Quadratic Maps: Mandelbrot and Julia Sets

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Some References

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Quadratic maps

- every $f : \mathbb{C} \to \mathbb{C}$ quadratic map $f(z) = \alpha z^2 + \beta z + \gamma$ with $\alpha \neq 0$ is conjugate to $f_c(z) = z^2 + c$
- can see explicitly conjugacy h: automorphism of $\mathbb{P}^1(\mathbb{C})$ that has to send ∞ to itself so of the form $h(z) = kz + \ell$

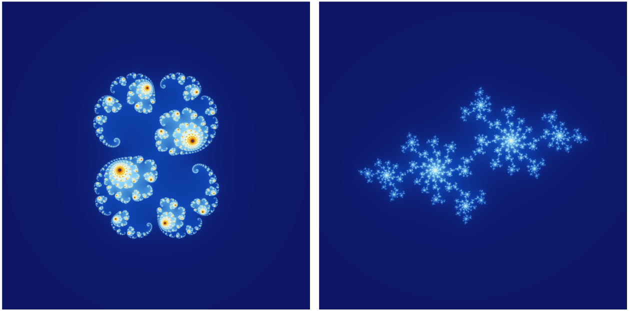
$$hf(z) = k(\alpha z^2 + \beta z + \gamma) + \ell \quad \text{and} \quad f_c h(z) = (kz + \ell)^2 + c$$

- equal for all $z \in \mathbb{C}$ when $k\alpha = k^2$, $k\beta = 2k\ell$, $k\gamma + \ell = \ell^2 + c$
- get $k = \alpha$, $\ell = \beta/2$ and $c = \alpha\gamma + \beta/2 - \beta^2/4$
- also $f_c(z) = z^2 + c$ is conjugate to a logistic map $p_\lambda(z) = \lambda z(1 - z)$ if $c = \lambda/2 - \lambda^2/4$
- f_c form better for looking at critical points, p_λ form better for fixed points

Julia Sets

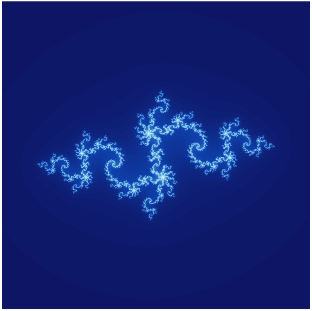
- $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ non-constant holomorphic function: rational function $f(z) = p(z)/q(z)$ ratio of complex polynomials
- assume p, q no common roots and at least one of them has $\deg > 1$
- Julia set $J(f)$: smallest set containing at least 3 points invariant under f
- closure of set of repelling periodic points of f
- if $f(z)$ is a polynomial the Julia set $J(f) =$ boundary of the set of points whose orbits under iterations of f remain bounded
- Example: $f(z) = z^2$ Julia set is the unit circle
- complement of the Julia set $J(f)$ is the Fatou set $F(f)$ components of $F(f)$ Fatou domains

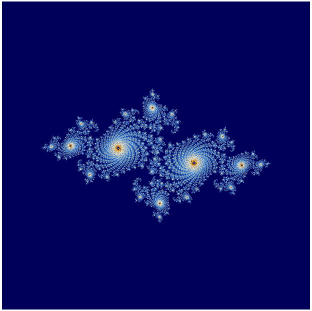
Julia Sets of Quadratic Maps $f_c(z) = z^2 + c$



Julia sets for $c = 0.285 + 0.01i$, $c = -0.70176 - 0.3842i$

Julia Sets of Quadratic Maps $f_c(z) = z^2 + c$





Julia sets for $c = -0.835 - 0.2321i$, $c = -0.7269 + 0.1889i$

Böttcher coordinate:

- $h(z)$ analytic function on $h : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$
- z_0 superattractive fixed point if $h'(z_0) = 0$
- $h(z) = z_0 + \alpha(z - z_0)^n + O((z - z_0)^{n+1})$ for some $n \geq 2$
- Böttcher equation: $F(h(z)) = (F(z))^n$
- solution $F(z)$ of Böttcher equation on a neighborhood of z_0: Böttcher coordinate
- the Böttcher coordinate conjugates $h(z)$ near the superattractive fixed point to the function z^n
- existence of solutions: Joseph Ritt 1920

Mandelbrot Set

- The subset of $\mathbb{C}$ given by values of the parameter c for which the Julia set $J(f_c)$ is connected
- Equivalent: $\mathcal{M}$ is the set of values of the parameter c for which the orbit $f_c^n(0)$ does not go to ∞

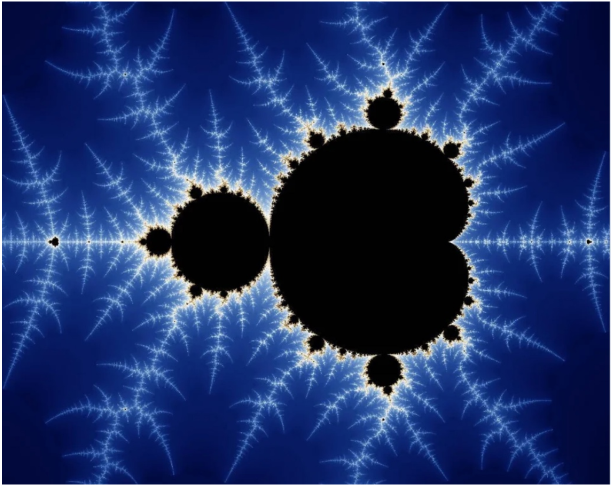
- Equivalence: if orbit of 0 does not go to ∞
  - f_c has critical points at 0 and ∞ and ∞ is an superattractive fixed point (that is $g_c(z) = 1/f_c(1/z)$ has superattractive fixed point at 0)
  - in basin of attraction $B(\infty)$ of point ∞ no other critical point of f_c
  - thus can extend Böttcher coordinate to all of $B(\infty)$
  - this shows $B(\infty)$ is homeomorphic to an open disc
  - complement $\mathbb{P}^1(\mathbb{C}) \smallsetminus B(\infty)$ then is connected
  - common boundary of these two regions $\partial B(\infty)$ closed and f-invariant
  - $B(\infty)$ and $\mathbb{P}^1(\mathbb{C}) \smallsetminus B(\infty)$ components of Fatou set and $\partial B(\infty) = J(f_c)$ Julia set

- Equivalence: if orbit of 0 goes to ∞ then Julia set $J(f)$ is totally disconnected (Cantor set)
    - rate of escape to infinity

$$\rho(z) = \lim_{k\to\infty} \frac{1}{d^k} \log_+ |f^k(z)|$$

      with $\log_+(x) = \log(x)$ for $x \geq 1$ and zero otherwise
    - by behavior z^d near ∞ in Böttcher coordinate
    - filled-in Julia set $\mathcal{K}(f) = \rho^{-1}(0) = \{z \,|\, f^n(z) \not\to \infty\}$
    - $\rho(f(z)) = d\,\rho(z)$ (using Böttcher coordinate)
    - shift map $\sigma : \Sigma_d^+ \to \Sigma_d^+$ (shift space on an alphabet of d letters)
    - if all finite critical points of f are in $B(\infty)$ then $f|_{J(f)}$ is topologically conjugate to the shift map $\sigma : \Sigma_d^+ \to \Sigma_d^+$ (in particular $J(f)$ homeomorphic to Cantor set)
    - for $f = f_c$ finite critical points just 0

Mandelbrot Set



Not obvious that the small self-similar islands off the main body are connected to it... but yes: the Mandelbrot set is connected (with filaments from main body to islands)

Mandelbrot Set is Connected (Douady and Hubbard)

- in fact show there is a conformal bijection between complement $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathcal{M}$ of Mandelbrot set and complement $\mathbb{P}^1(\mathbb{C}) \smallsetminus D$ of a disc
- Böttcher coordinates defines a bijection $\alpha_c : \mathbb{P}^1(\mathbb{C}) \smallsetminus \mathcal{K}(f_c) \to \mathbb{P}^1(\mathbb{C}) \smallsetminus D$ from complement of filled-in Julia set, using the map α_c that conjugates f_c to z^2
- then define map $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathcal{M} \to \mathbb{P}^1(\mathbb{C}) \smallsetminus D$ by taking $c \mapsto \alpha_c(c)$

Cardioid shape of the Mandelbrot set

- $\mathcal{M}_0 \subset \mathcal{M}$ set of values of c such that f_c has a superattractive fixed point (the Julia set of f_c is topologically a circle)
- $\mathcal{M}_0 = \{c = \lambda/2 - \lambda^2/4 \,|\, |\lambda| < 1\}$
- see this using the logistic map p_λ description of quadratic polynomials
- multipliers of fixed points of p_λ are λ and $2 - \lambda$
- $\lambda/2 - \lambda^2/4 = (2-\lambda)/2 - (2-\lambda)^2/4$
- $\mathcal{M}_0 = \{c = \lambda/2 - \lambda^2/4 \,|\, |\lambda| < 1 \text{ or } |2-\lambda| < 1\}$
- so $\mathcal{M}_0$ is a cardioid with cusp at $c = 1/4$
- parameterized by multiplier λ of the fixed point of f_c

Intersection of $\mathcal{M}$ with the real axis

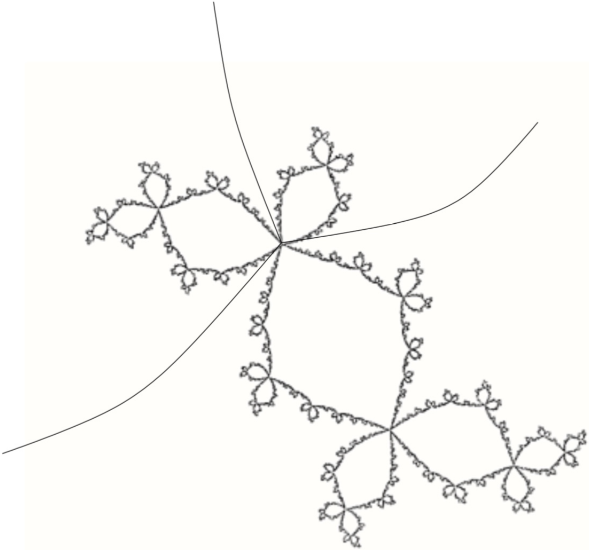
- $c > 1/4$ has $J(f_c)$ totally disconnected Cantor set
- $c = 1/4$ the map f_c has a neutral fixed point $z = 1/2$ with multiplier 1
- $-3/4 < c < 1/4$ the Julia set $J(f_c)$ is topologically a circle containing a dense set of repelling periodic orbits
- $c = -3/4$ neutral fixed point multiplier -1
- f_c has attractive period 2 orbit iff $|1 + c| < 1/4$
- $-5/4 < c < -3/4$: attracting period 2 orbit
- $-2 < c < -5/4$ sequence of period doubling like logistic map with transition to chaos (period 3 and all Sarkovsky ordering)
- for $c < -2$ again $J(f_c)$ is a Cantor set

External Rays

- under bijection $\alpha_c : \mathbb{P}^1(\mathbb{C}) \smallsetminus \mathcal{K}(f_c) \to \mathbb{P}^1(\mathbb{C}) \smallsetminus D$ radial lines (fixed angle) $\mathrm{agr}(z) = 2\pi\theta$ in the disc $\mathbb{P}^1(\mathbb{C}) \smallsetminus D$ become curves (external rays) in $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathcal{K}(f_c)$
- similarly under bijection $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathcal{M} \to \mathbb{P}^1(\mathbb{C}) \smallsetminus D$ radial lines $\mathrm{agr}(z) = 2\pi\theta$ in disc $\mathbb{P}^1(\mathbb{C}) \smallsetminus D$ become external rays in $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathcal{M}$
- these external rays are attached to some points of the boundary $J(f_c) = \partial\mathcal{K}(f_c)$ and to the boundary $\partial\mathcal{M}$ of the Mandelbrot set
- (Carlson–Gamelin, Douady–Hubbard) external ray with angle θ attached to point $c \in \partial\mathcal{M}$: if θ rational with odd denominator then f_c has a parabolic cycle, if θ rational with even denominator then critical point 0 of f_c strictly preperiodic

Internal Rays

- Mandelbrot set parameterized by λ with $|\lambda| < 1$, the multiplier at fixed point of f_c
- in disc $\{\lambda : |\lambda| < 1\}$ radial lines (fixed angle) $\mathrm{agr}(\lambda) = 2\pi\theta$
- these become lines inside the Mandelbrot set $\mathcal{M}$: internal rays
- an internal ray of angle θ is set of values c of parameter for which f_c has multiplier with argument $2\pi\theta$
- attached to some point on boundary $\partial\mathcal{M}$: value of c where multiplier equal $e^{2\pi i\theta}$
- case where internal ray inside main cardioid $\mathcal{M}_0$
- example: $\theta = 1/3$ point at boundary of internal ray is point where first period tripling in f_c happens
- corresponding external rays in the Julia set picture at this value of c with $\theta = 1/7, 2/7, 4/7$

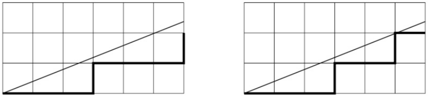


Dynamical plane (Julia) versus parameter plane (Mandelbrot)

- parameter plane: c endpoint of an internal ray in $\mathcal{M}_0$ with rational argument p/q
- dynamical plane: map f_c has neutral fixed point α with rotation number p/q
- $\mathcal{K}(f_c)$ with non-empty interior
- there are always two external rays that enclose a component of the interior of $\mathcal{K}(f_c)$ that contains critical value of slopes $\theta_\pm(p/q)$
- parameter plane: external rays with same slopes $\theta_\pm(p/q)$ land at $c \in \partial\mathcal{M}$
- dynamical plane: α with rotation number p/q so q external rays landing at α and action of f_c cyclically permutes them (since f_c quadratic action on angles by doubling so p/q-rotation orbit under doubling map $\theta \mapsto 2\theta$
- (Morse–Hendlund) unique such orbit for any rotation p/q

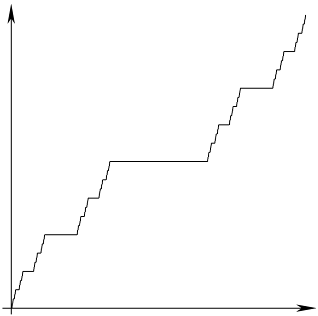
Devil's staircase: an algorithm for computing $\theta_\pm(p/q)$

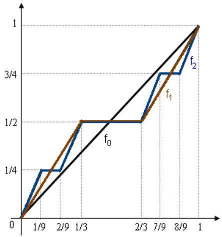
- line of slope p/q, staircases below (non-touching/touching)



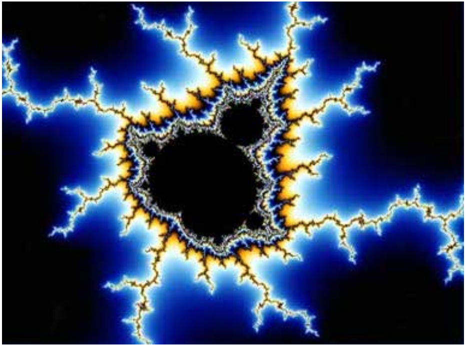
$$\theta_-(2/5) = .\overline{01001} = 9/31 \qquad \theta_+(2/5) = .\overline{01010} = 10/31$$

- rays with irrational angle ν: limits $\theta_\nu = \lim_{p/q\to\nu} \theta_\pm(p/q)$
- correspondence between internal angles and external angles for $\mathcal{M}_0$ given by a map: Devil's staircase
- Devil's staircases: continuous functions that are constant on a set of full measure without being globally constant, for example monotonically increasing on Cantor set and constant on all intervals in complement of Cantor set





An example of a Devil Staircase function: continuous function, constant on the intervals in the complement of the middle third Cantor set and monotonically increasing on the Cantor set, limit of piecewise linear functions



- existence of small repeated copies of the Mandelbrot sets in $\mathcal{M}$ from Misiurewicz cascade phenomenon
- Curtis McMullen, *The Mandelbrot set is universal*. In The Mandelbrot set, theme and variations, 1–17, London Math. Soc. Lecture Note Ser., 274, Cambridge Univ. Press, 2000