# Spectral Gravity models on multifractal Robertson–Walker cosmologies

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This talk is based on:

- Farzad Fathizadeh, Yeorgia Kafkoulis, Matilde Marcolli, *Bell polynomials and Brownian bridge in Spectral Gravity models on multifractal Robertson-Walker cosmologies*, arXiv:1811.02972

Other references

- A. Ball, M. Marcolli, *Spectral Action Models of Gravity on Packed Swiss Cheese Cosmology*, Classical and Quantum Gravity, 33 (2016), no. 11, 115018, 39 pp.

## Spectral Action

$$\mathcal{S}_\Lambda = \mathrm{Tr}(f(\frac{D}{\Lambda})) = \sum_{\lambda \in \mathrm{Spec}(D)} f(\frac{\lambda}{\Lambda})$$

- $D$ Dirac operator
- $\Lambda \in \mathbb{R}_+^*$ energy scale
- $f(x)$ test function (smooth approximation to cutoff function)

## Why a model of (Euclidean) Gravity?

- $M$ compact Riemannian 4-manifold

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

$$= \frac{48 f_4 \Lambda^4}{\pi^2} \int \sqrt{g}\, d^4x + \frac{96\, f_2 \Lambda^2}{24\pi^2} \int R \sqrt{g}\, d^4x$$

$$+ \frac{f_0}{10\pi^2} \int (\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}) \sqrt{g}\, d^4x$$

coefficients $a_0$, $a_2$ and $a_4$ cosmological, Einstein–Hilbert, and Weyl curvature $C^{\mu\nu\rho\sigma}$ and Gauss–Bonnet $R^* R^*$ gravity terms

## Asymptotic Expansion

- heat kernel expansion at $\tau \to 0^+$ for $D^2$ (Dirac Laplacian)

$$\mathrm{Tr}(e^{-\tau D^2}) \sim \sum_\alpha c_\alpha \tau^\alpha$$

- test function $f(x) = \int_0^\infty e^{-\tau x^2} d\mu(\tau)$ some measure $\mu$ normalized by $f(0) = \int_0^\infty d\mu(\tau)$
- asymptotic expansion of the spectral action (large $\Lambda$)

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\alpha<0} f_\alpha \, c_\alpha \, \Lambda^{-\alpha} + a_0 f(0) + \sum_{\alpha>0} f_\alpha \, c_\alpha \, \Lambda^{-\alpha}$$

- coefficients $f_\alpha$ given by

$$f_\alpha = \begin{cases} \int_0^\infty f(v) v^{-\alpha-1} dv & \alpha < 0 \\ (-1)^\alpha f^{(\alpha)}(0) & \alpha > 0, \ \alpha \in \mathbb{N} \end{cases}$$

Main Point: computing the expansion of the spectral action is the same problem as computing the coefficients of the heat kernel expansion of $D^2$

## Robertson–Walker spacetime

- Topologically $S^3 \times \mathbb{R}$
- Metric (Euclidean)

$$ds^2 = dt^2 + a(t)^2 d\sigma^2$$

scaling factor $a(t)$, round metric $d\sigma^2$ on $S^3$

- A.H. Chamseddine, A. Connes, *Spectral action for Robertson-Walker metrics*, J. High Energy Phys. (2012) N.10, 101

- form of Dirac-Laplacian $D^2$ for Robertson–Walker metric

$$D^2 = -(\frac{\partial}{\partial t} + \frac{3a'(t)}{2a(t)})^2 + \frac{1}{a(t)^2}(\gamma^0 D_3)^2 - \frac{a'(t)}{a^2(t)}\gamma^0 D_3$$

- $\gamma^0 D_3 = D_{S^3} \oplus -D_{S^3}$, Dirac operator on $S^3$
- Dirac spectrum on $S^3$

$$\mathrm{Spec}(D_{S^3}) = \{k + \frac{3}{2}\} \text{ multiplicities } \mu(k + \frac{3}{2}) = (k+1)(k+2)$$

• use basis of eigenfunctions of the Dirac operator on $S^3$ to decompose $D^2$ as direct sum of operators

$$H_n = -(\frac{d^2}{dt^2} - \frac{(n+\frac{3}{2})^2}{a^2} + \frac{(n+\frac{3}{2})a'}{a^2})$$

multiplicity $4(n+1)(n+2)$

• spectral action for test function $f(u) = e^{-su}$

$$\mathrm{Tr}(f(D^2)) \sim \sum_{n\geq 0} \mu(n)\, \mathrm{Tr}(f(H_n))$$

multiplicities $\mu(n) = 4(n+1)(n+2)$ and operator $H_n$

$$H_n = -\frac{d^2}{dt^2} + V_n(t),$$

$$V_n(t) = \frac{(n+\frac{3}{2})}{a(t)^2}((n+\frac{3}{2}) - a'(t))$$

Result of this approach (Chamseddine–Connes)

• to compute the spectral action for the Robertson–Walker metric need to evaluate the trace $\mathrm{Tr}(e^{-sH_n})$ which requires computing $e^{-sH_n}(t,t)$ (for coeffs prior to time integration)

Feynman–Kac formula

$$e^{-sH_n}(t,t) = \frac{1}{2\sqrt{\pi s}} \int \exp(-s \int_0^1 V_n(t + \sqrt{2s}\alpha(u))du)\, D[\alpha]$$

$D[\alpha]$ Brownian bridge integrals

Brownian bridge: Gaussian stochastic process characterized by the covariance

$$\mathbb{E}(\alpha(v_1)\alpha(v_2)) = v_1(1-v_2), \qquad 0 \le v_1 \le v_2 \le 1$$

Background reference for Brownian bridge and Feynman–Kac:

- Barry Simon, *Functional Integration and Quantum Physics*, Academic Press, 1979

## Brownian bridge and Heat Kernel

• Gaussian process $\{\alpha(s)\}_{0\leq s\leq 1}$ with covariance

$$\mathbb{E}(\alpha(s)\alpha(t)) = s(1-t) \text{ for } 0 \leq s \leq t \leq 1$$

• relation to Brownian motion: $\alpha(s) = b(s) - sb(1)$

• setting for Feynman-Kac formula: operator $H = H_0 + V$ with potential and heat kernel $e^{-sH}$

• Trotter product formula:

$$\langle f, e^{-sH} g\rangle = \lim_{n\to\infty} \langle f, (e^{-sH_0/n} e^{-iV/n})^n g\rangle$$

• consequence of relation between Brownian motion and heat kernel of $H_0$:

$$\langle f_0, e^{-s_1 H_0} f_1 \cdots e^{-s_n H_0} f_n\rangle = \int f_0(\omega(s_0)) \cdots f_n(\omega(s_n))\, D[\omega]$$

with $D[\omega]$ Wiener measure; $s_k = \sigma_k - \sigma_{k-1}$ and $0 \leq \sigma_0 < \sigma_1 < \cdots < \sigma_n$, with $L^\infty$ functions, and $\omega(s) = x + b(s)$

- use previous two to write

$$\langle f, e^{-sH} g\rangle = \lim_{n\to\infty} \int \overline{f(\omega(0))} g(\omega(s)) \exp(-\frac{s}{n}\sum_{j=0}^{n-1} V(\omega(sj/n)))\, D[\omega]$$

$$\frac{s}{n}\sum_{j=0}^{n-1} V(\omega(sj/n)) \to \int_0^s V(\omega(\sigma))\, d\sigma$$

- resulting Feynman-Kac formula:

$$\langle f, e^{-sH} g\rangle = \int \overline{f(\omega(0))} g(\omega(s)) \exp(-\int_0^s V(\omega(\sigma))\, d\sigma)\, D[\omega]$$

- this gives $(e^{-sH} f)(0) = \int \exp(-\int_0^s V(b(\sigma))\, d\sigma)\, f(b(s))\, D[b]$
- Brownian bridge reformulation:

$$e^{-sH} = \frac{1}{2\sqrt{\pi s}} \int \exp(-s \int_0^1 V_n(t + \sqrt{2s}\alpha(u)) du)\, D[\alpha]$$

Problem: technique used on Chamseddine–Connes for computing the Brownian bridge integrals becomes computationally intractable after the 10th or 12th term

New Method for computing the Brownian bridge integrals more efficiently and obtain the full expansion of the spectral action

Quick summary of results in our work:

- use this Brownian bridge computation to obtain explicit formula for all the coefficients $a_{2n}$ of the heat kernel expansion in terms of Bell polynomials
- consider isotropic non-homogeneous versions of Robertson–Walker spacetimes based on Apollonian packings of spheres (multifractal cosmologies)
- extend computation of the spectral action to these multifractal cases
- identify correction terms that *detect fractality*

## Brownian bridge integrals and expansion

- notation $A(t) = 1/a(t)$ and $B(t) = A(t)^2$ so potential $V_n$

$$V_n(t) = x^2 A(t)^2 + xA'(t) = x^2 B(t) + xA'(t), \qquad \text{with } x = n + 3/2$$

Integral in Feynman–Kac formula becomes

$$-s \int_0^1 V_n(t + \sqrt{2s}\,\alpha(v))\, dv = -x^2 U - xV$$

where

$$U = s \int_0^1 A^2 \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv = s \int_0^1 B \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv$$

$$V = s \int_0^1 A' \left(t + \sqrt{2s}\,\alpha(v)\right)\, dv$$

- in heat kernel spectral multiplicities (Dirac eigenvalues on $S^3$)

$$\sum_n \mu(n) \mathrm{Tr}(e^{-s H_n}) \qquad \mu(n) = 4(n+1)(n+2) \qquad H_n = -\frac{d^2}{dt^2} + V_n(t)$$

replace sum over multiplicities by an integration of a continuous variable (Poisson summation) $x = n + 3/2$

- including multiplicities: $f_s(x) := \left(x^2 - \frac{1}{4}\right) e^{-x^2 U - xV}$

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \frac{\sqrt{\pi}\, e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}}$$

Generating function for the full expansion of the spectral action

$$\frac{1}{\sqrt{\pi s}} \frac{\sqrt{\pi}\, e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}} = \frac{1}{\sqrt{s}} \frac{e^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}}$$

then consider Laurent series expansion in the variable $\tau = s^{1/2}$

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!} \tau^n \quad \text{and} \quad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!} \tau^n$$

$$u_n = B^{(n)}(t)\, 2^{n/2} x_n(\alpha) = \left( \sum_{k=0}^{n} \binom{n}{k} A^{(k)}(t) A^{(n-k)}(t) \right) 2^{n/2}\, x_n(\alpha)$$

$$v_n = A^{(n+1)}(t) 2^{n/2}\, x_n(\alpha)$$

$$x_k(\alpha) = \int_0^1 \alpha(v)^k\, dv$$

## resulting expansion

$$\mathrm{Tr}(\exp(-\tau^2 D^2)) \sim \sum_{M=0}^{\infty} \tau^{2M-4} \int a_{2M}(t)\, dt,$$

$$a_{2M}(t) = \int \left( \frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} \left( C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)} \right) \right) D[\alpha]$$

## coefficients and Bell polynomials

$$C_{2M}^{(r,m)} = \sum_{\substack{0\leq k,p\leq 2M \\ 0\leq n\leq M \\ 0\leq \beta\leq 2M-2n}} \left( \frac{\binom{-n+r}{k}\binom{2n+m}{p}\binom{2M-2n}{\beta}\, k!\, p!}{4^n\, n!\, (2M-2n)!} u_0^{-n+r-k} v_0^{2n+m-p} \times \right.$$

$$\left. B_{\beta,k}\,(u_1,\ldots,u_{\beta-k+1})\, B_{2M-2n-\beta,\,p}\,(v_1,\ldots,v_{2M-2n-\beta-p+1}) \right)$$

**Bell polynomials**: Faà di Bruno derivatives of composite functions

$$\frac{d^n}{dt^n} f(g(t)) = \sum_{m=1}^{n} f^{(m)}(g(t))\, B_{n,m}(g'(t), g''(t), \ldots, g^{(n-m+1)}(t))$$

## Structure of Brownian Bridge Integrals

Step 1: integrals of monomials on the standard simplex

$$\Delta^n = \{(v_1, v_2, \dots, v_n) \in \mathbb{R}^n : 0 \le v_1 \le v_2 \le \cdots \le v_n \le 1\}.$$

monomial $v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}$

$$\int_{\Delta^n} v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} \, dv_1 \, dv_2 \cdots dv_n =$$

$$\frac{1}{(k_1+1)(k_1+k_2+2)\cdots(k_1+k_2+\cdots+k_n+n)}$$

Similarly for $1 \le j_1 < j_2 < \cdots < j_k \le n$

$$\int_{\Delta^n} v_{j_1} v_{j_2} \cdots v_{j_k} \, dv_1 \, dv_2 \cdots dv_n = \frac{j_1(j_2+1)(j_3+2)\cdots(j_k+k-1)}{(n+k)!}$$

## Step 2: Brownian Bridge and integration on the simplex

- Using variance property of Brownian Bridge: $(v_1, v_2, \dots, v_{2n}) \in \Delta^{2n}$

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\, D[\alpha] = \sum v_{i_1}(1-v_{j_1})v_{i_2}(1-v_{j_2})\cdots v_{i_n}(1-v_{j_n})$$

summation over indices with $i_1 < j_1$, $i_2 < j_2$, $\dots$, $i_n < j_n$, and $\{i_1, j_1, i_2, j_2, \dots, i_n, j_n\} = \{1, 2, \dots, 2n\}$

- equivalently for $(v_1, v_2, \dots, v_{2n}) \in \Delta^{2n}$

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\, D[\alpha] =$$

$$\sum_{\sigma\in S^*_{2n}} v_{\sigma(1)}(1-v_{\sigma(2)})v_{\sigma(3)}(1-v_{\sigma(4)})\cdots v_{\sigma(2n-1)}(1-v_{\sigma(2n)})$$

$S^*_{2n}$ set of all permutations $\sigma$ in symmetric group $S_{2n}$ with $\sigma(1) < \sigma(2)$, $\sigma(3) < \sigma(4)$, $\dots$, $\sigma(2n-1) < \sigma(2n)$

## Brownian Bridge Integrals

- Notation: $\mathcal{J}_{k,n}$ = set of all $k$-tuples of integers $J = (j_1, j_2, \ldots, j_k)$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq n$; for $J \in \mathcal{J}_{k,n}$ and $\sigma \in S^*_{2n}$ define $\sigma_J(1)$, $\sigma_J(2)$, ..., $\sigma_J(n+k)$ by property that

$$\sigma_J(1) < \sigma_J(2) < \cdots < \sigma_J(n+k)$$

and that the set of such $\sigma_J$'s is given by
$\{\sigma_J(1) < \sigma_J(2) < \cdots < \sigma_J(n+k)\}$

$$= \{\sigma(1), \sigma(3), \ldots, \sigma(2n-1), \sigma(2j_1), \ldots, \sigma(2j_k)\}$$

$$x_k(\alpha) = \int_0^1 \alpha(v)^k \, dv$$

- Brownian Bridge Integrals

$$\int x_1(\alpha)^{2n} \, D[\alpha] = \int \Big( \int_0^1 \alpha(v) \, dv \Big)^{2n} \, D[\alpha] =$$

$$(2n)! \sum_{\sigma \in S^*_{2n}} \sum_{k=0}^{n} \sum_{J \in \mathcal{J}_{k,n}} (-1)^k \frac{\sigma_J(1) \, (\sigma_J(2)+1) \cdots (\sigma_J(n+k)+n+k-1)}{(3n+k)!}$$

# Monomial Brownian Bridge Integrals

- for $(v_1, v_2, \dots, v_n) \in \Delta^n$ and for $i_1, i_2, \dots, i_n \in \mathbb{Z}_{\geq 0}$ such that $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$

$$\int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n} \, D[\alpha] =$$

$$\binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!} \left( \sum \binom{|I|/2}{k_{m,j}} \sum_{r_1=0}^{K_1} \sum_{r_2=0}^{K_2} \cdots \sum_{r_n=0}^{K_n} \prod_{p=1}^{n} (-1)^{r_p} v_p^{i_p - r_p} \right),$$

with $I = (i_1, i_2, \dots, i_n)$, first summation over non-negative integers $k_{j,m}$, $j, m = 1, 2, \dots, n$ such that

$$\sum_{j,m=1}^{n} k_{j,m} = \frac{|I|}{2}, \qquad \sum_{m=1}^{n} (k_{j,m} + k_{m,j}) = i_j \text{ for all } j = 1, 2, \dots, n$$

and for each $m = 1, 2, \dots, n$,

$$K_m := k_{m,m} + \sum_{j=1}^{m-1} (k_{j,m} + k_{m,j})$$

# Sketch of proof

$$\int \exp\left(\sqrt{-1}\sum_{j=1}^{n} u_j \alpha(v_j)\right) D[\alpha] = \exp\left(-\frac{1}{2}\sum_{j,m=1}^{n} c_{j,m} u_j u_m\right)$$

where the terms $c_{j,m}$ are given by

$$c_{j,m} = v_j(1-v_m) \quad \text{if} \quad j \leq m, \quad \text{and} \quad c_{j,m} = v_m(1-v_j) \quad \text{if} \quad m \leq j$$

Expanding gives

$$\frac{(\sqrt{-1})^{i_1+i_2+\cdots+i_n}}{(i_1+i_2+\cdots+i_n)!}\binom{i_1+i_2+\cdots+i_n}{i_1, i_2, \ldots, i_n}\int \alpha(v_1)^{i_1}\alpha(v_2)^{i_2}\cdots\alpha(v_n)^{i_n}\, D[\alpha] =$$

$$\frac{(-1/2)^{(i_1+i_2+\cdots+i_n)/2}}{((i_1+i_2+\cdots+i_n)/2)!}\left(\text{Coefficient of } u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n} \text{ in } \Big(\sum_{j,m=1}^{n} c_{j,m}u_j u_m\Big)^{(i_1+i_2+\cdots+i_n)/2}\right)$$

$$= \frac{(-1/2)^{(i_1+i_2+\cdots+i_n)/2}}{((i_1+i_2+\cdots+i_n)/2)!}\sum\binom{(i_1+i_2+\cdots+i_n)/2}{k_{1,1}, k_{1,2}, \ldots, k_{1,n}, k_{2,1}, \ldots, k_{n,n}}\prod_{j,m=1}^{n} c_{j,m}^{k_{j,m}}$$

from which then can group terms as stated

## Shuffle Product

• for $(v_1, v_2, \ldots, v_n) \in \Delta^n$ and $i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\geq 0}$ with $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$

$$V^b(i_1, i_2, \ldots, i_n) := \int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n} \, D[\alpha]$$

• extend $V^b$ linearly to vector space generated by all words $(i_1, i_2, \ldots, i_n)$ in the letters $i_1, i_2, \ldots, i_n$

• Shuffle product $\alpha \sqcup\!\sqcup \beta$ of two words $\alpha = (i_1, i_2, \ldots, i_p)$ and $\beta = (j_1, j_2, \ldots, j_q)$ sum of $\binom{p+q}{p}$ words obtained by interlacing letters of these two words so that in each term the order of the letters of each word is preserved

• $2n = m_1 i_1 + m_2 i_2 + \cdots + m_r i_r$ even positive integer with $i_1, i_2, \ldots, i_r$ distinct positive integers and $m_1, m_2, \ldots, m_r$ positive integers

$$\int x_{i_1}(\alpha)^{m_1} x_{i_2}(\alpha)^{m_2} \cdots x_{i_r}(\alpha)^{m_r} \, D[\alpha] =$$

$$m! \int_{\Delta^{|m|}} V^b(\underbrace{(i_1, \ldots, i_1)}_{m_1} \sqcup \underbrace{(i_2, \ldots, i_2)}_{m_2} \sqcup \cdots \sqcup \underbrace{(i_r, \ldots, i_r)}_{m_r}) dv_1 dv_2 \cdots dv_{|m|}$$

$$m! = (m_1!)(m_2!) \cdots (m_r!), \qquad |m| = m_1 + m_2 + \cdots + m_r \, .$$

follows directly from writing

$$\int x_{i_1}(\alpha)^{m_1} x_{i_2}(\alpha)^{m_2} \cdots x_{i_r}(\alpha)^{m_r} \, D[\alpha]$$

$$= \int \Big( \int_0^1 \alpha(v_1)^{i_1} \, dv_1 \Big)^{m_1} \Big( \int_0^1 \alpha(v_2)^{i_2} \, dv_2 \Big)^{m_2} \cdots \Big( \int_0^1 \alpha(v_r)^{i_r} \, dv_r \Big)^{m_r} D[\alpha],$$

## Brownian Bridge Integrals in the Coefficients of the Spectral Action

$$\int C_{2M}^{(r,m)}\, D[\alpha] =$$

$$\sum \left( \frac{\binom{-n+r}{k}\binom{2n+m}{p} k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b\big(\underbrace{(1,\ldots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\ldots,2)}_{\lambda_2+\mu_2} \sqcup \cdots\big)\, dv_1 \cdots dv_{k+p} \right.$$

$$\left. \times B(t)^{-n+r-k}\, (A'(t))^{2n+m-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \right)$$

summation is over integers $0 \le k, p \le 2M, 0 \le n \le M$, $0 \le \beta \le 2M - 2n$, and over sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ of non-negative integers for each choice of $k, p, n, \beta$, such that $|\lambda|' = \beta, |\lambda| = k,\ |\mu|' = 2M - 2n - \beta, |\mu| = p$

# coefficients of the expansion of the spectral action of Robertson–Walker metric

$$a_{2M}(t) =$$

$$\frac{1}{2}\sum{}' \Bigg( \frac{\binom{-n-3/2}{k}\binom{2n}{p} k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b \big( \underbrace{(1,\dots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\dots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \big)\, dv_1 \cdots dv_{k+p} \times$$

$$B(t)^{-n-(3/2)-k} \left(A'(t)\right)^{2n-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \Bigg)$$

$$+\frac{1}{4}\sum{}'' \Bigg( \Bigg( \binom{-n-5/2}{k}\binom{2n+2}{p} B(t)^{-5/2}\left(A'(t)\right)^2 - \binom{-n-1/2}{k}\binom{2n}{p} B(t)^{-1/2} \Bigg) \times$$

$$\frac{k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b \big( \underbrace{(1,\dots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\dots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \big)\, dv_1 \cdots dv_{k+p} \times$$

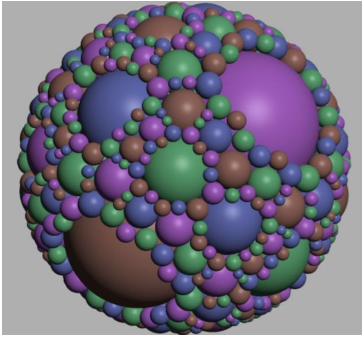
$$B(t)^{-n-k} \left(A'(t)\right)^{2n-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \Bigg)$$

summation $\sum'$ is over all integers $0 \le k, p \le 2M, 0 \le n \le M$, $0 \le \beta \le 2M - 2n$, and sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ of non-negative integers (for each choice of $k, p, n, \beta$) such that $|\lambda|' = \beta, |\lambda| = k$, $|\mu|' = 2M - 2n - \beta, |\mu| = p$; second summation $\sum''$ is over all integers $0 \le k, p \le 2M-2, 0 \le n \le M-1$, $0 \le \beta \le 2M - 2 - 2n$, over all sequences $\lambda = (\lambda_1, \lambda_2, \dots)$, $\mu = (\mu_1, \mu_2, \dots)$ of non-negative integers such that $|\lambda|' = \beta, |\lambda| = k$, $|\mu|' = 2M - 2 - 2n - \beta, |\mu| = p$

# Packed Swiss Cheese Cosmology

- $\mathcal{P}$ Apollonian packing of 3-spheres radii $\{a_{n,k} : n \in \mathbb{N}, k = 1, \ldots, 6 \cdot 5^{n-1}\}$
- iterative construction of packing: at $n$-th step $6 \cdot 5^{n-1}$ spheres $S^3_{a_{n,k}}$ are added
- spacetime that are isotropic but not homogeneous

$$ds^2_{n,k} = a^2_{n,k} \, (dt^2 + a(t)^2 \, d\sigma^2), \qquad n \in \mathbb{N}, k = 1, \ldots, 6 \cdot 5^{n-1}$$



Homogeneous non-isotropic spacetimes

• M.J. Rees, D.W. Sciama, *Large-scale density inhomogeneities in the universe*, Nature, Vol.217 (1968) 511–516.

• Proposed as explanation for possible fractal distribution of matter in galaxies, clusters, and superclusters

- A. Ball, M. Marcolli, *Spectral Action Models of Gravity on Packed Swiss Cheese Cosmology*, Classical and Quantum Gravity, 33 (2016), no. 11, 115018, 39 pp.
- F. Sylos Labini, M. Montuori, L. Pietroneo, *Scale-invariance of galaxy clustering*, Phys. Rep. Vol. 293 (1998) N. 2-4, 61–226.
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Spectral action model of gravity on $\mathcal{P} \times \mathbb{R}$: correction terms to Robertson–Walker spectral action that detect fractality

## Dirac operator on the multifractal geometry $\mathcal{P} \times \mathbb{R}$

• General setting of Spectral triple: $(\mathcal{A}, \mathcal{H}, D)$

1. unital associative algebra $\mathcal{A}$
2. represented as bounded operators on a Hilbert space $\mathcal{H}$
3. Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$

• prototype: $(C^\infty(M), L^2(M, S), \not{D}_M)$

• extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.) a good notion of Dirac operator $D$ and spectral action functional $\mathrm{Tr}(f(D/\Lambda))$

The spectral triple of a fractal geometry

- case of Sierpinski gasket: Christensen, Ivan, Lapidus
- similar case for $\mathcal{P}$ and $\mathcal{P}_Y$
- for $D$-dimensional sphere packing

$$\mathcal{P}_D = \{S^{D-1}_{a_{n,k}} \,:\, n \in \mathbb{N},\, 1 \leq k \leq (D+2)(D+1)^{n-1}\}$$

$$(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{\mathcal{P}_D}, \mathcal{D}_{\mathcal{P}_D}) = \oplus_{n,k}(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{S^{D-1}_{a_{n,k}}}, \mathcal{D}_{S^{D-1}_{a_{n,k}}})$$

- our goal: compute the spectral action expansion (or equivalently heat kernel expansion) for Dirac operator $\oplus_{n,k} D_{n,k}$ where $D_{n,k}$ is the Dirac operator on the rescaled Robertson–Walker metric

- two possible choices of associated Robertson–Walker metrics
  1. round scaling (of full 4-dim spacetime)

$$ds^2_{n,k} = a^2_{n,k}\,(dt^2 + a(t)^2\, d\sigma^2), \qquad n \in \mathbb{N}, k = 1, \dots, 6 \cdot 5^{n-1}$$

  2. non-round scaling (of spatial sections only)

$$ds^2_{n,k} = dt^2 + a(t)^2\, a^2_{n,k}\, d\sigma^2, \qquad n \in \mathbb{N}, k = 1, \dots, 6 \cdot 5^{n-1}$$

- $D_{n,k}$ resulting Dirac operators on $\mathbb{R} \times S^3_{a_{n,k}}$
- entire (multifractal) spacetime $\mathbb{R} \times \mathcal{P}$
- spectral triple for $\mathbb{R} \times \mathcal{P}$: $\mathcal{A}$ subalgebra of $C_0(\mathbb{R} \times \mathcal{P})$, Hilbert space $\mathcal{H} = \oplus_{n,k} \mathcal{H}_{n,k}$ with $\mathcal{H}_{n,k} = L^2(S_{a_{n,k}}, \mathbb{S})$ and Dirac

$$D = D_{\mathbb{R}\times\mathcal{P}} := \bigoplus_{n\in\mathbb{N}} \bigoplus_{k=1}^{6\cdot 5^{n-1}} D_{n,k}$$

Focus first on round scaling; comment later on other case

## Example: Packing of 4-Spheres

- round $S^4$ is a Robertson–Walker metric $dt^2 + a(t)^2 d\sigma^2$ with $a(t) = \sin t$ $(0 \le t \le \pi)$ and $d\sigma^2$ round metric on $S^3$
- spectrum of Dirac operator on $S_r^{D-1}$ radius $r > 0$

$$\mathrm{Spec}(D_{S_r^{D-1}}) = \left\{ \lambda_{\ell,\pm} = \pm r^{-1} \left( \frac{D-1}{2} + \ell \right) \mid l \in \mathbb{Z}_+ \right\}$$

multiplicities

$$\mathrm{m}_{\ell,\pm} = 2^{[\frac{D-1}{2}]} \binom{\ell + D}{\ell}.$$

- zeta function of Dirac operator

$$\zeta_D(s) = \mathrm{Tr}(|D_{S_r^4}|^{-s}) = \sum_{\ell,\pm} \mathrm{m}_{\ell,\pm} |\lambda_{\ell,\pm}|^{-s} = \frac{4}{3} \, r^s \, (\zeta(s-3) - \zeta(s-1))$$

$\zeta(s)$ Riemann zeta function

- fractal string zeta function $\zeta_{\mathcal{L}}(s) = \sum_{n,k} a_{n,k}^s$ of Apollonian packing $\mathcal{P}$ of $S^3_{a_{n,k}}$ with radii sequence $\mathcal{L} = \{a_{n,k}\}$
- resulting Dirac operator $\mathcal{D}_{\mathcal{P}}$ on associated packing of 4-spheres (each 3-sphere equator in a fixed hyperplane of a corresponding 4-sphere)
- zeta function of Dirac $\mathcal{D}_{\mathcal{P}}$ factors as product of zetas

$$\zeta_{\mathcal{D}_{\mathcal{P}}}(s) = \mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n,k} \frac{4}{3} a_{n,k}^s (\zeta(s-3) - \zeta(s-1))$$

$$= \zeta_{\mathcal{L}}(s) \zeta_{D_{S^4}}(s)$$

- Mellin transform relation between the zeta function of the Dirac operator and the heat-kernel of the Dirac Laplacian

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \frac{1}{\Gamma(s/2)} \int_0^\infty \mathrm{Tr}(e^{-t\mathcal{D}_{\mathcal{P}}^2})\, t^{s/2-1} dt$$

- use to compute spectral action leading terms from zeta function: $\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim$

$$f(0)\zeta_{\mathcal{D}_{\mathcal{P}}}(0) + f_2\Lambda^2 \frac{\zeta_{\mathcal{L}}(2)}{2} + f_4\Lambda^4 \frac{\zeta_{\mathcal{L}}(4)}{2} + \sum_{\sigma\in\mathcal{S}(\mathcal{L})} f_\sigma \Lambda^\sigma \frac{\zeta_{D_{S^4}}(\sigma)}{2} \mathcal{R}_\sigma$$

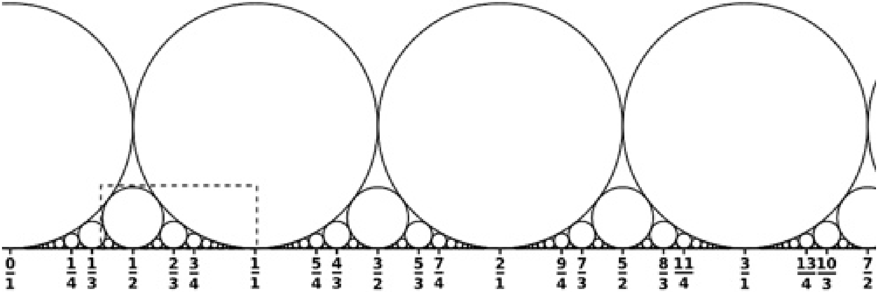
- $\mathcal{S}(\mathcal{L})$ set of poles of fractal string zeta $\zeta_{\mathcal{L}}(s)$ residues

$$\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)$$

- $\zeta_{\mathcal{L}}(2)$ and $\zeta_{\mathcal{L}}(4)$ replace radii $r^2$ and $r^4$ for a single sphere $S^4_r$: zeta regularization of $\sum_{n,k} a^2_{n,k}$ and $\sum_{n,k} a^4_{n,k}$

## Example: Lower Dimensional Apollonian Ford Circles

- Ford circles: tangent to the real line at points $(k/n, 0)$ with centers at points $(k/n, 1/(2n^2))$



- number of circles of radius $r_n = (2n^2)^{-1}$ is number of integers $1 \leq k \leq n$ coprime to $n$: multiplicity $m(r_n)$ given by Euler totient function

$$m(r_n) = \varphi(n),$$

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

product over the distinct prime numbers dividing $n$

- Dirichlet series generating function of the Euler totient function

$$\mathcal{D}_\varphi(s) = \sum_{n \geq 1} \frac{\varphi(n)}{n^s}$$

- fractal string zeta function

$$\zeta_{\mathcal{L}}(s) = \sum_{n\geq 1} \varphi(n)\,(2n^2)^{-s} = 2^{-s} \sum_{n\geq 1} \varphi(n)\, n^{-2s} = 2^{-s}\mathcal{D}_{\varphi}(2s)$$

- using $\varphi(p^k) = p^k - p^{k-1}$

$$1 + \sum_{k} \varphi(p^k) p^{-sk} = \frac{1 - p^{-s}}{1 - p^{1-s}}$$

- using Euler product formula

$$\mathcal{D}_{\varphi}(s) = \frac{\zeta(s-1)}{\zeta(s)}$$

- so fractal string zeta function of Ford circles

$$\zeta_{\mathcal{L}}(s) = 2^{-s}\, \frac{\zeta(2s-1)}{\zeta(2s)}$$

- build a packing of 4-spheres over the Ford circles: collection of 2-spheres with Ford circles as equators in a given hyperplane, then same as equators of a collection of 3-spheres and then of 4-spheres
- for the resulting packing of 4-spheres spectral action

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim \frac{11}{140}\, f(0) + \frac{f_1}{\pi^2}\Lambda + \frac{45\,\zeta(3)}{4\pi^4}\, f_2\Lambda^2 + \frac{4725\,\zeta(7)}{16\pi^8}\, f_4\Lambda^4$$

$$+ \sum_{k\in\mathbb{N}} \frac{2^{k+1} f_{-k}}{3}\, \frac{\zeta(-k-3)-\zeta(-k-1)}{\zeta(-2k-1)}\Lambda^{-k}$$

$$+ \sum_{\sigma=a+ib} \frac{2^{-a}\cos(b\log 2)}{3}\Re(Z_\sigma)\, r(f)_\sigma\, \cos(b\log\Lambda)\Lambda^a$$

$\sigma$ over nontrivial zeros of $\zeta(2s)$ with
$r(f)_\sigma = \int_0^\infty f(u)u^{a-1}\cos(bu)\,du$ and
$Z_\sigma = (\zeta(\sigma-3)-\zeta(\sigma-1))\zeta(2\sigma-1)$

## Tools for the general case: Singular expansions

- meromorphic function $\phi(z)$ with poles at $\mathcal{S} \subset \mathbb{C}$, Laurent series expansion at a pole $z_0 \in \mathcal{S}$

$$\phi(z) = \sum_{-N \leq k} c_k (z - z_0)^k$$

- singular element at $z_0 \in \mathcal{S}$

$$S(\phi, z_0) := \sum_{-N \leq k \leq 0} c_k (z - z_0)^k$$

- singular expansion of $\phi$

$$S_\phi(z) := \sum_{z \in \mathcal{S}} S(\phi, z)$$

- Example: for the Gamma function

$$\Gamma(z) \asymp \sum_{k \geq 0} \frac{(-1)^k}{k!} \frac{1}{z + k}$$

## Mellin transform and asymptotic expansion

- Mellin transform

$$\phi(z) = \mathcal{M}(f)(z) = \int_0^\infty f(\tau)\tau^{z-1} d\tau$$

- relation between asymptotic expansion at $u \to 0$ of a function $f(u)$ and singular expansion of its Mellin transform $\phi(z) = \mathcal{M}(f)(z)$
- small time asymptotic expansion

$$f(u) \sim_{u\to 0^+} \sum_{\alpha\in\mathcal{S},k_\alpha} c_{\alpha,k_\alpha} u^\alpha \log(u)^{k_\alpha}$$

- coefficients $c_{\alpha,k_\alpha}$ determined by singular expansion of Mellin transform

$$\mathcal{M}(f)(z) \asymp S_{\mathcal{M}(f)}(z) = \sum_{\alpha\in\mathcal{S},k_\alpha} c_{\alpha,k_\alpha} \frac{(-1)^{k_\alpha} k_\alpha!}{(s+\alpha)^{k_\alpha+1}}$$

- index $k_\alpha$ ranges over terms in singular element of $\phi(z) = \mathcal{M}(f)(z)$ at $z = \alpha$, up to order of pole at $\alpha$

# Mellin transform and summation asymptotics

- $R = \{r_n\}$ a sequence of radii $r_n \in \mathbb{R}_+^*$ so that $\zeta_R(z) = \sum_n r_n^{-z}$ converges for $\Re(z) > C$ for some $C > 0$
- function $f(\tau)$ with small time asymptotics

$$f(\tau) \sim \sum_N c_N \tau^N$$

- associated series

$$g_R(\tau) = \sum_n f(r_n \tau)$$

- then small time asymptotic expansion of $g_R(\tau)$

$$g_R(\tau) \sim_{\tau\to 0^+} \sum_N c_N \, \zeta_R(-N)\, \tau^N + \sum_{\sigma\in\mathcal{S}(\zeta_R)} \mathcal{R}_{R,\sigma}\, \mathcal{M}(f)(\sigma)\, \tau^{-\sigma}$$

with $\mathcal{S}(\zeta_R)$ poles of $\zeta_R(z)$

$$\mathcal{R}_{R,\sigma} := \mathrm{Res}_{z=\sigma}\zeta_R(z)$$

Sketch of proof:

- write associated series as

$$g_R(\tau) \sim \sum_{N,n} c_N r_n^N \tau^N = \sum_N \zeta_R(-N)\tau^N$$

- Mellin transform $\mathcal{M}(g)(z) = \int_0^\infty g(\tau)\tau^{z-1} d\tau$ gives

$$\mathcal{M}(g)(z) = (\sum_n r_n^{-z}) \int_0^\infty \sum_N c_N \, u^{N+z-1} du = \zeta_R(z) \cdot \mathcal{M}(f)(z)$$

- asymptotic expansion of $g_R(\tau)$ from Mellin transform $\mathcal{M}(g_R)(z)$ singular expansion

$$S_{\mathcal{M}(g_R)}(z) = \sum_{\sigma \in \mathcal{S}(\zeta_R)} \frac{\mathcal{R}_{R,\sigma}\, \mathcal{M}(f)(\sigma)}{z - \sigma} + \sum_{\sigma \in \mathcal{S}(\mathcal{M}(f))} \frac{\zeta_R(\sigma) c_\sigma}{z - \sigma}$$

- and from small time asymptotics of $f(\tau)$ know

$$S_{\mathcal{M}(f)}(z) = \sum_N \frac{c_N}{z + N}$$

## Feynman–Kac formula on $\mathbb{R} \times \mathcal{P}$ with $a_{n,k}^2 (dt^2 + a(t)^2 d\sigma^2)$

- on each $\mathbb{R} \times S^3_{a_{n,k}}$ decompose Dirac $D_{a_{n,k}}$ using operators

$$H_{m,n,k} = -a_{n,k}^{-2} \frac{d^2}{dt^2} + V_{m,n,k}(t)$$

$$V_{m,n,k} = \frac{(m+\frac{3}{2})}{a_{n,k}^2 \cdot a(t)^2} ((m + \frac{3}{2}) - a_{n,k} \cdot a'(t)$$

  as in Chamseddine–Connes

- Feynman–Kac formula

$$e^{-\tau^2 H_{m,n,k}}(t,t) = e^{-\frac{\tau^2}{a_{n,k}^2}(\frac{d^2}{dt^2} + a_{n,k}^2 V_{m,n,k})}(t,t)$$

$$= \frac{a_{n,k}}{2\sqrt{\pi}\tau} \int \exp(-\tau^2 \int_0^1 V_{m,n,k}(t + \sqrt{2}\frac{\tau}{a_{n,k}} \alpha(u))du) D[\alpha]$$

- Poisson summation to replace sum

$$\sum_m \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t)$$

with multiplicities $\mu(m)$ with the integral

$$\int_{-\infty}^{\infty} f_{\tau,\, n,k}(x)\, dx$$

$$f_{\tau,\, n,k}(x) = \left(x^2 - \frac{1}{4}\right) e^{-x^2 a_{n,k}^{-2} U - x a_{n,k}^{-1} V}$$

with $U$ and $V$ as in single sphere case

$$\sum_m \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t) =$$

$$\int \frac{a_{n,k}}{\tau} \left( \frac{e^{\frac{V^2}{4U}}}{4} (-a_{n,k} U^{-1/2} + 2 a_{n,k}^3 U^{-3/2} + a_{n,k}^3 V^2 U^{-5/2}) \right) D[\alpha]$$

$$= \int \frac{1}{\tau} \left( \frac{e^{\frac{V^2}{4U}}}{4} (-a_{n,k}^2 U^{-1/2} + a_{n,k}^4 (2U^{-3/2} + V^2 U^{-5/2})) \right) D[\alpha]$$

- same Taylor expansion method

$$e^{\frac{V^2}{4U}} U^r V^\ell = \tau^{2(r+\ell)} \sum_{M=0}^{\infty} a_{n,k}^{-M-2(r+\ell)} \, C_M^{(r,\ell)} \tau^M$$

with $C_M^{(r,\ell)}$ as in single sphere case $dt^2 + a(t)^2 d\sigma^2$

- resulting expansion

$$\sum_{n,k} \frac{1}{\tau} \left( \frac{e^{\frac{V^2}{4U}}}{4} (-a_{n,k}^2 U^{-1/2} + a_{n,k}^4 (2U^{-3/2} + V^2 U^{-5/2})) \right) =$$

$$\frac{1}{4} \sum_{M=0}^{\infty} \left( C_M^{(-5/2,2)} - C_M^{(-1/2,0)} \right) \zeta_{\mathcal{L}}(-M+2) \tau^{M-2}$$

$$+ \frac{1}{2} \sum_{M=0}^{\infty} C_M^{(-3/2,0)} \zeta_{\mathcal{L}}(-M+4) \tau^{M-4}.$$

- Feynman–Kac formula for the whole $\mathbb{R} \times \mathcal{P}$

$$\sum_{n,k} \sum_{m} \mu(m) e^{-\tau^2 H_{m,n,k}}(t,t) =$$

$$\sum_{M=0}^{\infty} \tau^{2M-4} \zeta_{\mathcal{L}}(-2M+4) \int \left( \frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} (C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}) \right) D[\alpha]$$

with only the term $\frac{1}{2} C_0^{(-3/2,0)}$ when $M = 0$

- obtained as a series

$$g_{\mathcal{L}}(\tau) = \sum_{n,k} f(a_{n,k}^{-1} \tau)$$

$$f(\tau) \sim \sum_{M} \tau^{2M-4} \int \left( \frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4} (C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}) \right) D[\alpha]$$

## Result: Spectral Action on Multifractal Robertson–Walker $\mathbb{R} \times \mathcal{P}$

$$\mathrm{Tr}(f(\mathcal{D}/\Lambda)) \sim$$

$$\sum_{M=0}^{\infty} \Lambda^{n_M} f_{n_M} \zeta_{\mathcal{L}}(n_M) \int \left( \frac{1}{2} C_{4-n_M}^{(-3/2,0)} + \frac{1}{4} (C_{2-n_M}^{(-5/2,2)} - C_{2-n_M}^{(-1/2,0)}) \right) D[\alpha]$$
$$+ \sum_{\sigma \in \mathcal{S}_{\mathcal{L}}} \tilde{f}(\sigma) \cdot f_\sigma \cdot \mathrm{Res}_{z=\sigma} \zeta_{\mathcal{L}} \cdot \Lambda^\sigma$$

$n_M = 4 - 2M$, set of poles $\mathcal{S}_{\mathcal{L}}$ of $\zeta_{\mathcal{L}}$ Mellin transform $\tilde{f}(z) = \mathcal{M}(f)(z)$ of $f(\tau) = \mathrm{Tr}(\exp(-\tau^2 D^2))$ with Dirac on $\mathbb{R} \times S^3$ with $dt^2 + a(t)^2 d\sigma^2$

Conclusion: presence of fractality detected by two types of effects

1. zeta regularization of coefficients $\zeta_{\mathcal{L}}(4-2M)$ in terms $\Lambda^{4-2M}$ (including effective gravitational and cosmological constant in top terms)
2. additional terms from non-real poles of order $\Lambda^{\Re\sigma}$ (and log periodic) with $3 < \Re\sigma = \dim_H \mathcal{P} < 4$ between cosmological and Einstein–Hilbert term

## Multifractal Robertson–Walker with non-round scaling

$dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

- rescaling $ds_a^2 = dt^2 + a^2 \cdot a(t)^2\, d\sigma^2$ with $a > 0$ gives $U \mapsto a^{-2}\, U$ and $V \mapsto a^{-1}\, V$
- this gives rescaling

$$\frac{1}{4}\sum_{M=0}^{\infty}(a^3\, C_M^{(-5/2,2)} - a\, C_M^{(-1/2,0)})\,\tau^{M-2} + \frac{1}{2}\sum_{M=0}^{\infty} a^3\, C_M^{(-3/2,0)}\,\tau^{M-4}$$

- expect presence of zeta regularized coefficients $\zeta_{\mathcal{L}}(3)$, $\zeta_{\mathcal{L}}(1)$
- to see this use a Mellin transform with respect to the "multiplicity variable" $x$ in $f_s(x)$

## Kummer confluent hypergeometric function

- notation: $a^{(n)} := a(a+1)\cdots(a+n-1)$ and $a^{(0)} := 1$
- Kummer confluent hypergeometric function defined by series

$$ {}_1F_1(a,b,t) = \sum_{n=0}^{\infty} \frac{a^{(n)} t^n}{b^{(n)} n!} $$

- solution of the Kummer equation

$$ t\frac{d^2 f}{dt^2} + (b-t)\frac{df}{dt} - af = 0. $$

## Mellin transform and hypergeometric function

- Mellin transform in the $x$-variable of the function

$$f_{s,-}(x) := f_s(x) = (x^2 - \frac{1}{4})e^{-x^2U-xV}$$

given by

$$\mathcal{M}((x^2 - \frac{1}{4})e^{-x^2U-xV})(z) = \frac{1}{8}U^{-(z+3)/2}\times$$

$$\Big(U^{1/2}\,\Gamma(\frac{z}{2})(-U\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}) + 2z\,{}_1F_1(\frac{z+2}{2},\frac{1}{2},\frac{V^2}{4U}))$$

$$+V\,\Gamma(\frac{z+1}{2})(U\,{}_1F_1(\frac{z+1}{2},\frac{3}{2},\frac{V^2}{4U}))-2(z+1)\,{}_1F_1(\frac{z+3}{2},\frac{3}{2},\frac{V^2}{4U}))\Big)$$

- similar expression for transform of $f_{s,+}(x) := (x^2 - \frac{1}{4})e^{-x^2U+xV}$

- multiplicity integral

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \int_0^{\infty} f_{s,-}(x)\, dx + \int_0^{\infty} f_{s,+}(x)\, dx$$

$$f_{s,\pm}(x) = (x^2 - \frac{1}{4}) e^{-x^2 U \pm x V}$$

- multiplicity integral as special value at $z = 1$ of Mellin

$$\int_{-\infty}^{\infty} f_s(x)\, dx = \mathcal{M}(f_{s,-})(z)|_{z=1} + \mathcal{M}(f_{s,+})(z)|_{z=1}$$

- Mellin transform $\mathcal{M}(f_{s,-})(z) + \mathcal{M}(f_{s,+})(z)$

$$= -\frac{1}{4}\, U^{-1-\frac{z}{2}}\, \Gamma(\frac{z}{2})(U\, {}_1F_1(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}) - 2z\, {}_1F_1(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}))$$

- value at $z = 1$

$$\left(-\frac{1}{4}\, U^{-(1+\frac{z}{2})}\, \Gamma(\frac{z}{2})(U\, {}_1F_1(\frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}) - 2z\, {}_1F_1(1 + \frac{z}{2}, \frac{1}{2}, \frac{V^2}{4U}))\right)|_{z=1} =$$

$$e^{\frac{V^2}{4U}} \frac{\sqrt{\pi}}{4}(-U^{-1/2} + 2U^{-3/2} + V^2 U^{-5/2})$$

## Effect of scaling

- notation:

$$H_\lambda(\tau,z) := \quad U^{-z/2}\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\lambda,\frac{V^2}{4U})$$

$$H(\tau,z) := \quad H_{1/2}(\tau,z) = U^{-z/2}\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U})$$

$$H_{\mathcal{L}}(\tau,z) := \quad U^{-z/2}\,\zeta_{\mathcal{L}}(z)\,\Gamma(z/2)\,{}_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}) = \zeta_{\mathcal{L}}(z)\,H(\tau,z)$$

- multiplicity integral with scaling $dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

$$\mathcal{M}(f_{s,n,k,-})(z) + \mathcal{M}(f_{s,n,k,+})(z) =$$

$$-\frac{1}{4}\,a_{n,k}^z\,U^{-\frac{z}{2}}\,\Gamma(\frac{z}{2})_1F_1(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U})$$

$$+a_{n,k}^{z+2}U^{1-\frac{z}{2}}\,\Gamma(1+\frac{z}{2})\,{}_1F_1(1+\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}))$$

$$-\frac{1}{4}a_{n,k}^z\,H(\tau,z) + a_{n,k}^{z+2}\,H(\tau,z+2).$$

- multiplicity integral over the full sphere packing $\mathbb{R} \times \mathcal{P}$

$$f_{\mathcal{P},s}(x) = \left(x^2 - \frac{1}{4}\right) \sum_{n,k} e^{-x^2 a_{n,k}^{-2} U - x a_{n,k}^{-1} V}$$

- as value of Mellin transform

$$\int_{-\infty}^{\infty} f_{\mathcal{P},s}(x)\, dx = \mathcal{M}(f_{\mathcal{P},s,-})(z)|_{z=1} + \mathcal{M}(f_{\mathcal{P},s,+})(z)|_{z=1}$$

$f_{\mathcal{P},s,\pm} = (x^2 - 1/4) \sum_{n,k} \exp(-x^2 a_{n,k}^{-2} U \pm x a_{n,k}^{-1} V)$

- Mellin transforms

$$\mathcal{M}(f_{\mathcal{P},s,-})(z) + \mathcal{M}(f_{\mathcal{P},s,+})(z) = -\frac{1}{4} H_{\mathcal{L}}(\tau, z) + H_{\mathcal{L}}(\tau, z+2)$$

- this shows one gets zeta regularized $\zeta_{\mathcal{L}}(3)$ and $\zeta_{\mathcal{L}}(1)$

Sketch of how to see the log periodic terms for non-round scaling $dt^2 + a_{n,k}^2 a(t)^2 d\sigma^2$

- $\tau$ expansion

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!} \tau^n, \quad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!} \tau^n$$

- gives expansion of confluent hypergeometric function

$$H_{\mathcal{L}}(\tau, z) = \zeta_{\mathcal{L}}(z)\Gamma(z/2) U^{-z/2} \sum_{n=0}^{\infty} \frac{(z/2)_n}{4^n \, n! \, (1/2)_n} V^{2n} U^{-n}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$ the rising factorial

- the term $U^{-z/2}$ contributes a term with $\tau^z$ times a power series in $\tau$, while confluent hypergeometric function contributes a power series in $\tau$

- to see why $\tau^z$ gives rise to log periodic terms in the spectral action consider simplified case
- product of the Mellin transforms $\mathcal{M}(f_1)(z) \cdot \mathcal{M}(f_2)(z)$ is Mellin transform of convolution

$$\mathcal{M}(f_1)(z) \cdot \mathcal{M}(f_2)(z) = \mathcal{M}(f_1 \star f_2)(z),$$

$$(f_1 \star f_2)(x) = \int_0^\infty f_1(\frac{x}{u}) f_2(u) \, \frac{du}{u}$$

- Mellin transform of a delta distribution

$$\tau^{z-1} = \mathcal{M}(\delta(x - \tau))$$

- Mellin transform of distribution $\Lambda_{\mathcal{P},\tau} := \sum_{n,k} \tau \, a_{n,k} \; \delta(x - \tau \cdot a_{n,k})$

$$\langle \sum_{n,k} \tau \, a_{n,k} \; \delta(x - \tau \cdot a_{n,k}), \phi(x) \rangle = \sum_{n,k} \tau \, a_{n,k} \; \phi(\tau \, a_{n,k})$$

given by

$$\tau^z \, \zeta_{\mathcal{L}}(z) = \mathcal{M}(\sum_{n,k} \tau \, a_{n,k} \; \delta(x - \tau \cdot a_{n,k}))$$

- given function $g(x)$
  will want $g_\gamma(x) := \mathcal{M}^{-1}(\Gamma(z/2)\, {}_1F_1(z/2, 1/2, \gamma))$

$$\mathcal{M}(\Lambda_{\mathcal{P},\tau})(z) \cdot \mathcal{M}(g)(z) = \mathcal{M}(\Lambda_{\mathcal{P},\tau} \star g)(z)$$

$$= \mathcal{M}(\sum_{n,k} \tau a_{n,k} \int_0^\infty \delta(u - \tau a_{n,k})\, g(\frac{x}{u})\, \frac{du}{u}) = \sum_{n,k} \mathcal{M}(g(\frac{x}{\tau \cdot a_{n,k}}))$$

- take $h_z(\tau) := \mathcal{M}(g(\frac{x}{\tau}))$

$$L_z(\tau) := \sum_{n,k} h_z(\tau \cdot a_{n,k})$$

- asymptotic expansion for this function through singular expansion of Mellin transform in $\tau$

$$\mathcal{M}_\tau(L_z(\tau))(\beta) = \zeta_{\mathcal{L}}(\beta) \cdot \mathcal{M}(h_z(\tau))(\beta)$$

- contributions from poles of $\zeta_{\mathcal{L}}(\beta)$ and of $\mathcal{M}(h_z(\tau))(\beta)$: log-periodic and zeta regularized terms as expected

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