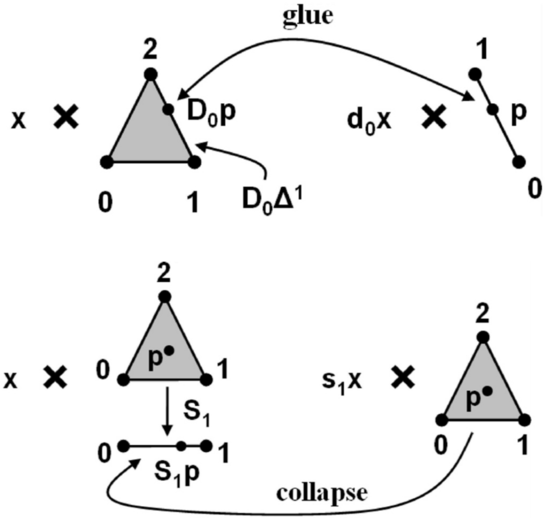
image of degeneracy s_1 degenerate 2-simplex image of collapse map S_1

- Simplicial set: functor $X : \Delta^{op} \to \mathcal{S}$ to the category of sets (contravariant functor from Δ)
- Realization: $|\Delta^n|$ the geometric simplex realization of combinatorial $\Delta^n = [n]$

$$|X| := \sqcup_n (X_n \times |\Delta^n|) \,/ \sim$$

modulo equivalence relation $(x, S_i(t)) \sim (s_i(x), t)$ and $(x, D_i(t)) \sim (d_i(x), t)$

- interpret as recipe for gluing the geometric simplexes $|\Delta^n|$ together according to the combinatorial scheme prescribed by the X_n so that faces and degeneracies match


glue
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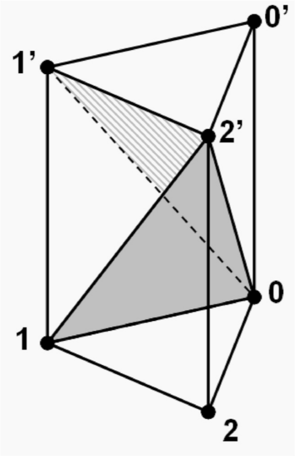
p
s_1x
p
s_1
0
1
0
1
s_1p
collapse

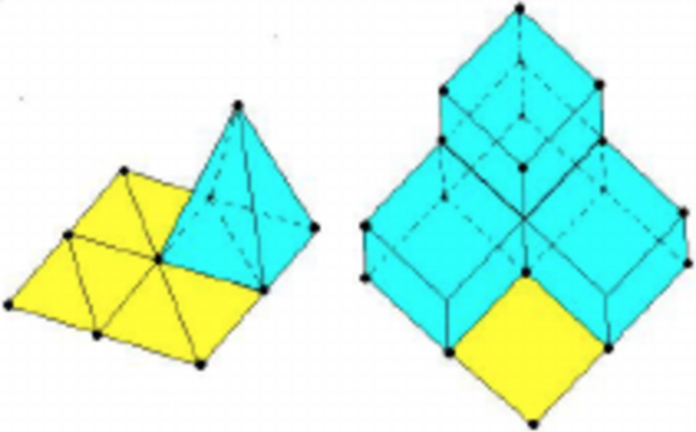
- Nerve: simplicial sets from categories
  - category Cat of small categories with functors as morphisms, nerve functor $\mathcal{N} : \mathrm{Cat} \to \Delta\mathcal{S}$ to the category of simplicial sets $\Delta\mathcal{S} = \mathrm{Func}(\Delta^{op}, \mathcal{S})$
  - for a small category $\mathcal{C}$ the nerve $\mathcal{N}(\mathcal{C})$ has a 0-simplex (vertex) for each object of $\mathcal{C}$, a 1-simplex (edge) for each morphism, a 2-simplex for each composition of two morphishs, a k-simplex for every chain of k composable morphisms
  - face maps: composition of two adjacent morphisms at the i-th place of a k-chain $d_i : \mathcal{N}_k(\mathcal{C}) \to \mathcal{N}_{k-1}(\mathcal{C})$ and degeneracies are insertions of the identity morphism at an object in the chain

• Products: product of simplexes is not a simplex but can be decomposed as a union of simplexes



Cubes behave better than simplexes with respect to products

Simplicial and cubical complexes



- Cubical sets in topology
  - $\mathcal{I}$ unit interval as combinatorial structure consisting of two vertices and an edge connecting them
  - $|\mathcal{I}| = [0,1]$ geometric realization: unit interval as topological space (subspace of $\mathbb{R}$)
  - $\mathcal{I}^n$ for the n-cube as combinatorial structure and $|\mathcal{I}^n| = [0,1]^n$ its geometric realization
  - $\mathcal{I}^0$ a single point
  - face maps $\delta_i^a : \mathcal{I}^n \to \mathcal{I}^{n+1}$, for $a \in \{0,1\}$ and $i = 1, \ldots, n$

$$\delta_i^a(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, a, t_i, \ldots, t_n)$$

  - degeneracy maps $s_i : \mathcal{I}^n \to \mathcal{I}^{n-1}$

$$s_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$$

- cubical relations for $i < j$

$$\delta_j^b \circ \delta_i^a = \delta_i^a \circ \delta_{j-1}^b \quad \text{and} \quad s_i \circ s_j = s_{j-1} \circ s_i$$

  and relations

$$\begin{array}{ll} \delta_i^a \circ s_{j-1} = s_j \circ \delta_i^a & i < j \\ s_j \circ \delta_i^a = 1 & i = j \\ \delta_{i-1}^a \circ s_j = s_j \circ \delta_i^a & i > j \end{array}$$

- Cube category: $\mathfrak{C}$ has objects $\mathcal{I}^n$ for $n \geq 0$ and morphisms generated by the face and degeneracy maps δ_i^a and s_i
- Cubical set: functor $C : \mathfrak{C}^{op} \to \mathcal{S}$ to the category of sets.
- notation: $C_n := C(\mathcal{I}^n)$

- variant of the cube category $\mathfrak{C}_c$ with additional degeneracy maps $\gamma_i : \mathcal{I}^n \to \mathcal{I}^{n-1}$ called connections

$$\gamma_i(t_1, \dots, t_n) = (t_1, \dots, t_{i-1}, \max\{t_i, t_{i+1}\}, t_{i+2}, \dots, t_n)$$

satisfying relations

$$\gamma_i \gamma_j = \gamma_j \gamma_{i+1},\ i \leq j; \quad s_j \gamma_i = \begin{cases} \gamma_i s_{j+1} & i < j \\ s_i^2 = s_i s_{i+1} & i = j \\ \gamma_{i-1} s_j & i > j \end{cases}$$

$$\gamma_j \delta_i^a = \begin{cases} \delta_i^a \gamma_{j-1} & i < j \\ 1 & i = j, j+1,\ a = 0 \\ \delta_j^a s_j & i = j, j+1,\ a = 1 \\ \delta_{i-1}^a \gamma_j & i > j+1. \end{cases}$$

- role of degeneracy maps: maps s_i identify opposite faces of a cube, additional degeneracies γ_i identify adjacent faces

- cubical set with connection: functor $C : \mathfrak{C}_c^{op} \to \mathcal{S}$ to the category of sets
- category of cubical sets has these functors as objects and natural transformations as morphisms
- so morphisms given by collection $\alpha = (\alpha_n)$ of morphisms $\alpha_n : C_n \to C'_n$ satisfying compatibilities $\alpha \circ \delta_i^a = \delta_i^a \circ \alpha$ and $\alpha \circ s_i = s_i \circ \alpha$ (and in the case with connection $\alpha \circ \gamma_i = \gamma_i \circ \alpha$)
- cubical nerve $\mathcal{N}_{\mathfrak{C}}\mathcal{C}$ of a category $\mathcal{C}$ is the cubical set with

$$(\mathcal{N}_{\mathfrak{C}}\mathcal{C})_n = \mathrm{Fun}(\mathcal{I}^n, \mathcal{C})$$

  with $\mathcal{I}^n$ the n-cube seen as a category with objects the vertices and morphisms generated by the 1-faces (edges), and $\mathrm{Fun}(\mathcal{I}^n, \mathcal{C})$ is the set of functors from $\mathcal{I}^n$ to $\mathcal{C}$
- when working with cubical sets with connection homotopy equivalent to simplicial nerve
- R. Antolini, *Geometric realisations of cubical sets with connections, and classifying spaces of categories*, Appl. Categ. Structures 10 (2002), no. 5, 481–494.

Building an analog of cubical sets using Menger spaces

- Menger spaces M^m_n are modelled on cubes, so want the same faces and degeneracy maps (and connections) as in the cube category
- additional important data: the self-similarity structure
- the iterated function system $\{f_1, \ldots, f_N\}$ given by the affine contraction maps that take the cube $\mathcal{I}^m$ to the N smaller cubes, scaled by a factor 3^{-m}, where $N = N(m, n)$ is the number of those subcubes that intersect the n-faces of $\mathcal{I}^m$
- Menger category $\mathfrak{M}$ with objects the M^m_n (or better their corresponding combinatorial spaces $\mathbb{M}^m_n$) and morphisms generated by the δ^a_i, s_i, γ_i, and the IFS maps f_k
- Menger sets: functors $M : \mathfrak{M}^{op} \to \mathcal{S}$ to the category of sets

- there is a good homology theory for cubical sets (introduced by Serre to study (co)homology of fibrations) see
  - S. Eilenberg, S. MacLane, *Acyclic models*, Amer. J. Math, 75 (1953) 189–199
  - W. Massey, *Singular homology theory*, Graduate Texts in Mathematics, Vol. 70, Springer 1980.
- it is also known that cubical sets with connections have the same homology theory as ordinary cubical sets (connections do not add any nontrivial cycles in the homology groups)
  - H. Barcelo, C. Greene, A.S. Jarrah, V. Welker, *Homology groups of cubical sets with connections*, arXiv:1812.07600
- What is the effect on homology of introducing the contraction maps of the IFS for the Menger spaces? directed system of homology groups of cubical sets?
- Similar approach with simplicial sets? (Sierpinski n-simplex instead of Menger space?)

What does this approach lead to?