Entropy and Dimension

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Shannon Entropy/Information

- bit memory storage unit = switch with two on/off positions = digit 0 or 1
- A set of switches with $N = 2^{\#A}$ positions
- possible states: write a number $m = \sum_{k=0}^{\#A-1} s_k \, 2^k$ in binary notation $s_k \in \{0, 1\}$
- need $\#A = \frac{\log N}{\log 2}$ bits to select one particular possible configuration
- $b = \log N$ measured in $\log 2$ units is the bit number
- if have probability p_i of an event i in a set $i \in \{1, \ldots, R\}$ such as a frequency of occurrence

$$p_i = \frac{N_i}{N}, \qquad N = \sum_{i=1}^{R} N_i$$

- number of bits required to identify a particular configuration α among all possible is $\log N$
- to select an α either select among all or first select which set of N_i elements it belongs to and then among these so $b_i + \log N_i = \log N$ hence $b_i = -\log p_i$
- Shannon information measure: the average of the b_i with respect to the probabilities p_i

$$\mathcal{I}(P) = \sum_{i=1}^{R} p_i \log p_i$$

- Shannon Entropy: $S(P) = -\mathcal{I}(P)$ ("negative information", in fact positive $S(P) \geq 0$)
- measure of knowledge of the observed about what event to expect knowing $P = (p_i)$ (least knowledge at the uniform distribution, most knowledge at the delta measures δ_i)
- if the events i are dynamical microstates of a physical system then it is the entropy in the thermodynamic sense

Khinchin Axioms and Shannon Entropy $\mathcal{I}_R(p_1, \ldots, p_R)$

- Khinchin Axioms

1. continuous function of $P = (p_1, \ldots, p_R)$
2. minimum at the uniform distribution (max for entropy):

$$\mathcal{I}_R(\frac{1}{R}, \ldots, \frac{1}{R}) \leq \mathcal{I}_R(P)$$

3. extendability: $\mathcal{I}_R(p_1, \ldots, p_R) = \mathcal{I}_{R+1}(p_1, \ldots, p_R, 0)$
4. extensivity (implies additivity on independent subsystems)

$$\mathcal{I}(P) = \mathcal{I}(P') + \sum_i p'_i \mathcal{I}(Q|i)$$

for a composite system $P = (p_{ij})$ with $p_{ij} = Q(j|i)\, p'_i$ with conditional probabilities $Q(j|i)$ of j given i with conditional information

$$\mathcal{I}(Q|i) = \sum_j Q(j|i) \log Q(j|i)$$

Note: case of independent subsystems $p_{ij} = p'_i p''_j$ gives $\mathcal{I}(P) = \mathcal{I}(P') + \mathcal{I}(P'')$

Axiomatic characterization of the Shannon Entropy

• family of functionals $\mathcal{I} = \{\mathcal{I}_R\}$ satisfying Khinchin axioms agree with the Shannon information up to a positive constant

$$\mathcal{I}(P) = C \cdot \sum_i p_i \log p_i, \quad \text{for some } C > 0$$

- at the uniform distribution: $p_{ij} = Q(j|i)\, p'_i$ with $p_{ij} = 1/N$ and $N = R \cdot r$ with $p'_i = 1/R$ and $Q(j|i) = 1/r$ obtain for $f(R) := \mathcal{I}_R(\frac{1}{R}, \ldots, \frac{1}{R})$ a function with $f(Rr) = f(R) + f(r)$ and continuous

$$f(R) = -C \cdot \log(R) \quad \text{for some } C \in \mathbb{R}^*$$

- also have $f(R) \geq f(R+1)$ by second and third axioms, so $C > 0$
- then from uniform to non-uniform: take p_{ij} and $Q(j|i)$ still uniform but p'_i arbitrary $f(N) = \mathcal{I}(P') + \sum_i p'_i f(N_i)$

$$\mathcal{I}(P') = -\sum_i p'_i (f(N_i) - f(N)) = C \sum_i p'_i \log p'_i$$

Rényi Entropy

• weaken the requirement of extensivity (non-extensive entropies) and replace only with additivity on statistically independent subsystems

$$p_{ij} = p'_i p''_j \;\Rightarrow\; \mathcal{I}(P) = \mathcal{I}(P') + \mathcal{I}(P'')$$

• then other solutions (not proportional to Shannon entropy): Rényi information

$$\mathcal{I}_\beta(P) = \frac{1}{\beta - 1} \log(\sum_{i=1}^{R} p_i^\beta)$$

Shannon Entropy as limit of Rényi Entropy

- $\mathcal{I}_\beta(P)$ defined for $\beta \in \mathbb{R}_+$ with $\beta \neq 1$
- limit when $\beta \to 1$: expand in $\epsilon = \beta - 1$

$$\sum_i p_i^{1+\epsilon} = \sum_i p_i \exp(\epsilon \log p_i) \sim \sum_i p_i (1 + \epsilon \log p_i)$$

$$= 1 + \epsilon \sum_i p_i \log p_i$$

so limit of the Rényi Entropy

$$\lim_{\epsilon \to 0} \mathcal{I}_{1+\epsilon}(P) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \log(1 + \epsilon \sum_i p_i \log p_i)$$

$$= \sum_i p_i \log p_i = \mathcal{I}(P)$$

Kullback–Leibler Divergence (Relative Entropy)

- given known probability distribution $P = (p_i)$ modified by some process to a new $Q = (q_i)$ with $q_i > 0$
- want to evaluate the information transfer of this process: $b_i(P) - b_i(Q) = \log(p_i/q_i)$
- estimate the mean value (in the known distribution)

$$KL(P|Q) := \sum_i p_i \log(p_i/q_i)$$

- non-negative because

$$\log x \geq 1 - \frac{1}{x} \;\Rightarrow\; \sum_i p_i \log \frac{p_i}{q_i} \geq \sum_i p_i (1 - \frac{p_i}{q_i}) = 0$$

- minimum value at 0 for $P = Q$ (again because $\log x > 1 - x^{-1}$ except at at $x = 1$ where equal)
- if uniform distribution $q_i = 1/R$ then $K(P|Q) = \mathcal{I}(P) + \log R$

Properties of Rényi Entropy

- monotonically increasing function: $\mathcal{I}_\beta(P) \leq \mathcal{I}_{\beta'}(P)$ when $\beta < \beta'$ for any P (so upper and lower bounds for Shannon entropy for $\beta > 1$ and $\beta < 1$)
- check monotonicity:

$$\frac{\partial \mathcal{I}_\beta(P)}{\partial \beta} = \frac{1}{(1-\beta)^2} \sum_i \mathbb{P}_i \log(\frac{\mathbb{P}_i}{p_i})$$

where escort probabilities

$$\mathbb{P}_i = \frac{p_i^\beta}{\sum_j p_j^\beta}$$

Kullback–Leibler Divergence is non-negative so monotonicity

- also another estimate for $\beta' > 0$ and $\beta\beta' > 0$

$$\frac{\beta - 1}{\beta}\mathcal{I}_\beta(P) \geq \frac{\beta' - 1}{\beta'}\mathcal{I}_{\beta'}(P)$$

- function x^σ convex for $\sigma > 1$ and concave for $0 < \sigma < 1$ so

$$(\sum_i a_i^\sigma) \geq \sum_i a_i^\sigma, \quad \forall \sigma > 1$$

$$(\sum_i a_i^\sigma) \leq \sum_i a_i^\sigma, \quad \forall 0 < \sigma < 1$$

take $a_i = p_i^\beta$ and $\sigma = \beta'/\beta$

$$(\sum_i p_i^\beta)^{\beta'/\beta} \geq \sum_i p_i^{\beta'} \quad \text{for } \beta' > \beta > 0$$

$$(\sum_i p_i^\beta)^{\beta'/\beta} \leq \sum_i p_i^{\beta'} \quad \text{for } \beta < \beta' < 0$$

- then taking $1/\beta'$ power (and then log)

$$(\sum_i p_i^\beta)^{1/\beta} \geq (\sum_i p_i^{\beta'})^{1/\beta'}$$

- monotonicity in β of

$$\Psi(\beta) := (1-\beta)\mathcal{I}_\beta = -\log \sum_i p_i^\beta$$

$$\Psi(\beta) \leq \Psi(\beta') \quad \text{for} \quad \beta' > \beta$$

because $p_i^\beta \geq p_i^{\beta'}$ and $-\log \sum_i p_i^\beta \leq -\log \sum_i p_i^{\beta'}$

- also have concavity in β

$$\frac{\partial^2 \Psi}{\partial \beta^2} \leq 0$$

Escort probabilities and statistical mechanics

- if write $p_i = \exp(-b_i)$ with $\sum_i p_i = 1$ (see later box-counting)
- then associated escort distribution

$$\mathbb{P}_i = \frac{p_i^\beta}{\sum_i p_i^\beta}$$

for $\beta \to \infty$ largest p_i dominates, for $\beta \to -\infty$ smallest

- analogy with statistical mechanics $\mathbb{P}_i = \exp(\Psi - \beta b_i)$ with $\Psi(\beta) = -\log Z(\beta)$ with partition function

$$Z(\beta) := \sum_i \exp(-\beta b_i) = \sum_i p_i^\beta$$

- Helmholtz free energy

$$F(\beta) := -\frac{1}{\beta} \log Z(\beta) = \frac{1}{\beta}\Psi(\beta)$$

- directly related to Rényi information

$$\mathcal{I}_\beta(P) = \frac{1}{\beta - 1} \log \sum_i p_i^\beta = -\frac{1}{\beta - 1}\Psi(\beta)$$

Entropy and Thermodynamics

- probabilities p_i of microstates of a physical system
- M_i value at state i of a random variable M: expectation value

$$\langle M \rangle_P = \sum_i M_i \, p_i$$

- max-ent principle: look for p_i's that maximize entropy
- "unbiased guess" in information theory: minimize information
- generalized canonical distribution: p_i such that

$$\delta \mathcal{I}(P) = \sum_i (1 + \log p_i) \delta p_i = 0$$

with $\sum_i M_i^\sigma \, \delta p_i = 0$ (all observables M^σ) and $\sum_i \delta p_i = 0$

- multiply these constraints by an arbitrary factor β_σ (Lagrange multipliers)

$$\sum_i (\log p_i - \Psi + \sum_\sigma \beta_\sigma M_i^\sigma) \delta p_i = 0$$

- interpret then as probabilities

$$\mathbb{P}_i = \exp(\Psi - \sum_\sigma \beta_\sigma M^\sigma)$$

by imposing normalization condition $\sum_i \mathbb{P}_i = 1$

- normalization condition gives

$$\Psi = -\log Z(\beta) \quad \text{for} \quad Z(\beta) = \sum_i \exp(-\sum_\sigma \beta_\sigma M_i^\sigma)$$

- Example: Gibbs distribution mean energy $M = E = (E_i)$ of a system in thermodynamic equilibrium

$$\mathbb{P}_i = \exp(\beta(F - E_i)) \quad \text{with} \quad F = \frac{1}{\beta}\Psi(\beta)$$

Helmholtz free energy at inverse temperature $\beta = 1/T$

$$Z(\beta) = \exp(-\beta F) = \sum_i \exp(-\beta E_i)$$

sum of microstates of the system with energies E_i

- entropy in the thermodynamic sense for such a system is

$$S = \beta(E - F)$$

- Shannon entropy agrees with (expectation value of) thermodynamic entropy

$$-\sum_i \mathbb{P}_i \log \mathbb{P}_i = -\sum_i e^{\beta(F - E_i)} \beta(F - E_i) = \langle S \rangle$$

Example: thermodynamic system modelled by coding space shift

- Potts model on a 1-dimensional lattice
  - system of spins at nodes of the lattice that can assume k possible states
  - case $k = 2$ up/down spins: Ising model
  - Hamiltonian $H_N(s_1, \dots, s_N)$ interactions of spins at different sites in a region of size N in lattice
  - probability of observing a microstate

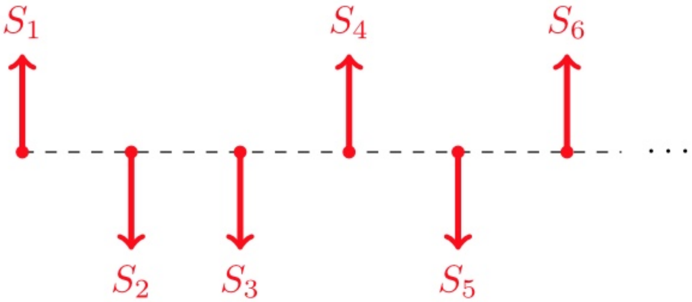
$$p_N(s_1, \dots, s_N) = \frac{1}{Z_N(\beta)} \exp(-\beta H_N(s_1, \dots, s_N))$$

  - partition function: sum over microstates

$$Z_N(\beta) = \sum_{s_1, \dots, s_n} \exp(-\beta H_N(s_1, \dots, s_N))$$

  - thermodynamic limit

$$f(\beta) = -\lim_{N \to \infty} \frac{1}{\beta N} \log Z_N(\beta)$$



Ising model on a one-dimensional lattice

- Coding of the Potts model dynamics
  - $A = (n_1, \dots, n_k)$ arbitrary subset of lattice site (not necessarily neighboring)
  - probability of observing spins $s_{n_1}, \dots, s_{n_k}$ at these sites

$$P_A(s_{n_1}, \dots, s_{n_k}) = \sum_{\mathcal{S}^*} p_N(s_1, \dots, s_N)$$

    with $\mathcal{S}^*$ = set of all configurations of spins $s_1, \dots, s_N$ except those at the sites of $A = (n_1, \dots, n_k)$
  - for simplicity take $A = (1, \dots, k)$ first k sites

$$P_k(s_1, \dots, s_k) = \sum_{s_{k+1}, \dots, s_N} p_N(s_1, \dots, s_k, s_{k+1}, \dots, s_N)$$

$$= \frac{1}{Z_N(\beta)} \sum_{s_{k+1}, \dots, s_N} \exp(-\beta H_N(s_1, \dots, s_N))$$

- reasonable assumption for Hamiltonian: for subsets $A \subset \{1, \ldots, N\}$

$$S_A := \prod_{n \in A} s_n, \quad \text{and} \quad H_N(s_1, \ldots, s_N) = -\sum_A J_A S_A$$

  for some coupling constants J_A
- special case: nearest neighbors interaction (only adjacent spins interact)
- under these assumptions then obtain

$$P_k(s_1, \ldots, s_k) = \frac{1}{Z_N(\beta)} \sum_{s_{k+1}, \ldots, s_N} \exp(-\beta H_N(s_1, \ldots, s_N))$$

$$= \frac{1}{Z_N(\beta)} \exp(-\beta H_k(s_1, \ldots, s_k))$$

- think of spin configurations $s_1, \dots, s_N$ on a lattice with N sites as words of length N on an alphabet of k letters
- if the probability distribution satisfies

$$P_k(s_1, \dots, s_k) = P_k(s_{1+a}, \dots, s_{k+a}) \quad \forall a \in \mathbb{Z}_+$$

then probability distribution on Σ_k^+ that is invariant under the shift map σ with these values on cylinder sets $\mathcal{C}(s_1, \dots, s_k)$
- this means Hamiltonian

$$H_k(s_1, \dots, s_k) = H_k(s_{1+a}, \dots, s_{k+a})$$

invariant under lattice translations

- Conditional probabilities for the Potts model

$$p_{1|k}(s_{k+1}|s_1,\ldots,s_k)$$

  - these satisfy the relation

$$p_{1|k}(s_{k+1}|s_1,\ldots,s_k)\, p_k(s_1,\ldots,s_k) = p_{k+1}(s_1,\ldots,s_k,s_{k+1})$$

  - Hamiltonians

$$H_{k+1}s_1,\ldots,s_k,s_{k+1}) - H_k(s_1,\ldots,s_k) = W_{1,k}(s_{k+1}|s_1,\ldots,s_k)$$

interaction energy of the $(k+1)$-st spin with the configuration $(s_1,\ldots,s_k)$ of the first k spins

- conditional probabilities

$$p_{1|k}(s_{k+1}|s_1,\ldots,s_k) = \frac{p_{k+1}(s_1,\ldots,s_k,s_{k+1})}{p_k(s_1,\ldots,s_k)}$$

$$= \frac{Z_k(\beta)}{Z_{k+1}(\beta)} \exp(-\beta(H_{k+1}(s_1,\ldots,s_{k+1}) - H_k(s_1,\ldots,s_k)))$$

$$\sim \exp(-\beta W_{1|k}(s_{k+1}|s_1,\ldots,s_k))$$

• 1d Potts model is prototypical example for applications of methods from statistical mechanics and thermodynamics to chaotic dynamical systems

Box-counting and Rényi entropy

- bounded set $E \subset \mathbb{R}^N$, say $E \subset [0,1]^N$
- probability measure μ on $[0,1]^N$ with support on E
- divide $[0,1]^N$ in boxes of equal size: cubes of side ϵ
- count number r of boxes that meet E in a subset of positive μ-measure

$$r \leq R \sim \epsilon^{-N}$$

total number of boxes in $[0,1]^N$

- $p_i = p_i(\epsilon)$ probability assigned to the i-th box B_i

$$p_i = \mu(E \cap B_i)$$

crowding index

$$\alpha_i(\epsilon) = \frac{\log p_i(\epsilon)}{\log \epsilon}$$

- it is also function of x point where the box is centered $\alpha(x,\epsilon)$

- pointwise dimension $\alpha(x) = \lim_{\epsilon\to 0} \alpha(x,\epsilon)$ if limit exists (local scaling exponent)
- in terms of "bits numbers" $p_i = \exp(-b_i)$

$$b_i = -\alpha_i(\epsilon) \log \epsilon$$

- escort distribution

$$\mathbb{P}_i = \exp(\Psi - \beta b_i)$$

$$\Psi(\beta) = -\log \sum_i \exp(-\beta b_i) = -(\beta - 1)\mathcal{I}_\beta(P)$$

- and partition function

$$Z(\beta) = \sum_i p_i^\beta = \sum_i \exp(-\beta b_i)$$

$$\mathcal{I}_\beta(P) = \frac{1}{\beta - 1} \log Z(\beta) = \frac{1}{\beta - 1} \log \sum_i p_i^\beta$$

Rényi (box-counting) dimensions

- the partition function $Z(\beta)$ for $p_i = p_i(\epsilon)$ diverges for $\epsilon \to 0$
- but it satisfies a power law with exponent that gives an associated dimension
- Rényi dimension

$$D(\beta) = \lim_{\epsilon \to 0} \frac{\mathcal{I}_\beta(P_\epsilon)}{\log \epsilon} = \lim_{\epsilon \to 0} \frac{1}{\log \epsilon} \frac{1}{\beta - 1} \log \sum_i p_i(\epsilon)^\beta$$

$$Z(\beta) \sim_{\epsilon \to 0} \epsilon^{(\beta-1)D(\beta)}$$

Meaning of Rényi Dimensions

- at $\beta = 0$ have $\mathcal{I}_0(P) = -\log r(\epsilon)$ with $r(\epsilon) =$ min number of boxes of size ϵ covering set E so $D(0)$ is box-counting dimension (with grid)

$$D(0) = -\lim_{\epsilon \to 0} \frac{\log r(\epsilon)}{\log \epsilon}$$

- Shannon entropy dimension: at $\beta = 1$ limit of Rényi entropies is Shannon entropy $\mathrm{Sh}(P) = \mathcal{I}(P) = -\sum_i p_i \log p_i$

$$D(1) = \lim_{\epsilon \to 0} \frac{1}{\log \epsilon} \sum_{i=1}^{r(\epsilon)} p_i(\epsilon) \log p_i(\epsilon) = -\lim_{\epsilon \to 0} \frac{\mathrm{Sh}(P_\epsilon)}{\log \epsilon}$$

- $D(2)$ is called correlation dimension: it estimates effects of propagation of errors in iterates of a chaotic dynamical system; shown by Yorke, Grebogi, Ott that for certain classes of chaotic dynamical systems average period length $\sim \Delta^{-D(2)/2}$ (where Δ is a measure of precision)
- limit $\beta \to \infty$ of $D(\beta)$ measures scaling properties of region of E where measure μ most concentrated
- limit $\beta \to -\infty$ of $D(\beta)$ regions where least concentrated

• Note: these Rényi dimensions $D(\beta) = D_\mu(\beta)$ depend also on the measure μ used to compute $p_i = \mu(E \cap B_i)$ for the boxes B_i

Properties of Rényi Dimensions

- positivity $D(\beta) \geq 0$
- monotonicity $D(\beta') \leq D(\beta)$ for $\beta' \geq \beta$
- other relation: for $\beta' \geq \beta$ and $\beta\beta' > 0$

$$\frac{\beta' - 1}{\beta'} D(\beta') \geq \frac{\beta - 1}{\beta} D(\beta)$$

- limiting cases

$$D(\beta) \leq \frac{\beta}{\beta - 1} D(\infty) \quad \text{for} \quad \beta > 1$$

$$D(\beta) \geq \frac{\beta}{\beta - 1} D(-\infty) \quad \text{for} \quad \beta < 0$$

All of these properties follow from the corresponding properties of the Rényi entropy

Thermodynamic relations when box size $\epsilon \to 0$

- take $V = -\log \epsilon$ so $V \to \infty$
- dynamically homogeneous system if for large V quantities like entropy S or observables M^σ become proportional to V
- especially so that for β fixed and $V \to \infty$ ratios S/V or M^σ/V remain finite
- continuum limit: formally replace summations by integrals

$$\Psi = -\log \int_{\alpha_{\min}}^{\alpha_{\max}} \exp(-\beta \alpha V)\, \gamma(\alpha)\, d\alpha$$

- density of states $\gamma(\alpha)\, d\alpha$ umber of boxes with crowding index between α and $\alpha + d\alpha$
- expect asymptotic scaling behavior $\gamma(\alpha) = \epsilon^{-f(\alpha)}$ for some function $f(\alpha)$
- if $\gamma(\alpha) \sim \epsilon^{-f(\alpha)}$

$$\Psi = -\log \int_{\alpha_{\min}}^{\alpha_{\max}} \exp((f(\alpha) - \beta\alpha)V)\, d\alpha$$

- Saddle point approximation method
  - if integrand has only one maximum in interval then as $V \to \infty$ integral concentrated near the maximum
  - in general: want to evaluate

$$\mathcal{I} = \int \exp(F(x)V) dx$$

for $V \to \infty$, with some smooth function $F(x)$ with single max at $x = x_0$ (e.g. $F(x) = -(x - x_0)^2$)

  - with $F'(x_0) = 0$ and $F''(x_0) < 0$

$$\mathcal{I} \sim \int \exp((F(x_0) + \frac{1}{2}(x - x_0)^2 F''(x_0))V) \, dx$$

$$= \left(\frac{2\pi}{V \, F''(x_0)} \right)^{1/2} \exp(F(x_0)V)$$

  - so have $-\log \mathcal{I} \sim -F(x_0)V$

- Entropy Density
  - take $F(\alpha) = f(\alpha) - \beta\alpha$
  - $b := \alpha V$ mean value of bit number $\sum_i b_i p_i$ with $b_i = -\alpha_i \log \epsilon$
  - with saddle point approximation

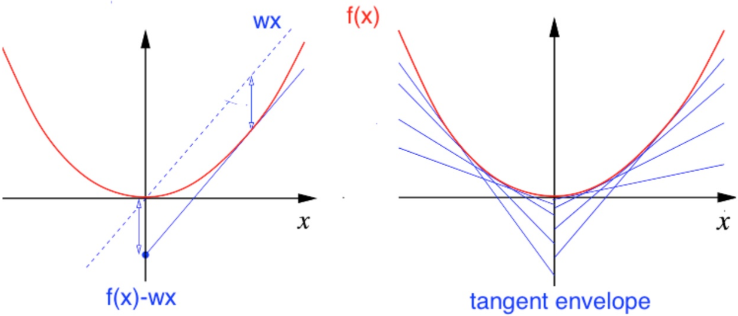
$$\Psi \sim (\beta\alpha - f(\alpha))V = \beta b - S$$

  - α mean crowding index is like a mean energy density so $\Psi = \beta F = \beta E - S = \beta\alpha V - S$
  - so function $f(\alpha)$ is entropy density

$$f(\alpha) = \lim_{V\to\infty} \frac{S}{V}$$

  - interpret $f(\alpha)$ as an estimate of the fractal dimension of a set of boxes of average pointwise dimension α
  - $f(\alpha) =$ spectrum of local dimensions (multifractal)

- Legendre transform
  - density $\tau(\beta)$
$$\tau(\beta) = \lim_{V\to\infty} \frac{\Psi}{V}$$
  - by previous relation of Ψ to Rényi entropy: function of Rényi dimension
$$\tau(\beta) = (\beta - 1)\, D(\beta)$$
  - Legedre transforms
$$S(b) = \beta b - \Psi(\beta) \quad \text{with} \quad \frac{d\Psi}{d\beta} = b, \quad \frac{dS}{d\beta} = \beta$$
$$f(\alpha) = \beta\alpha - \tau(\beta) \quad \text{with} \quad \frac{d\tau}{d\beta} = \alpha, \quad \frac{df}{d\alpha} = \beta$$
  - convex differentiable function $F(x)$ Legendre transform
$$F^*(w) := \sup_x (wx - F(x))$$



value of Legendre transform $F^*(w)$ is the negative of the y-intercept of the tangent line to the graph of F that has slope w

- take $\alpha(\beta)$ to be the value α where $\beta\alpha - f(\alpha)$ takes minimum
- from $\tau(\beta) = (\beta - 1)D(\beta)$ and Legendre transform get

$$\alpha(\beta) = D(\beta) + (\beta - 1)D'(\beta)$$

$$f(\alpha(\beta)) = D(\beta) + \beta(\beta - 1)D'(\beta)$$

- for $\beta = 0$ and $\beta = 1$ this gives

$$f(\alpha(0)) = D(0) = \alpha(0) + D'(0)$$

with $D(0)$ box-counting dimension

$$f(\alpha(1)) = D(1) = \alpha(1)$$

entropy dimension

Authenticating Jackson Pollock: an application of entropy and dimension

- R. Taylor, A. P. Micholich, and D. Jonas, *Fractal analysis of Pollock's drip paintings*, Nature 399 (1999) 422.
- S. Lyu, D. Rockmore, and H. Farid, *A digital technique for art authentication*, Proc. National Acad. of Science 101 (2004) 17006–17011.
- R. Taylor, R. Guzman, T. P. Martin, G. Hall, A. P. Micholich, D. Jonas, and C. A. Marlow, *Authenticating Pollock paintings using fractal geometry*, Pattern Recognition Letters 28 (2007) N.6, 695–702.
- D.J. Feng and Y. Wang, *A class of self-affine measures*, J. Fourier Anal. and Appl. 11 (2005) 107–124.
- J. Coddington, J. Elton, D. Rockmore, Y. Wang, *Multifractal analysis and authentication of Jackson Pollock paintings*

We focus here mostly on the approach of the last paper

Jackson Pollock's drip painting technique

Jackson Pollock, *N. 31*, 1950

in 2003 a set of 32 putative Pollock paintings were found by Alex Matter in the estate of his later father, Herbert Matter, who had been a friend of Jackson Pollock: are these authentic Pollock paintings or fakes?

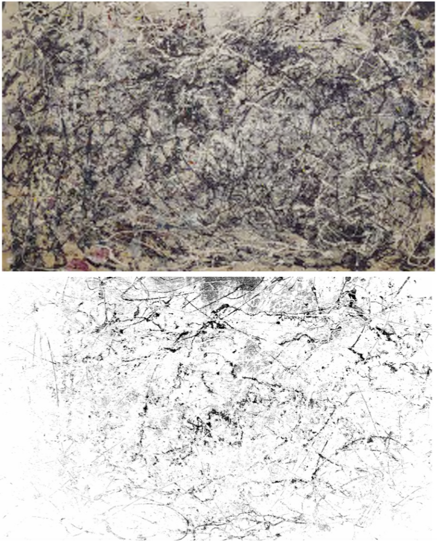
- there are several overlapping structures with different local dimensions: multifractal analysis separates out strata of varying local dimension and computes entropy dimension of each
- separate out different strata of the painting by image analysis
- use a box counting method: for $B_i \cap X \neq \emptyset$

$$S_\epsilon(\mu, X) = -\sum_{i=1}^{r(\epsilon)} \mu(B_i) \log \mu(B_i), \qquad C_{q,\epsilon}(X) = \sum_{i=1}^{r(\epsilon)} \mu(B_i)^q$$

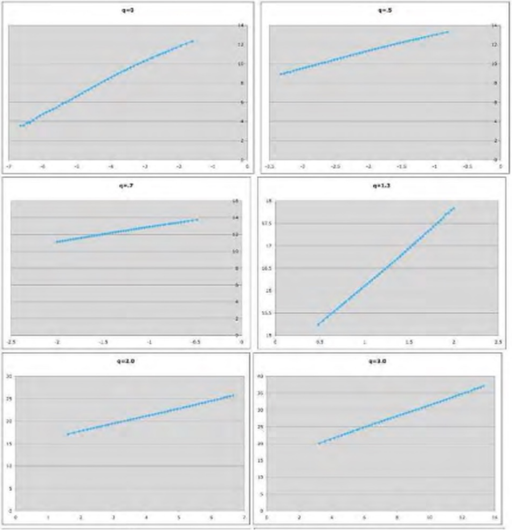
  where B_i are the grid boxes of size $\epsilon = 1/k$
- entropy (box-counting) dimension and Rényi (box-counting) dimensions

$$\dim_e(\mu, X) = D_\mu(1) = -\lim_{\epsilon \to 0} \frac{S_\epsilon(\mu, X)}{\log(\epsilon)}$$

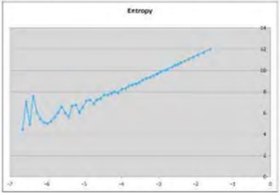
$$\dim_q(\mu, X) = D_\mu(\beta = q) = -\lim_{\epsilon \to 0} \frac{\log C_{q,\epsilon}(X)}{(1-q)\log \epsilon}$$

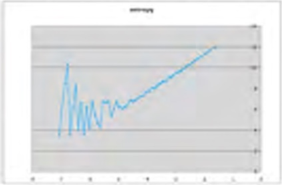


Jackson Pollock, *N. 1*, 1948 and extraction of a stratum



box-counting Rényi dimensions for different values of q for the stratum of Pollock's *N. 1*, 1948 (varying significantly, not a specific Pollock signature)





but entropy dimension seems to provide a more reliable signature and differs from the case of the disputed painting: this analysis suggests the paintings may be fakes