

Sharkovsky's Ordering and Chaos

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Ma4/104: Introduction to Fractal Geometry and Chaos

References for this lecture:

- K. Burns and B. Hasselblatt, *The Sharkovsky theorem: a natural direct proof*, Amer. Math. Monthly 118 (2011), no. 3, 229–244
- T.Y.Li and A.Yorke, *Period three implies chaos*, The American Mathematical Monthly, Vol. 82, No. 10. (Dec., 1975) 985–992
- O.M. Sharkovsky, *Co-existence of cycles of a continuous mapping of the line into itself*, Ukrain. Mat. Z. 16 (1964) 61–71
- B. Luque, L. Lacasa, F.J. Ballesteros, A. Robledo, *Feigenbaum Graphs: A Complex Network Perspective of Chaos*, PLoS ONE 6(9): e2241
- L. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, World Scientific, 2000

Period 3 implies Chaos

- T.Y.Li and A.Yorke, *Period three implies chaos*, The American Mathematical Monthly, Vol. 82, No. 10. (Dec., 1975) 985–992
 - considered the historical origin of Chaos Theory
 - if a continuous function $f : \mathcal{I} \rightarrow \mathcal{I}$ of an interval $\mathcal{I} \subset \mathbb{R}$ has a periodic point of period 3, then it has periodic points of any order $n \in \mathbb{N}$
 - there is also an uncountable subset of \mathcal{I} of points that are not even “asymptotically periodic”
 - this is a typical situation of *chaotic dynamics*
 - *chaos* = sensitive dependence on the initial conditions (starting points that are very close have very different behavior under iterates of the function)

Sharkovsky's theorem

- in fact the result of Li and Yorke is a special case of a previous much more general theorem of Sharkovsky
- Oleksandr M. Sharkovsky, *Co-existence of cycles of a continuous mapping of the line into itself*, Ukrainian Math. J. 16 (1964) 61–71
 - **Sharkovsky ordering**: there is an ordering of the natural numbers, different from usual ordering such that if a continuous function $f : \mathcal{I} \rightarrow \mathcal{I}$ has a periodic point of period n , then it has also periodic points of period k for all natural numbers k that follow n in the Sharkovsky ordering
 - the Sharkovsky ordering starts with 3 (hence Li–Yorke follows)

Sharkovsky Ordering

$3 \triangleright 5 \triangleright 7 \triangleright \dots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \dots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \dots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.$

- first all odd numbers in ascending order
- then all numbers $2 \cdot n$ with n odd in ascending order
- then all $2^2 \cdot n$ with n odd
- etc. for increasing powers $2^N \cdot n$ with ascending odd n
- then all powers of 2 in *descending* order
- last in the ordering is 1

odds, $2 \cdot$ odds, $2^2 \cdot$ odds, $2^3 \cdot$ odds, ..., $2^3 \cdot 1, 2^2 \cdot 1, 2 \cdot 1, 1.$

Some notation

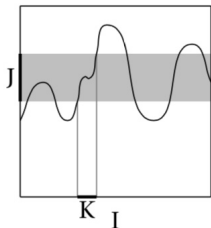
- an interval \mathcal{I} **covers** an interval \mathcal{J} (notation $\mathcal{I} \rightarrow \mathcal{J}$) with respect to continuous map f if $\mathcal{J} \subset f(\mathcal{I})$
- by intermediate value theorem $\mathcal{I} \rightarrow \mathcal{J}$ when f maps endpoints of \mathcal{I} to opposite sides of \mathcal{J}
- notation $\mathcal{I} \succrightarrow \mathcal{J}$ if $\mathcal{J} = f(\mathcal{I})$
- **\mathcal{O} -interval**: interval \mathcal{I} whose endpoints are part of a cycle (periodic orbit) \mathcal{O} of f
- **basic \mathcal{O} -interval**: \mathcal{O} -interval containing no other point of cycle \mathcal{O} besides endpoints (endpoints called adjacent in this case)
- knowing a cycle \mathcal{O} of f says how \mathcal{O} -intervals moved by the map (intermediate value theorem)
- in turn knowing how intervals are mapped gives other information about cycles

Intervals and Cycles

- if $[a_1, a_2] \rightarrow [a_1, a_2]$ (that is, $[a_1, a_2] \subseteq f([a_1, a_2])$) then f has a fixed point in $[a_1, a_2]$
 - since $[a_1, a_2] \rightarrow [a_1, a_2]$ there are $b_1, b_2 \in [a_1, a_2]$ with $f(b_1) = a_1$ and $f(b_2) = a_2$
 - so $f(b_1) - b_1 \leq 0 \leq f(b_2) - b_2$
 - intermediate value theorem: $f(x) - x = 0$ is satisfied somewhere between b_1 and b_2
- **points that follow a loop**
 - a loop of intervals \mathcal{J}_k for $k = 0, \dots, n - 1$ with $\mathcal{J}_k \rightarrow \mathcal{J}_{k+1}$ and $\mathcal{J}_{n-1} \rightarrow \mathcal{J}_0$
 - a point x follows the loop if $f^n(x) = x$ and $f^k(x) \in \mathcal{J}_k$ for all $k = 0, \dots, n - 1$

$$J_0 \overleftrightarrow{\cdots} J_{n-1}$$

- **existence of points that follow loops:** every loop has a point that follows it
 - if $\mathcal{I} \rightarrow \mathcal{J}$ there is a subinterval $\mathcal{K} \subseteq \mathcal{I}$ with $\mathcal{K} \rightarrow \mathcal{J}$

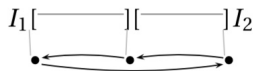
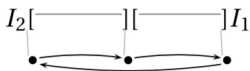


- thus have $\mathcal{K}_0 \rightarrow \mathcal{K}_1 \rightarrow \cdots \mathcal{K}_{n-1} \rightarrow \mathcal{J}_0$ with $\mathcal{K}_k \subseteq \mathcal{J}_k$
- $f^{\circ n}$ maps $\mathcal{K}_0 \subseteq \mathcal{J}_0$ to $\mathcal{J}_0 = f^{\circ n}(\mathcal{K}_0)$, so by previous fixed point statement there is $x \in \mathcal{K}_0$ with $f^{\circ n}(x) = x$
- this point x follows the loop

- **elementary loop** of n intervals: every point that follows it has period n (not a divisor of n)
- in particular existence of an elementary loop implies existence of a period n point
- what ensures existence of an elementary loop?
 - suppose loop is made of \mathcal{O} -intervals, when is it elementary?
 - points that follow loop are not in the cycle \mathcal{O} and the interior of \mathcal{J}_0 is disjoint from the other intervals $\mathcal{J}_1, \dots, \mathcal{J}_{n-1}$
 - there is a point x that follows the cycle, but it is not in \mathcal{O} so it must be in the interior of \mathcal{J}_0 , but then $f^{\circ k}(x) \in \mathcal{J}_k$ must be $f^{\circ k}(x) \neq x$ because the interior of \mathcal{J}_0 disjoint from \mathcal{J}_k so period cannot be shorter than n

Proof of Li-Yorke period three implies chaos

- period three intervals

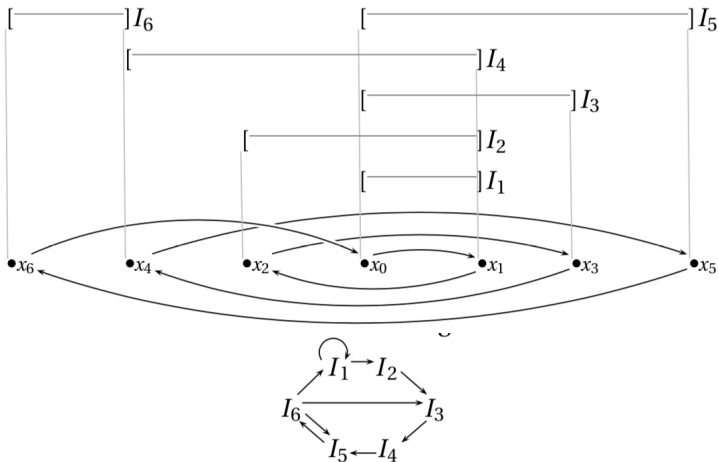


$$\Leftrightarrow I_1 \rightleftarrows I_2$$

- $I_1 \rightarrow I_1$ implies I_1 contains a fixed point
- by existence of points that follow the cycle there is a point of period 2 (endpoints are period three so they don't follow the cycle and interior of I_2 disjoint from I_1)
- consider a loop $I_2 \rightarrow I_1 \rightarrow I_1 \cdots \rightarrow I_1 \rightarrow I_2$ with $\ell - 1$ copies of I_1 with $\ell > 3$
- because endpoints are order 3 cycle \mathcal{O} they cannot stay in I_1 for more than two iterations of f
- so this is an elementary cycle of \mathcal{O} -intervals
- so it contains a periodic point of order exactly ℓ (and no divisor of ℓ) that follows the cycle

Another example: Period 7 implies periods of all following numbers in the Sharkovsky order

- just look at one possibility for \mathcal{O} -cycle and \mathcal{O} -intervals



- loop $I_1 \rightarrow I_1$ implies existence of a fixed point
- other loops that are elementary because of previous argument

$$I_6 \rightarrow I_5 \rightarrow I_6,$$

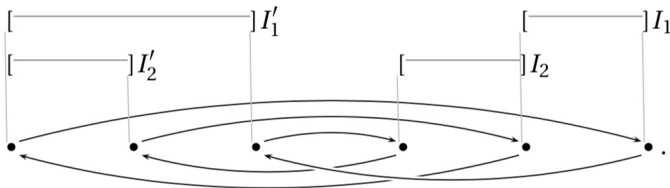
$$I_6 \rightarrow I_3 \rightarrow I_4 \rightarrow I_5 \rightarrow I_6,$$

$$I_6 \rightarrow I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow I_4 \rightarrow I_5 \rightarrow I_6,$$

$I_6 \rightarrow I_1 \rightarrow I_1 \rightarrow \dots \rightarrow I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow I_4 \rightarrow I_5 \rightarrow I_6$ with 3 or more copies of I_1 .

- these give existence of points of period 2, 4, 6, and then of any number that follows 7 in the Sharkovsky ordering

Another example (slightly different): Period 6 implies periods of all following numbers in Sharkovsky order



- new aspect here: a symmetry that maps the three points on the left to the three points on the right
- can use this to obtain information on the fixed points of f from the fixed points of f^2

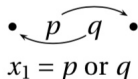
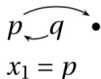
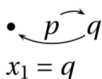
- f^2 has covering relations $I_1 \rightarrow I_1$, $I_1 \rightarrow I_2$ and $I_2 \rightarrow I_1$
- same as period three case, so f^2 has periodic points of all orders
- obtain periodic points for f from those of f^2
 - each occurrence of $I_1 \rightarrow$ in loops for f^2 replaced by $I_1 \rightarrow I'_1 \rightarrow$ for f
 - each occurrence of $I_2 \rightarrow$ for f^2 replaced by $I_2 \rightarrow I'_2 \rightarrow$ for f
 - so a k -loop for f^2 becomes a $2k$ -loop for f
 - these $2k$ loops for f are still elementary if corresponding k -loop for f^2 was, because if a point has period exactly k for f^2 and follows the loop, under f the point keeps alternating between the two sides of figure, so period is $2k$
 - obtain that f has all even periods (and a fixed point)
 - these are all numbers following 6 in the Sharkovsky ordering

Proof of Sharkovsky's Theorem

- assume \mathcal{O} is a cycle for f of order m
- show that f has periodic points of periods all numbers following m in Sharkovsky ordering
- **First Step:** points that switch sides
 - given non-trivial cycle \mathcal{O} , pick rightmost point p of \mathcal{O} such that $f(p) > p$; let q be next point of \mathcal{O} to the right of p , then $f(p) \geq q$ and $f(q) \leq p$
 - take $\mathcal{I} = [p, q]$, this interval satisfies the covering relation $\mathcal{I} \rightarrow \mathcal{I}$
 - pick a point c in the interior of \mathcal{I}
 - a point x switches sides (with respect to c) if x and $f(x)$ are on opposite sides of c
 - if all points of \mathcal{O} switch sides then \mathcal{O} is even (m is even)

• **Second Step:** assume that not all points of \mathcal{O} switch sides and construct a sequence x_0, \dots, x_k of points in \mathcal{O} that “spiral out as fast as possible”

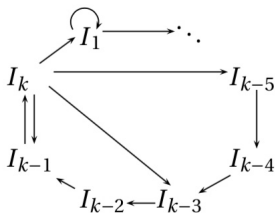
- start with x_0, x_1 the endpoints of \mathcal{I} with $f(x_1) \neq x_0$ (possible since $\mathcal{O} \neq \{x_0, x_1\}$ otherwise all points of \mathcal{O} switch sides): possibilities



- continue inductively: if all points of \mathcal{O} in the interval of endpoints x_i and c switch sides then x_{i+1} is point of \mathcal{O} in image of this interval furthest from c ; otherwise stop at x_i
- consecutive terms in this sequence on opposite sides of c
- also can see from the construction that x_{i+1} is further away from c than x_i
- all points in this sequence are distinct and the sequence terminates at some x_k with $k < m$ (otherwise orbit closes up)

• **Third Step:** assume that not all points of \mathcal{O} switch sides and show in this case all numbers ℓ that follow m in Sharkovsky order are periods **Sketch of proof:**

- use the previous sequence of points x_0, \dots, x_k to construct a sequence of intervals I_1, \dots, I_k satisfying the covering relations



- check also using previous argument that all these loops are elementary

$$I_1 \rightarrow I_1;$$

$$I_k \rightarrow I_{k-(l-1)} \rightarrow I_{k-(l-2)} \rightarrow \dots \rightarrow I_{k-2} \rightarrow I_{k-1} \rightarrow I_k \text{ for even } l \leq k;$$

$$I_k \rightarrow I_1 \rightarrow I_1 \rightarrow \dots \rightarrow I_1 \rightarrow I_2 \rightarrow \dots \rightarrow I_{k-1} \rightarrow I_k \text{ with } j \text{ occurrences of } I_1.$$

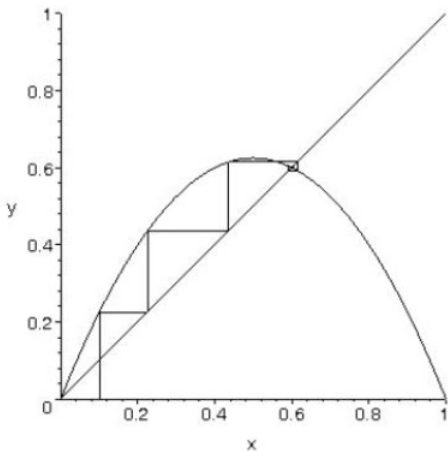
- the elementary loop $I_1 \rightarrow I_1$ implies existence of a fixed point
- the second family of elementary loops implies the existence of periodic points of all *even* orders $\ell \leq k$
- third family of elementary loops imply existence of periodic points of order any $\ell \geq k$ (except possibly m)
- note that these are all the numbers that follow m in the Sharkovsky ordering

- **Fourth Step:** the case where all points of \mathcal{O} switch sides
 - use an induction argument on the length m of \mathcal{O}
 - suppose proved for all numbers less than m (in the usual ordering of \mathbb{N})
 - consider an m -cycle \mathcal{O} : if there is a point that does not switch sides then result established by previous argument
 - if all points switch sides then case similar to the period 6 examples seen earlier: all points switch sides and m has to be even
 - the second iterate f^2 has an $m/2$ -cycle for which the result is proved (induction hypothesis)
 - so f^2 has periods all numbers following $m/2$ in Sharkovsky ordering
 - use the same argument as in the period 6 example to show f then has elementary $2k$ -loop for each elementary k -loop of f^2 so get result for m

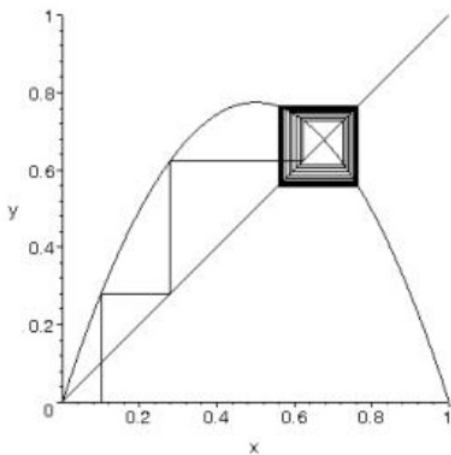
Logistic Map

$$f(x) = \mu x(1 - x)$$

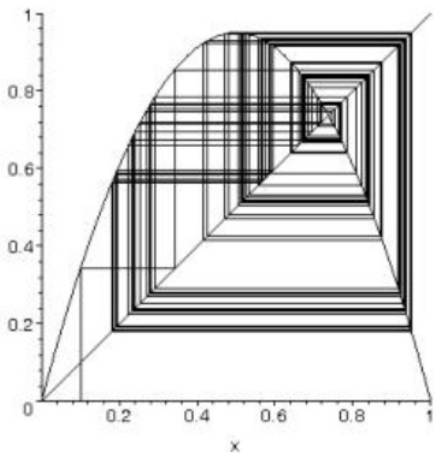
- orbits $f(x_0), f(f(x_0)), \dots, f^{\circ n}(x_0), \dots$
- fixed points and periodic points (depending on the value of the parameter $\mu > 0$)
- increase in complexity of periodic points structure then transition to chaos
- starts with a period doubling cascade (climbing the Sharkovsky ordering from the end)
- then reach a limit point of the parameter μ where transition to chaos (all periods)
- for higher values of μ substructures that reproduce the cascade



an orbit of the logistic map for $\mu = 2.5$: attractive fixed point

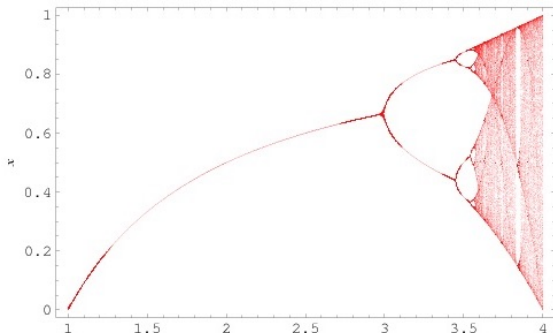


an orbit of the logistic map for $\mu = 3.1$: structure of attractor set, periodic orbits



an orbit of the logistic map for $\mu = 3.8$: chaotic dynamics

Logistic Equation Bifurcation Diagram



Plot as a function of the parameter μ the attractor set of the orbits $f^{\circ n}(x_0)$: for small μ single fixed point, then period 2 attractor appears, then period 4, ... cascade of period doubling and transition to chaotic region with attractor in bands rather than points and periodic points of all orders; above transition to chaos regions with splitting of bands into 2^n bands and orbits visiting different bands (like periodic orbits)

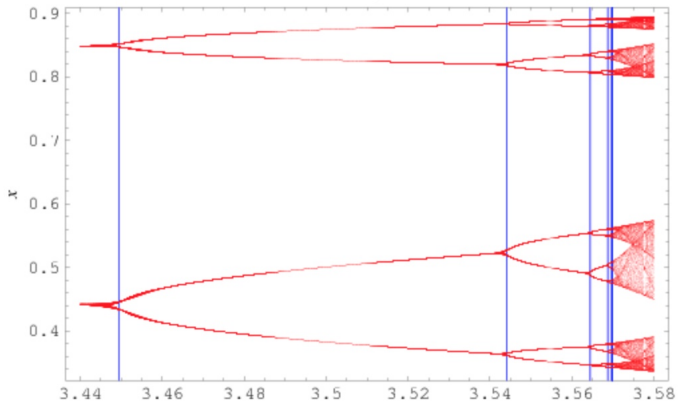
Compute where new periodic orbits appear as μ grows

- Example: $f(f(x)) = x$ gives an equation for the period two points

$$x_{\pm} = \frac{1}{2} \left((1 + \mu^{-1}) \pm \mu^{-1} \sqrt{(\mu - 3)(\mu + 1)} \right)$$

these x_{\pm} are real numbers only for $\mu \geq 3$ so occurrence of a period two orbit only possible for $\mu > 3$

- similarly can see the sequence μ_n of values of the parameter where orbits of period 2^n first appear



the sequence μ_n where period doubling occurs (blue lines)

Occurrence of period three

- to identify where there can be a non-trivial three-cycle (eliminating the trivial case of a fixed point)

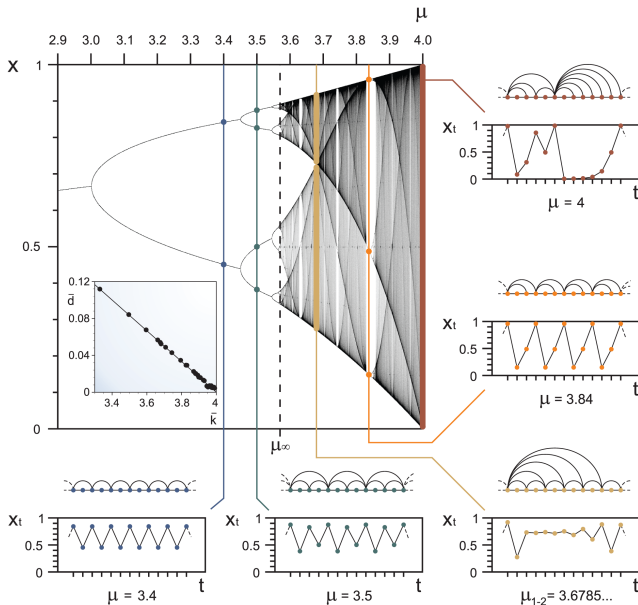
$$\frac{f^{\circ 3}(x) - x}{f(x) - x} = 0$$

- get polynomial equation in x with μ -dependent coefficients
- check when solutions are real numbers: for $\mu \leq \mu_F = 1 + 2\sqrt{2} = 3.828427 \dots$ imaginary roots
- for $\mu = \mu_F$ two of the roots become real
- μ_F solution to a discriminant equation $(\mu^2 - 5\mu + 7)^2(\mu^2 - 2\mu - 7)^3(1 + \mu + \mu^2)^2 = 0$
- at $\mu = \mu_F$ two roots coincide that then separate \Rightarrow three-cycle starts at μ_F
- a simpler derivation of the three-cycle onset in P. Saha, S.H. Strogatz, *The Birth of Period Three*, Math. Mag. 68 (1995) 42–47

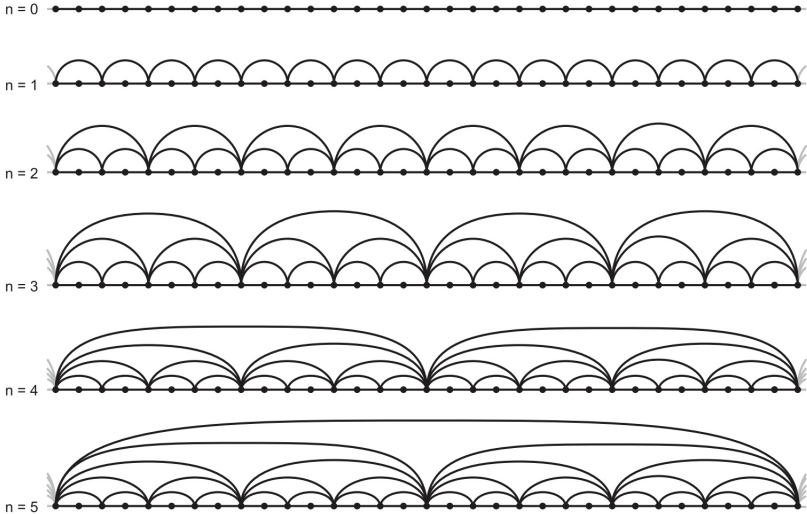
Feigenbaum Graphs

- given a sequence $\{x_k\}$ or a time series $\{x_t\}$
- one vertex for each datum k
- edge between vertices i and j if $x_i, x_j > x_n$ for all $i < n < j$
- for a fixed value of the parameter μ consider iterations of the logistic maps
- initial transient phase then approach the attractor
- form corresponding Feigenbaum graph with successive values of iterates

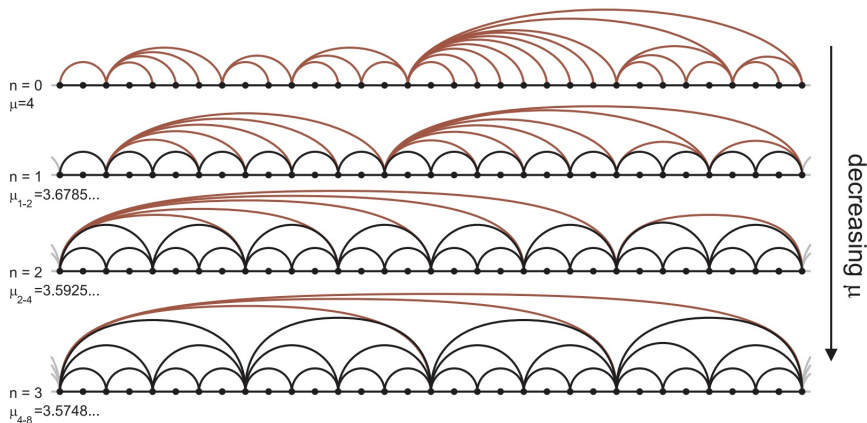
Feigenbaum Graphs of the Logistic Map



Period Doubling Cascade and Feigenbaum Graphs



Aperiodic Feigenbaum Graphs above the Transition to Chaos

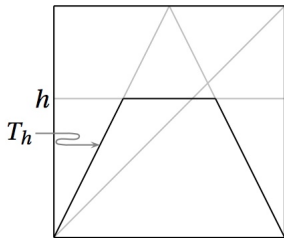


For $n > 0$ phase space partitioned into 2^n disconnected chaotic bands; the orbit visits each chaotic band in the same order as in the periodic region; order of visits gives structure in the Feigenbaum graphs (black edges) similar to the period-doubling cascade case

Tent Map realization of Sharkovsky's Theorem

- **Sharkovsky's Realization Theorem**: every tail of the Sharkovsky Ordering is the set of periods for a continuous map $f : \mathcal{I} \rightarrow \mathcal{I}$
- L. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, World Scientific, 2000
 - explicitly exhibiting a family of maps that realizes tails of the Sharkovsky Ordering as set of periods
 - **truncated tent maps** $T_h : [0, 1] \rightarrow [0, 1]$

$$T_h(x) = \min\left\{h, 1 - 2\left|x - \frac{1}{2}\right|\right\}, \quad \text{for } 0 \leq h \leq 1$$



sketch of proof of realization by truncated tent maps

- T_0 has a single fixed point at $x = 0$
- T_1 has a 3-cycle $\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$ hence all Sharkovsky ordering
- if $h \leq h'$ a cycle \mathcal{O} of T_h with $\mathcal{O} \subset [0, h)$ is a cycle of $T_{h'}$
- if $h \leq h'$ a cycle \mathcal{O} of $T_{h'}$ with $\mathcal{O} \subset [0, h]$ is a cycle of T_h
- $\max \mathcal{O} =$ largest x in the cycle \mathcal{O}

$$h(m) := \min\{\max \mathcal{O} \mid \mathcal{O} \text{ and } m\text{-cycle of } T_1\}$$

- T_h has an ℓ -cycle $\mathcal{O} \subset [0, h)$ iff $h(\ell) < h$ (previous property with $h' = 1$)
- orbit of $h(m)$ is an m -cycle for $T_{h(m)}$ and all other cycles for $T_{h(m)}$ contained in $[0, h(m))$
- $T_{h(m)}$ has an ℓ -cycle in $[0, h(m))$ for every ℓ following m in the Sharkovsky ordering and $h(\ell) < h(m)$
- $h(\ell) < h(m)$ iff ℓ follows m in Sharkovsky ordering
- for any $m \in \mathbb{N}$ the set of periods of $T_{h(m)}$ is the tail of the Sharkovsky ordering starting with m