Self-Similarity

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Contractions

- (X, d_X) and (Y, d_Y) metric spaces
  - Lipschitz map $f : X \to Y$ such that
    $$d_Y(f(x), f(y)) \leq C\, d_X(x, y)$$
    for all $x, y \in X$, *Lipschitz constant* $C > 0$
  - Lipschitz maps are continuous and in fact absolutely continuous: $\forall \epsilon > 0\ \exists \delta$ indep of $x \in X$ such that $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon)$
  - Contraction: Lipschitz map with Lipschitz constant $C < 1$

Complete metric spaces

- Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in metric space (X, d)

$$\forall \epsilon > 0 \;\; \exists N = N(\epsilon) \in \mathbb{N} \; : \quad d(x_n, x_m) < \epsilon, \quad \forall n, m \geq N$$

- by triangle inequality of metric every convergent sequence is Cauchy, but in general not viceversa
- complete metric space (X, d) is all Cauchy sequences in X converge
- Example: $\mathbb{Q}$ with $d(x, y) = |x - y|$ not complete but $\mathbb{R}$ completion

Contractions and fixed points

• (X, d) *complete* metric space, $f : X \to X$ contraction, then f has a unique *fixed point* $x \in X$ with $f(x) = x$

- pick an arbitrary point $x_0 \in X$ and consider the orbit under iterates $x_n = f(x_{n-1})$

$$x_n = f^n(x_0) = \underbrace{f \circ \cdots \circ f}_{n-\text{times}}(x_0)$$

- then have $d(x_{n+1}, x_n) \leq C^n d(x_1, x_0)$ by Lipschitz property
- for all $m > n$ then have

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq (C^n + C^{n+1} + \cdots + C^m) d(x_1, x_0)$$

$$\leq C^n(1 + \cdots + C^{m-n}) d(x_1, x_0) \leq \frac{C^n}{1 - C} d(x_1, x_0)$$

- by contraction property $C^n \to 0$ so $\{x_n = f^n(x_0)\}_n$ Cauchy sequence $d(x_m, x_n) \to 0$
- (X, d) complete so Cauchy sequence converges: $\exists x \in X$ with $\lim_{n\to\infty} x_n = x$
- also $\lim_{n\to\infty} x_{n+1} = x$ but $x_{n+1} = f(x_n)$ and by continuity $\lim_n f(x_n) = f(x)$ so fixed point $f(x) = x$
- uniqueness because if $x \neq y$ with $f(x) = x$ and $f(y) = y$ would have $0 \neq d(x, y) = d(f(x), f(y))$ but contraction so always $d(f(x), f(y)) < d(x, y)$ for $x \neq y$

- distance from fixed point
  - contraction fixed point $x = f(x)$, for all $y \in X$

$$d(x, y) \leq \frac{d(y, f(y))}{1 - C}$$

  - same as before start with $x_0 = y$

$$d(x_n, y) \leq \sum_{j=1}^{m} d(x_j, x_{j-1}) \leq (\sum_{j=0}^{n-1} C^j) d(f(y), y) \leq \frac{d(f(y), y)}{1 - C}$$

and $d(x_n, y) \to d(x, y)$ so same inequality for $d(x, y)$

Parametric version

- continuous $f : S \times X \to X$ parameter space S and (X, d) complete (and (S, d_S) metric space)
- for fixed $s \in S$

$$d(f(s, x_1), f(s, x_2)) \leq C\, d(x_1, x_2)$$

with $0 < C < 1$ independent of $s \in S$

- $f_s : X \to X$ by $f_s(x) = f(s, x)$
- unique fixed point x_s
- function $\phi : S \to X$ with $\phi(s) = x_s$ fixed point of f_s
- ϕ continuous: if $d_S(s, t) < \delta$ have $d(f_s(x), f_t(x)) < \epsilon$ by continuity of $f : S \times X \to X$; also $d(y, x_t) \leq (1 - C)^{-1} d(y, f_t(y))$ so that

$$d(x_s, x_t) \leq \frac{d(x_s, f_t(x_s))}{1 - C} = \frac{d(f_s(x_s), f_t(x_s))}{1 - C} \leq \frac{\epsilon}{1 - C}$$

Hausdorff metric on non-empty compact sets

• (X, d) complete metric space, $\mathcal{H}(X)$ set of all non-empty compact subsets of X

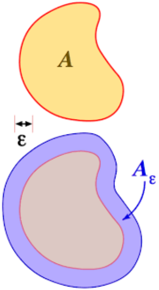
- for $A \in \mathcal{H}(X)$ and $\epsilon > 0$ set

$$A_\epsilon := \{x \in X \,|\, d(x,y) \leq \epsilon, \quad \text{for some } y \in A\}$$

- define $d(x, A) := \inf_{y \in A} d(x,y)$ (in fact min since A compact)

$$A_\epsilon = \{x \in X \,|\, d(x, A) \leq \epsilon\}$$

- ϵ-collar of A (or sometimes called ϵ-parallel body of A)



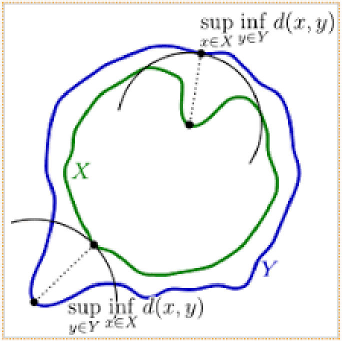
- construction of the Hausdorff distance, $A, B \in \mathcal{H}(X)$
  - $d(A,B) := \max_{x\in A} d(x,B)$ (*not symmetric*)

$$d(A,B) \le \epsilon \quad \text{iff} \quad A \subset B_\epsilon$$

  - symmetrize: $\delta(A,B) := \max\{d(A,B), d(B,A)\}$

$$\delta(A,B) = \inf\{\epsilon > 0 \,|\, A \subset B_\epsilon \text{ and } B \subset A_\epsilon\}$$

  - $(\mathcal{H}(X), \delta)$ is a metric space



- metric space properties
  - symmetric by construction
  - $\delta(A,B) = 0$ means every point of A at zero distance from B (in closure of B) but B compact hence closed so in B and viceversa so $A = B$
  - triangle inequality: A, B, C, for any $a \in A$

$$d(a,B) = \min_{b\in B} d(a,b) \leq \min_{b\in B}(d(a,c) + d(c,b))$$

for any $c \in C$

$$= d(a,c) + \min_{b\in B} d(c,b) = d(a,c) + d(c,B) \leq d(a,c) + d(C,B)$$

minimizing over $c \in C$ gives

$$d(a,B) \leq d(a,C) + d(C,B)$$

then take max over $a \in A$

Complete metric space $(\mathcal{H}(X), \delta)$

- sketch of argument for completeness:
  - A_n Cauchy sequence of non-empty compact sets in $\mathcal{H}(X)$

$$\delta(A_n, A_m) < \epsilon, \quad \forall n, m \geq N$$

  - define $A \subseteq X$ as set of points $x \in X$ such that $\exists x_n \in A_n$ with $x_n \to x$ in (X, d)
  - the set A is non-empty and compact
  - also $\lim_n A_n = A$ in the Hausdorff metric

Contractions in the Hausdorff metric

- $f : X \to X$ contraction with Lipschitz constant $0 < C < 1$
- define $F : \mathcal{H}(X) \to \mathcal{H}(X)$ by $F(A) = \{y \in X \,|\, y = f(x),\ x \in A\}$, compact since image of compact under continuous function
- Hausdorff distance

$$d(F(A), F(B)) = \max_{a \in A} \min_{b \in B} d(f(a), f(b))$$

$$\leq \max_{a \in A} \min_{b \in B} C\, d(a, b) = C\, d(A, B)$$

same for symmetric $d(F(B), F(A))$ so that contraction

$$\delta(F(A), F(B)) \leq C\, \delta(A, B)$$

More general contractions in the Hausdorff metric

- $\{f_1, \dots, f_n\}$ a family of contractions $f_i : X \to X$ on a complete metric space (X, d), with Lipschitz constants $\{C_1, \dots, C_n\}$, $0 < C_i < 1$
- define $F : \mathcal{H}(X) \to \mathcal{H}(X)$ by

$$F(A) = f_1(A) \cup \cdots \cup f_n(A)$$

- same argument as before: $F : \mathcal{H}(X) \to \mathcal{H}(X)$ is a contraction in the Hausdorff metric, show for two f_1, f_2 then inductively for n

$$\delta(F(A), F(B)) = \delta(f_1(A) \cup f_2(A), f_1(B) \cup f_2(B))$$

$$\leq \max\{\delta(f_1(A), f_1(B)), \delta(f_2(A), f_1(B))\} \leq \max\{C_1, C_2\}\delta(A, B)$$

contraction with constant $C = \max\{C_1, C_2\}$

- so $F(A) = f_1(A) \cup \cdots \cup f_n(A)$ contraction with constant $C = \max_i C_i$

Self-Similarity

- $\{f_1, \ldots, f_n\}$ a family of contractions $f_i : X \to X$ on a complete metric space (X, d) with constants $0 < C_i < 1$
- contraction $F : \mathcal{H}(X) \to \mathcal{H}(X)$ with $F(A) = f_1(A) \cup \cdots \cup f_n(A)$ and constant $C = \max C_i$
- since $(\mathcal{H}(X), \delta)$ also complete contraction $F : \mathcal{H}(X) \to \mathcal{H}(X)$ has a unique fixed point, a non-empty compact set $S \subseteq X$ such that $F(S) = S$,

$$S = f_1(S) \cup \cdots \cup f_n(S)$$

- self-similar set: $S \subseteq X$ non-empty compact set such that $S = f_1(S) \cup \cdots \cup f_n(S)$ for a family of contractions on (X, d) (fixed point of $F : \mathcal{H}(X) \to \mathcal{H}(X)$)

Convergence and construction of self-similar sets

- to construct self-similar sets consider a family of contractions $\{f_1, \dots, f_n\}$ on (X, d) and associated contraction $F : \mathcal{H}(X) \to \mathcal{H}(X)$ with $F(A) = f_1(A) \cup \cdots \cup f_n(A)$
- by fixed point theorem starting with any $A_0 \in \mathcal{H}(X)$ the iterations

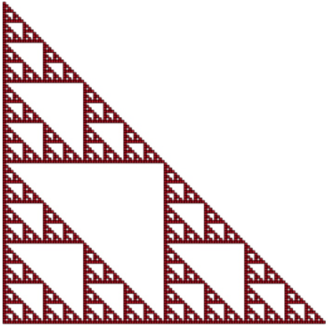
$$A_n := F^n(A_0) = \underbrace{F \circ \cdots \circ F}_{n-\text{times}}(A_0)$$

  converge to fixed point $S = F(S)$ self-similar set
- so $F^n(A_0)$ give good approximate description of self-similar set S
- note that depending on the choice of A_0 convergence to S may be slow or fast, so $F^n(A_0)$ may be a good or a bad approximation of S depending on A_0
- iterated function system (IFS) $\{f_1, \dots, f_n\}$ in many examples given by affine maps

Issues with Speed of Convergence

- fixed point theorem ensures for any choice of initial set $A_0 \in \mathcal{H}(X)$ the iterations $F^n(A_0)$ for an iterated function system $\{f_1, \dots, f_n\}$ converge to fixed point $A = F(A)$
- but... speed of convergence can be very different depending on the choice of the initial set A_0
- there is usually a "good choice" of A_0, which is suggested by the form of the contractions f_i, for which after only a few iteration $F(A_0)$, $F^2(A_0) \cdots$ one can see a very good approximation of the fixed point A
- other choices of A_0 may require many iterates before one can see a good approximation of A
- so in explicit constructions of fractals the choice of a good A_0 is an essential part

Example: Sierpinski gasket



The Sierpinski Gasket is obtained from IFS $\mathcal{F} = \{f_1, f_2, f_3\}$ where

$$f_1(x) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$f_2(x) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix},$$

$$f_3(x) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}.$$

Each f_i is a contraction with $\lambda = \frac{1}{2}$.

Starting with initial set A_0 given by the large triangle gives a very good approximation after just two or three iterations

Self-similarity dimension

- $A = F(A) = f_1(A) \cup \cdots \cup f_n(A)$ fixed point with $\{f_1, \ldots, f_n\}$ contractions on (X, d) with Lipschitz constants $\{\lambda_1, \ldots, \lambda_n\}$ with $0 < \lambda_k < 1$
- the function $j(s) := \sum_{k=1}^n \lambda_k^s$ for $s \in \mathbb{R}_+$ is monotonically decreasing with $j(0) = n > 1$ and $\lim_{s \to \infty} j(s) = 0$
- so there is a *unique* point $s_0 > 0$ where $j(s_0) = 1$
- this unique solution of

$$\sum_{k=1}^n \lambda_k^s = 1$$

  is the self-similarity dimension $s_0 = \dim_{\text{self-sim}}(K)$
- relation of self-similarity dimension to Hausdorff dimension will be discussed later
- Note a possible problem: same K can be realized with different sets of contractions so $\dim_{\text{self-sim}}(K)$ is really $\dim_{\text{self-sim}}(K, F)$ depending on F not only on K
- redundant set $\{f_1, \ldots, f_n\}$ gives larger $\dim_{\text{self-sim}}(K, F)$

Example: unequal intervals Cantor set

• Examples taken from Andrejs Treibergs (U. Utah)
http://www.math.utah.edu/∼treiberg/FractalSlides.pdf

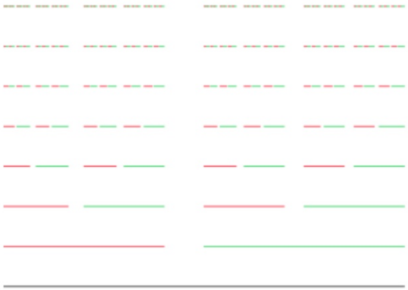


Figure: Cantor Set with Unequal Intervals

This Cantor set is obtained from IFS $\mathcal{F} = \{f_1, f_2\}$ on $\mathbb{R}$ where

$$f_1(x) = .4x,$$
$$f_2(x) = .5x + .5$$

Each f_i's are contractions with $\lambda_1 = .4$ and $\lambda_2 = .5$.

Example: Sierpinski Gasket

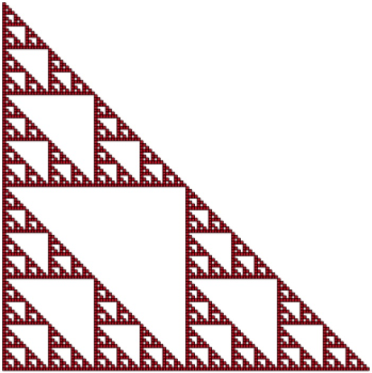


Figure: Sierpinski Gasket

The Sierpinski Gasket is obtained from IFS $\mathcal{F} = \{f_1, f_2, f_3\}$ where

$$f_1(z) = \frac{1}{2}z,$$
$$f_2(z) = \frac{1}{2}z + \frac{1}{2},$$
$$f_3(z) = \frac{1}{2}z + \frac{i}{2}.$$

Each f_i is a contraction with $\lambda = \frac{1}{2}$. Thus

$$1 = 3\left(\frac{1}{2}\right)^s$$

or $\dim_H(A) = \dfrac{\ln 3}{\ln 2} \cong 1.58$

computation of self-similarity dimension: solution $s \geq 0$ of $\sum_i \lambda_i^s = 1$ contraction rates λ_i (we'll see later why same as Hausdorff dimension)

Example: Koch Snowflake

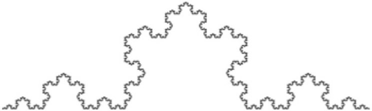


Figure: von Koch Curve

The von Koch Curve is obtained from IFS $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$ where in complex notation $z = x + iy$,

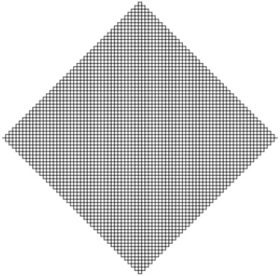
$$f_1(z) = \frac{1}{3}z,$$
$$f_2(z) = \frac{e^{\pi i/3}}{3}z + \frac{1}{3}$$
$$f_3(z) = \frac{e^{-\pi i/3}}{3}z + \frac{e^{\pi i/3}+1}{3}$$
$$f_4(z) = \frac{1}{3}z + \frac{2}{3}.$$

Each contraction has $\lambda = \frac{1}{3}$. Thus

$$1 = 4\left(\frac{1}{3}\right)^s$$

or $\dim_H(A) = \dfrac{\ln 4}{\ln 3} \cong 1.26$

Example: Peano Curve



The Peano Curve is obtained from IFS $\mathcal{F} = \{f_1, \ldots, f_9\}$ where

$$f_1(z) = \frac{1}{3}z,$$
$$f_2(z) = \frac{i}{3}z + \frac{1}{3}$$
$$f_3(z) = \frac{1}{3}z + \frac{1+i}{3}$$

$$f_4(z) = -\frac{i}{3}z + \frac{2+i}{3}$$
$$f_5(z) = -\frac{1}{3}z + \frac{2}{3}$$
$$f_6(z) = -\frac{i}{3}z + \frac{1}{3}$$
$$f_7(z) = \frac{1}{3}z + \frac{1-i}{3}$$
$$f_8(z) = \frac{i}{3}z + \frac{2-i}{3}$$
$$f_9(z) = \frac{1}{3}z + \frac{2}{3}$$

The contractions all have $\lambda_i = \frac{1}{3}$. Thus

$$1 = 9\left(\frac{1}{3}\right)^s$$

or $\dim_H(A) = \frac{\ln 9}{\ln 3} = 2.$

Example: Levy Dragon Curve

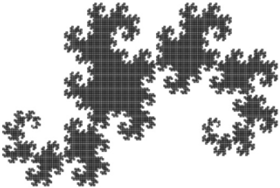


Figure: Levy Dragon

Levy's Dragon Curve is obtained from IFS $\mathcal{F} = \{f_1, f_2\}$ where

$$f_1(z) = -\frac{1+i}{2}z + \frac{1+i}{2}$$
$$f_2(z) = \frac{1-i}{2}z + \frac{1+i}{2}$$

Both contractions have $\lambda_i = \frac{1}{\sqrt{2}}$. Thus

$$1 = 2\left(\frac{1}{\sqrt{2}}\right)^s$$

or $\dim_H(A) = \dfrac{\ln 2}{\ln \sqrt{2}} = 2.$

Example: Minkowski Curve

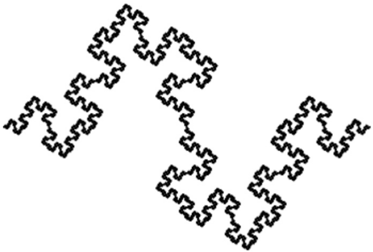


Figure: Minkowski Curve

$$f_1(z) = \frac{1}{4}z,$$
$$f_2(z) = \frac{i}{4}z + \frac{1}{4}$$
$$f_3(z) = \frac{1}{4}z + \frac{1+i}{4}$$

The Minkowski Curve is obtained from IFS $\mathcal{F} = \{f_1, \ldots, f_8\}$ where

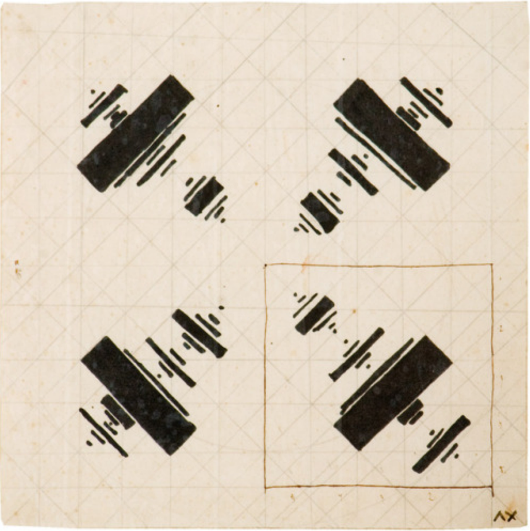
$$f_4(z) = -\frac{i}{4}z + \frac{2+i}{4}$$
$$f_5(z) = -\frac{i}{4}z + \frac{1}{2}$$
$$f_6(z) = \frac{1}{4}z + \frac{2-i}{4}$$
$$f_7(z) = \frac{i}{4}z + \frac{3-i}{4}$$
$$f_8(z) = \frac{i}{4}z + \frac{3}{4}$$

All $\lambda_i = \frac{1}{4}$. Thus

$$1 = 8\left(\frac{1}{4}\right)^s$$

or $\dim_H(A) = \frac{\ln 8}{\ln 4} = 1.5.$

What about this?



Lazar Khidekel, *Kinetic Elements of Suprematism*, 1920