Feynman integrals and periods in configuration spaces

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Abstract. We describe two different constructions related to Feynman amplitudes in configuration spaces. One of them corresponds to the physical Feynman amplitude, while the second problem is, in a suitable sense, a complexification. We show that the two problems, while similar in form, lead to very different mathematical techniques and results. In the first case, we use analytic methods related to the expansion of the Green function of the Laplacian in Gegenbauer polynomials, while in the second case we use algebro-geometric techniques related to wonderful compactifications of configuration spaces. In the first case, we identify a contribution to the Feynman integral, which has a natural description in representation theoretic terms, which is expressible in terms of Mordell–Tornheim and Apostol–Vu series, even when the remaining terms of the integral might not be expressible in terms of multiple zeta values. In the second case, we can always express the resulting integral as an integral of an algebraic differential form over a variety whose motive is mixed Tate, even though the argument does not directly ensure that the algebraic differential form will be defined over the rationals.

1. Introduction

The past decade has seen the development of a new field of inquiry, aimed at understanding the geometric and algebraic structure underlying regularization and renormalization of Feynman integrals as well as the number theoretic properties of the residues of Feynman graphs obtained in perturbative quantum field theory. In particular, an unexpected connection between quantum field theory and Grothendieck's theory of motives of algebraic varieties was uncovered. The main idea behind this relation is that (regularized) integrals associated to Feynman graphs can be interpreted as periods of some algebraic varieties, and the nature of the motive of the variety can predict what class of numbers can arise as periods. An especially interesting class of numbers that occur as periods are the multiple zeta values, which occur as periods of an especially interesting class of motives, the mixed Tate motives. Thus, within the general framework of the relation between quantum field theory and motives, specific questions have often revolved around whether the specific motives are mixed Tate, and the periods expressible as combinations of multiple zeta values. This area of research has grown rapidly in the past

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few years, hence the list of relevant references is extensive. Without any claim of completeness, we refer the reader to some of the relevant literature, starting with the paper of Bloch, Esnault and Kreimer [**22**], including several other contributions to the subject: [**7**], [**8**], [**9**], [**10**], [**24**], [**27**], [**30**], [**31**], [**32**], [**33**], [**40**], [**44**], [**67**].

In the perturbative expansion in Feynman graphs of a Euclidean scalar massless quantum field theory, one can compute the contributions of individual Feynman graphs either in configuration or in momentum space. A good part of the literature on the subject of motives and periods in quantum field theory has focused on the momentum space picture, and especially the Feynman parametric form, where the contribution of a Feynman graph is interpreted as the (possibly divergent) integral of an algebraic differential form on the complement of a hypersurface (the graph hypersurface) in a projective space. In this setting, the relevant motivic question is the nature of the motive of the graph hypersurface. General results of Belkale and Brosnan [**16**] show that the graph hypersurfaces additively span a localized version of the Grothendieck ring of varieties, hence they can be arbitrarily far from the mixed Tate case, and more concrete recent results of Doryn, Brown and Doryn, and Schnetz, provided explicit non-mixed-Tate examples, [**30**], [**31**], [**44**].

In this paper, we will focus on the dual picture in configuration space. We will consider both the physical case of the Feynman amplitude and some mathematical generalizations that show similar but different behavior. Clearly, momentum and configuration space approaches are dual pictures, related by a Fourier transform of the propagators, as we will show more precisely in Section 3.4 below. However, Fourier transform is not an algebraic operation, hence there is no a priori reason why it should preserve the nature of motives and periods, other than a general philosophical belief in the consistency of the universe. In other words, Fourier transform preserves the L^2 but not the L^1 norm, hence it does not preserve period integrals. These also take place in different dimensions: due to momentum conservation, integration takes place in a space of dimension $b_1(\Gamma)$ in the momentum space formulation, and in dimension $\# V_\Gamma$ in the configuration space case. All this simply means that one cannot just appeal directly to Fourier duality to explain the equivalence of the two pictures, in terms of the arithmetic information encoded in the Feynman integrals. One has to develop a mathematical argument that shows what happens in both cases and how they are related.

1.1. Integrals in configuration spaces.

In the present paper we will consider two different cases of integrals in configuration spaces. The first case, which we refer to as *the real case*, corresponds to the physically relevant Feynman integral, while the second case, which we call *the complexified case* is a simple mathematical generalization. We find it instructive to compare these two cases, because although they at first look very similar, they lead to different mathematical techniques and different results.

In both cases, for consistency, we use the following general setting and notation. Let Γ be a Feynman graph of the given Euclidean scalar massless quantum field theory. We assume chosen an orientation of Γ. Let $D \in \mathbb{N}$ be the spacetime dimension of the theory. Let X be a D-dimensional algebraic variety, which contains a dense big cell $\mathbb{A}^D \subset X$. Let $\mathrm{Conf}_\Gamma(X) = X^{V_\Gamma} \smallsetminus \cup_{e\in E_\Gamma} \Delta_e$ denote the configuration space of the graph Γ, where $\Delta_e = \{x = (x_v) \,|\, x_{s(e)} = x_{t(e)}\}$ are the diagonals. The geometry of these configuration spaces and of their wonderful compactifications is described in [**36**], [**61**], [**62**].

Note: here and elsewhere in the paper, for a space X and a finite set S we use the standard notation $X^S = \mathrm{Maps}(S, X) \cong X^{\#S}$, where $\#S$ is the cardinality, and the identification is through a choice of a bijection $S \to \{1, \dots, \#S\}$. Thus, in the above, any such bijection identifies $X^{V_\Gamma} \cong X^N$ with $N = \#V_\Gamma$.

1.1.1. *The real case.* In this case, we associate to the graph Γ the differential form of middle-dimensional degree DN, with $N = \#V_\Gamma$,

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v,$$

where $D = 2\lambda + 2$, and where $\| \cdot \|$ is the Euclidean norm on $\mathbb{A}^{DN}(\mathbb{C})$.

This a $\mathcal{C}^\infty$-differential form with singularities along diagonals $x_{s(e)} = x_{t(e)}$. The domain of integration in this case is the middle dimensional locus of real points

$$\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}.$$

For later purposes, let us emphasize the following two facts:

(1) ω_Γ is not an algebraic form;
(2) ω_Γ is not a closed form on $\mathrm{Conf}_\Gamma(X)$.

1.1.2. *The complexified case.* In this other case we consider the variety $Z = X \times X$ with projection $p : Z \to X$, $p : z = (x, y) \mapsto x$. We associate to the graph Γ a differential form of middle-dimensional degree $2DN$, with $N = \#V_\Gamma$,

$$\omega_\Gamma^{(Z)} = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v$$

where $\|x_{s(e)} - x_{t(e)}\| = \|p(z)_{s(e)} - p(z)_{t(e)}\|$. As chain of integration, in this case, we consider the middle dimensional locus

$$\sigma^{(Z,y)} = X^{V_\Gamma}(\mathbb{C}) = X^{V_\Gamma} \times \{y = (y_v)\} \subset Z^{V_\Gamma} = X^{V_\Gamma} \times X^{V_\Gamma}$$

for a fixed $y = (y_v \mid v \in V_\Gamma)$. In this case, we have the following situation:

(1) $\omega_\Gamma^{(Z)}$ is not an algebraic form;
(2) $\omega_\Gamma^{(Z)}$ is a closed form on $\mathrm{Conf}_\Gamma(X) \times X^{V_\Gamma}$.

1.2. Algebraic versus smooth differential forms. In the typical physical situation, the relevant integral is over a locus of real coordinates, corresponding to the physical momenta, or positions. It is in general convenient, in order to apply geometric methods, to extend the differential form from the real locus to an ambient space with complex coordinates. In general, there are inequivalent ways of doing this, which agree on the physically relevant locus of real coordinates. Since the extension to complex coordinates is an artifact and the values on the real locus are all that matters to the computation of the integral, the result is independent of the chosen extension. However, different extensions have different properties and lead to different methods of dealing with the question of the arithmetic nature of the Feynman integral.

1.2.1. *Algebraic formulations.* Typical algebraic formulations of Feynman integrals extend the integration form from real to complex variables using a quadratic form (algebraic) instead of a Euclidean norm. For example, in momentum space amplitudes, for $n = \#E_\Gamma$, one writes the amplitude at

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^{n} \epsilon_{v,i} k_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n,$$

where $\epsilon_{v,i}$ and $\epsilon_{v,j}$ are incidence matrices for internal and external edges, that express the linear constraints of momentum conservation at vertices, and the denominator is written as a product of quadratic form (in the case with non-zero mass m)

$$q_e(k_e) = \sum_{j=1}^{D} k_{e,j}^2 + m^2.$$

In configuration space, with mass $m = 0$, the analogous approach would correspond to an integral, for $N = \#V_\Gamma$,

$$U(\Gamma) = \int \frac{1}{Q_1 \cdots Q_n} d^D x_{v_1} \cdots d^D x_{v_N},$$

where the quadratic forms in the denominator are given by

$$Q_e(x_{s(e)}, x_{t(e)}) = \sum_{j=1}^{D} (x_{s(e),j} - x_{t(e),j})^2.$$

This point of view for configuration space integrals was discussed in [**36**].

The advantage of this algebraic approach to extending the form from real to complex coordinates is that one directly deals with an algebraic differential form. Thus, the interpretation of the Feynman integral as a period (modulo the convergence issue), is completely transparent, as it is manifestly written as an integral of an algebraic differential form on a locus defined by algebraic equations inside an algebraic variety.

The disadvantage of this method is that the singularities of the algebraic differential form occur along a hypersurface defined by the vanishing of the product of quadratic forms $Q_1 \cdots Q_n$, and typically the motive of this hypersurface is difficult to control.

1.2.2. *Non-algebraic formulation.* The non-algebraic formulation, which is our main focus in the present paper, extends the form from real to complex coordinates using a Euclidean norm. The resulting form

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v$$

is manifestly not an algebraic differential form. It is a smooth $\mathcal{C}^\infty$-form on the complement of the diagonals Δ_e, where it is singular.

The advantage of this approach is that now the locus of singularities of the form is not a hypersurface, but a union of diagonals, which, as we will see more in detail, is much easier to control in geometric and motivic terms.

The obvious disadvantage lies in the fact that, since the form is not algebraic, one cannot directly describe the integral as a period, and one should either use cohomological methods for replacing the form with an algebraic one in the same cohomology class (closed form case) or proceed via some explicit computation that identifies the result as a period by direct inspection (non-closed form case).

1.3. Structure of the paper. Section 2 gives a quick overview of the questions and methods developed to investigate the relation between Feynman integrals and motives from different viewpoints.

Section 3 introduces algebraic varieties $X^{\mathbf{V}_\Gamma}$ and $Z^{\mathbf{V}_\Gamma}$ that describe configuration spaces associated to a Feynman graph Γ, and smooth differential forms on these varieties, with singularities along diagonals, that extend the Feynman rules and Feynman propagators in configuration spaces to varieties containing a dense affine space that represents the "physical space" on which the Feynman diagram integration lives.

We consider three possible variants of the geometry, and of the corresponding differential form, respectively given by the forms (3.2), (3.9) and (3.12), because this will allow us to present different methods of regularization and integration and show, in different ways, how the relation between Feynman integrals and periods of mixed Tate motives arises in the setting of configuration spaces.

These include the "real case" and "complexified case" discussed above, as well as another variant that is related to some interesting integral kernels we will discuss in Section 8.

The geometric setting builds upon our previous work [**36**], and we will be referring to that source for several of the algebro-geometric arguments we need to use here. In terms of the Feynman amplitudes themselves, the main difference between the approach followed in this paper and the one of [**36**], as we mentioned above, is in the non-algebraic nature of the differential form. In our previous work [**36**] we extended the Feynman propagator to the configuration space $X^{\mathbf{V}_\Gamma}$ as an algebraic differential form, which then had singularities not only along the diagonals, but also along a quadric Z_Γ (the configuration space analog of the graph hypersurfaces describing singularities of Feynman amplitudes in momentum space). In this paper, we extend the Feynman propagator to a $\mathcal{C}^\infty$ (non-algebraic) differential form on $X^{\mathbf{V}_\Gamma}$, which then has singularities only along the diagonals.

In the real case, we will deal with the resulting integrals through analytic methods, and identify a term that is expressible through multiple zeta values. In the complexified case, we will appeal to cohomological arguments to replace the form with an algebraic one. In this second case, we will also use the fact that, having singularities only along diagonals allows us to keep a good control over the motive, which does not leave the mixed Tate class.

Section 4 deals with the "real case" (the physical Feynman amplitude). We consider the first form (3.2) as the Feynman amplitude and its restriction to the affine space $\mathbb{A}^{D\mathbf{V}_\Gamma}(\mathbb{R})$ in the real locus in the graph configuration space $X^{\mathbf{V}_\Gamma}$, with $X = \mathbb{P}^D(\mathbb{C})$. We first describe a setting generalizing a recent construction, by Spencer Bloch, of cycles in the configuration spaces $X^{\mathbf{V}_\Gamma}$. Our setting assigns a middle-dimensional relative cycle, with boundary on the union of diagonals, to any acyclic orientation on the graph. We then consider the development of the propagator into Gegenbauer orthogonal polynomials. This is a technique for the calculation of Feynman diagrams in configuration spaces (known as x-space technique) that is well established in the physics literature since at least the early ’80s, [**38**]. We split these x-space integrals into contributions parameterized by the chains described above. The decompositions of the chains of integration, induces a corresponding decomposition of the Feynman integral into an angular and a radial part. In the case of dimension $D = 4$, and for polygon graphs these simplify into integrals over simplexes of polylogarithm functions, which give rise to zeta values. For more general graphs, each star of a vertex contributes a certain combination of integrals of triple integrals of spherical harmonics. These can be expressed in terms of isoscalar coefficients. We compute explicitly, in the case $D = 4$ the top term of this expansion, and show that, when pairs of half edges are joined together to form an edge of the graph, one obtains a combination of nested sums that can be expressed in terms of Mordell–Tornheim and Apostol–Vu series. This result means that, even when the overall Feynman amplitude would contain non-mixed Tate periods, one can isolate a contribution, which is defined in representation theoretic terms, which remains a mixed Tate period.

Starting with §5, we consider what we called the “complexified case”. Here we deal with a different integration form, given by (3.9), which is a generalization of the Feynman amplitude previously considered. This is defined after performing a doubling of the dimension of the relevant configuration spaces $Z^{\mathbf{V}_\Gamma} \simeq (X \times X)^{\mathbf{V}_\Gamma}$. The advantage of passing to this formulation is that one is then dealing with a closed form, unlike the case of (3.2) considered in the previous section, hence cohomological arguments become available. We first describe the simple modifications to the geometry of configuration spaces, with respect to the results of our previous work [**36**], which are needed in order to deal with this doubling of dimension. We introduce the wonderful compactification $F(X, \Gamma)$ of the configuration space $\overline{Z^{\mathbf{V}_\Gamma}}$, which works pretty much as in the case of the wonderful compactification $\overline{\mathrm{Conf}_\Gamma(X)}$ of $X^{\mathbf{V}_\Gamma}$ described in [**36**], [**61**], [**62**], with suitable modifications. The purpose of passing to the wonderful compactification is to realize the complement of the diagonals in the configuration space explicitly as the complement of a union of divisors intersecting transversely, so that we can describe de Rham cohomology classes in terms of representatives that are algebraic forms with logarithmic poles along the divisors. We also discuss, in this section, the properties of the motive of the wonderful compactification and we verify that, if the underlying variety X is a Tate motive, the wonderful compactification also is, and so are the unions and intersections of the divisors obtained in the construction. We also describe iterated Poincaré residues of these forms along the intersections of divisors, according to the general theory of iterated Poincaré residues of forms with logarithmic poles, [**2**], [**3**], [**42**], [**52**].

In §6, we consider a first regularization method for the Feynman integral based on the amplitude (3.9), which is essentially a cutoff regularization and is expressed in terms of principal-value currents and the theory of Coleff–Herrera residue currents, [**18**], [**39**]. We show that, when we regularize the Feynman integral as a principal value current, the ambiguity is expressed by residue currents supported on the intersections of the divisors of the wonderful compactification, which are related to the iterated Poincaré residues described in §5. In particular, when evaluated on algebraic test differential forms on these intersections, the currents describing the ambiguities take values that are periods of mixed Tate motives.

In §7, we introduce a more directly algebro-geometric method of regularization of both the Feynman amplitude form (3.9) and of the chain of integration, which is based on the *deformation to the normal cone* [**46**], which we use to separate the chain of integration from the locus of divergences of the form. We check again that the motive of the deformation space constructed using the deformation to the normal cone remains mixed Tate, and we show once again that the regularized Feynman integral obtained in this way gives rise to a period.

Section 8 is an Appendix, where we present additional material that can be read independently of the rest of the paper (or can be skipped by readers more directly interested in the main results on periods). Its main purpose is to explain the relation between the Feynman amplitudes (3.2), (3.9) and (3.12) and real and complex Green functions and the Bochner–Martinelli integral kernel operator. In particular, we show that the form of the Feynman amplitude leads naturally to a Bochner–Martinelli type theorem, adapted to the geometry of graph configuration spaces, which combines the usual Bochner–Martinelli theorem on complex manifolds [**53**] and the notion of harmonic functions and Laplacians on graphs.

2. Motives and Periods in Feynman Integrals

This introductory section was added at the request of the volume editors as a short overview of the occurrence of motives and periods in quantum field theory.

The empirical observation of the pervasive presence of multiple zeta values in Feynman integral of massless scalar field theories illustrated in the extensive analysis carried out by Broadhurst and Kreimer [**28**] suggested a possible deeper relation between periods and Feynman integrals. Indeed, it was known by [**51**] that multiple zeta values can be realized as periods of motives (moduli spaces of curves of genus zero) that are mixed Tate, and it was conjectured and later proved by Brown [**29**] that indeed all periods of mixed Tate motives over $\mathbb{Z}$ are $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of multiple zeta values.

At the level of formal categorical structures, the work of Connes and the second author [**40**] (see also Chapter 1 of [**41**]), showed that the divergences of a scalar field theory can be organized into a Tannakian category of irregular-singular differential systems, whose Tannakian Galois group has the same kind of pro-unipotent structure as the motivic Galois groups of mixed Tate motives. While this approach is also suggestive of a direct relation between quantum field theory and motives, it does not provide a way to associate a specific motive to a specific Feynman integral.

In order to analyze motives and periods arising from individual Feynman integrals, one can consider different ways of writing the (possibly divergent) integrals associated to individual Feynman graphs. For a (massive) Euclidean scalar field theory in dimension D, the unrenormalized Feynman integral of a Feynman graph is given, up to a divergent Γ-factor, by the expression

$$C \int \frac{\prod_{v\in V(\Gamma)} \delta(\sum_{e\in E_{int}(\Gamma)} \epsilon_{v,e} k_e + \sum_{j\in E_{ext}(\Gamma)} \epsilon_{v,j} p_j)}{\prod_{e\in E_{int}(\Gamma)} q_e(k_e)} \prod_{e\in E_{int}(\Gamma)} \frac{d^D k_e}{(2\pi)^D}$$

where $q_e(k_e)$ is a quadratic form in the momentum variables k_e and the mass parameters m_e associated to the internal edges $e \in E_{int}(\Gamma)$,

$$q_e(k_e) = \sum_{i=1}^{D} k_{e,i}^2 + m_e^2.$$

The constant is given by

$$C = \prod_v \lambda_v (2\pi)^{-D}$$

with λ_v the coupling constant of the $\phi^{\iota(v)}$ term in the Lagrangian, with $\iota(v)$ the valence of the vertex v, and where p_j are the momenta associated to the external edges $e_j \in E_{ext}(\Gamma)$ of the graph Γ. The matrix $(\epsilon_{v,e})_{v\in V(\Gamma), e\in E_{int}(\Gamma)}$ is the incidence matrix with entries ± 1 or 0 according to whether $v = s(e)$, or $v = t(e)$ or $v \notin \partial(e)$, and similarly for $\epsilon_{v,j}$, with e_j running over the external edges. The explicit derivation of the perturbative expansion with this form of the Feynman integrals from the Lagrangian of the quantum field theory and the formal infinite dimensional integral is recalled briefly in the first chapter of [**65**], or in the first chapter of [**41**], see also standard quantum field theory textbooks like [**20**], [**57**].

Under the assumption that the parameters p_j and λ_v are rational numbers, the integral above (if convergent) could be seen as a period of the complement of a union of quadrics. It is important to notice that, in general, in the *massive* case with $m = (m_e) \neq 0$, one does not expect to find mixed Tate motives even for very simple graphs like the sunset graph and other so called banana graphs (two vertices with a set of parallel edges between them). Indeed, it was shown in [**23**], [**23**], [**85**] that non-mixed-Tate motives and periods occur in the case of the sunset and the 3-banana graph. This massive case should be contrasted with the massless case, where the banana graphs provide a very simple example where one can compute explicitly the motive (at least at the level of classes in the Grothendieck ring) and see easily that it is mixed Tate, [**6**].

Indeed the case of the massless Feynman integrals is significantly different from the massive case, in terms of their motivic properties and one encounters mixed-Tate motives for all graphs up to a significant size. The integral computation, in the massless case, can be reformulated in Feynman's parametric form (see [**20**], [**71**], and also Chapter 3 of [**65**]). When rewritten in terms of Feynman parameters, the Feynman integrals for a massless field theory take the form

$$C \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}} \omega_n$$

where $C \in \mathbb{Q}$ is a constant, p are the external momenta, $n = \# E_{int}(\Gamma)$ and $\ell = b_1(\Gamma)$, with locus of integration σ_n the n-simplex, and volume form ω_n. The integrand is a rational function with polynomials Ψ_Γ and P_Γ consisting of the Kirchhoff (or first Symanzik) polynomial

$$\Psi_\Gamma(t) = \sum_{T\subset\Gamma} \prod_{e\notin T} t_e$$

and the second Symanzik polynomial

$$P_\Gamma(t,p) = \sum_{C\subset\Gamma} s_C \prod_{s\in C} t_e$$

with the sum over cit-sets C of Γ (collections of $b_1(\Gamma)+1$ edges consisting of the complement of a spanning tree plus one edge), with s_C a quadratic function of the external momenta. In the case where all the parameters are rational and $-n + D\ell/2 \geq 0$, the integral (if convergent) would be identified with a period of the complement of the graph hypersurface $X_\Gamma = \{\Psi_\Gamma = 0\}$.

Kontsevich conjectured that the motives of the graph hypersurfaces X_Γ would always be mixed Tate, which would explain the occurrence of multiple zeta values. This conjecture was disproved by Belkale and Brosnan [**16**], who showed that the graph hypersurfaces generate the localization of the Grothendieck ring at the multiplicative subset S generated by $\mathbb{L}^n - \mathbb{L}$, where $\mathbb{L} = [\mathbb{A}^1]$ is the Lefschetz motive, as a module over the localization $S^{-1}\mathbb{Z}[\mathbb{L}]$. This implies that, for sufficiently large graph, not only the classes of the graph hypersurfaces will leave the mixed Tate world but they will become arbitrarily complicated as motives. This can be contrasted with the remarkably simple behavior of the Grothendieck classes of the graph hypersurfaces in the quotient of the Grothendieck ring obtained by killing the Lefschetz motive, as described in [**11**].

Even though the general result of [**16**] shows that the classes of the graph hypersurfaces are not always mixed-Tate, explicit examples are difficult to come by and only occur for graphs that are considered to be very large from the quantum field theory perspective. Indeed, it was first proved by Stembridge [**81**] that all graphs up to 12 edges contribute mixed Tate motives, and the first explicit non-mixed-Tate examples, found by Doryn [**44**] and Schnetz [**75**] occur only at 14 and 16 edges.

Most of the results on the relation between quantum fields and motives have been obtained so far using the Feynman parametric form of the integrals and studying the graph hypersurfaces, and the problem of divergences has been approached in terms of blowups and deformations of the graph hypersurface, see [**22**], [**24**], [**66**]. Other methods involved mapping the integrals to the complement of determinant hypersurfaces and their compactifications, [**7**], [**67**]. In all of these cases, one is dealing with the formulation of Feynman integrals in momentum variables. In this paper instead, following the approach developed in [**36**], [**37**], we consider instead Feynman integrals written in terms of position variables in configuration spaces.

3. Feynman amplitudes in configuration spaces and complexifications

In the following we let X be a D-dimensional smooth projective variety, which contains a dense subset isomorphic to an affine space $\mathbb{A}^D$, whose set of real points $\mathbb{A}^D(\mathbb{R})$ we identify with Euclidean D-dimensional spacetime. For instance, we can take $X = \mathbb{P}^D$. We assume that D is even and we write $D = 2\lambda + 2$.

3.1. Feynman graphs.

A *graph* Γ is a one-dimensional finite CW-complex. We denote the set of vertices of Γ by V_Γ, the set of edges by E_Γ, and the boundary map by $\partial_\Gamma : E_\Gamma \to (V_\Gamma)^2$.

When an orientation is chosen on Γ, we write $\partial_\Gamma(e) = \{s(e), t(e)\}$, the source and target vertices of the oriented edge e.

A *looping* edge is an edge for which the two boundary vertices coincide and *multiple edges* are edges between the same pair of vertices. We assume that our graphs have no looping edges.

3.1.1. *Induced subgraphs and quotient graphs.* A subgraph $\gamma \subseteq \Gamma$ is called an *induced subgraph* if its set of edges E_γ is equal to $\partial_\Gamma^{-1}((V_\gamma)^2)$, that is, γ is determined by a choice of a set of vertices $V_\gamma \subseteq V_\Gamma$ with all the edges of Γ on the same set of vertices V_γ. We will denote by $\mathbf{SG}(\Gamma)$ the set of all induced subgraphs of Γ and by

$$\mathbf{SG}_k(\Gamma) = \{\gamma \in \mathbf{SG}(\Gamma) \mid |V_\gamma| = k\}, \tag{3.1}$$

the subset $\mathbf{SG}_k(\Gamma) \subseteq \mathbf{SG}(\Gamma)$ of all induced subgraphs with k vertices.

For $\gamma \in \mathbf{SG}(\Gamma)$, we denote by $\Gamma//\gamma$ the graph obtained from Γ by shrinking each connected component of γ to a separate vertex. The quotient graph $\Gamma//\gamma$ does not have looping edges since we consider only induced subgraphs.

3.2. Feynman amplitude.

When computing Feynman integrals in configuration space, one considers singular differential forms on X^{V_Γ}, integrated on the real locus of this space.

Definition 3.1. The Euclidean massless Feynman amplitude in configuration space is given by the differential form

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v. \tag{3.2}$$

The form (3.2) defines a $\mathcal{C}^\infty$-differential form on the configuration space

$$\mathrm{Conf}_\Gamma(X) = X^{V_\Gamma} \smallsetminus \cup_{e \in E_\Gamma} \Delta_e, \tag{3.3}$$

with singularities along the diagonals

$$\Delta_e = \{(x_v \,|\, v \in V_\Gamma) \mid x_{s(e)} = x_{t(e)}\}. \tag{3.4}$$

Remark 3.2. The diagonals (3.4) corresponding to edges between the same pair of vertices are the same, consistently with the fact that the notion of degeneration that defines the diagonals is based on “collisions of points” and not on contraction of the edges connecting them. This suggests that we may want to exclude graphs with multiple edges. However, multiple edges play a role in the definition of Feynman amplitudes (see Definition 3.1 above and §4 of [**36**]) hence we allow this possibility. On the other hand, the definition of configuration space is void in the presence of looping edges, since the diagonal Δ_e associated to a looping edge is the whole space X^{V_Γ}, and its complement is empty. Thus, our assumption that graphs have no looping edges is aimed at excluding this geometrically trivial case. Looping edges can also occur in Feynman graphs, for instance in tadpole graphs, but the geometry of the configuration spaces $\mathrm{Conf}_\Gamma(X)$ is just not suitable for treating these cases.

Remark 3.3. As we already discussed in the Introduction, our choice of the (non-algebraic) Euclidean norm $\|x_{s(e)} - x_{t(e)}\|^2$ in the Feynman propagator differs from the customary choice of propagator (see for instance [**36**]) where the algebraic quadratic form $(x_{s(e)} - x_{t(e)})^2$ is used instead, but these two expressions agree on the locus $X^{V_\Gamma}(\mathbb{R})$ of real points, which is the chain of integration of the Feynman amplitude. The latter choice gives a manifestly algebraic differential form, but at the cost of introducing a hypersurface $\mathcal{Z}_\Gamma$ in X^{V_Γ} where the singularities of the form occur, which makes it difficult to control explicitly the nature of the motive. Our choice here only gives a $\mathcal{C}^\infty$ differential form, but the domain of definition is now simply $\mathrm{Conf}_\Gamma(X)$, whose geometry is much easier to understand than that of $\mathrm{Conf}_\Gamma(X) \smallsetminus \mathcal{Z}_\Gamma$, see [**36**].

Formally (before considering the issue of divergences), the Feynman integral is obtained by integrating the form (3.2) on the locus of real points of the configuration space.

Definition 3.4. The chain of integration for the Feynman amplitude is taken to be the set of real points of this configuration space,

$$\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}. \tag{3.5}$$

The form (3.2) defines a top form on σ_Γ. There are two sources of divergences in integrating the form (3.2) on σ_Γ: the intersection between the chain of integration and the locus of divergences of ω_Γ,

$$\sigma_\Gamma \cap \cup_e \Delta_e = \cup_e \Delta_e(\mathbb{R})$$

and the behavior at infinity, on $\Delta_\infty := X \smallsetminus \mathbb{A}^D$. In physics terminology, these correspond, respectively, to the *ultraviolet* and *infrared* divergences.

We will address these issues in §4 below.

3.3. Complexification and other variations upon a theme. In addition to the form (3.2) considered above, we will also consider other variants, which will allow us to discuss different possible methods to address the question of periods and in particular the occurrence of multiple zeta values in generalizations of Feynman integrals in configuration spaces. As mentioned in the Introduction, comparing two apparently very similar cases of the Feynman amplitude (3.2) and of the complexified version (3.9) will allow us to illustrate how these lead to the use of very different mathematical methods.

3.3.1. *Complexified case.* In this setting, instead of the configuration space X^{V_Γ} and its locus of real points $\sigma = X^{V_\Gamma}(\mathbb{R})$, we will work with a slightly different space, obtained as follows.

As above, let X be a smooth projective variety, and Z denote the product $X \times X$. Let $p : Z \to X$, $p : z = (x, y) \mapsto x$ be the projection. This trivial doubling of the space X and projection are introduced purely as a simple artifact to ensure that (as in the real case) the differential form and locus of integration we will be considering in this complexified setting will still be middle-dimensional. In essence, the "complexification" consists of the fact that we will be replacing real affine spaces $\mathbb{A}^D(\mathbb{R})$ in the chain of integration with complex $\mathbb{A}^D(\mathbb{C})$.

Given a graph Γ, the *configuration space* $F(X, \Gamma)$ of Γ in Z is the complement

$$Z^{V_\Gamma} \setminus \bigcup_{e \in E_\Gamma} \Delta_e^{(Z)} \cong (X \times X)^{V_\Gamma} \setminus \bigcup_{e \in E_\Gamma} \Delta_e^{(Z)}, \tag{3.6}$$

in the cartesian product $Z^{V_\Gamma} = \{(z_v \mid v \in V_\Gamma)\}$ of the *diagonals associated to the edges* of Γ,

$$\Delta_e^{(Z)} \cong \{(z_v \mid v \in V_\Gamma) \in Z^{V_\Gamma} \mid p(z_{s(e)}) = p(z_{t(e)})\}. \tag{3.7}$$

The relation between the configuration space $F(X, \Gamma)$ and the previously considered $\mathrm{Conf}_\Gamma(X)$ of (3.3) is described by the following.

Lemma 3.5. *The configuration space* $F(X, \Gamma)$ *is isomorphic to*

$$F(X, \Gamma) \simeq \mathrm{Conf}_\Gamma(X) \times X^{V_\Gamma}, \tag{3.8}$$

and the diagonals (3.7) *and related to those of* (3.4) *by* $\Delta_e^{(Z)} \cong \Delta_e \times X^{V_\Gamma}$.

PROOF. The map realizing the identification (3.8) is induced by the identification $Z = X \times X$ through (3.6), so that

$$F(X,\Gamma) = Z^{V_\Gamma} \setminus \bigcup_{e\in E_\Gamma} \Delta_e^{(Z)}$$

$$= (X\times X)^{V_\Gamma} \setminus \bigcup_{e\in E_\Gamma} \Delta_e^{(Z)} = (X^{V_\Gamma} \smallsetminus \bigcup_{e\in E_\Gamma} \Delta_e) \times X^{V_\Gamma},$$

since $\Delta_e^{(Z)} = \Delta_e \times X^{V_\Gamma}$. □

3.3.2. *Amplitudes in the complexified case.* We assume that the smooth projective variety Z contains a dense open set isomorphic to affine space $\mathbb{A}^{2D}$, with coordinates $z = (x_1, \dots, x_D, y_1, \dots, y_D)$.

In principle, our setting could also be generalized by just requiring that the variety X (hence $Z = Z \times X$) contain a dense open subset Ω that is isomorphic to an open set in an affine space $\mathbb{A}^D$, that is, by considering the case of a rational X. However, in this more general case one will have to consider Green functions for the Laplacian on the domain Ω, for a suitable elliptic boundary value problem, and those will in general be less explicitly accessible than the Green functions on the whole affine space $\mathbb{A}^D$.

DEFINITION 3.6. Given a Feynman Γ with no looping edges, we define the corresponding Feynman amplitude (weight) as

$$\omega_\Gamma^{(Z)} = \prod_{e\in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v\in V_\Gamma} dx_v \wedge d\bar{x}_v, \tag{3.9}$$

where $\|x_{s(e)} - x_{t(e)}\| = \|p(z)_{s(e)} - p(z)_{t(e)}\|$ and where the differential forms dx_v and $d\bar{x}_v$ denote, respectively, the holomorphic volume form $dx_{v,1} \wedge \cdots \wedge dx_{v,D}$ and its conjugate. The chain of integration is given, in this case, by the range of the projection

$$\sigma^{(Z,y)} = X^{V_\Gamma} \times \{y = (y_v)\} \subset Z^{V_\Gamma} = X^{V_\Gamma} \times X^{V_\Gamma}, \tag{3.10}$$

for a fixed choice of a point $y = (y_v \mid v \in V_\Gamma)$.

The form (3.9) is a closed $\mathcal{C}^\infty$ differential form on $F(X,\Gamma)$ of degree $2\dim_{\mathbb{C}} X^{V_\Gamma} = \dim_{\mathbb{C}} Z^{V_\Gamma} = 2DN$, with $N = \#V_\Gamma$, hence it defines a cohomology class in the group $H^{2DN}(F(X,\Gamma))$, and it gives a top form on the locus $\sigma^{(Z,y)}$.

In this case, the divergences of $\int_{\sigma^{(Z,y)}} \omega_\Gamma^{(Z)}$ are coming from the union of the diagonals $\cup_{e\in E_\Gamma} \Delta_e^{(Z)}$ and from the divisor at infinity

$$\Delta_{\infty,\Gamma}^{(Z)} := \sigma^{(Z,y)} \smallsetminus \mathbb{A}^{DN}(\mathbb{C}) \times \{y\}. \tag{3.11}$$

with $N = \#V_\Gamma$. We will address the behavior of the integrand near the loci $\cup_{e\in \mathbf{E}_\Gamma} \Delta_e^{(Z)}$ and $\Delta_{\infty,\Gamma}^{(Z)}$ and the appropriate regularization of the Feynman amplitude and the chain of integration in §5 below. When convenient, we will choose coordinates so as to identify the affine space $\mathbb{A}^{DN}(\mathbb{C}) \times \{y\} \subset \sigma^{(Z,y)}$ with a real affine space $\mathbb{A}^{2DN}(\mathbb{R})$, with $N = \#V_\Gamma$.

3.3.3. *Complexified amplitudes and complex Green forms.* A variant of the amplitude $\omega_\Gamma^{(Z)}$ of (3.9), which we will discuss briefly in §8, is related to the complex Green forms. In this setting, instead of (3.9) one considers the closely related form

$$\hat{\omega}_\Gamma = \prod_{e\in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v\in V_\Gamma} \sum_{k=1}^{D} (-1)^{k-1} dx_{v,[k]} \wedge d\bar{x}_{v,[k]}, \tag{3.12}$$

where the forms $dx_{v,[k]}$ and $d\bar{x}_{v,[k]}$ denote

$$dx_{v,[k]} = dx_{v,1} \wedge \cdots \wedge \widehat{dx_{v,k}} \wedge \cdots \wedge dx_{v,D},$$

$$d\bar{x}_{v,[k]} = d\bar{x}_{v,1} \wedge \cdots \wedge \widehat{d\bar{x}_{v,k}} \wedge \cdots \wedge d\bar{x}_{v,D},$$

respectively, with the factor $dx_{v,k}$ and $d\bar{x}_{v,k}$ removed.

Notice how, unlike the form considered in (3.9), which is defined on the affine $\mathbb{A}^{2DN} \subset Z^{V_\Gamma}$, with $N = \#V_\Gamma$, the form (3.12) has the same degree of homogeneity $2D-2$ in the numerator and denominator, when the graph Γ has no multiple edges, hence it is invariant under rescaling of the coordinates by a common non-zero scalar factor.

3.3.4. *Distributional interpretation.* In the cases discussed above, the amplitudes defined by (3.2) and (3.9) can be given a distributional interpretation, as a pairing of a distribution

$$\prod_{e\in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{\alpha}}, \tag{3.13}$$

where α is either $2\lambda = D-2$ or $2D-2$, or $2D$, and test forms given in the various cases, respectively, by

1. forms $\phi(x_v) \bigwedge_{v\in V_\Gamma} dx_v$, where ϕ is a test function in $\mathcal{C}^\infty(\mathbb{A}^{DN}(\mathbb{R}))$, with $N = \#V_\Gamma$;
2. forms $\phi(z_v) \bigwedge_{v\in V_\Gamma} dx_v \wedge d\bar{x}_v$, with $\phi(z_v) = \phi(p(z_v)) = \phi(x_v)$ a test function in $\mathcal{C}^\infty(X^{V_\Gamma}(\mathbb{C}))$.

This distributional interpretation will be used later in the paper, when we introduce Principal Value Currents, see Definition 6.6.

3.3.5. *Relation between complexified amplitudes.* We have introduced here above two simple mathematical generalization of the Feynman amplitude (3.2), respectively given by (3.9) and (3.12). These are related by a simple operation.

Let τ_Γ denote the form

$$\tau_\Gamma = \bigwedge_{v\in \mathbf{V}_\Gamma} \left(\sum_{k=1}^{D} (-1)^{D+k} dx_{v,k} \wedge d\bar{x}_{v,k} \right). \tag{3.14}$$

Lemma 3.7. *The forms* $\omega_\Gamma^{(Z)}$ *and* $\hat{\omega}_\Gamma$ *of* (3.9) *and* (3.12) *are related by*

$$\omega_\Gamma^{(Z)} = \tau_\Gamma \wedge \hat{\omega}_\Gamma. \tag{3.15}$$

Proof. We have

$$\bigwedge_{w\in \mathbf{V}_\Gamma} \left(\sum_{j=1}^{D} (-1)^{D-1} (-1)^{j-1} dx_{w,j} \wedge d\bar{x}_{w,j} \right) \wedge \bigwedge_{v\in \mathbf{V}_\Gamma} \left(\sum_{k=1}^{D} (-1)^{k-1} dx_{v,[k]} \wedge d\bar{x}_{v,[k]} \right)$$

$$= \bigwedge_{v\in \mathbf{V}_\Gamma} dx_v \wedge d\bar{x}_v.$$

□

3.4. Amplitudes and Fourier transform. We expand here, briefly, on the comment made in the Introduction regarding amplitudes in momentum and configuration space and Fourier duality between the propagators. This was discussed in greater detail in [**37**].

In real momentum space, with momentum variables $k_e \in \mathbb{A}^D(\mathbb{R})$, for each edge $e \in E_\Gamma$, the form of the Feynman amplitude is a product of edge-propagators

$$\frac{1}{(m^2 + \|k_e\|^2)}.$$

In configuration space, as we have seen above, one considers in (3.2) a product of propagators of the form

$$G_{0,\mathbb{R}}(x_{s(e)} - x_{t(e)}) := \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}}, \quad \text{with} \quad D = 2\lambda + 2, \tag{3.16}$$

in the massless case. The massive case (which we will not consider in this paper) has a somewhat more complicated expression

$$G_{m,\mathbb{R}}(x_{s(e)} - x_{t(e)}) := \frac{m^\lambda}{(2\pi)^{(\lambda+1)}} \|x_{s(e)} - x_{t(e)}\|^{-\lambda} \mathcal{K}_\lambda(m\|x_{s(e)} - x_{t(e)}\|),$$

with $\mathcal{K}_\nu(z)$ the modified Bessel function.

The expression $G_{0,\mathbb{R}}(x_{s(e)} - x_{t(e)})$ of (3.16) is the Green function of Laplacian on $\mathbb{A}^D(\mathbb{R})$, with $D = 2\lambda + 2$:

$$G_\mathbb{R}(x, y) = \frac{1}{\|x - y\|^{2\lambda}}$$

Similarly the expression $G_{m,\mathbb{R}}(x_{s(e)} - x_{t(e)})$ above is fundamental solution of Helmholtz equation $(\Delta + m^2)G = \delta$.

For test functions $\varphi \in \mathcal{S}(\mathbb{R}^D)$, Fourier transform of convolution with the Green function gives

$$\widehat{(G_{0,\mathbb{R}} \star \varphi)}(k) = \frac{4\pi^{D/2}}{\Gamma(\lambda)} \frac{1}{\|k\|^2} \widehat{\varphi}(k)$$

$$\widehat{(G_{m,\mathbb{R},} \star \varphi)}(k) = \frac{1}{(m^2 + \|k\|^2)} \widehat{\varphi}(k).$$

This shows that, in the real case, the amplitude (3.2) we are considering is indeed the dual picture, under Fourier transform, of the momentum space Feynman integral.

When we pass from the Feynman amplitude (3.2) to the complexified amplitude (3.9), we replace the real Laplacian on $\mathbb{A}^D(\mathbb{R})$ with the complex Laplacian on $\mathbb{A}^D(\mathbb{C})$,

$$\Delta = \sum_{k=1}^D \frac{\partial^2}{\partial x_k \partial \bar{x}_k}.$$

The Green function of the complex Laplacian is given by

$$G_\mathbb{C}(x, y) = \frac{-(D-2)!}{(2\pi i)^D \|x - y\|^{2D-2}}, \tag{3.17}$$

which is the expression that replaces (3.16) when passing from (3.2) to (3.9). This shows why (3.9) can be regarded as a natural generalization of (3.2).

4. Feynman amplitudes: integration over the real locus

In this section we consider the physical Feynman amplitudes ω_Γ, defined as in (3.2), and the domain σ_Γ defined in (3.5). We give an explicit formulation of the integral in terms of an expansion of the real Green functions $\|x_{s(e)} - x_{t(e)}\|^{-2\lambda}$ in Gegenbauer polynomials, based on a technique well known to physicists (the x-space method, see [**38**]). We consider the integrand restricted to the real locus $X(\mathbb{R})^{\mathbf{V}_\Gamma}$, and express it in polar coordinates, separating out an angular integral and a radial integral. We identify a natural subdivision of the domain of integration into chains that are indexed by acyclic orientations of the graph. In the special case of dimension $D = 4$, we express the integrand in terms closely related to multiple polylogarithm functions.

Spencer Bloch recently introduced a construction of cycles in the relative homology $H_*(X^{\mathbf{V}_\Gamma}, \cup_e \Delta_e)$ of the graph configuration spaces, that explicitly yield multiple zeta values as periods [**21**]. We use here a variant of his construction, which will have a natural interpretation in terms of the x-space method for the computation of the Feynman amplitudes in configuration spaces.

4.0.1. *Directed acyclic graph structures.* We recall some basic facts about directed acyclic orientations on graphs.

Definition 4.1. Let Γ be a finite graph without looping edges. Let $\Omega(\Gamma)$ denote the set of edge orientations on Γ such that the resulting directed graph is a directed acyclic graph.

It is well known that all finite graphs without looping edges admit such orientations. In fact, the number of possible orientations that give it the structure of a directed acyclic graph are given by $(-1)^{\mathbf{V}_\Gamma} P_\Gamma(-1)$, where $P_\Gamma(t)$ is the chromatic polynomial of the graph Γ, see [**79**].

The following facts about directed acyclic graphs are also well known (see for instance §2.6 of [**58**]), and we recall them here for later use.

- Each orientation $\mathbf{o} \in \Omega(\Gamma)$ determines a partial ordering on the vertices of the graph Γ, by setting $w \geq_{\mathbf{o}} v$ whenever there is an oriented path of edges from v to w in the directed graph $(\Gamma, \mathbf{o})$.
- In every directed acyclic graph there is at least a vertex with no incoming edges and at least a vertex with no outgoing edges.

4.0.2. *Relative cycles from directed acyclic structures.* Given a graph Γ we consider the space $X^{\mathbf{V}_\Gamma}$. On the dense subset $\mathbb{A}^D(\mathbb{R}) = X(\mathbb{R}) \smallsetminus \Delta_\infty(\mathbb{R})$ of the chain of integration σ_Γ, we use polar coordinates with $x_v = r_v \omega_v$, with $r_v \in \mathbb{R}_+$ and $\omega_v \in S^{D-1}$.

Definition 4.2. Let $\mathbf{o} \in \Omega(\Gamma)$ be an acyclic orientation. Define the chain associated to the acyclic orientation $\mathbf{o}$ of Γ as

$$\Sigma_{\mathbf{o}} := \{(x_v) \in X^{\mathbf{V}_\Gamma}(\mathbb{R}) \,:\, r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}. \tag{4.1}$$

It is a middle dimensional relative homology class in $X^{\mathbf{V}_\Gamma}$.

Points on a diagonal $\Delta_e = \{x_{s(e)} = x_{t(e)}\}$ have $r_{s(e)} = r_{t(e)}$, hence the locus $\cup_e \Delta_e$ of singularities of the Feynman amplitudes is contained in the boundary $\partial\Sigma_{\mathbf{o}}$.

The following simple observation will be useful in the Feynman integral calculation we describe later in this section.

Lemma 4.3. *Let* $\mathbf{o} \in \Omega(\Gamma)$ *be an acyclic orientation and* $\Sigma_{\mathbf{o}}$ *the chain defined in* (4.1)*. Then* $\Sigma_{\mathbf{o}} \smallsetminus \cup_v \{r_v = 0\}$ *is a bundle with fiber* $(S^{D-1})^{\mathbf{V}_\Gamma}$ *over a base*

$$\overline{\Sigma}_{\mathbf{o}} = \{(r_v) \in (\mathbb{R}^*_+)^{\mathbf{V}_\Gamma} \, : \, r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}. \tag{4.2}$$

Proof. This is immediate from the polar coordinate form $x_v = r_v \omega_v$, with $r_v \in \mathbb{R}^*_+$ and $\omega_v \in S^{D-1}$. □

4.1. Gegenbauer polynomials and angular integrals.

One of the techniques developed by physicists to compute Feynman amplitudes, by passing from momentum to configuration space, relies on the expansion in Gegenbauer polynomials, see for instance [**38**] and the recent [**72**].

The Gegenbauer polynomials (or ultraspherical polynomials) are defined through the generating function

$$\frac{1}{(1-2tx+t^2)^\lambda} = \sum_{n=0}^\infty C_n^{(\lambda)}(x) t^n, \tag{4.3}$$

for $|t| < 1$. For $\lambda > -1/2$, they satisfy

$$\int_{-1}^1 C_n^{(\lambda)}(x) C_m^{(\lambda)}(x) \, (1-x^2)^{\lambda-1/2} dx = \delta_{n,m} \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}. \tag{4.4}$$

We use what is known in the physics literature as the x-space method (see [**38**]) to reformulate the integration involved in the Feynman amplitude calculation in a way that involves the relative chains of Definition 4.2.

Theorem 4.4. *In even dimension* $D = 2\lambda + 2$*, the integral* $\int_{\sigma_\Gamma} \omega_\Gamma$ *of the form* (3.2) *on the chain* σ_Γ *can be rewritten in the form*

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\Sigma_{\mathbf{o}}} \prod_{e\in\mathbf{E}_\Gamma} r_{t_{\mathbf{o}}(e)}^{-2\lambda} \left(\sum_n (\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}})^n C_n^{(\lambda)}(\omega_{s_{\mathbf{o}}(e)} \cdot \omega_{t_{\mathbf{o}}(e)}) \right) dV, \tag{4.5}$$

for some positive integers $m_{\mathbf{o}}$*, and with volume element*

$$dV = \prod_v d^D x_v = \prod_v r_v^{D-1} dr_v \, d\omega_v.$$

Proof. We write the integral in polar coordinates, with

$$d\omega = \sin^{D-2}(\phi_1)\sin^{D-3}(\phi_2)\cdots\sin(\phi_{D-2}) d\phi_1 \cdots d\phi_{D-1}$$

the volume element on the sphere S^{D-1} and $d^D x_v = r_v^{D-1} dr_v \, d\omega_v$.

In dimension $D = 2\lambda + 2$, by (4.3), the Newton potential has an expansion in Gegenbauer polynomials, so that

$$\begin{aligned} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} &= \frac{1}{\rho_e^{2\lambda}(1 + (\frac{r_e}{\rho_e})^2 - 2\frac{r_e}{\rho_e}\omega_{s(e)} \cdot \omega_{t(e)})^{\lambda}} \\ &= \rho_e^{-2\lambda} \sum_{n=0}^{\infty} (\frac{r_e}{\rho_e})^n C_n^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}), \end{aligned} \tag{4.6}$$

where $\rho_e = \max\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and $r_e = \min\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and with $\omega_v \in S^{D-1}$.

We can subdivide the integration into open sectors where, for each edge, either $r_{s(e)} < r_{t(e)}$ or the converse holds, so that each term $\rho_e^{-2\lambda}$ is $r_{t(e)}^{-2\lambda}$ (or $r_{s(e)}^{-2\lambda}$) and each term $(r_e/\rho_e)^n$ is $(r_{s(e)}/r_{t(e)})^n$ (or its reciprocal). In other words, let $\mathbf{b}$ denote an assignment of either $r_{s(e)} < r_{t(e)}$ or $r_{s(e)} > r_{t(e)}$ at each edge, which we write simply as $\mathbf{b}(r_{s(e)}, r_{t(e)})$, and let

$$\bar{\mathcal{R}}_{\mathbf{b}} = \{(r_v) \in (\mathbb{R}_+^*)^{\mathbf{V}_\Gamma} \mid \mathbf{b}(r_{s(e)}, r_{t(e)}) \text{ for } e \in \mathbf{E}_\Gamma\}$$

and $\mathcal{R}_{\mathbf{b}} = \bar{\mathcal{R}}_{\mathbf{b}} \times (S^{D-1})^{\mathbf{V}_\Gamma}$. Then we identify the domain of integration with $\cup_{\mathbf{b}} \mathcal{R}_{\mathbf{b}}$, up to a set of measure zero. The set $\mathcal{R}_{\mathbf{b}}$ is empty unless the assignment $\mathbf{b}$ defines a strict partial ordering of the vertices of Γ, in which case $\mathbf{b}$ determines an acyclic orientation $\mathbf{o} = \mathbf{o}(\mathbf{b})$ of Γ, as described in Definition 4.1. In this case, then, the chain of integration corresponding to the sector $\mathcal{R}_{\mathbf{b}}$ is the chain $\Sigma_{\mathbf{o}}$ of Definition 4.2, with $\rho_e = r_{t_{\mathbf{o}}(e)}$ and $r_e = r_{s_{\mathbf{o}}(e)}$. Thus, the domain of integration can be identified with $\cup_{\mathbf{o} \in \Omega(\Gamma)} m_{\mathbf{o}} \, \Sigma_{\mathbf{o}}$, where $m_{\mathbf{o}}$ is a multiplicity, taking into account the fact that different strict partial orderings may define the same acyclic orientation (for example, the cyclic graph with four vertices has 24 different possible vertex sequences and 14 acyclic orientations). ☐

The integral (4.5) can be approached by first considering a family of angular integrals

$$\mathcal{A}_{(n_e)_{e \in \mathbf{E}_\Gamma}} = \int_{(S^{D-1})^{\mathbf{V}_\Gamma}} \prod_e C_{n_e}^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}) \prod_v d\omega_v, \tag{4.7}$$

labelled by all choices of integers n_e for $e \in \mathbf{E}_\Gamma$. The evaluation of these angular integrals will lead to an expression $\mathcal{A}_{n_e}$ in the n_e, so that one obtains a radial integral

$$\sum_{\mathbf{o} \in \Omega(\Gamma)} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \prod_{e \in \mathbf{E}_\Gamma} \mathcal{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) \, \prod_v r_v^{D-1} dr_v \tag{4.8}$$

where

$$\mathcal{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) = r_{t_{\mathbf{o}}(e)}^{-2\lambda} \sum_{n_e} \mathcal{A}_{n_e} (\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}})^{n_e}. \tag{4.9}$$

4.2. Polygons and polylogarithms.

We first discuss the very simple example of a polygon graph, where one sees polylogarithms and zeta values arising in the expression (4.9) and its integration on the domains $\bar{\Sigma}_{\mathbf{o}}$. In the following subsections we will analyze the more general structure of these integrals for more complicated graphs.

4.2.1. *The angular integral for polygons in arbitrary dimension.* The angular integral for polygon graphs has the following explicit expression.

Proposition 4.5. *Let Γ be a polygon with k edges. Then the angular integral (4.7) depends on a single variable $n \in \mathbb{N}$ and is given by*

$$\mathcal{A}_n = \left(\frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)(n+\lambda)} \right)^k \cdot \dim \mathcal{H}_n(S^{2\lambda+1}), \tag{4.10}$$

and $\mathcal{H}_n(S^{2\lambda+1})$ is the space of harmonic functions of degree n on the sphere $S^{2\lambda+1}$.

Proof. The angular integral, in this case, is simply given by

$$\int_{(S^{D-1})^{\mathbf{V}_\Gamma}} C^{(\lambda)}_{n_1}(\omega_{v_k}\cdot\omega_{v_1}) C^{(\lambda)}_{n_2}(\omega_{v_1}\cdot\omega_{v_2}) \cdots C^{(\lambda)}_{n_k}(\omega_{v_{k-1}}\cdot\omega_{v_k}) \prod_{v\in\mathbf{V}_\Gamma} d\omega_v,$$

which is independent of the orientation. We then use the fact that the Gegenbauer polynomials satisfy ([**14**] Vol.2, Lemma 4, §11.4)

$$\int_{S^{D-1}} C^{(\lambda)}_m(\omega_1\cdot\omega) C^{(\lambda)}_n(\omega\cdot\omega_2) d\omega = \delta_{n,m} \frac{\lambda\, Vol(S^{D-1})}{n+\lambda} C^{(\lambda)}_n(\omega_1\cdot\omega_2), \tag{4.11}$$

with $Vol(S^{D-1}) = 2\pi^{\lambda+1}/\Gamma(\lambda+1)$, for $D = 2\lambda+2$. Thus, we obtain

$$\mathcal{A}_n = \left(\frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)} \right)^{k-1} \frac{1}{(n+\lambda)^{k-1}} \int_{S^{D-1}} C^{(\lambda)}_n(\omega\cdot\omega)\, d\omega, \tag{4.12}$$

where $n = n_1 = \cdots = n_k$ and where the remaining integral is just

$$\int_{S^{D-1}} C^{(\lambda)}_n(\omega\cdot\omega) d\omega = C^{(\lambda)}_n(1)\, Vol(S^{D-1}).$$

The value of $C^{(\lambda)}_n(1)$ can be seen using the fact that the Gegenbauer polynomials are related to the *zonal spherical harmonics* (see §4 of [**80**] and also [**49**], [**70**], [**87**]) $Z^{(n)}_{\omega_1}(\omega_2)$ by

$$C^{(\lambda)}_n(\omega_1\cdot\omega_2) = c_{D,n}\, Z^{(n)}_{\omega_1}(\omega_2), \tag{4.13}$$

for $D = 2\lambda+2$, with $\omega_1, \omega_2 \in S^{D-1}$, where the coefficient $c_{D,n}$ is given by

$$c_{D,n} = \frac{Vol(S^{D-1})\,(D-2)}{2n+D-2}. \tag{4.14}$$

In turn, the zonal spherical harmonics are expressed in terms of an orthonormal basis $\{Y_j\}$ of the Hilbert space $\mathcal{H}_n(S^{D-1})$ of spherical harmonics on S^{D-1} of degree n, as

$$Z^{(n)}_{\omega_1}(\omega_2) = \sum_{j=1}^{\dim \mathcal{H}_n(S^{D-1})} Y_j(\omega_1)\overline{Y_j(\omega_2)}. \tag{4.15}$$

The dimension of the space $\mathcal{H}_n(S^{D-1})$ of spherical harmonics is given by

$$\dim \mathcal{H}_n(S^{D-1}) = \binom{D-1+n}{n} - \binom{D-3+n}{n-2}.$$

Using (4.15), we then have

$$\int_{S^{D-1}} C^{(\lambda)}_n(\omega\cdot\omega) d\omega = c_{D,n} \int_{S^{D-1}} Z^{(n)}_\omega(\omega) d\omega$$

$$= c_{D,n} \sum_{j=1}^{\dim \mathcal{H}_n(S^{D-1})} \int_{S^{D-1}} |Y_j(\omega)|^2 d\omega = c_{D,n} \dim \mathcal{H}_n(S^{D-1}),$$

which gives $C_n^{(\lambda)}(1) = \frac{2\lambda \dim \mathcal{H}_n(S^{D-1})}{2(n+\lambda)}$, for $D = 2\lambda + 2$. □

4.2.2. *Polygon amplitudes in dimension four.* We now specialize to the case where $D = 4$ (hence $\lambda = 1$) and we show how one obtains integrals of polylogarithm functions

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

Proposition 4.6. *Let Γ be a polygon with k edges and let $D = 4$. Then the Feynman amplitude is given by the integral*

$$(2\pi^2)^k \sum_{\mathbf{o}} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \mathrm{Li}_{k-2}(\prod_i \frac{r_{w_i}^2}{r_{v_i}^2}) \prod_v r_v \, dr_v, \tag{4.16}$$

where the vertices v_i and w_i are the sources and tails of the oriented paths determined by $\mathbf{o}$.

Proof. We write the terms in the integrand (4.5) as

$$\prod_{e \in \mathbf{E}_\Gamma} \rho_e^{-2\lambda} \left(\sum_n (\frac{r_e}{\rho_e})^n \; C_n^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}) \right) = \tag{4.17}$$

$$(\rho_1 \cdots \rho_k)^{-2\lambda} \sum_{n_1,\ldots,n_k} (\frac{r_1}{\rho_1})^{n_1} \cdots (\frac{r_k}{\rho_N})^{n_k} \; C_{n_1}^{(\lambda)}(\omega_{s(e_1)} \cdot \omega_{t(e_1)}) \cdots C_{n_k}^{(\lambda)}(\omega_{s(e_k)} \cdot \omega_{t(e_k)})$$

and we perform the angular integral as in (4.10). In the case $D = 4$ we have

$$\dim \mathcal{H}_n(S^3) = \binom{n+3}{n} - \binom{n+1}{n-2} = (n+1)^2.$$

Thus, the angular integral of Proposition 4.5 becomes

$$\mathcal{A}_n = \frac{(2\pi^2)^k}{(n+1)^{k-2}}. \tag{4.18}$$

We then write the radial integrand as in (4.8). An acyclic orientation $\mathbf{o} \in \Omega(\Gamma)$, subdivides the polygon Γ into oriented paths γ_i such that $s_{\gamma_i} = s_{\gamma_{i-1}}$ and $t_{\gamma_i} = t_{\gamma_{i+1}}$ or $s_{\gamma_i} = s_{\gamma_{i+1}}$ and $t_{\gamma_i} = t_{\gamma_{i-1}}$. Correspondingly, the set of vertices is subdivided into $\mathbf{V}_\Gamma = \{v_i\} \cup \{w_i\} \cup \{v \notin \{v_i, w_i\}\}$ with v_i the sources and w_i the tails of the oriented paths and the remaining vertices partitioned into internal vertices of each oriented path. We then have

$$\mathcal{F}(r_{s(e)}, r_{t(e)}) = (2\pi^2)^k (\prod_e r_{t(e)}^{-2}) \sum_{n \geq 0} \frac{1}{(n+1)^{k-2}} (\prod_i \frac{r_{v_i}^2}{r_{w_i}^2})^n$$

$$= (2\pi^2)^k (\prod_e r_{t(e)}^{-2}) (\prod_i \frac{r_{w_i}^2}{r_{v_i}^2}) \sum_{n \geq 1} \frac{1}{n^{k-2}} (\prod_i \frac{r_{v_i}^2}{r_{w_i}^2})^n.$$

Since each w_i is counted twice as target of an edge and the internal vertices of the oriented paths are counted only once, we obtain

$$\prod_e \mathcal{F}(r_{s(e)}, r_{t(e)}) \prod_v r_v^3 dv = (2\pi^2)^k \cdot \mathrm{Li}_{k-2}(\prod_i \frac{r_{w_i}^2}{r_{v_i}^2}) \prod_v r_v dr_v. \tag{4.19}$$

$\square$

4.2.3. *Zeta values.* After a cutoff regularization, these integrals produce combinations of zeta values with coefficients that are rationals combinations of powers of $2\pi i$. To see how this happens, we look explicitly at the contribution of an acyclic orientations of the polygon consisting of just two oriented paths γ_1 and γ_2 with source v and target w, respectively with k_1 and k_2 internal vertices. The other summands can be handled similarly. By changing variables to $t = r_v^2/r_w^2$, $t_i = r_{v_i}^2/r_w^2$ for v_i the internal edges of γ_1 and $s_i = r_{v_i}^2/r_w^2$ for v_i the internal edges of γ_2, we obtain

$$\bigwedge_{v\in \mathbf{V}_\gamma} r_v\, dr_v = \pm 2^{1-k} r_w^{2k+1} dr_w\, dt \bigwedge_i dt_i \wedge ds_i.$$

After factoring out a divergence along $\Delta_\infty = X \smallsetminus \mathbb{A}^4$, coming from the integration of the r_w term, which gives a pole along the divisor Δ_∞, one obtains an integral of the form

$$2\pi^{2k} \int_{\bar\Sigma_1\cap\bar\Sigma_2} \mathrm{Li}_{k-2}(t)\ dt \prod_i dt_i ds_i,$$

where $\bar\Sigma_{\mathbf{o}} = \bar\Sigma_1 \cap \bar\Sigma_2$ with $\bar\Sigma_1 = \{(t, t_i, s_i)\,|\, t \leq t_1 \leq \cdots \leq t_{k_1-1} \leq 1\}$ and $\bar\Sigma_2 = \{(t, t_i, s_i)\,|\, t \leq s_1 \leq \cdots \leq s_{k_2-1} \leq 1\}$. One can use the relation [**45**]

$$\int x^m \mathrm{Li}_n(x)\, dx = \frac{1}{m+1} x^{m+1} \mathrm{Li}_n(x) - \frac{1}{m+1} \int x^m \mathrm{Li}_{n-1}(x) dx, \tag{4.20}$$

to reduce the integral to a combination of zeta values. Further results on zeta values and multiple zeta values, beyond the relations in [**45**], can be found in [**1**], [**26**].

We will return to discuss cutoff regularizations in §5.8 and §6, in the case of the complexified amplitudes.

4.3. Stars of vertices and isoscalars.

To see how more complicated expressions can arise in the integrands, which eventually lead to the presence of multiple zeta values, it is convenient to regard graphs as being built out of stars of vertices pasted together by suitably matching the half edges, where by the *star of a vertex* we mean a single vertex v of valence k with k half-edges e_j attached to it. One can then built the Feynman integral by first identifying the contribution of the star of a vertex, which is obtained by integrating in the variables r_v and ω_v of the central vertex v, and results in a function of variables r_{v_j} and ω_{v_j}, for each of the edges e_j. One then obtains the integral for the graph, which gives a number (possibly after a regularization), by matching the half edges and identifying the corresponding variables and integrating over them.

We first introduce the analog of the angular integral (4.7) for the case of a graph with half-edges. While in the case of a usual graph, where all half-edges are paired to form edges, the angular integral (4.7) is a number, in the case with open (unpaired) half-edges it is a function of the variables of the half edges, which we denote by $\mathcal{A}_{\underline{n}}(\underline{\omega})$, where $\underline{n} = (n_1, \ldots, n_\ell)$ and $\underline{\omega} = (\omega_1, \ldots, \omega_\ell)$ are vectors of integers $n_j \in \mathbb{N}$, for each half-edge e_j, and of variables $\omega_j \in S^{D-1}$. We will sometime denote these variables by ω_{v_j}, where v_j simply denotes the end of the half-edge e_j. We will equivalently use the notation $\mathcal{A}_{\underline{n}}(\underline{\omega})$ or $\mathcal{A}_{(n_j)}(\omega_{v_j})$. In the case of the star of a vertex, the angular integral is of the form

$$\mathcal{A}_{\underline{n}}(\underline{\omega}) = \int_{S^{D-1}} \prod_j C^{(\lambda)}_{n_j}(\omega_j \cdot \omega)\, d\omega, \text{ with } \underline{n} = (n_j)_{e_j \in \mathbf{E}_\Gamma},\ \underline{\omega} = (\omega_j)_{e_j \in \mathbf{E}_\Gamma}. \tag{4.21}$$

Lemma 4.7. *Let Γ be the star of a valence k vertex v. Then the angular integral (4.21) is given by the function*

$$\mathcal{A}_{(n_j)}(\omega_{v_j}) = c_{D,n_1} \cdots c_{D,n_k}\ \tilde{\mathcal{A}}_{(n_j)}(\omega_{v_j}),$$

$$\tilde{\mathcal{A}}_{(n_j)}(\omega_{v_j}) = \sum_{\ell_1,\dots,\ell_k} \overline{Y^{(n_1)}_{\ell_1}(\omega_1) \cdots Y^{(n_k)}_{\ell_k}(\omega_k)} \int_{S^{D-1}} Y^{(n_1)}_{\ell_1}(\omega) \cdots Y^{(n_k)}_{\ell_k}(\omega)\ d\omega, \tag{4.22}$$

where $\{Y^{(n)}_\ell\}_{\ell=1,\dots,d_n}$ is an orthonormal basis of the space $\mathcal{H}_n(S^{D-1})$ of spherical harmonics of degree n, and $d_n = \dim \mathcal{H}_n(S^{D-1})$, with the coefficients $c_{D,n}$ as in (4.14).

Proof. Consider first the angular integral as written in (4.7). Using (4.11), we can write each edge contribution $C^{(\lambda)}_{n_e}(\omega_{s(e)} \cdot \omega_{t(e)})$ in (4.7) in terms of an integral

$$\frac{n_e + \lambda}{\lambda \mathrm{Vol}(S^{D-1})} \int_{S^{D-1}} C^{(\lambda)}_{n_e}(\omega_{s(e)} \cdot \omega) C^{(\lambda)}_{n_e}(\omega \cdot \omega_{t(e)})\, d\omega.$$

Thus, we can think of the edge contribution $C^{(\lambda)}_{n_e}(\omega_{s(e)} \cdot \omega_{t(e)})$ as obtained by a contraction (integration on a common variable) of contributions $C^{(\lambda)}_{n_e}(\omega_{s(e)} \cdot \omega)$ and $C^{(\lambda)}_{n_e}(\omega \cdot \omega_{t(e)})$ associated to two half-edges, with a factor depending on n_e, λ, D. This gives us an angular integral associated to a star of half-edges, which is of the form (4.21). Using the relation (4.13), (4.15) between the Gegenbauer polynomials and the spherical harmonics, we then rewrite the angular integral (4.21) in the form (4.22). □

Thus, the evaluation of the angular integrals (4.22) for stars of vertices relates to the well known problem of evaluating coupling coefficients for spherical harmonics,

$$\langle Y^{(n_1)}_{\ell_1}, \dots, Y^{(n_k)}_{\ell_k} \rangle_D := \int_{S^{D-1}} Y^{(n_1)}_{\ell_1}(\omega) \cdots Y^{(n_k)}_{\ell_k}(\omega)\ d\omega. \tag{4.23}$$

In the following we will be using the standard labeling of the basis $\{Y^{(n)}_\ell\}$ where, for fixed n, the indices ℓ run over a set of $(D-2)$-tuples

$$(m_{D-2}, m_{D-1}, \dots, m_2, m_1) \quad \text{with} \quad n \geq m_{D-2} \geq \cdots \geq m_2 \geq |m_1|.$$

The spherical harmonics $Y^{(n)}_\ell$ have the symmetry

$$\overline{Y^{(n)}_\ell} = (-1)^{m_1} Y^{(n)}_{\bar{\ell}}, \tag{4.24}$$

where, for $\ell = (m_{D-2}, m_{D-1}, \dots, m_2, m_1)$, one has

$$\bar{\ell} := (m_{D-2}, m_{D-1}, \dots, m_2, -m_1).$$

In the simplest case of a tri-valent vertex, these coefficients are also referred to as the Gaunt coefficients, and have been extensively studied, see for instance [**5**], [**59**], [**77**]. The Gaunt coefficients arising from the integration of three harmonic functions determine the coefficients of the expansion formula

$$Y^{(n_1)}_{\ell_1} Y^{(n_2)}_{\ell_2} = \sum_{n,\ell} \mathcal{K}_{D,n_i,n,\ell_i,\ell}\; Y^{(n)}_{\ell} \tag{4.25}$$

that expresses the product of two harmonic functions in terms of a linear combination of other harmonic functions, with the cases where some of the factors are conjugated taken care of by the symmetry (4.24). In the more general case (4.23) one can therefore repeatedly apply (4.25), hence we focus here on the example of the star of a tri-valent vertex.

The Gaunt coefficients $\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2}, Y^{(n_3)}_{\ell_3}\rangle_D$ can be computed via Racah's factorization lemma ([**5**], [**59**]) in terms of *isoscalar factors* and the Gaunt coefficients for $D-1$, according to

$$\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2}, Y^{(n_3)}_{\ell_3}\rangle_D = \begin{pmatrix} n_1 & n_2 & n_3 \\ n'_1 & n'_2 & n'_3 \end{pmatrix}_{D:D-1} \langle Y^{(n'_1)}_{\ell'_1}, Y^{(n'_2)}_{\ell'_2}, Y^{(n'_3)}_{\ell'_3}\rangle_{D-1}, \tag{4.26}$$

where $\ell_i = (n'_i, \ell'_i)$ with $n'_i = m_{D-2,i}$ and $\ell'_i = (m_{D-3,i}, \dots, m_{1,i})$. An explicit expression of the isoscalar factors

$$\begin{pmatrix} n_1 & n_2 & n_3 \\ n'_1 & n'_2 & n'_3 \end{pmatrix}_{D:D-1} \tag{4.27}$$

is given in [**5**], [**59**]. We will discuss this more in detail in §4.4 and §4.5 below.

4.3.1. *Gaunt versus Clebsch–Gordon.* There is some inconsistency of terminology in the literature on spherical harmonics and coupling coefficients between what is referred to as "Gaunt coefficients" and what is termed "Clebsch–Gordon coefficients". The two notions are very closely related, but the respective explicit expressions considered in the literature vary by a factor and the terminology adopted is not always consistent. For example, the definition of Gaunt coefficients adopted in [**77**], as the integral of three spherical harmonics, differs from the notion of Gaunt coefficient used in [**89**] (see equation (10) of [**89**]) by a factor as in equation (21) of [**77**]. In turn, the Gaunt coefficients considered in [**89**] differ from the Clebsch–Gordon coefficients computed for instance in [**86**] as indicated in equation (46) of [**89**]. Given the frequent inconsistencies of terminology in the literature, we urge the reader to check for the specific conventions followed by specific authors and not simply consider Gaunt and Clebsch–Gordon as interchangeable synonyms.

4.4. Gluing two stars along an edge. We now consider the effect of patching together two trivalent stars by gluing two half edges with matching orientations.

LEMMA 4.8. *Let $\mathcal{A}_{n,n_1,n_2}(\omega,\omega_1,\omega_2)$ and $\mathcal{A}_{n',n_3,n_4}(\omega',\omega_3,\omega_4)$ be the angular integrals associated to two trivalent stars, as in Lemma 4.7. Then the angular integral of the graph obtained by joining the two stars at an edge is*

$$\mathcal{A}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}, n) = \sum_{\ell_i} \prod_{i=1}^{4} c_{D,n_i}\, \overline{Y^{(n_i)}_{\ell_i}}(\omega_i)\, \mathcal{K}_{n_i,\ell_i}(n), \tag{4.28}$$

with coefficients $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$ *given by*

$$\mathcal{K}_{n_i,\ell_i}(n) = c^2_{D,n} \sum_{\ell=1}^{d_n} \langle Y^{(n)}_\ell, Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} \rangle_D \cdot \langle Y^{(n)}_\ell, Y^{(n_3)}_{\ell_3}, Y^{(n_4)}_{\ell_4} \rangle_D. \tag{4.29}$$

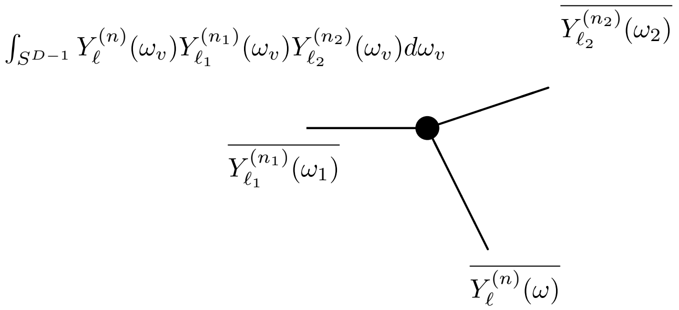
Proof. As shown in Lemma 4.7 above, using (4.11) one reduces the contribution $C^{(\lambda)}_n(\omega_v \cdot \omega_{v'})$ of an edge e with $s(e) = v$ and $t(e) = v'$ to an integration along an angular variable ω of a product $C^{(\lambda)}_n(\omega_v \cdot \omega) C^{(\lambda)}_n(\omega \cdot \omega_{v'})$ (with a proportionality factor that depends on D, n, λ). One should think here of the variable ω as associated to a half-edge attached to the vertex v and contracted with the corresponding variable associated to a half-edge attached to the vertex v' when the half edges are joined to form the edge e. That is the meaning of (4.11). By rewriting the Gegenbauer polynomials in terms of spherical harmonics one can identify, as shown in Lemma 4.7, the contribution to the angular integral of a star of half-edges associated to a vertex, in the form written in (4.22). In particular, consider a trivalent star of half-edges attached to a vertex. According to Lemma 4.7, the expression associated to this trivalent star is of the form

$$\overline{Y^{(n)}_\ell(\omega) Y^{(n_1)}_{\ell_1}(\omega_1) Y^{(n_2)}_{\ell_2}(\omega_2)} \int_{S^{D-1}} Y^{(n)}_\ell(\omega_v) Y^{(n_1)}_{\ell_1}(\omega_v) Y^{(n_2)}_{\ell_2}(\omega_v) d\omega_v,$$

where $\omega_v \in S^{D-1}$ is the angular variable associated to the vertex v, and $\omega, \omega_1, \omega_2$ are the angular variables associated to the three half-edges. This can be visualized graphically as in the figure, with the Gaunt coefficient

$$\langle Y^{(n)}_\ell, Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} \rangle_D = \int_{S^{D-1}} Y^{(n)}_\ell(\omega_v) Y^{(n_1)}_{\ell_1}(\omega_v) Y^{(n_2)}_{\ell_2}(\omega_v) d\omega_v$$

associated to the vertex v of the star and the spherical harmonics $\overline{Y^{(n)}_\ell(\omega)}$, $\overline{Y^{(n_1)}_{\ell_1}(\omega_1)}$ and $\overline{Y^{(n_2)}_{\ell_2}(\omega_2)}$ associated to each of the half-edges, respectively.



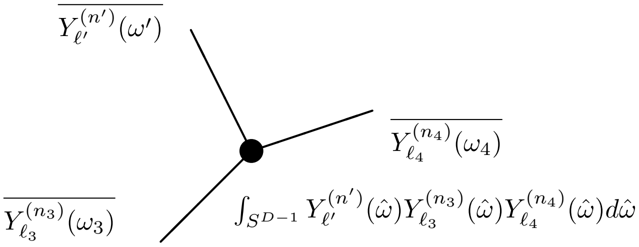
Consider then a second trivalent star, for which the expression of Lemma 4.7 will be of the form

$$\overline{Y^{(n')}_{\ell'}(\omega') Y^{(n_3)}_{\ell_3}(\omega_3) Y^{(n_4)}_{\ell_4}(\omega_4)} \int_{S^{D-1}} Y^{(n')}_{\ell'}(\omega_{v'}) Y^{(n_3)}_{\ell_3}(\omega_{v'}) Y^{(n_4)}_{\ell_4}(\omega_{v'}) d\omega_{v'}.$$

We again represent this graphically with the Gaunt coefficient

$$\langle Y^{(n')}_{\ell'}, Y^{(n_3)}_{\ell_3}, Y^{(n_4)}_{\ell_4} \rangle = \int_{S^{D-1}} Y^{(n')}_{\ell'}(\omega_{v'}) Y^{(n_3)}_{\ell_3}(\omega_{v'}) Y^{(n_4)}_{\ell_4}(\omega_{v'}) d\omega_{v'}$$

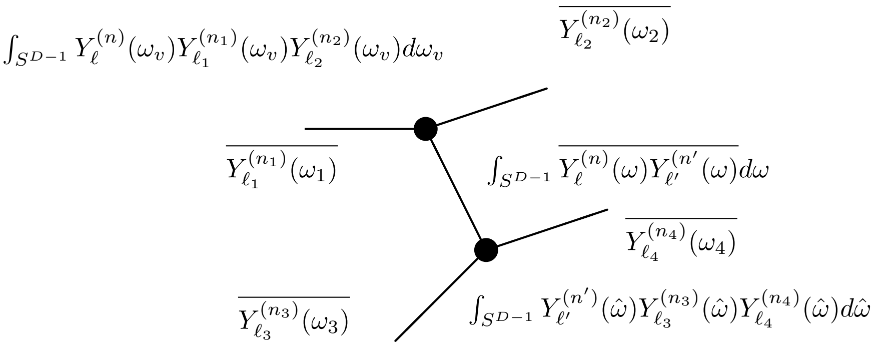
associated to the vertex v' of the star and with the spherical harmonics $\overline{Y_{\ell'}^{(n')}(\omega')}$, $\overline{Y_{\ell_3}^{(n_3)}(\omega_3)}$ and $\overline{Y_{\ell_4}^{(n_4)}(\omega_4)}$ placed on the half-edges, as in the figure.



Gluing together the half-edges labeled, respectively, by the spherical harmonics $\overline{Y_{\ell}^{(n)}(\omega)}$ and $\overline{Y_{\ell'}^{(n')}(\omega')}$ corresponds to the integration of (4.11), which corresponds to

$$\int_{S^{D-1}} \overline{Y_{\ell}^{(n)}(\omega)Y_{\ell'}^{(n')}(\omega')}\, d\omega.$$

This is represented graphically in the following picture.



Thus, the angular integral $\mathcal{A}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}, n)$ is obtained by integrating along the variables of the matched half edges,

$$\mathcal{A}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}, n) = c_{D,n}c_{D,n'}(\prod_{i=1}^{4} c_{D,n_i}) \cdot \tilde{\mathcal{A}}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}, n),$$

where $\tilde{\mathcal{A}}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}, n)$ is given by

$$\int_{S^{D-1}} d\omega \sum_{\ell,\ell',\ell_i} \overline{Y^{(n)}_{\ell}(\omega) Y^{(n')}_{\ell'}(\omega)} \prod_i \overline{Y^{(n_i)}_{\ell_i}(\omega_i)} \begin{pmatrix} n & n_1 & n_2 \\ \ell & \ell_1 & \ell_2 \end{pmatrix}_D \begin{pmatrix} n' & n_3 & n_4 \\ \ell' & \ell_3 & \ell_4 \end{pmatrix}_D$$

where we used the shorthand notation

$$\begin{pmatrix} n & n_1 & n_2 \\ \ell & \ell_1 & \ell_2 \end{pmatrix}_D := \langle Y^{(n)}_{\ell}, Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} \rangle_D. \tag{4.30}$$

Using the orthogonality relations for the spherical harmonics, this gives

$$(\prod_{i=1}^{4} c_{D,n_i}) \sum_{\ell_1,\ell_2,\ell_3,\ell_4} \overline{Y^{(n_1)}_{\ell_1}(\omega_1) Y^{(n_2)}_{\ell_2}(\omega_2) Y^{(n_3)}_{\ell_3}(\omega_3) Y^{(n_4)}_{\ell_4}(\omega_4)} \;\; \mathcal{K}_{\underline{n},\underline{\ell}}(n),$$

with the coefficients as in (4.29). ☐

Remark 4.9. Note that, since according to (4.11) each edge e carries an integral of a product of Gegenbauer polynomials $C^{(\lambda)}_n(\omega_v \cdot \omega) C^{(\lambda)}_n(\omega \cdot \omega_{v'})$, when viewed as a contraction of two half-edges this corresponds to viewing stars of vertices as carrying a Gaunt coefficient at the vertex (of order equal to the valence of the vertex) and spherical harmonics attached to the half edges, as in (4.22). Thus, for the case of gluing two trivalent stars along one edge we only need triple Gaunt coefficients and not quadruple ones, which would instead occur in the presence of valence four vertices.

The coefficients $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$ are usually very involved to compute explicitly (see (3.3), (3.6) and (4.7) of [**5**]). However, some terms simplify greatly in the case $D = 4$, and that will allow us to show the occurrence of functions closely related to multiple polylogarithm functions in §4.5 below. For later use, we give here the explicit computation, in dimension $D = 4$, of the coefficients $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$, in the particular case with $\underline{\ell} = 0$.

Proposition 4.10. *In the case where $D = 4$, the coefficient $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$ with $\ell_i = 0$ has the form*

$$\mathcal{K}^{(D=4)}_{\underline{n},\underline{0}}(n) = (\prod_{i=1}^{4} \frac{1}{(n_i+1)^{1/2}}) \frac{4\pi^4}{(n+1)^3}, \tag{4.31}$$

in the range where $n + n_1 + n_2$ and $n + n_3 + n_4$ are even and the inequalities $|n_j - n_k| \leq n_i \leq n_j + n_k$ hold for (n_i, n_j, n_k) equal to (n, n_1, n_2) or (n, n_3, n_4) and transpositions, and are equal to zero outside of this range.

Proof. We use the fact that ([**5**], (4.9) and [**59**], (22) and (23)) the coefficients $\langle Y^{(n_1)}_0, Y^{(n_2)}_0, Y^{(n_3)}_0 \rangle_D$ are zero outside the range where

$$\sum_i n_i \quad \text{is even and} \quad |n_j - n_k| \leq n_i \leq n_j + n_k, \tag{4.32}$$

while within this range they are given by the expression

$$\epsilon_D \frac{1}{\Gamma(D/2)} \left(\frac{(J+D-3)!}{(D-3)!\,\Gamma(J+D/2)} \prod_i \frac{(n_i + \frac{D}{2} - 1)\,\Gamma(J - n_i + \frac{D}{2} - 1)}{d^{(D)}_{n_i}\,(J-n_i)!} \right)^{1/2}, \tag{4.33}$$

where ϵ_D is a sign, $J = \frac{1}{2}\sum_i n_i$, and $d_{n_i}^{(D)} = \dim \mathcal{H}_{n_i}(S^{D-1})$. In the particular case where $D = 4$, the expression (4.33) reduces to

$$\begin{pmatrix} n_1 & n_2 & n_3 \\ 0 & 0 & 0 \end{pmatrix}_4 = \epsilon_4 \prod_i \frac{(n_i+1)^{1/2}}{(d_{n_i}^{(4)})^{1/2}} = \epsilon_4 \prod_i (n_i+1)^{-1/2},$$

using again the fact that $\dim \mathcal{H}_n(S^3) = (n+1)^2$. Thus, we obtain

$$\mathcal{K}^{(D=4)}_{n_i,\ell_i=0}(n) = c^2_{4,n} \begin{pmatrix} n & n_1 & n_2 \\ 0 & 0 & 0 \end{pmatrix}_4 \begin{pmatrix} n & n_3 & n_4 \\ 0 & 0 & 0 \end{pmatrix}_4 =$$

$$= \left(\frac{2\,Vol(S^3)}{2(n+1)}\right)^2 \frac{1}{(n+1)} \prod_{i=1}^4 \frac{1}{(n_i+1)^{1/2}} = \prod_{i=1}^4 (n_i+1)^{-1/2} \frac{4\pi^4}{(n+1)^3}.$$

□

4.5. Gluing stars of vertices. We now consider the full integrand, including the radial variables and again look at the effect of gluing together two half edges of two trivalent stars. We will see that one can explicitly identify the leading term in the resulting expression in the integrand with a function closely related to multiple polylogarithms.

Lemma 4.11. *Consider the star of a trivalent vertex, and let* $D = 4$. *After a change of variables* $t_i = r_{v_i}/r$, *with* $r = r_v$ *for* v *the central vertex of the star, the integrand (4.8), for an orientation* $\mathbf{o}$, *can be written as an expression* $\mathcal{I}_{\mathbf{o}}(r, t_1, t_2, t_3, \omega_1, \omega_2, \omega_3)\, dr dt_1 dt_2 dt_3$ *of the form*

$$r^9 \prod_{i=1}^3 t_i^{\alpha_i} \sum_{n_1,n_2,n_3} \mathcal{A}_{(n_1,n_2,n_3)}(\omega_1,\omega_2,\omega_3) t_1^{\epsilon_1 n_1} t_2^{\epsilon_2 n_2} t_3^{\epsilon_3 n_3}\, dr \prod_{i=1}^3 dt_i, \tag{4.34}$$

where $\alpha_i = 1$ *and* $\epsilon_i = 1$ *if the half-edge* e_i *is outgoing in the orientation* $\mathbf{o}$ *and* $\alpha_i = 3$ *and* $\epsilon_i = -1$ *if it is incoming, and where* $\mathcal{A}_{\underline{n}}(\underline{\omega}) = \mathcal{A}_{(n_j)}(\omega_j)$ *is the angular integral of Lemma 4.7 with* $D = 4$.

Proof. The integrand of (4.8), for the case of a trivalent star, is of the form

$$\prod_{i=1}^3 \mathcal{F}(r_{s(e_i)}, r_{t(e_i)}) r^3 dr \prod_{i=1}^3 r_{v_i}^3 dr_{v_i}, \tag{4.35}$$

with

$$\prod_{i=1}^3 \mathcal{F}(r_{s(e_i)}, r_{t(e_i)}) = (\prod_{i=1}^3 r_{t(e_i)}^{-2}) \sum_{\underline{n}} \mathcal{A}_{\underline{n}}(\underline{\omega}) \left(\frac{r_{s(e_1)}}{r_{t(e_1)}}\right)^{n_1} \left(\frac{r_{s(e_2)}}{r_{t(e_2)}}\right)^{n_2} \left(\frac{r_{s(e_3)}}{r_{t(e_3)}}\right)^{n_3}.$$

When combined with the volume form as in (4.35), this can be rewritten as

$$r^{\alpha_0} r_1^{\alpha_1} r_2^{\alpha_2} r_3^{\alpha_3} \sum_{\underline{n}} \mathcal{A}_{\underline{n}}(\underline{\omega}) \left(\frac{r_1}{r}\right)^{\epsilon_1 n_1} \left(\frac{r_2}{r}\right)^{\epsilon_2 n_2} \left(\frac{r_3}{r}\right)^{\epsilon_3 n_3} dr\, dr_1 dr_2 dr_3, \tag{4.36}$$

where the exponents α_i are given by the table

	$\mathbf{o}_0$	$\mathbf{o}_1$	$\mathbf{o}_2$	$\mathbf{o}_3$
α_0	-3	3	-1	1
α_1	1	3	1	1
α_2	1	3	3	1
α_3	1	3	3	3

where the orientation $\mathbf{o}_0$ has all the half-edges of the star pointing outward, $\mathbf{o}_1$ all pointing inward, $\mathbf{o}_2$ has e_1 outward and e_2 e_3 inward and $\mathbf{o}_3$ has e_1 and e_2 outward and e_3 inward. All the other cases are obtained by relabeling of indices. After we change variables to $t_i = r_{v_i}/r$, we obtain $dr \wedge \wedge_{i=1}^3 dr_i = r^3 dr \wedge \wedge_{i=1}^3 dt_i$ and (4.36) becomes

$$\mathcal{I}_{\mathbf{o}}(r,(t_i),\underline{\omega}) = r^9 t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \sum_{\underline{n}} \mathcal{A}_{\underline{n}}(\underline{\omega})\ (t_1)^{\epsilon_1 n_1}\ (t_2)^{\epsilon_2 n_2}\ (t_3)^{\epsilon_3 n_3}\ dr\, dt_1 dt_2 dt_3. \tag{4.37}$$

□

We can now perform the gluing of two stars by matching an oriented half-edge of one trivalent star to an oriented half edge of the other, so that one obtains an oriented edge. We rewrite (4.37) as

$$\mathcal{I}_{\mathbf{o}}(r,(t_i),\underline{\omega}) = \tilde{\mathcal{I}}_{\mathbf{o}}((t_i),\underline{\omega})\, r^9\, dr \tag{4.38}$$

where we set

$$\tilde{\mathcal{I}}_{\mathbf{o}}((t_i),\underline{\omega}) = t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \sum_{\underline{n}} \mathcal{A}_{\underline{n}}(\underline{\omega})\ (t_1)^{\epsilon_1 n_1}\ (t_2)^{\epsilon_2 n_2}\ (t_3)^{\epsilon_3 n_3}\ dt_1 dt_2 dt_3. \tag{4.39}$$

Lemma 4.12. *The gluing of two stars by matching an oriented half-edge corresponds to an integration*

$$\mathcal{I}_{\mathbf{o}}(t_i,\omega_i) := \int_{\bar{\Sigma}} \int_{S^{D-1}} \tilde{\mathcal{I}}_{\mathbf{o}}(t,t_1,t_2,\omega,\omega_1,\omega_2) \tilde{\mathcal{I}}_{\mathbf{o}}(t,t_3,t_4,\omega,\omega_3,\omega_4)\, dt d\omega, \tag{4.40}$$

where the domain of integration $\bar{\Sigma} = \bar{\Sigma}(t_1,t_2,t_3,t_4)$ *for the variable* t *is given by*

$$\bar{\Sigma} = \cap_{i,j: t(e_i)=s(e), s(e_j)=t(e)} \{ t \,|\, t_i \le t \le t_j \},$$

and a cutoff regularized divergence

$$\int_0^\Lambda \mathcal{I}_{\mathbf{o}}(t_i,\omega_i)\, r^9\, dr \sim_{\Lambda\to\infty} \frac{\Lambda^{10}}{10}\, \mathcal{I}_{\mathbf{o}}(t_i,\omega_i). \tag{4.41}$$

Proof. Gluing of two stars by matching an oriented half-edge means integrating

$$\int_0^\infty \int_{\bar{\Sigma}} \int_{S^{D-1}} \mathcal{I}_{\mathbf{o}}(r,t,t_1,t_2,\omega,\omega_1,\omega_2) \mathcal{I}_{\mathbf{o}}(r,t,t_3,t_4,\omega,\omega_3,\omega_4)\, dr\, dt\, d\omega. \tag{4.42}$$

This gives rise to an overall divergent factor (4.41) arising from the integration of (4.42) in the variable r, which can be taken care of by a cutoff regularization. The integration in the remaining variables gives the integrand $\mathcal{I}(t_1,t_2,t_3,t_4,\omega_1,\omega_2,\omega_3,\omega_4)$, as the result of gluing two trivalent stars by matching oriented half-edges to form an oriented edge e, which is given by (4.40). □

In the following we write $\underline{t} = (t_1,t_2,t_3,t_4)$ and similarly for $\underline{\omega}$, $\underline{n}$ and $\underline{\ell}$.

By combining (4.28) with (4.22), we can rephrase (4.40) in terms of isoscalars. This gives a decomposition of $\mathcal{I}_{\mathbf{o}}(t_i,\omega_i)$ into a sum of terms of the form

$$\mathcal{I}_{\mathbf{o}}(t_i,\omega_i) = \sum_{\underline{n},\underline{\ell}} \mathcal{I}_{\mathbf{o},\underline{\mathbf{n}},\underline{\ell}}(t_i,\omega_i).$$

We denote by $\mathcal{I}_{\mathbf{o},0}(\underline{t},\underline{\omega})$ the leading term

$$\mathcal{I}_{\mathbf{o},\mathbf{0}}(t_i,\omega_i) = \sum_{\underline{n}} \mathcal{I}_{\mathbf{o},\underline{\mathbf{n}},\underline{\mathbf{0}}}(t_i,\omega_i), \tag{4.43}$$

involving only the isoscalars with all $\ell_i = 0$. We have the following result computing the terms $\mathcal{I}_{\mathbf{o},0}(t_i,\omega_i)$.

Lemma 4.13. *In the case $D=4$, the integrands $\mathcal{I}_{\mathbf{o},0}(\underline{t},\underline{\omega})$ are given by*

$$\mathcal{I}_{\mathbf{o},0}(\underline{t},\underline{\omega}) = \sum_{\underline{n}} (\prod_{i=1}^{4} c_{D,n_i} \overline{Y_0^{(n_i)}(\omega_i)} \frac{t_i^{\alpha_i+\epsilon_i n_i}\, dt_i}{(n_i+1)^{1/2}}) \int_{\bar{\Sigma}} t^4\, dt \sum_n \frac{4\pi^2}{(n+1)^3} t^{\epsilon n}, \tag{4.44}$$

where the sum over the indices n and $\underline{n}$ is restricted by the constraints $n+n_1+n_2$ and $n+n_3+n_4$ are even and the inequalities $|n_j-n_k| \leq n_i \leq n_j+n_k$ hold for (n_i,n_j,n_k) equal to (n,n_1,n_2) or (n,n_3,n_4) and transpositions.

Proof. Using (4.40) and (4.28), (4.29), (4.34), we obtain for $\mathcal{I}_{\mathbf{o}}(\underline{t},\underline{\omega})$ the expression

$$\sum_{\underline{n}} \sum_{\underline{\ell}} (\prod_{i=1}^{4} c_{D,n_i} \overline{Y_{\ell_i}^{(n_i)}(\omega_i)} t_i^{\alpha_i+\epsilon_i n_i}\, dt_i) \int_{\bar{\Sigma}} t^4\, dt \sum_n \mathcal{K}_{\underline{n},\underline{\ell}}(n)\, t^{\epsilon n}.$$

The expression (4.44) then follows directly from the form (4.31) of the coefficients $\mathcal{K}_{\underline{n},\underline{0}}^{(D=4)}(n)$. The factor t^4 in the integral comes from the exponents $\alpha=1$ and $\alpha=3$ of the two half edges, which have matching orientations. □

Notice that, without the constraints on the summation range of the indices n, n_i, we would obtain again an integral of the general form (4.20), involving polylogarithm functions $\mathrm{Li}_s(t^\epsilon)$, with $s=3=k-2$ as in the case of polygons analyzed above. However, because not all values of n, n_i are allowed and one needs to impose the constraints of the form (4.32), one obtains more interesting expressions. We first introduce some notation.

4.5.1. *Summation domains and even condition.* In the following we let $\mathcal{R}$ denote a domain of summation for integers $(n_1,\ldots,n_k)$. We consider in particular the cases

$$\begin{aligned} \mathcal{R} = \mathcal{R}_P^{(k)} &:= \{(n_1,\ldots,n_k)\,|\, n_i>0,\ \ i=1,\ldots,k\} \\ \mathcal{R} = \mathcal{R}_{MP}^{(k)} &:= \{(n_1,\ldots,n_k)\,|\, n_k > \cdots > n_2 > n_1 > 0\} \\ \mathcal{R} = \mathcal{R}_T^{(3)} &:= \{(n_1,n_2,n_3)\,|\, n_2>n_1,\ \ n_2-n_1<n_3<n_2+n_1\}. \end{aligned} \tag{4.45}$$

We denote by $\mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k)$ the associated series

$$\mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{(n_1,\ldots,n_k)\in\mathcal{R}} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}. \tag{4.46}$$

In the first two cases of (4.45), this is, respectively, a product of polylogarithms $\mathrm{Li}^{\mathcal{R}_P}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \prod_j \mathrm{Li}_{s_j}(z_j)$ and and a multiple polylogarithm

$$\mathrm{Li}^{\mathcal{R}_{MP}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \mathrm{Li}_{s_1,\ldots,s_k}(z_1,\ldots,z_k).$$

We will discuss the third case more in detail below. We then define

$$\begin{aligned} \mathrm{Li}^{\mathcal{R},\mathrm{even}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) &:= \tfrac{1}{2}\left(\mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) + \mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(-z_1,\ldots,-z_k)\right) \\ \mathrm{Li}^{\mathcal{R},\mathrm{odd}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) &:= \tfrac{1}{2}\left(\mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) - \mathrm{Li}^{\mathcal{R}}_{s_1,\ldots,s_k}(-z_1,\ldots,-z_k)\right). \end{aligned} \tag{4.47}$$

The odd $\mathrm{Li}^{\mathcal{R},\mathrm{odd}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k)$ is a direct generalization of the Legendre χ function, while the even $\mathrm{Li}^{\mathcal{R},\mathrm{even}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k)$ corresponds to summing only over those indices in $\mathcal{R}$ whose sum is even,

$$\mathrm{Li}^{\mathcal{R},\mathrm{even}}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{(n_1,\ldots,n_k)\in\mathcal{R},\, \sum_i n_i \in 2\mathbb{N}} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}. \tag{4.48}$$

More generally, one can also consider summations of the form

$$\mathrm{Li}^{\mathcal{R},\mathcal{E}_1,\ldots,\mathcal{E}_k}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{(n_1,\ldots,n_k)\in\mathcal{R},\, n_i\in\mathcal{E}_i} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}, \tag{4.49}$$

where, for each $i=1,\ldots,k$, $\mathcal{E}_i = 2\mathbb{N}$ or $\mathcal{E}_i = \mathbb{N}\smallsetminus 2\mathbb{N}$, that is, some of the summation indices are even and some odd.

4.5.2. *Matching half-edges.* We now illustrate in one sufficiently simple and explicit case, what the leading $\ell = 0$ term looks like when all the half-edges of stars are joined together. We look at the case of two stars of trivalent vertices with the half edges pairwise joined, that is, the 3-banana graph (two vertices and three parallel edges between them).

PROPOSITION 4.14. *In the case of $D=4$, consider the graph with two vertices and three parallel edges between them. The $\underline{\ell}=0$ amplitude $\mathcal{I}_{\mathbf{o},0}$ is given by*

$$\mathcal{I}_{\mathbf{o},0} = \int_0^1 t^9\,(2^6\mathrm{Li}^{\mathcal{R}_{MP},odd,even}_{6,3}(t,t) + 2\mathrm{Li}^{\mathcal{R}_T,even}_{3,3,3}(t,t,t))\,dt. \tag{4.50}$$

PROOF. There is a unique acyclic orientation of this graph, with the three edges oriented in the same direction. Thus, there is a single variable $t\in[0,1]=\bar{\Sigma}$ in the integrand of $\mathcal{I}_{\mathbf{o},0}$, and the latter has the form

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}} \mathcal{K}_{n_1,n_2,n_3} t^{n_1+n_2+n_3},$$

where the coefficients $\mathcal{K}_{n_1,n_2,n_3}$ are given by

$$\mathcal{K}_{n_1,n_2,n_3} = \frac{c^2_{4,n_1}c^2_{4,n_2}c^4_{4,n_3}}{(n_1+1)(n_2+1)(n_3+1)} = \frac{(4\pi^2)^3}{(n_1+1)^3(n_2+1)^3(n_3+1)^3},$$

according to Proposition 4.10, and the fact that all the half-edges of the two trivalent stars are matched. The summation domain $\mathcal{D}$ is given by

$$\mathcal{D} = \{(n_1,n_2,n_3)\,|\, n_i\geq 0\ |n_j-n_k|\leq n_i\leq n_j+n_k,\ \sum_i n_i \text{ even}\}.$$

We subdivide this into separate domains $\mathcal{D}=\mathcal{D}_1\cup\mathcal{D}_2\cup\mathcal{D}_3\cup\mathcal{D}_4\cup\mathcal{D}_5$, where

$$\begin{aligned}
&\mathcal{D}_1 = \{n_1=0,\ n_2\geq 0,\ n_3=n_2\}\\
&\mathcal{D}_2 = \{n_2=0,\ n_1>0,\ n_3=n_1\}\\
&\mathcal{D}_3 = \{n_1>0,\ n_2=n_1,\ 0\leq n_3\leq 2n_1,\ n_3 \text{ even }\}\\
&\mathcal{D}_4 = \{0<n_2<n_1,\ n_1-n_2\leq n_3\leq n_1+n_2,\ \textstyle\sum_i n_i \text{ even }\}\\
&\mathcal{D}_5 = \{0<n_1<n_2,\ n_2-n_1\leq n_3\leq n_1+n_2,\ \textstyle\sum_i n_i \text{ even }\}.
\end{aligned}$$

We have

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_1} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = t^{-2}\sum_{n\geq 1}\frac{t^{2n}}{n^6} = t^{-2}\mathrm{Li}_6(t^2),$$

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_2} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = t^{-2}\sum_{n\geq 2}\frac{t^{2n}}{n^6} = t^{-2}\mathrm{Li}_6(t^2)-1$$

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_3} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = \sum_{n>0, 0\leq \ell\leq n}\frac{t^{2(n+\ell)}}{(2\ell+1)^3(n+1)^6}$$

$$= -1 + 2^6 t^{-3} \sum_{n\geq 0,\, 0\leq \ell\leq n} \frac{t^{2n+2+2\ell+1}}{(2\ell+1)^3(2n+2)^6}$$

$$= -1 + 2^6 t^{-3} \sum_{\substack{0<m_1<m_2\\ m_1 \text{ odd },\, m_2 \text{ even}}} \frac{t^{m_1+m_2}}{m_1^6 m_2^3} = -1 + 2^6 t^{-3}\mathrm{Li}_{6,3}^{\mathcal{R}_{MP},\text{odd},\text{even}}(t,t)$$

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_4} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = t^{-3}\mathrm{Li}_{3,3,3}^{\mathcal{R}_T,\text{even}}(t,t,t) + 1 - t^{-2}\mathrm{Li}_6(t^2)$$

$$\sum_{(n_1,n_2,n_3)\in\mathcal{D}_5} \frac{t^{n_1+n_2+n_3}}{(n_1+1)^3(n_2+1)^3(n_3+1)^3} = t^{-3}\mathrm{Li}_{3,3,3}^{\mathcal{R}_T,\text{even}}(t,t,t) + 1 - t^{-2}\mathrm{Li}_6(t^2),$$

where in the last two cases the term $t^{-2}\mathrm{Li}_6(t^2)-1$ corresponds to the summation over $m_2 = 1$, $m_1 > 1$ and $m_3 = m_1$ (respectively, $m_1 = 1$, $m_2 > 1$, $m_3 = m_2$), with $m_i = n_1 + 1$. The integrand has a factor of t^4 for each edge, as in Lemma 4.13, which gives a power of t^{12} that combines with the t^{-3} factor in the result of the sum of the terms above to give the t^9 factor in (4.50). ☐

For more general graphs, where more vertices and more stars are involved, one gets summations involving several "triangular conditions" $|n_j - n_k| \leq n_i \leq n_j + n_k$ around each vertex, and the integrand can correspondingly be expressed in terms of series with a higher depth. Moreover, notice that we have focused here on the leading terms $\mathcal{K}^{D=4}_{\underline{n},\underline{\ell}=0}(n)$ only. When one includes all the other terms $\mathcal{K}_{\underline{n},\underline{\ell}}(n)$ with $\ell_i \neq 0$, the expressions become much more involved, as these coefficients are expressed in terms of the isoscalars (4.27) and of the standard $3j$-symbols for $SO(3)$, through the factorization (4.26) that expresses the triple $SO(4)$ Gaunt coefficients in terms of isoscalar factors and triple $SO(3)$ Gaunt coefficients. The isofactors are known explicitly [**5**], [**59**] so the computation can in principle be carried out in full, but it becomes much more cumbersome.

Next we show that the functions $\mathrm{Li}^{\mathcal{R}_T}_{s_1,s_2,s_3}(z_1,z_2,z_3)$ that appear in these Feynman amplitude computations can be related, via the Euler–Maclaurin summation formula, to some well known generalizations of multiple zeta values and multiple polylogarithms.

4.5.3. *Mordell–Tornheim and Apostol–Vu series.* We consider two generalizations of the multiple polylogarithm series, which arise in connection to the Mordell–Tornheim and the Apostol–Vu multiple series. The Mordell–Tornheim multiple series is given by [**69**], [**83**]

$$\zeta_{MT,k}(s_1,\ldots,s_k;s_{k+1}) = \sum_{(n_1,\ldots,n_k)\in\mathcal{R}_P^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}}, \tag{4.51}$$

with an associated multiple polylogarithm-type function

$$\mathrm{Li}^{MT}_{s_1,\dots,s_k;s_{k+1}}(z_1,\dots,z_k;z_{k+1}) = \sum_{(n_1,\dots,n_k)\in\mathcal{R}^{(k)}_P} \frac{z_1^{n_1}\cdots z_k^{n_k} z_{k+1}^{(n_1+\cdots+n_k)}}{n_1^{s_1}\cdots n_k^{s_k}(n_1+\cdots+n_k)^{s_{k+1}}}. \tag{4.52}$$

Similarly, the Apostol–Vu multiple series [**12**] is defined as

$$\zeta_{AV,k}(s_1,\dots,s_k;s_{k+1}) = \sum_{(n_1,\dots,n_k)\in\mathcal{R}^{(k)}_{MP}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}}, \tag{4.53}$$

and we consider the associated multiple polylogarithm-type series

$$\mathrm{Li}^{AV}_{s_1,\dots,s_k;s_{k+1}}(z_1,\dots,z_k;z_{k+1}) = \sum_{(n_1,\dots,n_k)\in\mathcal{R}^{(k)}_{MP}} \frac{z_1^{n_1}\cdots z_k^{n_k} z_{k+1}^{(n_1+\cdots+n_k)}}{n_1^{s_1}\cdots n_k^{s_k}(n_1+\cdots+n_k)^{s_{k+1}}}. \tag{4.54}$$

4.5.4. *Euler–Maclaurin formula.* A way to understand better the behavior of the functions (4.48) with $\mathcal{R} = \mathcal{R}^{(3)}_T$ that appear in this result, is in terms of the Euler–Maclaurin summation formula.

LEMMA 4.15. *Let $f(t) = x^t t^{-s}$. Then*

$$f^{(k)}(t) = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}\binom{s+k-j-1}{k-j}(k-j)!\,t^{-(s+k-j)}x^t\,\log(x)^j. \tag{4.55}$$

PROOF. Inductively, we have

$$f^{(k)}(t) = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} s(s+1)\cdots(s+k-j-1)\,t^{-(s+k-j)}\,x^t\,\log(x)^j,$$

where $s(s+1)\cdots(s+k-j-1) = \binom{s+k-j-1}{k-j}(k-j)!$. □

The Euler–Maclaurin summation formula gives

$$\begin{aligned}\sum_{n=a}^{b} f(n) &= \int_a^b f(t)dt + \frac{1}{2}(f(b)+f(a)) \\ &+\sum_{k=2}^{N}\frac{b_k}{k!}(f^{(k-1)}(b)-f^{(k-1)}(a)) \\ &-\int_a^b \frac{B_N(t-[t])}{N!}f^{(N)}(t)\,dt,\end{aligned} \tag{4.56}$$

where b_k are the Bernoulli numbers and B_k the Bernoulli polynomials. We then have the following result.

PROPOSITION 4.16. *Consider the series $\mathrm{Li}^{\mathcal{R}}_{s_1,s_2,s_3}(z_1,z_2,z_3)$ defined as in (4.46), with $\mathcal{R} = \mathcal{R}^{(3)}_T$. When applying the Euler–Maclaurin formula to the innermost sum, the summation terms in (4.56) give rise to terms of the form*

$$\pm F_{j,k}(s_3,z_3)\ \mathrm{Li}^{AV}_{s_1,s_2;s_{3+k-j}}(z_1,z_2;z_3) \tag{4.57}$$

or

$$\pm F_{j,k}(s_3,z_3)\ \mathrm{Li}^{MT}_{s_1,s_{3+k-j};s_2}(z_1,z_2;z_3), \tag{4.58}$$

where

$$F_{j,k}(s,z) = \frac{b_k}{k!}\binom{k}{j}\binom{s+k-j-1}{k-j}(k-j)!\ \log(z)^j \tag{4.59}$$

Proof. For $\mathrm{Li}^{\mathcal{R}}_{s_1,s_2,s_3}(z_1,z_2,z_3)$, with $\mathcal{R} = \mathcal{R}^{(3)}_T$, the summation

$$\sum_{n_2-n_1<n_3<n_2+n_1} \frac{z_3^{n_3}}{n_3^{s_3}} \tag{4.60}$$

can be expressed, using Lemma 4.15, through the Euler–Maclaurin summation formula (4.56). Up to a sign, each summation term in the right-hand-side of (4.56) is the product of a function of z_3 of the form $F_{j,k}(s_3,z_3)$, as in (4.59), and a term of the form

$$\frac{z_3^{n_2+n_1}}{(n_2+n_1)^{s_3+k-j}} \quad \text{or} \quad \frac{z_3^{n_2-n_1}}{(n_2-n_1)^{s_3+k-j}}.$$

When inserted back into the summation on the remaining indices $n_2 > n_1$, this gives summations of the form

$$\sum_{n_2>n_1>0} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_1+n_2}}{n_1^{s_1} n_2^{s_2} (n_1+n_2)^{s_3+k-j}}, \tag{4.61}$$

in the first case, or in the second case, after a change of variables $m = n_2 - n_1$, $n = n_1$ in the indices

$$\sum_{n>0,m>0} \frac{z_1^{n} z_3^{m} z_2^{n+m}}{n^{s_1} m^{s_3+k-j} (n+m)^{s_2}}, \tag{4.62}$$

which are respectively of the form (4.52) and (4.54). □

4.6. Conclusion. In this section we have seen that, in the "real case" of the physical Feynman amplitude (3.2), in dimension $D = 4$, one can identify a specific contribution to the integral, which, after applying (4.26) that expresses the $SO(4)$ Gaunt coefficients in terms of isoscalars and $SO(3)$ Gaunt coefficients, is described naturally in representation theoretic terms as the $\ell = 0$ (deepest) term in the $SO(3)$ isofactors. This term contributes an integral that is always expressible in terms of summations like Mordell-Tornheim and Apostol-Vu series, which are ultimately expressible in terms of multiple zeta values. Thus, even when, for sufficiently large graphs for which the Feynman integral would involve non-mixed Tate periods, this specific term remains within the class of mixed Tate periods. We do not have, at present, a direct motivic interpretation of this term.

Note, however, that here we only see this mixed Tate term at the level of the period integrals and not at the level of the motive. Thus, for instance, we can keep track of graph operations such as pasting two half edges through what happens to the integral, as illustrated for instance in Proposition 4.14, but not at the level of the motive itself. So, in particular, what we see in terms of the integrals does not directly imply any kind of "motivic decomposition" in the motive that would split off a mixed Tate summand. The question of whether anything of that sort may be the case would require a different investigation.

5. Wonderful compactifications and complexified amplitudes

We move now to discuss a different problem of regularization and renormalization of integrals in configuration spaces, based on our generalization of the Feynman amplitude (3.2) given by the complexified version (3.9) introduced in §3.3.2. As in Definition 3.6, the locus of integration is, in this case, the complex variety $X^{V_\Gamma} \times \{y = (y_v)\}$, for a fixed choice of a point $y = (y_v)$, inside the configuration space Z^{V_Γ} with $Z = X \times X$.

We described in detail in our previous work [**36**] the geometry of the wonderful compactifications of the configuration spaces $\mathrm{Conf}_\Gamma(X)$. We recall here the main definitions and statements, adapted from $\mathrm{Conf}_\Gamma(X)$ to $F(X,\Gamma)$. The arguments are essentially the same as in [**36**].

5.1. Arrangements of diagonals.

A *simple arrangement* of subvarieties of a smooth quasi-projective ambient variety Y is a finite collection of nonsingular closed subvarieties $\mathcal{S} = \{S_i \subset Y, i \in I\}$ such that

- all nonempty intersections $\bigcap_{i\in J} S_i$ for $J \subset I$ are in the collection $\mathcal{S}$.
- for any pair $S_i, S_j \in \mathcal{S}$, the intersection $S_i \cap S_j$ is *clean*, that is, the tangent bundle of the intersection is the intersection of the restrictions of the tangent bundles.

5.1.1. *Diagonals of induced subgraphs and their arrangement.* For each induced and not necessarily connected subgraph $\gamma \subseteq \Gamma$, the corresponding (poly)diagonal is

$$\Delta_\gamma^{(Z)} = \{z = (z_v) \in Z^{V_\Gamma} \mid p(z_v) = p(z_{v'}) \text{ for } \{v, v'\} = \partial_\Gamma(e),\ e \in E_\gamma\}. \tag{5.1}$$

We have the following simple description.

Lemma 5.1. *The diagonal $\Delta_\gamma^{(Z)}$ is isomorphic to $X^{V_{\Gamma//\gamma}} \times X^{V_\Gamma}$, and has dimension*

$$\dim \Delta_\gamma^{(Z)} = \dim X^{V_{\Gamma//\gamma}} \times X^{V_\Gamma} = (2N_\Gamma - N_\gamma + b_0(\gamma)) \dim(X) \tag{5.2}$$

with $N_\Gamma = \#V_\Gamma$, and with $b_0(\gamma)$ the number of connected components of γ.

We then observe that the diagonals form an arrangement of subvarieties. This is the analog of Lemma 5 of [**36**].

Lemma 5.2. *For a given graph Γ, the collection*

$$\mathcal{S}_\Gamma = \{\Delta_\gamma^{(Z)} \mid \gamma \in \mathbf{SG}(\Gamma)\}, \tag{5.3}$$

with $\mathbf{SG}(\Gamma)$ the set of all induced subgraphs of Γ, is a simple arrangement of diagonal subvarieties in Z^{V_Γ}.

Proof. Let γ_1 and γ_2 be a pair of induced subgraphs. If $\gamma_1 \cap \gamma_2 = \emptyset$, then $\gamma = \gamma_1 \cup \gamma_2$ is in $\mathbf{SG}(\Gamma)$, and the corresponding diagonal $\Delta_\gamma^{(Z)}$ is given by the intersection $\Delta_{\gamma_1}^{(Z)} \cap \Delta_{\gamma_2}^{(Z)}$. On the other hand, if $\gamma_1 \cap \gamma_2 \neq \emptyset$, we consider the connected components γ_j of the union γ. Then, the intersection $\Delta_{\gamma_1}^{(Z)} \cap \Delta_{\gamma_2}^{(Z)}$ can be written as $\cap_j \Delta_{i(\gamma_j)}^{(Z)}$ where $i(\gamma_j)$ is the smallest induced subgraph of Γ containing γ_j. All diagonals are smooth and the ideal sheaf of intersection $\Delta_\gamma^{(Z)}$ is the direct sum of the ideal sheaves of the intersecting diagonals $\Delta_{\gamma_j}^{(Z)}$. By Lemma 5.1 of [**62**], their intersections are clean. □

5.1.2. *Building set of the arrangements of diagonals.* A subset $\mathcal{G} \subset \mathcal{S}$ is called a *building set* of the simple arrangement $\mathcal{S}$ if for any $S \in \mathcal{S}$, the minimal elements in $\{G \in \mathcal{G} : G \supseteq S\}$ intersect transversely and the intersection is S.

A $\mathcal{G}$-building set for the arrangement $\mathcal{S}_\Gamma$ can be identified by considering further combinatorial properties of graphs. A graph Γ is *2-vertex-connected* (or *biconnected*) if it cannot be disconnected by the removal of a single vertex along with the open star of edges around it. The graph consisting of a single edge is assumed to be biconnected. We then have the analog of Proposition 1 of [**36**].

Proposition 5.3. *For a given graph Γ, the set*

$$\mathcal{G}_\Gamma = \{\Delta_\gamma^{(Z)} \mid \gamma \subseteq \Gamma \text{ induced, biconnected}\} \tag{5.4}$$

is a $\mathcal{G}$-building set for the arrangement $\mathcal{S}_\Gamma$ of (5.3).

Proof. The intersection of a bi-connected subgraph of Γ with an induced subgraph is either empty or a union of bi-connected induced subgraphs attached at cut-vertices. We decompose induced subgraphs into bi-connected components. The diagonals corresponding to these bi-connected components are the minimal elements in the collection $\mathcal{S}_\Gamma$. For a pair of bi-connected induced subgraphs γ_1, γ_2 with $\gamma = \gamma_1 \cup \gamma_2$, we have the following equalities due to Lemma 5.1; $\dim \Delta_{\gamma_1}^{(Z)} + \dim \Delta_{\gamma_2}^{(Z)} - \dim \Delta_\gamma^{(Z)} = \dim(X^{V_{\Gamma//\gamma_1}} \times X^{V_\Gamma}) + \dim(X^{V_{\Gamma//\gamma_2}} \times X^{V_\Gamma}) - \dim(X^{V_{\Gamma//\gamma}} \times X^{V_\Gamma}) = 2\dim(X)N_\Gamma = \dim Z^{V_\Gamma}$, with $N_\Gamma = \#V_\Gamma$. These guarantee the transversality of the intersection $\Delta_{\gamma_1}^{(Z)} \cap \Delta_{\gamma_2}^{(Z)}$. □

5.2. The wonderful compactifications of arrangements. To be able to analyze the residues of Feynman integrals, we need a compactification $Z[\Gamma]$ of the configuration space $F(X,\Gamma)$ satisfying certain properties. In particular, $Z[\Gamma]$ must contain $F(X,\Gamma)$ as the top dimensional stratum, and the complement $Z[\Gamma]\setminus F(X,\Gamma)$ of this principal stratum must be a union of transversally intersecting divisors in $Z[\Gamma]$. The transversality is essential for the use of iterated Poincaré residues, which we will discuss in §7 below.

There is a smooth wonderful compactification $Z[\Gamma]$ of the configuration space $F(X,\Gamma)$ which is a generalization of the Fulton–MacPherson compactification [**48**]. The construction is completely analogous to the construction of the wonderful compactifications $\overline{\mathrm{Conf}_\Gamma(X)}$ considered in our previous paper [**36**]. Again, we illustrate here briefly what changes in passing from the case of $\mathrm{Conf}_\Gamma(X)$ to the case of $F(X,\Gamma)$.

5.2.1. *The iterated blowup description.* As in the case of $\mathrm{Conf}_\Gamma(X)$ (see §2.3 of [**36**]), the wonderful compactification $Z[\Gamma]$ is obtained by an iterated sequence of blowups. The following is the direct analog of Proposition 2 of [**36**].

Let $N = \#V_\Gamma$ and let $\mathcal{G}_{k,\Gamma} \subseteq \mathcal{G}_\Gamma$ be the subcollection

$$\mathcal{G}_{k,\Gamma} = \{\Delta_\gamma^{(Z)} \mid \gamma \in \mathbf{SG}_k(\Gamma) \text{ and biconnected}\} \quad \text{for} \quad k = 1, \dots, N-1. \tag{5.5}$$

Let $Y_0 = Z^{V_\Gamma}$ and let Y_k be the blowup of Y_{k-1} along the (iterated) dominant transform of $\Delta_\gamma^{(Z)} \in \mathcal{G}_{N-k+1,\Gamma}$. If Γ is itself biconnected, then Y_1 is the blowup of Y_0 along the deepest diagonal $\Delta_\Gamma^{(Z)}$, and otherwise $Y_1 = Y_0$. Similarly, we have

$Y_k = Y_{k-1}$ if there are no biconnected induced subgraphs with exactly $N-k+1$ vertices. The resulting sequence of blowups

$$Y_{N-1} \to \cdots \to Y_2 \to Y_1 \to Z^{V_\Gamma} \tag{5.6}$$

does not depend on the order in which the blowups are performed along the (iterated) dominant transforms of the diagonals $\Delta_\gamma^{(Z)}$, for $\gamma \in \mathcal{G}_{N-k+1,\Gamma}$, for a fixed k. Thus, the intermediate varieties Y_k in the sequence (5.6) are all well defined. The variety Y_{N-1} obtained through this sequence of iterated blowups is called the wonderful compactification;

$$Z[\Gamma] := Y_{N-1}. \tag{5.7}$$

Note that $Z[\Gamma]$ is a smooth quasi-projective variety as can be seen through its iterated blow-up construction, see [**62**].

5.2.2. *Divisors and their intersections.* A flag $\mathcal{F}$ in the arrangement $\mathcal{S}_\Gamma$ consists of a sequence $\hat{\Delta}_{\gamma_1} \subseteq \hat{\Delta}_{\gamma_2} \subseteq \cdots \hat{\Delta}_{\gamma_r}$ of (poly)diagonals associated to disjoint unions of induced subgraphs γ_i. Recall from [**62**] and Definition 3.7.1 of [**68**] that a $\mathcal{G}_\Gamma$-*nest* $\mathcal{N}$ for the arrangement $\mathcal{S}_\Gamma$ is a collection $\{\gamma_1, \ldots, \gamma_\ell\}$ of biconnected induced subgraphs with the property that there exists a flag $\mathcal{F}$ such that $\mathcal{N}$ is the set of all the $\mathcal{G}_\Gamma$-factors of the flag. The $\mathcal{G}_\Gamma$-factors of an element $\hat{\Delta}_{\gamma_i}$ of the flag are defined as the minimal elements of the building set $\mathcal{G}_\Gamma$ that contain $\hat{\Delta}_{\gamma_i}$.

The flags $\mathcal{F}$ of the arrangement $\mathcal{S}_\Gamma$ are in one-to-one correspondence with the forests of nested subgraphs $\mathcal{T}_\mathcal{F}$ (Proposition 3 of [**36**] and Proposition 3.7.3 of [**68**]). The latter are finite collections of rooted trees, where each component is a finite tree, where the vertices are labelled by connected induced subgraphs of Γ, in such a way that, if there is an edge from a vertex v_j to a vertex v_i (oriented away from the root), then the corresponding connected induced subgraphs satisfy $\gamma_i \subset \gamma_j$. Graphs assigned to vertices on different branches of a tree, or on different trees in the forest, are disjoint.

Given a subgraph $\gamma \subseteq \Gamma$, we denote by $\iota(\gamma)$ the smallest induced subgraph containing γ. By Proposition 3.7.3 of [**68**], given two biconnected induced subgraphs γ, γ', the set $\mathcal{N} = \{\Delta_\gamma, \Delta_{\gamma'}\}$ is a $\mathcal{G}_\Gamma$-nest if and only the graphs satisfy one of the following conditions:

1. $\gamma \cap \gamma' = \emptyset$;
2. $\iota(\gamma \cup \gamma')$ is not biconnected;
3. $\gamma \subset \gamma'$ or $\gamma' \subset \gamma$.

Remark 5.4. In [**62**] and in Proposition 3 of [**36**], the $\mathcal{G}_\Gamma$-nests are characterized in terms of a list of properties similar to the above, with the possibility of $\iota(\gamma \cup \gamma')$ not biconnected replaced by $\gamma \cap \gamma' = \{v\}$, a single vertex. However, the latter condition does not always correspond to a $\mathcal{G}_\Gamma$-nest, as observed by Martin and Sabbah, [**68**], [**73**]. This fact does not affect the results recalled below on the strata of the wonderful compactification.

We then have the following analog of Proposition 4 of [**36**] and Proposition 3.8.1 of [**68**].

Proposition 5.5. *For a given biconnected induced subgraph* $\gamma \subseteq \Gamma$*, let* $D_\gamma^{(Z)}$ *be the divisor obtained as the iterated dominant transform of* $\Delta_\gamma^{(Z)}$ *in the iterated* *blowup construction* (5.6) *of* $Z[\Gamma]$*. Then*

$$Z[\Gamma] \setminus F(X,\Gamma) = \bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma^{(Z)}. \tag{5.8}$$

The divisors $D_\gamma^{(Z)}$ *have the property that*

$$D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_\ell}^{(Z)} \neq \emptyset \Leftrightarrow \{\gamma_1, \dots, \gamma_\ell\} \ \text{ is a } \mathcal{G}_\Gamma\text{-nest.} \tag{5.9}$$

5.3. Motives of wonderful compactifications. As in the case of the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ analyzed in [**36**], one can obtain the explicit formula for the motive of the compactifications $Z[\Gamma]$ directly from the formula for the motive of blow-ups and the iterated construction of §5.2.1.

We first introduce the following notation as in [**36**], [**61**]. Given a $\mathcal{G}_\Gamma$-nest $\mathcal{N}$, and a biconnected induced subgraph γ such that $\mathcal{N}' = \mathcal{N} \cup \{\gamma\}$ is still a $\mathcal{G}_\Gamma$-nest, we set

$$r_\gamma = r_{\gamma,\mathcal{N}} := \dim(\cap_{\gamma' \in \mathcal{N}: \gamma' \subset \gamma} \Delta_{\gamma'}) - \dim \Delta_\gamma, \tag{5.10}$$

$$M_{\mathcal{N}} := \{(\mu_\gamma)_{\Delta_\gamma \in \mathcal{G}_\Gamma} \, : \, 1 \leq \mu_\gamma \leq r_\gamma - 1, \ \mu_\gamma \in \mathbb{Z}\}, \tag{5.11}$$

$$\|\mu\| := \sum_{\Delta_\gamma \in \mathcal{G}_\Gamma} \mu_\gamma. \tag{5.12}$$

We write here $\mathfrak{m}(X)$ for the motive in the Voevodsky category. This corresponds to the notation M_{gm} of [**88**].

The following result is the analog of Proposition 8 of [**36**], see also [**61**] for the formulation in the case of Chow motives.

Proposition 5.6. *Let* X *be a smooth projective variety. The Voevodsky motive* $\mathfrak{m}(Z[\Gamma])$ *of the wonderful compactification is given by*

$$\mathfrak{m}(Z[\Gamma]) = \mathfrak{m}(Z^{V_\Gamma}) \oplus \bigoplus_{\mathcal{N} \in \mathcal{G}_\Gamma\text{-nests}} \ \bigoplus_{\mu \in M_{\mathcal{N}}} \mathfrak{m}(X^{V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}} \times X^{V_\Gamma})(\|\mu\|)[2\|\mu\|] \tag{5.13}$$

where $\Gamma/\delta_{\mathcal{N}}(\Gamma)$ *is the quotient* $\Gamma/\delta_{\mathcal{N}}(\Gamma) := \Gamma//(\gamma_1 \cup \cdots \cup \gamma_r)$ *for the* $\mathcal{G}_\Gamma$*-nest* $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$.

Proof. Let $\widetilde{Y} \to Y$ be the blow-up of a smooth scheme Y along a smooth closed subscheme $V \subset Y$. Then $\mathfrak{m}(\widetilde{Y})$ is canonically isomorphic to

$$\mathfrak{m}(Y) \oplus \bigoplus_{k=1}^{\mathrm{codim_Y(V)}-1} \mathfrak{m}(V)(k)[2k],$$

see Proposition 3.5.3 in [**88**]. The result then follows by applying this blow-up formula for Voevodsky's motives to the iterated blow-up construction given in Section 5.2.1. □

A simplified form of (5.13) (in the case of the Grothendieck class) was given in Theorem 4.3.3 of [**68**].

We obtain from Proposition 5.6 the following simple corollary (see §3.2 of [**36**]).

Corollary 5.7. *If the motive of the smooth projective variety X is mixed Tate, then the motive of $Z[\Gamma]$ is mixed Tate, for all graphs Γ. Moreover, the exceptional divisors $D_\gamma^{(Z)}$ associated to the biconnected induced subgraphs $\gamma \subseteq \Gamma$ and the intersections $D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_\ell}^{(Z)}$ associated to the $\mathcal{G}_\Gamma$-nests $\{\gamma_1, \ldots, \gamma_\ell\}$ are also mixed Tate.*

Proof. This is an immediate consequence of the construction of $Z[\Gamma]$ since the motive of $Z[\Gamma]$ depends upon the motive of X only through products, Tate twists, sums, and shifts. All these operations preserve the subcategory of mixed Tate motives. The reason why the intersections $D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_\ell}^{(Z)}$ are also mixed Tate is because one has an explicit stratification, as described in [**36**] and [**62**], where one has a description of the intersections of diagonals in terms of configuration spaces of quotient graphs and by repeated use of the blowup formula for motives. □

Remark 5.8. One can also see easily that, if the variety X is defined over $\mathbb{Z}$, then so is $Z[\Gamma]$ and so are the $D_\gamma^{(Z)}$ and their unions and intersections. Moreover, all these varieties then satisfy the unramified criterion of §3.5 and Proposition 3.10 of [**50**].

5.4. Complexified amplitude and wonderful compactifications. We now consider the form $\omega_\Gamma^{(Z)}$ defined as in (3.9) and discuss its behavior when pulled back from Z^{V_Γ} to the wonderful compactification $Z[\Gamma]$.

5.4.1. *Loci of divergence.* For massless scalar Euclidean field theories, the pole locus $\{\omega_\Gamma^{(Z)} = \infty\}$ in $Z^{\mathbf{V}_\Gamma}$ is

$$\mathcal{Z}_\Gamma := \left\{ \prod_{e \in E_\Gamma} \| p(z_{v_{s(e)}}) - p(z_{v_{t(e)}}) \|^2 = 0 \right\}. \tag{5.14}$$

Since we have used the Euclidean norm in the definition of the amplitudes, rather than an algebraic quadratic form, the locus of divergences is simply a union of $\Delta_e^{(Z)} = \Delta_e \times X^{V_\Gamma}$, hence we can restate the above as follows.

Lemma 5.9. *The divergent locus of the density $\omega_\Gamma^{(Z)}$ of (3.9) in Z^{V_Γ} is given by the union $\bigcup_{e \in E_\Gamma} \Delta_e^{(Z)}$.*

5.4.2. *Order of singularities in the blowups.* Let $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ denote the pullbacks of the form $\omega_\Gamma^{(Z)}$ of (3.9) to the iterated blowups of $Z^{\mathbf{V}_\Gamma}$ along the (dominant transforms of) the diagonals $\Delta_\gamma^{(Z)}$, for $\gamma \subset \Gamma$ a biconnected induced subgraph.

Proposition 5.10. *Let Γ be a connected graph. Then for every biconnected induced subgraph $\gamma \subset \Gamma$, the pullback $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ of $\omega_\Gamma^{(Z)}$ to the blowup along the (dominant transform of) $\Delta_\gamma^{(Z)}$ has singularities of order*

$$\begin{aligned} \mathrm{ord}_\infty(\pi_\gamma^*(\omega_\Gamma), D_\gamma^{(Z)}) = &\quad (2D-2)\#E_\gamma - 2D(\#V_\gamma - 1) + 2 \\ = &\quad 2Db_1(\gamma) - 2\#E_\gamma + 2 \end{aligned} \tag{5.15}$$

along the exceptional divisors $D_\gamma^{(Z)}$ in the blowup. Here $b_1(\gamma)$ denotes the first Betti number of graph γ.

Proof. Let $m = DN_\Gamma$, with $N_\Gamma = \#V_\Gamma$ and $L \subset \mathbb{A}^{2m}$ be the coordinate subspace given by the equations $\{x_1 = \cdots = x_k = 0\}$, and $\pi : \widetilde{\mathbb{A}}^{2m} \to \mathbb{A}^{2m}$ be the blowup along $L \subset \mathbb{A}^{2m}$. If one chooses the coordinates w_i in the blow up, so that $w_i = x_i$, for $i = k, \ldots, 2m$, and $w_i w_k = x_i$ for $i < k$. The exceptional divisor given by $w_k = 0$ in these coordinates. Then, one obtains

$$\pi^*(dx_1 \wedge d\bar{x}_1 \wedge \cdots \wedge dx_m \wedge d\bar{x}_m) = |w|^{2(k-1)} dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_m \wedge d\bar{w}_m.$$

This form has a zero of order $2 \cdot (\mathrm{codim}(L) - 1)$ along the exceptional divisor of the blowup.

The codimension of the diagonal $\Delta_\gamma \subset X^{V_\Gamma}$ associated to a connected subgraph $\gamma \subset \Gamma$ is $D\ (N_\gamma - 1)$, with $N_\gamma = \#V_\gamma$ On the other hand, the form $\omega_\Gamma^{(Z)}$ has singularity along $\Delta_\gamma^{(Z)}$ of order $(2D-2)\#E_\gamma$. Hence,

$$\begin{aligned} \mathrm{ord}_\infty(\pi_\gamma^*(\omega_\Gamma^{(Z)}), D_\gamma^{(Z)}) &= (\text{order of } \infty) - (\text{order of zeros}) \\ &= (2D-2)n_\gamma - 2D(N_\gamma - 1) + 2. \end{aligned}$$

with $n_\gamma = \#E_\gamma$ and $N_\gamma = \#V_\gamma$. □

Remark 5.11. Note that the orders of pole are different from the case of the form ω_Γ on $\overline{\mathrm{Conf}_\Gamma(X)}$, see §4.3 of [**36**]. In particular, note that we have order of singularity $(2D-2)n_\gamma$ along $\Delta_\gamma^{(Z)}$, with $n_\gamma = \#E_\gamma$, because the form $\omega_\Gamma^{(Z)}$ of (3.9) is built using the Green functions of the complex Laplacian, (3.17), which has exponent $2D-2$ instead of the exponent $D-2$ of the real case. In particular, in the real case of the physical Feynman amplitude one has the familiar log divergent condition for graphs, given by the relation $(D-2)n_\gamma = D(N_\gamma - 1)$, with $N_\gamma = \#V_\gamma$, or equivalently $n_\gamma = b_1(\gamma)D/2$. In the complexified case we are considering here, this relation is replaced by the condition $(D-1)n_\gamma = D(N_\gamma - 1)$, or equivalently $n_\gamma = b_1(\gamma)D$. This is compatible with the usual log divergent condition, since in the complexified case D is the complex dimension.

On the basis of the remark here above, we make the following definition, which replaces the usual definition of log divergent graphs of the real case.

Definition 5.12. A graph Γ is primitive $\mathbb{C}$-log divergent if it satisfies the following conditions:

(1) $\#E_\Gamma = Db_1(\Gamma)$
(2) all subgraphs $\gamma \subseteq \Gamma$ satisfy $\#E_\gamma \geq Db_1(\gamma)$.

Lemma 5.9 and Proposition 5.10 then imply the following.

Corollary 5.13. *Let* $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ *denote the pullback of* $\omega_\Gamma^{(Z)}$ *to the wonderful compactification* $Z[\Gamma]$. *The divergence locus of* $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ *in* $Z[\Gamma]$ *is given by the union of divisors*

$$\mathcal{D}_\Gamma = \bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma^{(Z)}. \tag{5.16}$$

5.5. Chain of integration and divergence locus. When pulling back the form $\omega_\Gamma^{(Z)}$ along the projection $\pi_\Gamma : Z[\Gamma] \to Z^{V_\Gamma}$, one also replaces the chain of integration $\sigma_\Gamma^{(Z,y)} = X^{V_\Gamma} \times \{y\}$ of (3.10) with $\tilde{\sigma}_\Gamma^{(Z,y)} \subset Z[\Gamma]$ with $\pi_\Gamma(\tilde{\sigma}_\Gamma^{(Z,y)}) = \sigma_\Gamma^{(Z,y)}$, which gives

$$\tilde{\sigma}_\Gamma^{(Z,y)} = \overline{\mathrm{Conf}}_\Gamma(X) \times \{y\} \subset Z[\Gamma] = \overline{\mathrm{Conf}}_\Gamma(X) \times X^{V_\Gamma}. \tag{5.17}$$

LEMMA 5.14. *The chain of integration* $\tilde{\sigma}_\Gamma^{(Z,y)}$ *of* (5.17) *intersects the locus of divergence* (5.16) *in*

$$\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma \times \{y\} \subset \overline{\mathrm{Conf}}_\Gamma(X) \times \{y\}. \tag{5.18}$$

PROOF. This follows directly from Corollary 5.13 and (5.17). □

Notice that, since $\mathcal{G}_\Gamma$-factors intersect transversely (see Proposition 2.8 of [**62**] and Proposition 4 of [**36**]), the intersection (5.18) of the chain of integration $\tilde{\sigma}_\Gamma^{(Z,y)}$ with the locus of divergence consists of a union of divisors D_γ inside X^{V_Γ} intersecting transversely, with $D_{\gamma_1} \cap \cdots \cap D_{\gamma_\ell} \neq \emptyset$ whenever $\{\gamma_1, \dots, \gamma_\ell\}$ form a $\mathcal{G}_\Gamma$-nest (see [**36**] and [**62**]).

5.6. Smooth and algebraic differential forms. Consider the restriction of the amplitude $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ to the chain $\tilde{\sigma}_\Gamma^{(Z,y)}$. It is defined on the complement of the divergence locus, namely on

$$\tilde{\sigma}_\Gamma^{(Z,y)} \smallsetminus \left(\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma \times \{y\} \right) \simeq \overline{\mathrm{Conf}}_\Gamma(X) \smallsetminus \left(\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma \right), \tag{5.19}$$

which is a copy of $\mathrm{Conf}_\Gamma(X)$ inside $Z[\Gamma]$. The form $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ is a closed form of top dimension on this domain.

Our next goal is to replace the form $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ with a cohomologous form with logarithmic poles along the divisor $\mathcal{D}_\Gamma = \cup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} \mathcal{D}_\gamma^{(Z)}$ of the compactification. As we see below, this could be done easily if one remains within the class of $\mathcal{C}^\infty$-forms, by directly applying the Griffiths-Schmid theorem of [**54**]. However, in view of identifying our regularized integrals with periods, we want something stronger, namely an *algebraic* form with logarithmic poles. We first need to recall some general well known facts about de Rham cohomology and forms with logarithmic poles.

5.6.1. *Forms with logarithmic poles.* Let $\mathcal{X}$ be a smooth projective variety of dimension m and let $\mathcal{D}$ be a union of smooth hypersurfaces intersecting transversely (strict normal crossings divisor). Let $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{D}$ and let $j : \mathcal{U} \hookrightarrow \mathcal{X}$ denote the inclusion. Let $\mathcal{M}^\star_{\mathcal{D},\mathcal{X}}$ denote the meromorphic de Rham complex, namely the complex of meromorphic forms on $\mathcal{X}$ with poles (of arbitrary order) along $\mathcal{D}$. Let $\Omega^\star_{\mathcal{X}}(\log \mathcal{D})$ denote the complex of differential forms on $\mathcal{X}$ with logarithmic poles along $\mathcal{D}$. We recall some general facts about cohomology classes, algebraic forms, and forms with logarithmic poles:

(1) **Grothendieck Comparison Theorem**, [**55**]: the natural morphism (de Rham morphism)

$$\mathcal{M}^\star_{\mathcal{D},\mathcal{X}} \to Rj_* \mathbb{C}_{\mathcal{U}}$$

is a quasi-isomorphism. This implies that the de Rham cohomology $H^\star_{dR}(\mathcal{U})$ is computed by the hypercohomology of the meromorphic de Rham complex. If $\mathcal{U}$ is affine, then the hypercohomology can be replaced by the ordinary cohomology of the complex of global sections.

(2) **Logarithmic Comparison Theorem**, [**43**]: for $\mathcal{D}$ a simple normal crossings divisor, the natural morphism

$$\Omega^\star_{\mathcal{X}}(\log \mathcal{D}) \to \mathcal{M}^\star_{\mathcal{D},\mathcal{X}}$$

is also a quasi-isomorphism. Combined with the previous statement, this implies that the de Rham cohomology $H^\star_{dR}(\mathcal{U})$ is computed by the hypercohomology of the logarithmic de Rham complex,

$$H^\star_{dR}(\mathcal{U}) \simeq \mathbb{H}^\star(\mathcal{X}, \Omega^\star_{\mathcal{X}}(\log \mathcal{D})). \tag{5.20}$$

However, even in the affine case, it is not always possible to replace the hypercohomology in (5.20) with ordinary cohomology and global sections, see Example 5.15.

(3) **Stein open sets** (Lemma 2.5 of [**34**]): the Logarithmic Comparison Theorem is equivalent to the statement that, for all Stein open sets $\mathcal{V} \subset \mathcal{X}$, there are isomorphisms $H^\star(\Gamma(\mathcal{V}, \Omega^\star_{\mathcal{X}}(\log \mathcal{D}))) \simeq H^\star_{dR}(\mathcal{V} \smallsetminus \mathcal{D})$. Namely, the hypercohomology in (5.20) can be replaced with ordinary cohomology of global sections when restricted to Stein open sets.

(4) **Hodge filtration**, [**43**]: there is a mixed Hodge structure on $\mathcal{U}$ with Hodge filtration given by

$$F^p H^k_{dR}(\mathcal{U}) = \mathrm{Im}(\mathbb{H}^k(\mathcal{X}, \Omega^{\geq p}_{\mathcal{X}}(\log \mathcal{D})) \to \mathbb{H}^k(\mathcal{X}, \Omega^\star_{\mathcal{X}}(\log \mathcal{D}))).$$

The de Rham cohomology classes that lie in the term $F^m H^m_{dR}(\mathcal{U})$ of the Hodge filtration can be realized by global forms with logarithmic poles, since there is an isomorphism

$$F^m H^m_{dR}(\mathcal{U}) = H^0(\mathcal{X}, \Omega^m_{\mathcal{X}}(\log \mathcal{D})). \tag{5.21}$$

This follows from the degeneration at the E_1 term of the Frölicher spectral sequence associated to the Hodge filtration (see Proposition 5.16 of [**67**], which we reproduce below as Proposition 5.16).

(5) **Griffiths-Schmid Theorem**, [**54**]: Let $\Omega_{\mathcal{C}^\infty(\mathcal{X})}(\log \mathcal{D})$ denote the $\mathcal{C}^\infty$-logarithmic de Rham complex, with $\mathcal{D}$ a simple normal crossings divisor in a complex smooth projective variety $\mathcal{X}$. Then there is an isomorphism of cohomologies $H^*_{dR}(\mathcal{U}) = H^*(\Omega_{\mathcal{C}^\infty(\mathcal{X})}(\log \mathcal{D}))$.

Example 5.15. The hypercohomology in (5.20) cannot, in general, be replaced by cohomology of global sections, even assuming that $\mathcal{U}$ is affine. Let $\mathcal{U}$ be the complement of a finite set of n points in a smooth projective curve $\mathcal{X}$. Then $\dim H^1_{dR}(\mathcal{U}) = 2g + n - 1$, while by Riemann–Roch the space of global sections of the sheaf of logarithmic differentials has dimension $g + n - 1$.

Let $\mathcal{X}$ be a smooth projective variety of dimension $n = \dim \mathcal{X}$, with $\mathcal{D}$ a simple normal crossings divisor with affine complement $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{D}$. The Hodge filtration $F^p H^k_{dR}(\mathcal{U})$ is induced by the naive filtration on $\Omega^\star_{\mathcal{X}}(\log \mathcal{D})$.

The following fact is very well known, see [**43**]. We recall it here simply for the reader's convenience.

Proposition 5.16. *The Hodge filtration satisfies*

$$F^n H^n_{dR}(\mathcal{U}) = H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}(\log \mathcal{Z})). \tag{5.22}$$

Proof. Given a complex $K^\star$ that comes from a double complex $K^{p,q}$, the spectral sequence of a filtration F with

$$F^p K^n = \oplus_{r\geq p, r+s=n} K^{r,s}$$

is given by

$$E_0^{p,q} = \mathrm{Gr}_p^F K^{p+q} = F^p K^{p+q}/F^{p+1}K^{p+q} = K^{p,q}$$
$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_p^F K^\star) = H^q(K^{p,\star})$$
$$E_\infty^{p,q} = \mathrm{Gr}_p^F H^{p+q}(K^\star).$$

In particular, the Frölicher spectral sequence associated to the Hodge filtration $F^p H^k_{dR}(\mathcal{U})$ has

$$E_1^{p,q} = H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}(\log \mathcal{D}))$$
$$E_\infty^{p,q} = F^p H^{p+q}_{dR}(\mathcal{U})/F^{p+1}H^{p+q}_{dR}(\mathcal{U}).$$

In particular, $E_1^{n,0} = H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}(\log \mathcal{D}))$ and $E_\infty^{n,0} = F^n H^n_{dR}(\mathcal{U})$. Deligne proved in [**43**] that, if $\mathcal{D}$ is normal crossings, then this spectral sequence degenerates at the E_1 term. This gives (5.22). □

5.7. Order of pole and Hodge filtration. It is in general difficult to estimate where a given class lies in the Hodge filtration. However, in the case of meromorphic forms, one can estimate the position in the Hodge fitration in terms of a filtration on the complex of meromorphic forms determined by the order of pole. More precisely, if $\mathcal{X}$ is a smooth projective variety and $\mathcal{D} \subset \mathcal{X}$ a simple normal crossings divisor, and $\mathcal{M}^\star_{\mathcal{D},\mathcal{X}}$ denotes the complex of meromorphic differential forms, one defines the polar filtration by taking $P^k\mathcal{M}^m_{\mathcal{D},\mathcal{X}}$ to consist of meromorphic m-forms with poles of order at most $m-k+1$, if $m-k\geq 0$ and zero otherwise. Deligne showed in §II.3, Proposition 3.13 of [**42**] and Proposition 3.1.11 of [**43**], that the natural morphism

$$(\Omega^\star_{\mathcal{X}}(\log \mathcal{D}), F^\star) \to (\mathcal{M}^\star_{\mathcal{D},\mathcal{X}}, P^\star)$$

is a filtered quasi-isomorphism.

Theorem 5.17. *Let Γ be a primitive $\mathbb{C}$-log divergent graph in the sense of Definition 5.12. Then there is a meromorphic form with logarithmic poles $\beta_\Gamma^{(Z)}$ such that $\omega_\Gamma^{(Z)}$ can be replaced, up to coboundaries, by a form $\eta_\Gamma^{(Z)} = \epsilon\, \beta_\Gamma^{(Z)} \wedge \bar\beta_\Gamma^{(Z)}$, where ϵ is a sign.*

Proof. We proceed as in Proposition 5.10. It suffices to check on a local model, with local coordinates. In $\mathbb{A}^{2m}$ with coordinates $\{x_1, \ldots, x_{2m}\}$ consider a subspace of codimension k, given by $L = \{x_1 = \cdots = x_k = 0\}$, for some $k < m$. Consider a smooth differential form

$$\omega = \prod_{j\in E} \frac{1}{\|x_j\|^{2r}} \bigwedge_{i=1}^k dx_i \wedge d\bar x_i \wedge \theta_k,$$

where $\theta_k = dx_{k+1} \wedge d\bar x_{k+1} \wedge \cdots \wedge dx_m \wedge d\bar x_m$. The set E is a subset (with repetitions) of the coordinates x_i with $i = 1, \ldots, k$. The form ω can be rewritten, for convenience, as a product $\omega = \epsilon\, \alpha \wedge \bar{\alpha}$, where ϵ is a sign and α is the meromorphic form

$$\alpha = \left(\prod_{j\in E} \frac{1}{x_j^r} \bigwedge_{i=1}^{k} dx_i \right) \wedge dx_{k+1} \wedge \cdots \wedge dx_m .$$

Let $\pi : \tilde{\mathbb{A}}^{2m} \to \mathbb{A}^{2m}$ be the blowup of $\mathbb{A}^{2m}$ along L. On $\tilde{\mathbb{A}}^{2m}$ we have local coordinates w_i with $w_i = x_i$ for $i \geq k$ and $w_i w_k = x_i$ for $i < k$, where $\{w_k = 0\}$ is the exceptional divisor of the blowup. The pullback gives

$$\pi^*(\bigwedge_{i=1}^{k} dx_i) = w_k^{k-1} \bigwedge_{i=1}^{k} dw_i$$

and $\pi^*(dx_{k+1} \wedge \cdots \wedge dx_m) = dw_{k+1} \wedge \cdots \wedge dw_m$, hence the pull back $\pi^*(\omega) = \epsilon\pi^*(\alpha) \wedge \overline{\pi^*(\alpha)}$ has a meromorphic form $\pi^*(\alpha)$ with a pole of order $re - k + 1$ along the exceptional divisor, where $e = \#E$. The form $\pi^*(\alpha)$, which is of degree m and pole of order $re - k + 1$ is therefore in the term P^ℓ of the polar filtration, with $\ell = m - (re - k + 1) + 1 = m - re + k$. This implies that its class is in the term of the Hodge filtration of order at least $m - re + k$.

When we apply this to the form $\omega_\Gamma^{(Z)}$ and the blowup along a diagonal $\Delta_\gamma^{(Z)}$ of codimension $k = D(N_\gamma - 1)$, with $N_\gamma = \#V_\gamma$, and with $r = D-1$ and $e = e_\gamma = \#E_\gamma$, we obtain $\omega_\Gamma^{(Z)} = \epsilon\, \alpha_\Gamma^{(Z)} \wedge \bar{\alpha}_\Gamma^{(Z)}$, where $\alpha_\Gamma^{(Z)}$ is a meromorphic form of degree DN_Γ, with poles of order $(D-1)e_\gamma - D(N_\gamma - 1) + 1$ along the divisor $D_\gamma^{(Z)}$. Thus, the class of the form $\alpha_\Gamma^{(Z)}$ is in the term $F^{DN_\Gamma + D(N_\gamma - 1) - (D-1)e_\gamma} H^{DN_\Gamma}(\mathcal{U})$ of the Hodge filtration, unless the pole order is lowered by cancellations due to exact forms, which would result in a higher level in the filtration. In particular, if the graph Γ is $\mathbb{C}$-log divergent, then the subgraphs $\gamma \subseteq \Gamma$ satisfies $D(N_\gamma - 1) - (D-1)e_\gamma \geq 0$, see Remark 5.11 and Definition 5.12. Then the meromorphic form $\alpha_\Gamma^{(Z)}$ is in $F^{DN_\Gamma} H^{DN_\Gamma}(\mathcal{U})$. In this case, by the general result on the Hodge filtration and the Frölicher spectral sequence mentioned in (5.21) above, we can replace $\alpha_\Gamma^{(Z)}$ by a global DN_Γ-form $\beta_\Gamma^{(Z)}$ with logarithmic poles along $D_\gamma^{(Z)}$. One then checks that, if the meromorphic form $\alpha_\Gamma^{(Z)}$ can be replaced, up to a coboundary, by $\beta_\Gamma^{(Z)}$, then the closed form $\omega_\Gamma^{(Z)}$ can also be replaced, up to a coboundary, by $\eta_\Gamma^{(Z)} = \epsilon\beta_\Gamma^{(Z)} \wedge \bar{\beta}_\Gamma^{(Z)}$. Indeed, if $\alpha_\Gamma^{(Z)} = \beta_\Gamma^{(Z)} + d\xi_\Gamma^{(Z)}$ then $\omega_\Gamma^{(Z)} - \eta_\Gamma^{(Z)} = \epsilon_1 \beta_\Gamma \wedge d\bar{\xi}_\Gamma^{(Z)} + \epsilon_2 d\xi_\Gamma^{(Z)} \wedge \bar{\beta}_\Gamma^{(Z)} + \epsilon_3 d\xi_\Gamma^{(Z)} \wedge d\bar{\xi}_\Gamma^{(Z)}$, where the ϵ_i are signs. Since $d\alpha_\Gamma^{(Z)} = d\beta_\Gamma^{(Z)} = 0$, the expression above is a coboundary. $\square$

5.8. Toy model regularization.

Consider the simple toy model example of the differential form

$$\omega = \frac{dz \wedge d\bar{z}}{\|z\|^2} = \frac{dz}{z} \wedge \frac{d\bar{z}}{\bar{z}} \tag{5.23}$$

on $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$.

Lemma 5.18. *A cutoff regularization of the integral on* $\mathbb{C}^*$ *of* (5.23) *has a logarithmic divergence in the cutoff scale, with a coefficient expressible in terms of a period integral of the form with logarithmic poles* dz/z.

Proof. Using polar coordinates $z = re^{i\theta}$ we can write ω as

$$\omega = -2i\,\frac{dr}{r} \wedge d\theta.$$

The integral on $\mathbb{C}^*$ has divergences at both $r = 0$ and $r \to \infty$: integrating with cutoffs leads to

$$-2i\int_0^{2\pi} d\theta \int_\epsilon^\Lambda \frac{dr}{r} = -4\pi i \log(\frac{\Lambda}{\epsilon}).$$

The coefficient is an integer multiple of the period of the meromorphic form with logarithmic poles

$$2\pi i = \int_{S^1} \frac{dz}{z}. \tag{5.24}$$

□

Note that the divergent integral $\int_{\mathbb{C}^*}\omega$ also admits other forms of regularization:

1. Gaussian cutoff:
$$\mathcal{I}_\epsilon(a) = \int_\epsilon^\infty \frac{dr}{r} e^{-ar} = \mathrm{Ei}(-a\epsilon) \sim \log(a\epsilon) + \gamma - a\epsilon + \frac{(a\epsilon)^2}{4} + \cdots ,$$
with Ei the exponential integral function;
2. Pauli–Villars regularization: $\mathcal{I}(a) - \mathcal{I}(b) = \int_0^\infty \frac{dr}{r}(e^{-ar} - e^{-br}) = \log(\frac{a}{b})$;
3. Dimensional regularization:
$$\mathcal{I}_s(a) = \mu^{2s}\int_0^\infty \frac{dr}{r^{1-s}} e^{-ar} = -\Gamma(s)(\frac{a}{\mu^2})^{-s} \sim \frac{1}{s} + \gamma + \log\frac{a}{\mu^2} + \cdots$$

As a similar, higher dimensional, toy model, we consider the case of the $2n$-form

$$\omega = \frac{dz_1}{z_1} \wedge \frac{d\bar{z}_1}{\bar{z}_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \wedge \frac{d\bar{z}_n}{\bar{z}_n} \tag{5.25}$$

on the torus $\mathbb{G}_m^n$ inside the affine space $\mathbb{A}^n$. As before, we can regularize the integral of ω on $\mathbb{G}_m^n$ with a cutoff regularization (or with one of the other regularization methods listed above).

Lemma 5.19. *A cutoff regularization of the integral on $\mathbb{G}_m(\mathbb{C})^n$ of* (5.25) *has a logarithmic divergence in the cutoff scale, with a coefficient expressible in terms of a period integral of an n-form with logarithmic poles η, integrated on an n-dimensional locus Σ (a real torus) inside the Leray coboundary $\mathcal{L}(\mathcal{D}) = \cup_i \mathcal{L}(D_i)$ of the divisor $\mathcal{D} = \cup_i D_i$ of coordinate hyperplanes $D_i = \{z_i = 0\}$.*

Proof. One simply applies the previous lemma coordinatewise and obtains a product integral with one logarithmic divergence in each variable. The coefficient of this divergence is (up to a sign) of the form $2^n\int_\Sigma \eta$, where $\eta = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ and $\Sigma = T^n = S^1 \times \cdots \times S^1$. We view $\Sigma \subset \cup_i \mathcal{L}(D_i)$ where $\mathcal{L}(D_i)$ is the boundary of a unit disk tubular neighborhood of D_i. □

With this toy model case in mind, we see how one can approach the computation of a regularized integral of the form $\beta_\Gamma^{(Z)} \wedge \bar{\beta}_\Gamma^{(Z)}$ obtained in Theorem 5.17, with the period (5.24) replaced by period integrals of iterated Poincaré residues of the form with logarithmic poles $\beta_\Gamma^{(Z)}$, on components and intersections of components of the divisor $\mathcal{D}_\Gamma^{(Z)}$ of the compactification, or equivalently integrals of the form $\beta_\Gamma^{(Z)}$ along iterated Leray coboundaries. We will discuss this procedure in the coming sections.

For the remaining of this section, we discuss how one can proceed, in the case of graphs that are not $\mathbb{C}$-log divergent, to use forms with logarithmic poles.

5.9. Cellular structures, torifications, and constructible torifications.

We now sketch a possible approach for the more general case, where the graph Γ is not necessarily $\mathbb{C}$-log divergent. In this case, unlike the previous result of Theorem 5.17, we cannot appeal to the Hodge filtration result to replace the meromorphic form $\pi_\Gamma^*(\alpha_\Gamma^{(Z)})$ in the pullback $\pi_\Gamma^*(\omega_\Gamma^{(Z)}) = \epsilon \pi_\Gamma^*(\alpha_\Gamma^{(Z)}) \wedge \pi_\Gamma^*(\bar{\alpha}_\Gamma^{(Z)})$ with a meromorphic form $\beta_\Gamma^{(Z)}$ with logarithmic poles. Thus, we need to proceed in a different way. A useful observation, for this case, is the "Stein open sets" method mentioned in §5.6.1, based on Lemma 2.5 of [**34**]. More precisely, we will show that if a smooth projective variety $\mathcal{X}$ has a decomposition into cells (cellular structure), or more generally into tori (torifications, or the certain weaker structures of constructible torifications) then the restriction of a meromorphic form α to each top dimensional set in the decomposition can be replaced by a form β with logarithmic poles. We will then use results of [**15**], [**64**] on the existence of (constructible) torifications on the compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$, under the assumption that X has a cell decomposition.

As before, suppose given a smooth projective variety $\mathcal{X}$ with a normal crossings divisor $\mathcal{D}$, and a form $\omega = \alpha \wedge \bar{\alpha}$, where α is a meromorphic form on $\mathcal{X}$ with poles along $\mathcal{D}$. The first observation (see Proposition 5.8 of [**67**]) is that, if $\mathcal{X}$ has a cellular decomposition, with top dimensional cells $\mathcal{X}_k$, then by Lemma 2.5 of [**34**] the restrictions $\alpha|_{\mathcal{X}_k}$ can be replaced with cohomologous meromorphic forms β_k on $\mathcal{X}_k$ with logaritmic poles along $\mathcal{D}$. These forms do not necessarily match on the boundaries of the cells, due to the existence of nontrivial Čech cocycles: the hypercohomology cannot be replaced globally by cohomology, but it is possible locally in the Stein open sets $\mathcal{X}_k$. The regularized integral of ω on $\mathcal{X} \smallsetminus \mathcal{D}$ can then be replaced by a sum of regularized integrals $\sum_k \int_{X_k \smallsetminus \mathcal{D}} \omega|_{X_k}$. After replacing $\omega|_{X_k}$ with $\beta_k \wedge \bar{\beta}_k$, one can apply, within each cell X_k the regularization procedure modeled on the toy model case of §5.8. This results in a logarithmic divergence with coefficient a period integral $\int_{X_k \cap \Sigma} \beta_k$. The nature of this period now depends on how the divisor $\mathcal{D}$ and its Leray coboundary $\mathcal{L}(\mathcal{D})$ intersect each cell X_k and the nature of the motives $\mathfrak{m}(\mathcal{D} \cap X_k)$.

In order to apply this method to our situation we need a further step. Indeed, the compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ do not necessarily admit a cellular structure, even assuming that the variety X does. However, an argument similar to the one outlined above applies to other decompositions of the variety.

A *torification* of a variety $\mathcal{X}$ is a decomposition of $\mathcal{X}$ into a disjoint union of tori $T_j = \mathbb{G}_m^{n_j}$ (see [**63**] for more details). By the same argument as above, if T_k denotes the subcollection of top dimensional tori, the restrictions $\alpha|_{T_k}$ can be replaced by meromorphic forms with logarithmic poles along $\mathcal{D}$, again by Lemma 2.5 of [**34**]. As in the previous case, the forms with logarithmic poles β_k on T_k do not necessarily extend to a single form η on all of $\mathcal{X} \smallsetminus \mathcal{D}$. One can proceed, as in the case of cellular decompositions, to compute regularized integrals on each T_k. In each case the coefficient of the logarithmic divergence will be a period integral that depends on the nature of the motive $\mathfrak{m}(\mathcal{D} \cap T_k)$.

A further level of generality is the notion of constructible torification introduced in [**64**]. In this case, one only requires that the variety $\mathcal{X}$ admits a decomposition into a disjoint union of pieces that are constructible sets obtained, starting from $\mathbb{G}_m$ by repeated applications of products, disjoint unions, and complements. Roughly speaking, in a constructible torification the building blocks are complements of tori inside other tori.

Under the assumption that X is a smooth projective variety with a cellular decomposition (in particular, for a projective space $X = \mathbb{P}^D$), Lemma 8.1 of [**64**] shows that the blowup of X^{V_Γ} along a diagonal Δ_γ admits a torification. When one iterates the blowup construction, one obtains a (constructible) torification on $\overline{\mathrm{Conf}}_\Gamma(X)$. The top dimensional pieces C_k of this decompositions are, up to sets of measure zero, complements of hypersurfaces in top dimensional tori, to which Lemma 2.5 of [**34**] can still be applied. The resulting periods, which appear as coefficients of the logarithmic divergences in the regularized integrals on the top dimensional pieces of this decomposition, depend on the nature of the motives $\mathfrak{m}(\mathcal{D} \cap C_k)$.

6. Regularization and residue integration

In this section, we describe a regularization of the Feynman integral

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)} \smallsetminus \mathcal{D}_\Gamma} \eta_\Gamma^{(Z)}, \tag{6.1}$$

which is based on a generalization of the toy model case of §5.8, applied to the form with logarithmic poles $\eta_\Gamma^{(Z)} = \beta_\Gamma^{(Z)} \wedge \bar{\beta}_\Gamma^{(Z)}$ described in the previous section. We present the regularization in distributional terms, using the theory of principal values and residue currents. This will show that one can express the resulting residue integrals in the regularization procedure in terms of the iterated Poincaré residues, or equivalently in terms of an integration along a locus in a Leray coboundary of an intersections of divisors $D_\gamma^{(Z)}$.

As we will discuss more in detail in the proof of Proposition 6.1 below, the Leray coboundary of a hypersurface is dual to the Poincaré residue map and assigns to a k-dimensional Σ in the hypersurface a $k+1$-dimensional chain in the hypersurface complement that is a circle bundle over Σ. The Leray coboundary of a transverse intersection of r hypersurfaces is dual to the iterated residue map and assigns to a Σ in the intersection of the hypersurfaces a T^r-torus bundle over Σ. The precise definition of iterated residues and Leray conboundaries and their main properties are also recalled in the proof of Proposition 6.1. For a general treatment of Leray cobondaries and their properties, we refer the reader to [**13**].

6.1. Iterated Poincaré residues. One can associate to a holomorphic differential form β_Γ with logarithmic poles along $\mathcal{D}_\Gamma$ a Poincaré residue on each nonempty intersection of a collection of divisors $D_\gamma^{(Z)}$ that corresponds to a $\mathcal{G}_\Gamma$-nest $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$.

PROPOSITION 6.1. *Let* $N_\Gamma = \# V_\Gamma$. *Let* β_Γ *be a meromorphic* DN_Γ*-form with logarithmic poles along* $\mathcal{D}_\Gamma$. *For every* $\mathcal{G}_\Gamma$*-nest* $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$, *there is a Poincaré residue* $\mathcal{R}_\mathcal{N}(\beta_\Gamma)$, *which defines a cohomology class in* $H^{DN_\Gamma - r}(V_\mathcal{N})$, *on the complete* *intersection* $V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$. *The pairing of* $\mathcal{R}_{\mathcal{N}}(\eta_\Gamma)$ *with an* $(DN_\Gamma - r)$-*cycle* $\Sigma_{\mathcal{N}}$ *in* $V_{\mathcal{N}}^{(Z)}$ *is equal to*

$$\int_{\Sigma_{\mathcal{N}}} \mathcal{R}_{\mathcal{N}}(\beta_\Gamma) = \frac{1}{(2\pi i)^r} \int_{\mathcal{L}_{\mathcal{N}}(\Sigma_{\mathcal{N}})} \beta_\Gamma, \tag{6.2}$$

where $\mathcal{L}_{\mathcal{N}}(\Sigma_{\mathcal{N}})$ *is the* DN_Γ-*cycle in* $Z[\Gamma]$ *given by an iterated Leray coboundary of* $\Sigma_{\mathcal{N}}$, *which is a* T^r-*torus bundle over* $\Sigma_{\mathcal{N}}$. *Under the assumption that the variety* X *is a mixed Tate motive, the integrals* (6.2) *are periods of mixed Tate motives.*

Proof. As shown in Proposition 5.5, the divisors $D_\gamma^{(Z)}$ in $Z[\Gamma]$ have the property that

$$V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)} \neq \emptyset \Leftrightarrow \{\gamma_1, \ldots, \gamma_r\} \quad \text{is a } \mathcal{G}_\Gamma - \text{nest}, \tag{6.3}$$

with transverse intersections.

Consider the first divisor $D_{\gamma_1}^{(Z)}$ in the $\mathcal{G}_\Gamma$-nest $\mathcal{N}$, and a tubular neighborhood $N_{\Gamma,\gamma_1} = N_{Z[\Gamma]}(D_{\gamma_1}^{(Z)})$ of $D_{\gamma_1}^{(Z)}$ in N_{Γ,γ_1}. This is a unit disk bundle over $D_{\gamma_1}^{(Z)}$ with projection $\pi : N_{\Gamma,\gamma_1} \to D_{\gamma_1}^{(Z)}$ and with $\sigma : D_{\gamma_1}^{(Z)} \hookrightarrow N_{\Gamma,\gamma_1}$ the zero section. The Gysin long exact sequence in homology gives

$$\cdots \to H_k(N_{\Gamma,\gamma_1} \smallsetminus D_{\gamma_1}^{(Z)}) \xrightarrow{\iota_*} H_k(N_{\Gamma,\gamma_1})$$

$$\xrightarrow{\sigma^!} H_{k-2}(D_{\gamma_1}^{(Z)}) \xrightarrow{\pi^!} H_{k-1}(N_{\Gamma,\gamma_1} \smallsetminus D_{\gamma_1}^{(Z)}) \to \cdots$$

where $\pi^!$ is the Leray coboundary map, which assigns to a chain Σ in $D_{\gamma_1}^{(Z)}$ the homology class in $N_{\Gamma,\gamma_1} \smallsetminus D_{\gamma_1}^{(Z)}$ of the boundary $\partial\pi^{-1}(\Sigma)$ of the disk bundle $\pi^{-1}(\Sigma)$ over Σ, which is an S^1-bundle over Σ. Its dual is a morphism

$$\mathcal{R}_{\gamma_1} : H^{k+1}(N_{\Gamma,\gamma_1} \smallsetminus D_{\gamma_1}^{(Z)}) \to H^k(D_{\gamma_1}^{(Z)}),$$

which is the residue map. The iterated residue map is obtained by considering the complements $\mathcal{U}_0 = N_{\Gamma,\gamma_1} \smallsetminus D_{\gamma_1}^{(Z)}$ and

$$\mathcal{U}_1 = D_{\gamma_1}^{(Z)} \smallsetminus \bigcup_{1<k\leq r} D_{\gamma_k}^{(Z)},$$

$$\mathcal{U}_2 = (D_{\gamma_1}^{(Z)} \cap D_{\gamma_2}^{(Z)}) \smallsetminus \bigcup_{2<k\leq r} D_{\gamma_k}^{(Z)},$$

and so on. One obtains a sequence of maps

$$H^k(\mathcal{U}_0) \xrightarrow{\mathcal{R}_{\gamma_1}} H^{k-1}(\mathcal{U}_1) \xrightarrow{\mathcal{R}_{\gamma_2}} H^{k-2}(\mathcal{U}_2) \to \cdots \xrightarrow{\mathcal{R}_{\gamma_r}} H^{k-r}(V_{\mathcal{N}}^{(Z)}).$$

The composition $\mathcal{R}_{\mathcal{N}} = \mathcal{R}_{\gamma_r} \circ \cdots \circ \mathcal{R}_{\gamma_1}$ is the iterated residue map. Because the residue map is dual to Leray coboundary, under the pairing of homology and cohomology one obtains

$$\langle \mathcal{R}_{\mathcal{N}}(\beta), \Sigma \rangle = \langle \beta, \mathcal{L}_{\mathcal{N}}(\Sigma) \rangle,$$

where $\mathcal{L}_{\mathcal{N}} = \mathcal{L}_{\gamma_1} \circ \cdots \circ \mathcal{L}_{\gamma_r}$ is the compositions of the Leray coboundary maps of the divisors $D_{\gamma_k}^{(Z)}$. The resulting $\mathcal{L}_{\mathcal{N}}(\Sigma)$ is therefore, by construction, a T^r-torus bundle over Σ. At the level of differential forms, the residue map $\mathcal{R}_{\gamma_1}$ is given by integration along the circle fibers of the S^1-bundle $\partial\pi^{-1}(\Sigma) \to \Sigma$. Thus, the pairing $\langle \mathcal{R}_{\mathcal{N}}(\beta), \Sigma \rangle$ contains a $2\pi i$ factor, coming from the integration of a form df/f, with f the local defining equation of the hypersurface, along the circle fibers. This means that, when writing the pairings in terms of differential forms, one obtains (6.2). As shown in [**36**], if $\mathfrak{m}(X)$ is mixed Tate, the divisors D_γ and their intersections in $\overline{\mathrm{Conf}}_\Gamma(X)$ are mixed Tate motives, and so are the $D_\gamma^{(Z)}$ and their intersections $V_{\mathcal{N}}^{(Z)}$ in $Z[\Gamma]$. $\square$

Given a $\mathcal{G}_\Gamma$-nest $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$ let $\mathcal{V}_{\mathcal{N}} = D_{\gamma_1} \cap \cdots \cap D_{\gamma_r}$ be the intersection of the corresponding divisors in $\overline{\mathrm{Conf}}_\Gamma(X)$. The residues $\mathcal{R}_{\mathcal{N}}(\beta_\Gamma)$ of Proposition 6.1 pair with a $DN_\Gamma - r$-dimensional cycles $\Sigma_{\mathcal{N}} \subset \mathcal{V}_{\mathcal{N}} \times \{y\} \subset \mathcal{V}_{\mathcal{N}}^{(Z)}$, with $N_\Gamma = \# V_\Gamma$,

$$\langle \mathcal{R}_{\mathcal{N}}(\beta_\Gamma), \Sigma_{\mathcal{N}} \rangle = \int_{\Sigma_{\mathcal{N}}} \mathcal{R}_{\mathcal{N}}(\beta_\Gamma) = \frac{1}{(2\pi i)^r} \int_{\mathcal{L}^r(\Sigma_{\mathcal{N}})} \beta_\Gamma. \tag{6.4}$$

We can then reinterpret and extend the toy model case of §5.8 in the following way. We write $\eta_\Gamma = \beta_\Gamma \wedge \bar{\beta}_\Gamma$ (ignoring a possible sign), with β_Γ holomorphic with logarithmic poles along $\mathcal{D}_\Gamma$. We will also ignore some rational constant factors in the computation, as they will not affect the nature of the resulting periods.

COROLLARY 6.2. *Suppose given a $\mathcal{G}_\Gamma$-nest $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$ as above, with $\mathcal{V}_{\mathcal{N}} = D_{\gamma_1} \cap \cdots \cap D_{\gamma_r}$. There is a regularization of the integral* (6.1)*, such that, its restriction to a neighborhood of $\mathcal{V}_{\mathcal{N}}$ is an iterated Poincaré residue.*

PROOF. When integrating the $2DN_\Gamma$-form η_Γ in a neighborhood $\mathcal{U}$ of $\mathcal{V}_{\mathcal{N}}$, we have (up to a sign)

$$\int_{\mathcal{U}} \eta_\Gamma = \int_{\mathcal{U}} \eta \wedge \beta_{\mathcal{N}} \wedge \bar{\beta}_{\mathcal{N}},$$

where we can locally write the form in coordinates

$$\beta_{\mathcal{N}} = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_r}{f_r},$$

with f_j the function that locally defines the divisor D_{γ_j}, and with η a $(2DN_\Gamma - r)$-form which is regular along $\mathcal{V}_{\mathcal{N}}$. The integral of $\beta_{\mathcal{N}} \wedge \bar{\beta}_{\mathcal{N}}$ can be modeled as in the toy model case of §5.8. Thus, up to a rational factor, the integral of $\beta_{\mathcal{N}} \wedge \bar{\beta}_{\mathcal{N}}$ can be regularized as in §5.8, so that it results in a periods integral

$$\int_{T^r} \beta_{\mathcal{N}},$$

where the torus T^r is the fiber of the Leray coboundary $\mathcal{L}^r(\mathcal{V}_{\mathcal{N}})$, identified with a T^r-bundle over $\mathcal{V}_{\mathcal{N}}$, while the remaining term η is integrated along $\mathcal{V}_{\mathcal{N}}$-directions. Indeed, the chain of integration $\tilde{\sigma}_\Gamma^{(Z,y)} = \overline{\mathrm{Conf}}_\Gamma(X) \times \{y\}$ intersects the loci $\mathcal{V}_{\mathcal{N}}^{(Z)} = \mathcal{V}_{\mathcal{N}} \times X^{V_\Gamma}$ along $\mathcal{V}_{\mathcal{N}} \times \{y\}$, where the $\mathcal{V}_{\mathcal{N}}$ are the intersections $\mathcal{V}_{\mathcal{N}} = D_{\gamma_1} \cap \cdots \cap D_{\gamma_r}$ of the divisors D_{γ_k} in $\overline{\mathrm{Conf}}_\Gamma(X)$. Thus, $\Sigma = \mathcal{V}_{\mathcal{N}} \times \{y\}$ defines a $2DN_\Gamma - r$-dimensional cycle in $\mathcal{V}_{\mathcal{N}}^{(Z)}$, with $N_\Gamma = \# V_\Gamma$, which can be paired with the form η of degree $2DN_\Gamma - r$ on $\mathcal{V}_{\mathcal{N}}^{(Z)}$. Thus, we write the integral equivalently (up to a rational factor) as an residue integral

$$\int_{\mathcal{L}^r(\mathcal{V}_{\mathcal{N}})} \eta \wedge \beta_{\mathcal{N}}.$$

$\square$

Remark 6.3. The reason for passing to the wonderful compactification $Z[\Gamma]$ and pulling back the form $\omega_\Gamma^{(Z)}$ along the projection $\pi_\Gamma : Z[\Gamma] \to Z^{V_\Gamma}$ is in order to pass to a setting where the locus of divergence is described by divisors D_γ intersecting transversely in $\overline{\mathrm{Conf}}_\Gamma(X)$, while the diagonals Δ_γ in X^{V_Γ} have higher codimensions and do not in general intersect transversely. These transversality issues are discussed in more detail in [**36**]. There is a generalization of the theory of forms with logarithmic poles and Poincaré residues [**74**], that extends the case of [**42**] of normal crossings divisors to more general classes of divisors, but in this more general setting the Poincaré residue gives meromorphic instead of holomorphic forms.

6.2. Current-regularization of complexified amplitudes. We review here briefly some well known facts about residue and principal value currents and we apply them to reinterpret the regularization procedure described above for the integral (6.1) in terms of currents and principal values.

6.2.1. *Residue currents and Mellin transforms.* Recall that, for a single smooth hypersurface defined by an equation $\{f = 0\}$, the residue current $[Z_f]$, supported on the hypersurface, is defined as

$$[Z_f] = \frac{1}{2\pi i}\bar\partial[\frac{1}{f}] \wedge df := \frac{1}{2\pi i}\bar\partial\partial \log |f|^2.$$

This is known as the Poincaré–Lelong formula. It can also be seen as a limit

$$\int_{Z_f} \varphi = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{|f|=\epsilon} \frac{df}{f} \wedge \varphi.$$

A generalization is given by the Coleff–Herrera residue current [**39**], associated to a collection of functions $\{f_1, \dots, f_r\}$. Under the assumption that these define a complete intersection $V = \{f_1 = \cdots = f_r = 0\}$, the residue current

$$\mathcal{R}_f = \bar\partial[\frac{1}{f_1}] \wedge \cdots \wedge \bar\partial[\frac{1}{f_r}] \tag{6.5}$$

is obtained as a limit

$$\mathcal{R}_f(\varphi) = \lim_{\delta\to 0} \int_{T_{\epsilon(\delta)}(f)} \frac{\varphi}{f_1 \cdots f_r},$$

with $T_{\epsilon(\delta)}(f) = \{|f_k| = \epsilon_k(\delta)\}$, with the limit taken over "admissible paths" $\epsilon(\delta)$, which satisfy the properties

$$\lim_{\delta\to 0} \epsilon_r(\delta) = 0, \qquad \lim_{\delta\to 0} \frac{\epsilon_k(\delta)}{(\epsilon_{k+1}(\delta))^\ell} = 0,$$

for $k = 1, \dots, r$ and any positive integer ℓ. The test form φ is a $(2n-r)$-form of type $(n, n-r)$, where $2n$ is the real dimension of the ambient space, and the residual current obtained in this way is a $(0, r)$-current. For more details, see §3 of [**18**] and [**84**]. Notice that, while in general one cannot take products of distributions, the Coleff–Herrera product (6.5) is well defined for residue currents, as well as between residue and principal value currents.

Moreover, the Mellin transform

$$\Gamma_f^\varphi(\lambda) = \int_{\mathbb{R}_+^r} \mathcal{I}_f^\varphi(\epsilon)\, \epsilon^{\lambda - I}\, d\epsilon, \tag{6.6}$$

with

$$\mathcal{I}_f^{\varphi}(\epsilon) = \int_{T_\epsilon(f)} \frac{\varphi}{f_1 \cdots f_r}$$

and with

$$\epsilon^{\lambda - I}\, d\epsilon = \epsilon_1^{\lambda_1 - 1} \cdots \epsilon_r^{\lambda_r - 1}\, d\epsilon_1 \wedge \cdots \wedge d\epsilon_r,$$

can also be written, as in [**18**], [**84**], as

$$\Gamma_f^{\varphi}(\lambda) = \frac{1}{(2\pi i)^r} \int_{\mathcal{X}} |f|^{2(\lambda - I)} \overline{df} \wedge \varphi, \tag{6.7}$$

where the integration is on the ambient variety $\mathcal{X}$ and where

$$|f|^{2(\lambda-I)} = |f_1|^{2(\lambda_1 - 1)} \cdots |f_r|^{2(\lambda_r - 1)}, \quad \text{and} \quad \overline{df} = \overline{df_1} \wedge \cdots \wedge \overline{df_r}.$$

When $\{f_1, \ldots, f_r\}$ define a complete intersection, the function $\lambda_1 \cdots \lambda_r \Gamma_f^{\varphi}(\lambda)$ is holomorphic in a neighborhood of $\lambda = 0$ and the value at zero is given by the residue current ([**18**], [**84**])

$$\mathcal{R}_f(\varphi) = \lambda_1 \cdots \lambda_r \Gamma_f^{\varphi}(\lambda)|_{\lambda=0}. \tag{6.8}$$

Equivalently, (6.7) and (6.8) can also be written as

$$\lim_{\lambda \to 0} \lambda \Gamma_f^{\varphi}(\lambda) = \lim_{\lambda \to 0} \frac{1}{(2\pi i)^r} \int_{\mathcal{X}} \frac{\bar\partial |f_r|^{2\lambda_r} \wedge \cdots \wedge \bar\partial |f_1|^{2\lambda_1}}{f_r \cdots f_1} \wedge \varphi, \tag{6.9}$$

where the factor λ on the left-hand-side stands for $\lambda_1 \cdots \lambda_r$ as in (6.8).

The Poincaré–Lelong formula, in this more general case of a complete intersection defined by a collection $\{f_1, \ldots, f_r\}$, expresses the integration current Z_f as

$$[Z_f] = \frac{1}{(2\pi i)^r} \bar\partial [\frac{1}{f_r}] \wedge \cdots \wedge \bar\partial [\frac{1}{f_1}] \wedge df_1 \wedge \cdots \wedge df_r. \tag{6.10}$$

The correspondence between residue currents and the Poincaré residues on complete intersections, discussed above in §6.1, is given for instance in Theorem 4.1 of [**4**].

6.2.2. *Principal value current.* The principal value current $[1/f]$ of a single holomorphic function f can be computed as [**56**], [**76**]

$$\langle [\frac{1}{f}], \phi \rangle = \lim_{\epsilon \to 0} \int_{|f| > \epsilon} \frac{\phi\, d\zeta \wedge d\bar\zeta}{f}, \tag{6.11}$$

where ϕ is a test function. More generally, for $\{f_1, \ldots, f_r\}$ as above, the principal value current is given by

$$\langle [\frac{1}{f}], \phi \rangle = \lim_{\epsilon \to 0} \int_{N_\epsilon(f)} \frac{\phi}{f_r \cdots f_1}, \tag{6.12}$$

with ϕ a test form and with

$$N_\epsilon(f) = \{ |f_k| > \epsilon_k \}. \tag{6.13}$$

More generally we will use the following notation.

Definition 6.4. Given a (p,q)-form η on an m-dimensional smooth projective variety $\mathcal{X}$, with poles along an effective divisor $\mathcal{D} = D_1 \cup \cdots D_r$, where the components D_k are smooth hypersurfaces defined by equations $f_k = 0$, the principal value current $PV(\eta)$ is defined by

$$\langle PV(\eta), \phi\rangle = \lim_{\epsilon\to 0} \int_{N_\epsilon(f)} \eta \wedge \phi, \tag{6.14}$$

for an $(m-p, m-q)$ test form ϕ, with $N_\epsilon(f)$ defined as in (6.13).

The following simple Lemma describes how the principal value current and the residue currents are related.

Lemma 6.5. *When the test form ϕ is modified to $\phi + \bar\partial\psi$, the principal value current satisfies*

$$\langle [\frac{1}{f}], \phi + \bar\partial\psi\rangle = \langle [\frac{1}{f}], \phi\rangle - \langle \bar\partial[\frac{1}{f}], \psi\rangle, \tag{6.15}$$

where $\bar\partial[1/f]$ is the residue current $\mathcal{R}_f$ of (6.5).

Proof. By Stokes theorem, we have

$$\langle [\frac{1}{f}], \bar\partial\psi\rangle = \lim_{\epsilon\to 0}\int_{|f|>\epsilon} \frac{\bar\partial\psi}{f} = \lim_{\epsilon\to 0} - \int_{|f|=\epsilon} \frac{\psi}{f} = -\langle \bar\partial[\frac{1}{f}], \psi\rangle.$$

□

We now return to the case of the complexified amplitudes (3.9) and describe the corresponding regularization and ambiguities.

6.2.3. *Principal value and complexified amplitude.* We can regularize the complexified amplitude given by the integral (6.1), interpreted in the distributional sense, as in §3.3.4, using the principal value current.

Definition 6.6. The principal value regularization of the amplitude (6.1) is given by the current $PV(\eta_\Gamma^{(Z)})$ defined as in (6.14),

$$\langle PV(\eta_\Gamma^{(Z)}), \varphi\rangle = \lim_{\epsilon\to 0}\int_{N_\epsilon(f)} \varphi\ \eta,$$

for a test function φ.

We can also write the regularized integral in the following form.

Lemma 6.7. *The regularized integral satisfies*

$$\langle PV(\eta_\Gamma^{(Z)}), \varphi\rangle = \lim_{\lambda\to 0}\int_{\tilde\sigma_\Gamma^{(Z,y)}} |f_n|^{2\lambda_n}\cdots |f_1|^{2\lambda_1} \eta_\Gamma^{(Z,y)}\ \varphi$$

where $n = n_\Gamma$ is the cardinality $n_\Gamma = \#\mathcal{G}_\Gamma$ of the building set $\mathcal{G}_\Gamma$ and the f_k are the defining equations of the $D_\gamma^{(Z)}$ in $\mathcal{G}_\Gamma$

Proof. The form $\eta_\Gamma^{(Z)}$ has poles along the divisor $\mathcal{D}_\Gamma = \cup_{\Delta_\gamma^{(Z)}\in\mathcal{G}_\Gamma} D_\gamma^{(Z)}$. Thus, if we denote by f_k, with $k = 1,\dots,n$ with $n = \#\mathcal{G}_\Gamma$ the defining equations of the $D_\gamma^{(Z)}$, we can write the principal value current in the form

$$\lim_{\lambda\to 0}\int_{\tilde\sigma_\Gamma^{(Z,y)}} \frac{|f_n|^{2\lambda_n}\cdots |f_1|^{2\lambda_1}}{f_n\cdots f_1} h\,\varphi,$$

with h an algebraic from without poles and φ is a test function. □

6.2.4. *Pseudomeromorphic currents.* If $\{f_1,\ldots,f_n\}$ define a complete intersection $V=\{f_1=\cdots=f_r=0\}$ in a smooth projective variety $\mathcal{X}$, an *elementary pseudomeromorphic current* is a current of the form

$$C_{r,n} := [\frac{1}{f_n}]\wedge\cdots\wedge[\frac{1}{f_{r+1}}]\wedge\bar\partial[\frac{1}{f_r}]\wedge\cdots\wedge\bar\partial[\frac{1}{f_1}], \tag{6.16}$$

for some $1\leq r\leq n$, where the products of principal value and residue currents are well defined Coleff–Herrera products and the resulting current is commuting in the principal value factors and anticommuting in the residue factors. These distributions also have a Mellin transform formulation (see [**18**]) as

$$\langle C_{r,n},\phi\rangle = \lim_{\lambda\to 0}\frac{1}{(2\pi i)^r}\int_{\mathcal{X}}\prod_{k=r+1}^{n}\frac{|f_k|^{2\lambda_k}}{f_k}\bigwedge_{j=1}^{r}\bar\partial\left(\frac{|f_j|^{2\lambda_j}}{f_j}\right)\wedge\phi. \tag{6.17}$$

6.3. Ambiguities of regularized integrals. We can use the formalism of residue currents recalled above to describe the ambiguities in the principal value regularization of the complexified amplitudes of Definition 6.6.

6.3.1. *Complexified amplitude and residue currents.* As in §6.1, consider a $\mathcal{G}_\Gamma$-nest $\{\gamma_1,\ldots,\gamma_r\}$ and the associated intersection $V^{(Z)}_{\mathcal{N}}=D^{(Z)}_{\gamma_1}\cap\cdots\cap D^{(Z)}_{\gamma_r}$. Also let $n_\Gamma=\#\mathcal{G}_\Gamma$ and let f_k, for $k=1,\ldots,n_\Gamma$ be the defining equations for the $D^{(Z)}_\gamma$, for γ ranging over the subgraphs defining the building set $\mathcal{G}_\Gamma$. For $\epsilon=(\epsilon_k)$, we define

$$\tilde\sigma^{(Z,y)}_{\Gamma,\epsilon} := \tilde\sigma^{(Z,y)}_{\Gamma}\cap N_\epsilon(f), \tag{6.18}$$

with $N_\epsilon(f)$ defined as in (6.13). The principal value regularization of Definition 6.6 can then be written as

$$\langle PV(\eta^{(Z,y)}_\Gamma),\varphi\rangle = \lim_{\epsilon\to 0}\int_{\tilde\sigma^{(Z,y)}_{\Gamma,\epsilon}}\varphi\,\eta^{(Z,y)}_\Gamma,$$

where the limit is taken over admissible paths.

Similarly, given a $\mathcal{G}_\Gamma$-nest $\mathcal{N}=\{\gamma_1,\ldots,\gamma_r\}$, we introduce the notation

$$\tilde\sigma^{(Z,y)}_{\Gamma,\mathcal{N},\epsilon} := \tilde\sigma^{(Z,y)}_{\Gamma}\cap T_{\mathcal{N},\epsilon}(f)\cap N_{\mathcal{N},\epsilon}(f), \tag{6.19}$$

where $T_{\mathcal{N},\epsilon}(f)=\{|f_k|=\epsilon_k,\ k=1,\ldots,r\}$ and $N_{\mathcal{N},\epsilon}(f)=\{|f_k|>\epsilon,\ k=r+1,\ldots,n\}$, where we have ordered the n subgraphs γ in $\mathcal{G}_\Gamma$ so that the first r belong to the nest $\mathcal{N}$.

PROPOSITION 6.8. *For a $\mathcal{G}_\Gamma$-nest $\mathcal{N}=\{\gamma_1,\ldots,\gamma_r\}$, as above, the limit*

$$\lim_{\epsilon\to 0}\int_{\tilde\sigma^{(Z,y)}_{\Gamma,\mathcal{N},\epsilon}}\varphi\,\eta^{(Z,y)}_\Gamma \tag{6.20}$$

determines a pseudomeromorphic current, whose residue part is an iterated residue supported on $V^{(Z)}_{\mathcal{N}}=D^{(Z)}_{\gamma_1}\cap\cdots\cap D^{(Z)}_{\gamma_r}$, obtained as in Corollary 6.2.

PROOF. Let f_k, with $k=1,\ldots,n$ with $n=\#\mathcal{G}_\Gamma$, be the defining equations of the $D^{(Z)}_\gamma$. We assume the subgraphs in $\mathcal{G}_\Gamma$ are ordered so that the first r belong to the given $\mathcal{G}_\Gamma$-nest $\mathcal{N}$. We can then write the current (6.20) in the form $\langle C_{r,n},h\varphi\rangle$, where $C_{r,n}$ is the elementary pseudomeromorphic current of (6.16) and h is algebraic without poles. Using a local coordinates model, as in Corollary 6.2 one can identify the resulting residue current with the regularization computed there. □

6.3.2. *Residue currents as ambiguities.* With the same setting as in Proposition 6.8, we then have the following characterization of the ambiguities of the principal value regularization.

PROPOSITION 6.9. *The ambiguities in the current-regularization $PV(\eta_\Gamma^{(Z,y)})$ are given by iterated residues supported on the intersections $V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$, of divisors corresponding to $\mathcal{G}_\Gamma$-nests $\mathcal{N} = \{\gamma_1, \ldots, \gamma_r\}$.*

PROOF. As above, we have

$$\langle PV(\eta_\Gamma^{(Z,y)}), \varphi \rangle = \lim_{\epsilon \to 0} \int_{\tilde{\sigma}_{\Gamma,\epsilon}^{(Z,y)}} \varphi\, \eta_\Gamma^{(Z,y)} = \lim_{\lambda \to 0} \int_{\tilde{\sigma}_\Gamma^{(Z,y)}} \frac{|f_n|^{2\lambda_n} \cdots |f_1|^{2\lambda_1}}{f_n \cdots f_1}\, h\, \varphi$$

If we replace the form $h\,\varphi$ with a form $h\varphi + \bar{\partial}_{\mathcal{N}}\psi$, where $\mathcal{N}$ is a $\mathcal{G}_\Gamma$-nest and the notation $\bar{\partial}_{\mathcal{N}}\psi$ means a form

$$\bar{\partial}_{\mathcal{N}}\psi := \psi_n \cdots \psi_{r+1}\, \bar{\partial}\psi_r \wedge \cdots \wedge \bar{\partial}\psi_1,$$

for test functions ψ_k, $k = 1, \ldots, n$, we obtain a pseudomeromorphic current

$$\langle PV(\eta_\Gamma^{(Z,y)}), \bar{\partial}_{\mathcal{N}}\psi \rangle = \langle [\frac{1}{f_n}] \wedge \cdots \wedge [\frac{1}{f_{r+1}}] \wedge \bar{\partial}[\frac{1}{f_r}] \wedge \cdots \wedge \bar{\partial}[\frac{1}{f_1}], \psi \rangle,$$

with $\psi = \psi_n \cdots \psi_1$. Notice then that the residue part

$$\langle \bar{\partial}[\frac{1}{f_r}] \wedge \cdots \wedge \bar{\partial}[\frac{1}{f_1}], \psi \rangle = \mathcal{R}_{\mathcal{N}}(\psi) \tag{6.21}$$

is an iterated residue supported on $V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$. □

By the results of §6.1, and the relation (6.21) between residue currents and iterated Poincaré residues (see [**4**]), when evaluated on algebraic test forms on the varieties $V_{\mathcal{N}}^{(Z)}$, these ambiguities can be expressed in terms of periods of mixed Tate motives. In other words, the mixed Tate nature of the period is not changed by the presence of these ambiguities, as these are themselves expressible in terms of mixed Tate periods of intersections of components of the boundary divisor.

7. Other regularization methods

We now discuss a regularization method for the evaluation of the integrals of the complexified amplitudes,

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)}} \pi_\Gamma^*(\omega_\Gamma^{(Z)}),$$

with the pullback $\pi_\Gamma^*(\omega_\Gamma^{(Z)})$ to $Z[\Gamma]$ as in Corollary 5.13. The chain of integration $\tilde{\sigma}_\Gamma^{(Z,y)}$ is as in (5.17), obtained from the form $\omega_\Gamma^{(Z)}$ and the chain $\sigma_\Gamma^{(Z)}$ of Definition 3.6. The geometric method of regularization we describe in this section is based on the *deformation to the normal cone.*

A general method of regularization consists of deforming the chain of integration so that it no longer intersects the locus of divergences. We first describe briefly why this cannot be done directly within the space $Z[\Gamma]$ considered above, and then we introduce a simultaneous deformation of the form and of the space where integration happens, so that the integral can be regularized according to the general method mentioned above.

7.1. Deformations and linking. To illustrate where the problem arises, if one tries to deform the chain of integration away from the locus of divergence in $Z[\Gamma]$, consider the local problem near a point $z \in D_\gamma^{(Z)}$ in the intersection of $\tilde{\sigma}_\Gamma^{(Z,y)}$ with one of the divisors in the divergence locus of the form $\pi_\Gamma^*(\omega_\Gamma)$. Near this point, the locus of divergence is a product $D_\gamma \times X^{V_\Gamma}$. We look at the intersection of the integration chain $\tilde{\sigma}_\Gamma^{(Z,y)}$ with a small tubular neighborhood T_ϵ of $D_\gamma \times X^{V_\Gamma}$. We have

$$\tilde{\sigma}_\Gamma^{(Z,y)} \cap \partial T_\epsilon = \partial \pi_\epsilon^{-1}(D_\gamma) \times \{y\},$$

with $\pi_\epsilon : T_\epsilon(D_\gamma) \to D_\gamma$ the projection of the 2-disc bundle and $\partial\pi_\epsilon^{-1}(D_\gamma)$ a circle bundle, locally isomorphic to $D_\gamma \times S^1$. Thus, locally, $\tilde{\sigma}_\Gamma^{(Z,y)} \cap T_\epsilon$ looks like a ball $B^{2D|\mathbf{V}_\Gamma|} \times \{0\}$ inside a ball B^{4DN_Γ}, with $N_\Gamma = \# V_\Gamma$. Locally, we can think of the problem of deforming the chain of integration in a neighborhood of the divergence locus as the question of deforming a ball $B^{2DN_\Gamma} \times \{0\}$ leaving fixed the boundary $S^{2DN_\Gamma - 1} \times \{0\}$ inside a ball B^{4DN_Γ} so as to avoid the locus $\{0\} \times B^{2DN_\gamma}$ that lies in the divergence locus, with $N_\gamma = \# V_\gamma$. However, we find the following simple topological fact.

Lemma 7.1. *The spheres $S^{2DN_\Gamma - 1} \times \{0\}$ and $\{0\} \times S^{2DN_\gamma - 1}$ are linked inside the sphere $S^{4DN_\Gamma - 1}$.*

Proof. This can be seen, for instance, by computing their Gauss linking integral (see [**78**])

$$\mathrm{Lk}(M,N) = \frac{1}{\mathrm{Vol}(S)} \int_{M \times N} \frac{\Omega_{k,\ell}(\alpha)}{\sin^n(\alpha)} [x, dx, y, dy] \tag{7.1}$$

with $M = S^k \times \{0\}$, $N = \{0\} \times S^\ell$, $S = S^n$, and with $k = \ell = 2D|\mathbf{V}_\Gamma| - 1$ and $n = 4D|\mathbf{V}_\Gamma| - 1$, where

$$\Omega_{k,\ell}(\alpha) := \int_{\theta=\alpha}^{\pi} \sin^k(\theta - \alpha) \sin^\ell(\theta) d\theta,$$

$$\alpha(x,y) := \mathrm{dist}_{S^n}(x,y), \quad x \in M,\, y \in N,$$

$$[x, dx, y, dy] := \det(x, \frac{\partial x}{\partial s_1}, \dots, \frac{\partial x}{\partial s_k}, y, \frac{\partial y}{\partial t_1}, \dots, \frac{\partial y}{\partial t_\ell})\, ds\, dt,$$

with x, y the embeddings of S^k and S^ℓ in S^n and s, t the local coordinates on S^k and S^ℓ. Then one can see (§4 of [**78**]) that in S^n with $n = k + \ell + 1$ the linking number is $\mathrm{Lk}(S^k \times \{0\}, \{0\} \times S^\ell) = 1$. □

This type of problem can be easily avoided by introducing a simultaneous deformation of the form $\pi_\Gamma^*(\omega_\Gamma)$ and of the space $Z[\Gamma]$ as we show in the following.

7.2. Form extension. We first embed the configuration space Z^{V_Γ} as the fiber over zero in a one parameter family $Z^{V_\Gamma} \times \mathbb{P}^1$. We extend the form $\omega_\Gamma^{(Z)}$ to $Z^{V_\Gamma} \times \mathbb{P}^1$ using the additional coordinate $\zeta \in \mathbb{P}^1$ to alter the differential form in a suitable way.

Definition 7.2. The extension of the complexified amplitude $\omega_\Gamma^{(Z)}$ on the space $Z^{V_\Gamma} \times \mathbb{P}^1$ is given by

$$\tilde{\omega}_\Gamma^{(Z)} = \prod_{e \in E_\Gamma} \frac{1}{(\|x_{s(e)} - x_{t(e)}\|^2 + |\zeta|^2)^{D-1}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v \wedge d\zeta \wedge d\bar{\zeta}, \tag{7.2}$$

where ζ is the local coordinate on $\mathbb{P}^1$.

Lemma 7.3. *The divergent locus* $\{\tilde{\omega}_\Gamma^{(Z)} = \infty\}$ *of the form* (7.2) *on* $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$ *is given by the locus* $\cup_{e\in\mathbf{E}_\Gamma}\Delta_e^{(Z)} \subset Z^{\mathbf{V}_\Gamma} \times \{0\}$.

Proof. The locus of divergence is the intersection of $\{\zeta = 0\}$ and the union of the products $\Delta_e^{(Z)} \times \mathbb{P}^1 = \{x_{s(e)} - x_{t(e)} = 0\}$. $\square$

Notice that we have introduced in the form (7.2) an additional variable of integration, $d\zeta \wedge d\bar{\zeta}$. The reason for shifting the degree of the form will become clear later in this section (see §7.6 below), where we see that, when using the deformation to the normal cone, the chain of integration $\sigma_\Gamma^{(Z,y)}$ is also extended by an additional complex dimension to $\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1$, of which one then takes a proper transforms and deforms it inside the deformation to the normal cone. In terms of the distributional interpretation of the complexified amplitudes of §3.3.4, the relation between the form (7.2) and the original amplitude (3.9) can be written as

$$\omega_\Gamma^{(Z)} = \int \prod_{e\in\mathbf{E}_\Gamma} \frac{\delta(\zeta=0)}{(\|x_{s(e)} - x_{t(e)}\|^2 + |\zeta|^2)^{D-1}} \bigwedge_{v\in\mathbf{V}_\Gamma} dx_v \wedge d\bar{x}_v \wedge d\zeta \wedge d\bar{\zeta}, \tag{7.3}$$

where the distributional delta constraint can be realized as a limit of normalized integrations on small tubular neighborhoods of the central fiber $\zeta = 0$ in the trivial fibration $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$.

7.3. Deformation to the normal cone. The deformation to the normal cone is the natural algebro-geometric replacement for tubular neighborhoods in smooth geometry, see [**46**]. We use it here to extend the configuration space Z^{V_Γ} to a trivial fibration $Z^{V_\Gamma} \times \mathbb{P}^1$ and then replacing the fiber over $\{0\} \in \mathbb{P}^1$ with the wonderful compactification $Z[\Gamma]$. This will allow us to simultaneously regularize the form and the chain of integration. For simplicity we illustrate the construction for the case where the graph Γ is itself biconnected.

Proposition 7.4. *Let* Γ *be a biconnected graph. Starting with the product* $Z^{V_\Gamma} \times \mathbb{P}^1$*, a sequence of blowups along loci parameterized by the* $\Delta_\gamma^{(Z)} \times \{0\}$*, with* γ *induced biconnected subgraphs yields a variety* $\mathcal{D}(Z[\Gamma])$ *fibered over* $\mathbb{P}^1$ *such that the fiber over all points* $\zeta \in \mathbb{P}^1$ *with* $\zeta \neq 0$ *is still equal to* Z^{V_Γ}*, while the fiber over* $\zeta = 0$ *has a component equal to the wonderful compactification* $Z[\Gamma]$ *and other components given by projectivizations* $\mathbb{P}(C \oplus 1)$ *with* C *the normal cone of the blowup locus.*

Proof. We start with the product $Z^{V_\Gamma} \times \mathbb{P}^1$. We then perform the first blowup of the iterated sequence of §5.2.1 on the fiber over $\zeta = 0$ namely we blowup the locus $\Delta_\Gamma^{(Z)} \times \{0\}$, with $\Delta_\Gamma^{(Z)}$ the deepest diagonal, inside $Z^{\mathbf{V}_\Gamma} \times \mathbb{P}^1$. (Note that this is where we are using the biconnected hypothesis on Γ, otherwise the first blowup may be along induced biconnected subgraphs with a smaller number of vertices.) The blowup $\mathrm{Bl}_{\Delta_\Gamma^{(Z)}\times\{0\}}(Z^{V_\Gamma} \times \mathbb{P}^1)$ is equal to $Z^{V_\Gamma} \times (\mathbb{P}^1 \smallsetminus \{0\})$ away from $\zeta = 0$, while over the point $\zeta = 0$ it has a fiber with two components. One of the components is isomorphic to the blowup of Z^{V_Γ} along $\Delta_\Gamma^{(Z)}$, that is, $\mathrm{Bl}_{\Delta_\Gamma^{(Z)}}(Z^{V_\Gamma}) = Y_1$, with the notation of §5.2.1. The other component is equal to $\mathbb{P}(C_{Z^{V_\Gamma}}(\Delta_\Gamma^{(Z)}) \oplus 1)$ where $C_{Z^{V_\Gamma}}(\Delta_\Gamma^{(Z)})$ is the normal cone of $\Delta_\Gamma^{(Z)}$ in Z^{V_Γ}. Since $\Delta_\Gamma^{(Z)} \simeq X \times X^{V_\Gamma}$ is smooth,

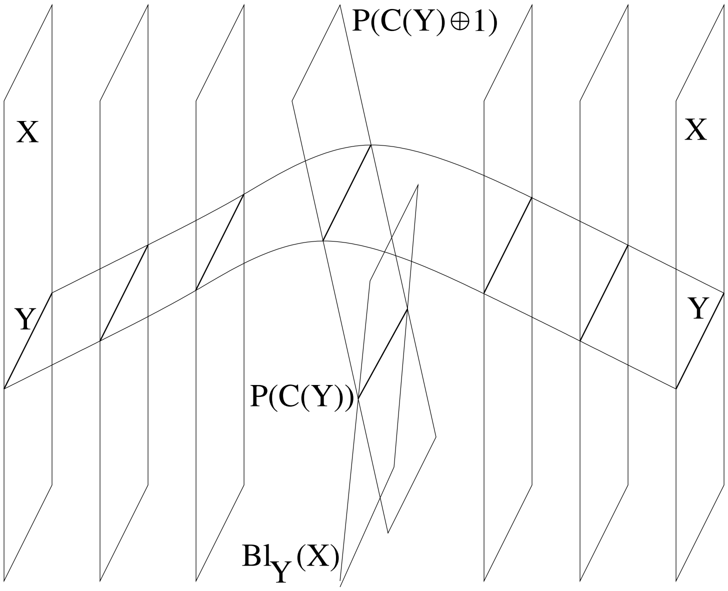


FIGURE 1. Deformation to the normal cone.

the normal cone is the normal bundle of $\Delta_\Gamma^{(Z)}$ in Z^{V_Γ}. The two Cartier divisors Y_1 and $\mathbb{P}(C_{Z^{V_\Gamma}}(\Delta_\Gamma^{(Z)}) \oplus 1)$ meet along $\mathbb{P}(C_{Z^{V_\Gamma}}(\Delta_\Gamma^{(Z)}))$. We can then proceed to blow up the further loci $\Delta_\gamma^{(Z)}$ with $\gamma \in \mathcal{G}_{n-1,\Gamma}$ inside the special fiber $\tilde{\pi}^{-1}(0)$ in $\mathrm{Bl}_{\Delta_\Gamma^{(Z)} \times \{0\}}(Z^{V_\Gamma} \times \mathbb{P}^1)$, where

$$\tilde{\pi} : \mathrm{Bl}_{\Delta_\Gamma^{(Z)} \times \{0\}}(Z^{V_\Gamma} \times \mathbb{P}^1) \to Z^{V_\Gamma} \times \mathbb{P}^1$$

is the projection. These loci lie in the intersection of the two components of the special fiber $\tilde{\pi}^{-1}(0)$. Thus, at the next stage we obtain a variety that again agrees with $Z^{V_\Gamma} \times (\mathbb{P}^1 \smallsetminus \{0\})$ away from the central fiber, while over $\zeta = 0$ it now has a component equal to Y_2 and further components coming from the normal cone after this additional blowup. After iterating this process as in §5.2.1 we obtain a variety that has fiber Z^{V_Γ} over all points $\zeta \neq 0$ and over $\zeta = 0$ it has a component equal to the wonderful compactification $Z[\Gamma]$ and other components coming from normal cones. ☐

Notice that one can also realize the iterated blowup of §5.2.1 as a single blowup over a more complicated locus and perform the deformation to the normal cone for that single blowup. We proceed as in Proposition 7.4, as it will be easier in this way to follow the effect that this deformation has on the motive.

The main reason for introducing the deformation to the normal cone, as we discuss more in detail in §7.6 below, is the fact that it will provide us with a natural mechanism for deforming the chain of integration away from the locus of divergences. The key idea is depicted in Figure 1, where one considers a variety $\mathcal{X}$ and the deformation $\mathrm{Bl}_{\mathcal{Y}\times\{0\}}(\mathcal{X} \times \mathbb{P}^1)$. If $\pi : \mathrm{Bl}_{\mathcal{Y}\times\{0\}}(\mathcal{X} \times \mathbb{P}^1) \to \mathbb{P}^1$ denotes the projection, the special fiber $\pi^{-1}(0)$ has two components, one given by the blowup

$\mathrm{Bl}_{\mathcal{Y}}(\mathcal{X})$ of $\mathcal{X}$ along $\mathcal{Y}$ and the other is the normal cone $\mathbb{P}(C_{\mathcal{X}}(\mathcal{Y})\oplus 1)$ of $\mathcal{Y}$ inside $\mathcal{X}$. The two components meet along $\mathbb{P}(C_{\mathcal{X}}(\mathcal{Y}))$. As shown in §2.6 of [**47**], one can use the deformation to the normal cone to deform $\mathcal{Y}$ to the zero section of the normal cone. Thus, given a subvariety $\mathcal{Z}\subset\mathcal{Y}$ the proper transform $\overline{\mathcal{Z}\times\mathbb{P}^1}$ in $\mathrm{Bl}_{\mathcal{Y}\times\{0\}}(\mathcal{X}\times\mathbb{P}^1)$ gives a copy of $\mathcal{Z}$ inside the special fiber $\pi^{-1}(0)$ lying in the normal cone component, see Figure 1.

7.4. Deformation and the motive.

7.4. Deformation and the motive. We check that passing from the space Z^{V_Γ} to the deformation $\mathcal{D}(Z[\Gamma])$ described in Proposition 7.4 does not alter the nature of the motive.

It is easy to see that this is the case at the level of virtual motives, that is, classes in the Grothendieck ring of varieties.

Proposition 7.5. *If the class $[X]$ in the Grothendieck ring of varieties $K_0(\mathcal{V})$ is a virtual mixed Tate motive, that is, it lies in the subring $\mathbb{Z}[\mathbb{L}]$ generated by the Lefschetz motive $\mathbb{L}=[\mathbb{A}^1]$, then the class of $\mathcal{D}(Z[\Gamma])$ is also in $\mathbb{Z}[\mathbb{L}]$.*

Proof. As shown in Proposition 7.4, the space $\mathcal{D}(Z[\Gamma])$ is a fibration over $\mathbb{P}^1$, which is a trivial fibration over $\mathbb{P}^1\smallsetminus\{0\}$ with fiber Z^{V_Γ}. By inclusion-exclusion, we can write the class $[\mathcal{D}(Z[\Gamma])]$ in $K_0(\mathcal{V})$ as a sum of the class of the fibration over $\mathbb{P}^1\smallsetminus\{0\}$, which is

$$[Z^{V_\Gamma}\times(\mathbb{P}^1\smallsetminus\{0\})]=[X]^{2N_\Gamma}\,\mathbb{L},$$

with $N_\Gamma=\#V_\Gamma$, and the class $[\pi^{-1}(0)]$ of the fiber over $\zeta=0$, with $\pi:\mathcal{D}(Z[\Gamma])\to\mathbb{P}^1$ the fibration. The component $[X]^{2N_\Gamma}\mathbb{L}$ is in $\mathbb{Z}[\mathbb{L}]$ if the class $[X]\in\mathbb{Z}[\mathbb{L}]$ as we are assuming, so we need to check that the class $[\pi^{-1}(0)]$ is also in $\mathbb{Z}[\mathbb{L}]$. The locus $\pi^{-1}(0)$ is constructed in a sequence of steps as shown in Proposition 7.4. At the first step, we are dealing with the deformation to the normal cone $\mathrm{Bl}_{\Delta_\Gamma^{(Z)}}(Z^{V_\Gamma}\times\mathbb{P}^1)$ and the fiber over zero is the union of Y_1 and $\mathbb{P}(C_{Z^{V_\Gamma}}(\Delta_\Gamma^{(Z)})\oplus 1)$, intersecting along $\mathbb{P}(C_{Z^{V_\Gamma}}(\Delta_\Gamma^{(Z)}))$. Since $\Delta_\Gamma^{(Z)}\simeq X\times X^{V_\Gamma}$ is smooth and a Tate motive, $\mathbb{P}(C_{Z^{V_\Gamma}}(\Delta_\Gamma^{(Z)})\oplus 1)$ is a projective bundle over a Tate motive so it is itself a Tate motive. So is $\mathbb{P}(C_{Z^{V_\Gamma}}(\Delta_\Gamma^{(Z)}))$, for the same reason. So is also Y_1 because of the blowup formula for Grothendieck classes [**19**],

$$[Y_1]=[Z^{V_\Gamma}]+\sum_{k=1}^{\mathrm{codim}(\Delta_\Gamma^{(Z)}\times\{0\})-1}[\Delta_\Gamma^{(Z)}]\,\mathbb{L}^k.$$

By the inclusion-exclusion relations in the Grothendieck ring, it then follows that if the two components of the fiber over zero are in $\mathbb{Z}[\mathbb{L}]$ and the class of their intersection also is, then so is also the class of the union, which is the class of the fiber itself. At the next step the fiber over zero is blown up again, this time along the (dominant transforms of) $\Delta_\gamma^{(Z)}$ with $\gamma\in\mathcal{G}_{n-1,\Gamma}$. Each of these is a blowup of a variety whose class is a virtual mixed Tate motive along a locus whose class is also a virtual mixed Tate motive, hence repeated application of the blowup formula in the Grothendieck ring and an argument analogous to the one used in the first step shows that the Grothendieck class of the fiber over zero is also in $\mathbb{Z}[\mathbb{L}]$. $\square$

We can then, with a similar technique, improve the result from the level of Grothendieck classes to the level of motives.

Proposition 7.6. *If the motive* $\mathfrak{m}(X)$ *of the variety* X *is mixed Tate, then the motive* $\mathfrak{m}(\mathcal{D}(Z[\Gamma])$ *of the deformation* $\mathcal{D}(Z[\Gamma])$ *is also mixed Tate.*

Proof. As in the case of the Grothendieck classes, it suffices to check that, at each step in the construction of $\mathcal{D}(Z[\Gamma])$, the result remains inside the category of mixed Tate motives. It is clear that, if $\mathfrak{m}(X)$ is mixed Tate, then $\mathfrak{m}(Z)$, $\mathfrak{m}(Z^{V_\Gamma})$ and $\mathfrak{m}(Z^{V_\Gamma}\times\mathbb{P}^1)$ also are. At the next step, we use the blowup formula for Voevodsky motives (Proposition 3.5.3 of [**88**]) and we obtain

$$\mathfrak{m}(\mathrm{Bl}_{\Delta_\Gamma^{(Z)}\times\{0\}}(Z^{V_\Gamma}\times\mathbb{P}^1)) =$$

$$\mathfrak{m}(Z^{V_\Gamma}\times\mathbb{P}^1))\oplus \bigoplus_{k=1}^{\operatorname{codim}(\Delta_\Gamma^{(Z)}\times\{0\})-1} \mathfrak{m}(\Delta_\Gamma^{(Z)})(k)[2k].$$

This implies that $\mathfrak{m}(\mathrm{Bl}_{\Delta_\Gamma^{(Z)}\times\{0\}}(Z^{V_\Gamma}\times\mathbb{P}^1))$ is mixed Tate if $\mathfrak{m}(X)$ is. The successive steps are again obtained by blowing up loci $\Delta_\gamma^{(Z)}$ whose motive $\mathfrak{m}(\Delta_\gamma^{(Z)})$ is mixed Tate, inside a variety whose motive is mixed Tate by the previous step, hence repeated application of the blowup formula for motives yields the result. ☐

The analog of Remark 5.8 also holds for the motive $\mathfrak{m}(\mathcal{D}(Z[\Gamma])$.

7.5. Amplitude form on the deformation.

Let $\tilde{\omega}_\Gamma^{(Z)}$ be the regularized form defined in (7.2). In order to allow room for a regularization of the chain of integration, we pull it back to the deformation to the normal cone described above.

Definition 7.7. The regularization of the form $\omega_\Gamma^{(Z)}$ on the deformation space $\mathcal{D}(Z[\Gamma])$ is the pullback

$$\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)}), \tag{7.4}$$

where $\tilde{\pi}_\Gamma:\mathcal{D}(Z[\Gamma])\to Z^{V_\Gamma}\times\mathbb{P}^1$ is the projection and $\tilde{\omega}_\Gamma^{(Z)}$ is the extension (7.2) of the complexified amplitude to $Z^{V_\Gamma}\times\mathbb{P}^1$.

The locus of divergence $\{\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})=\infty\}$ inside the deformation space $\mathcal{D}(Z[\Gamma])$ is then given by the following.

Lemma 7.8. *The locus of divergence of the regularized Feynman amplitude* $\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$ *on the space* $\mathcal{D}(Z[\Gamma])$ *is a union of divisors inside the central fiber,*

$$\bigcup_{\Delta_\gamma^{(Z)}\in\mathcal{G}_\Gamma} D_\gamma^{(Z)}\subset\pi^{-1}(0), \tag{7.5}$$

where $\pi:\mathcal{D}(Z[\Gamma])\to\mathbb{P}^1$ *is the projection of the fibration.*

Proof. When pulling back the regularized form $\tilde{\omega}_\Gamma^{(Z)}$ from $Z^{\mathbf{V}_\Gamma}\times\mathbb{P}^1$ to $\mathcal{D}(Z[\Gamma])$, the poles of $\tilde{\omega}_\Gamma^{(Z)}$ along the diagonals $\Delta_\gamma^{(Z)}\times\{0\}$ yield (as in Proposition 5.10 and Corollary 5.13) poles along the divisors $D_\gamma^{(Z)}$, contained in the central fiber $\pi^{-1}(0)$ at $\zeta=0$ of $\mathcal{D}(Z[\Gamma])$. ☐

7.6. Deformation of the chain of integration. We now describe a regularization of the chain of integration, based on the deformation to the normal cone.

Proposition 7.9. *The proper transform of the chain* $\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1$ *inside* $\mathcal{D}(Z[\Gamma])$ *gives a deformation of the chain of integration, which does not intersect the locus of divergences of the form* $\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$.

Proof. Consider the chain $\sigma_\Gamma^{(Z,y)} = X^{V_\Gamma} \times \{y\}$ of (3.10), inside Z^{V_Γ}. Extend it to a chain $\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1$ inside $Z^{V_\Gamma} \times \mathbb{P}^1$. Let $\overline{\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1}$ denote the proper transform in the blowup $\mathcal{D}(Z[\Gamma])$. Then, as illustrated in Figure 1, we obtain a deformation of $\sigma_\Gamma^{(Z,y)}$ inside the normal cone component of the special fiber $\pi^{-1}(0)$ in $\mathcal{D}(Z[\Gamma])$ that is separated from the intersection with the component given by the blowup $Z[\Gamma]$. ☐

7.7. Regularized integral. Using the deformation of the chain of integration and of the form, one can regularize the integral by

$$\int_{\Sigma_\Gamma^{(Z,y)}} \tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)}), \tag{7.6}$$

where $\Sigma_\Gamma^{(Z,y)}$ denotes the $(2N_\Gamma + 2)$-chain on $\mathcal{D}(Z[\Gamma])$, with $N_\Gamma = \# V_\Gamma$, obtained as in Proposition 7.9. As in (7.3), one also has a corresponding integral on the intersection of the deformed chain $\Sigma^{(Z,y)}$ with the central fiber, which we can write as

$$\int_{\Sigma_\Gamma^{(Z,y)}} \delta(\pi^{-1}(0))\ \tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)}).$$

7.7.1. *Behavior at infinity.* The regularization (7.6) described above avoids divergences along the divisors $D_\gamma^{(Z)}$ in $Z[\Gamma]$. It remains to check the behavior at infinity, both in the $\mathbb{P}^1$-direction added in the deformation construction, and along the locus $\mathcal{D}_\infty$ in $\mathcal{D}(Z[\Gamma])$ defined, in the intersection of each fiber $\pi^{-1}(\zeta)$ with the chain of integration $\Sigma_\Gamma^{(Z,y)}$, by $\Delta_{\Gamma,\infty} := X^{V_\Gamma} \smallsetminus \mathbb{A}^{DN_\Gamma}$, with $N_\Gamma = \# V_\Gamma$.

Proposition 7.10. *The integral* (7.6) *is convergent at infinity when* $D > 2$.

Proof. For the behavior of (7.6) when $\zeta \to \infty$ in $\mathbb{P}^1$, we see that the form behaves like $r^{-2D+2}\, r dr$, where $r = |\zeta|$. This gives a convergent integral for $2D - 3 > 1$. For the behavior at $\Delta_{\Gamma,\infty}$, consider first the case where a single radial coordinate $r_v = |x_v| \to \infty$. In polar coordinates, we then have a radial integral $r_v^{-(2D-2)\# E_{\Gamma,v}}\, r^{D-1} dr$, where $E_{\Gamma,v} = \{ e \in E_\Gamma \,|\, v \in \partial(e) \}$ is the valence $\upsilon(v)$ of the vertex v. This gives a convergent integral when $(2D-2)\upsilon(v) - D + 1 > 1$. Since $\upsilon(v) \geq 1$ and $2D - 2 \geq 0$, we have $(2D-2)\# E_{\Gamma,v} - D + 1 \geq D - 1$, so the condition is satisfied whenever $D > 2$. More generally, one can have several $r_v \to \infty$. The strongest constraint comes from the case that behaves like $r^{-(2D-2)\sum_v \upsilon(v)} r^{DN_\Gamma - 1}$, with $N_\Gamma = \# V_\Gamma$. In this case the convergence condition is given by $(2D-2)\upsilon_\Gamma - DN_\Gamma > 0$, where $\upsilon_\Gamma = \sum_{v \in V_\Gamma} \upsilon(v)$. Again we have $\upsilon_\Gamma \geq N_\Gamma$, and we obtain

$$(2D-2)\upsilon_\Gamma - DN_\Gamma \geq (D-2)N_\Gamma > 0,$$

whenever $D > 2$. In this case the condition for convergence at $|\zeta| \to \infty$ is also satisfied. ☐

8. Appendix: Amplitudes and Bochner–Martinelli kernels

In this Appendix we present some additional complementary material. Using the relation between the Feynman amplitude ω_Γ of (3.2) and the Green functions of the Laplacian, we compute $d\omega_\Gamma$ in terms of an integral kernel associated to the graph Γ and the affine space $\mathbb{A}^{DN_\Gamma} \subset X^{V_\Gamma}$, with $N_\Gamma = \#V_\Gamma$. We also discuss the relation between the complexified amplitude $\omega_\Gamma^{(Z)}$ of (3.9) and the Bochner–Martinelli kernel.

8.1. Real Green functions and differentials. As before, we consider the Green function of the real Laplacian on $\mathbb{A}^D(\mathbb{R})$, with $D = 2\lambda + 2$, given by

$$G_{\mathbb{R}}(x,y) = \frac{1}{\|x-y\|^{2\lambda}}. \tag{8.1}$$

Consider then the differential form $\omega = G_{\mathbb{R}}(x,y)\, dx \wedge dy$. This corresponds to the Feynman amplitude (3.2) in the case of the graph consisting of a single edge, with configuration space $(X \times X) \smallsetminus \Delta$, with $\Delta = \{(x,y) \,|\, x = y\}$ the diagonal.

Lemma 8.1. *The form $\omega = G_{\mathbb{R}}(x,y)\, dx \wedge dy$ is not closed. Its differential is given by*

$$d\omega = -\lambda \sum_{k=1}^{D} \frac{(x_k - y_k)}{\|x-y\|^D} \left(dx \wedge dy \wedge d\bar{x}_k - dx \wedge dy \wedge d\bar{y}_k\right). \tag{8.2}$$

Proof. We have

$$d\omega = \bar{\partial}\omega = \sum_{k=1}^{D} \frac{\partial G_{\mathbb{R}}}{\partial \bar{x}_k} dx \wedge dy \wedge d\bar{x}_k + \sum_{k=1}^{D} \frac{\partial G_{\mathbb{R}}}{\partial \bar{y}_k} dx \wedge dy \wedge d\bar{y}_k.$$

We then see that

$$\frac{\partial \|x-y\|^{-2\lambda}}{\partial \bar{x}_k} = -\lambda \frac{(x_k - y_k)}{\|x-y\|^D}, \quad \text{and} \quad \frac{\partial \|x-y\|^{-2\lambda}}{\partial \bar{y}_k} = \lambda \frac{(x_k - y_k)}{\|x-y\|^D},$$

so that we obtain (8.2). □

8.2. Feynman amplitudes and integral kernels on graphs. We first consider the form defining the Feynman amplitude ω_Γ of (3.2). It is not a closed form. In fact, we compute here explicitly its differential $d\omega_\Gamma$ in terms of some integral kernels associated to graphs.

Recall that, given a graph Γ and a vertex $v \in V_\Gamma$, the graph $\Gamma \smallsetminus \{v\}$ has

$$V_{\Gamma \smallsetminus \{v\}} = V_\Gamma \smallsetminus \{v\}, \quad \text{and} \quad E_{\Gamma \smallsetminus \{v\}} = E_\Gamma \smallsetminus \{e \in E_\Gamma \,|\, v \in \partial(e)\}, \tag{8.3}$$

that is, one removes a vertex along with its star of edges.

Definition 8.2. Suppose given a graph Γ and a vertex $v \in V_\Gamma$. Let $N_\Gamma = \#V_\Gamma$. Let $\epsilon_v = (-1)^{N_\Gamma} \epsilon'_v$, where ϵ'_v is the sign determined by

$$\epsilon'_v \,(\bigwedge_{w \neq v} dx_w) \wedge dx_v = \bigwedge_{v' \in \mathbf{V}_\Gamma} dx_{v'}.$$

Let $S_{\Gamma,v}$ and $T_{\Gamma,v}$ be the expressions defined as

$$S_{\Gamma,v} = \prod_{e \in E_\Gamma : v = s(e)} \frac{1}{\|x_v - x_{t(e)}\|^{2\lambda}}, \qquad T_{\Gamma,v} = \prod_{e \in E_\Gamma : v = t(e)} \frac{1}{\|x_{s(e)} - x_v\|^{2\lambda}}. \tag{8.4}$$

Also denote by $\kappa_{v,w,k}$ the expression

$$\kappa_{v,w,k} := \frac{(x_{v,k} - x_{w,k})}{\|x_v - x_w\|^D}. \tag{8.5}$$

We define the differential form

$$\kappa^{\mathbb{R}}_{\Gamma,v} := \epsilon_v \sum_{k=1}^{D} \left(S_{\Gamma,v} \sum_{e:v=s(e)} T_{\Gamma\smallsetminus e,v} \kappa_{v,t(e),k} - T_{\Gamma,v} \sum_{e:v=t(e)} S_{\Gamma\smallsetminus e,v} \kappa_{s(e),v,k} \right) dx_v \wedge d\bar{x}_{v,k}, \tag{8.6}$$

in the coordinates $x = (x_v \,|\, v \in V_\Gamma)$. Given an oriented graph Γ, we then consider the integral kernel

$$\mathcal{K}_{\mathbb{R},\Gamma} = \lambda \sum_{v \in \mathbf{V}_\Gamma} \omega_{\Gamma\smallsetminus\{v\}} \wedge \kappa^{\mathbb{R}}_{\Gamma,v}, \tag{8.7}$$

where $\omega_{\Gamma\smallsetminus\{v\}}$ is the form (3.2) for the graph (8.3).

Proposition 8.3. *The differential $d\omega_\Gamma$ is the integral kernel $\mathcal{K}_{\mathbb{R},\Gamma}$ of* (8.7).

Proof. First observe that, for ω_Γ the Feynman amplitude of (3.2), we have $d\omega_\Gamma = \bar{\partial}\omega_\Gamma$, that is,

$$d\omega_\Gamma = \sum_{v \in V_\Gamma} \sum_{k=1}^{D} (-1)^{N_\Gamma} \frac{\partial}{\partial \bar{x}_{v,k}} \omega_\Gamma \wedge d\bar{x}_{v,k},$$

with $N_\Gamma = \# V_\Gamma$. We introduce the notation

$$\upsilon_{\Gamma,v} := \prod_{e: v \notin \partial(e)} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}}. \tag{8.8}$$

We can then write, for any given $v \in V_\Gamma$,

$$\prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} = \upsilon_{\Gamma,v} S_{\Gamma,v} T_{\Gamma,v},$$

and we obtain

$$\partial_{\bar{x}_{v,k}} \left(\prod_e \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \right) = \upsilon_{\Gamma,v} S_{\Gamma,v} \partial_{\bar{x}_{v,k}} T_{\Gamma,v} + \upsilon_{\Gamma,v} T_{\Gamma,v} \partial_{\bar{x}_{v,k}} S_{\Gamma,v}.$$

We also have

$$\frac{\partial}{\partial \bar{x}_{v,k}} T_{\Gamma,v} = \lambda \sum_{e:v=t(e)} T_{\Gamma\smallsetminus e,v} \frac{-(x_{s(e),k} - x_{v,k})}{\|x_{s(e)} - x_v\|^D}$$

$$\frac{\partial}{\partial \bar{x}_{v,k}} S_{\Gamma,v} = \lambda \sum_{e:v=s(e)} S_{\Gamma\smallsetminus e,v} \frac{(x_{v,k} - x_{t(e),k})}{\|x_v - x_{t(e)}\|^D},$$

where we used the fact that, for $z, w \in \mathbb{A}^D$, one has

$$\frac{\partial}{\partial \bar{w}_k} \frac{1}{\|z - w\|^{2\lambda}} = -\frac{\lambda\,(w_k - z_k)}{\|z - w\|^D}.$$

This gives

$$\partial_{\bar{x}_{v,k}} \left(\prod_e \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \right) =$$

$$\lambda \upsilon_{\Gamma,v} \left(\sum_{v=s(e)} S_{\Gamma \smallsetminus e,v} T_{\Gamma,v} \kappa_{v,t(e),k} - \sum_{v=t(e)} S_{\Gamma,v} T_{\Gamma \smallsetminus e,v} \kappa_{s(e),v,k} \right).$$

Thus, we find

$$d\omega_\Gamma = \lambda (-1)^{\mathbf{V}_\Gamma} \sum_{v \in \mathbf{V}_\Gamma} \upsilon_{\Gamma,v} \bigwedge_{w \neq v} dx_w \wedge \kappa^{\mathbb{R}}_{\Gamma,v}.$$

We then identify the term $\upsilon_{\Gamma,v} \wedge_{w \neq v} dx_w$ with the form $\omega_{\Gamma \smallsetminus \{v\}}$. □

It is easy to see that this recovers (8.2) in the case of the graph consisting of two vertices and a single edge between them.

8.3. Complex Green functions and the Bochner–Martinelli kernel.

On $\mathbb{A}^D(\mathbb{C}) \subset X$ the complex Laplacian

$$\Delta = \sum_{k=1}^{D} \frac{\partial^2}{\partial x_k \partial \bar{x}_k}.$$

has a fundamental solution of the form ($D > 1$)

$$G_{\mathbb{C}}(x,y) = \frac{-(D-2)!}{(2\pi i)^D \|x-y\|^{2D-2}}. \tag{8.9}$$

The Bochner–Martinelli kernel is given by

$$\mathcal{K}_{\mathbb{C}}(x,y) = \frac{(D-1)!}{(2\pi i)^D} \sum_{k=1}^{D} (-1)^{k-1} \frac{\bar{x}_k - \bar{y}_k}{\|x-y\|^{2D}} d\bar{x}_{[k]} \wedge dx, \tag{8.10}$$

where we write

$$dx = dx_1 \wedge \cdots \wedge dx_D \quad \text{and} \quad d\bar{x}_{[k]} = d\bar{x}_1 \wedge \cdots \wedge \widehat{d\bar{x}_k} \wedge \cdots \wedge d\bar{x}_D,$$

with the k-th factor removed. The following facts are well known (see [**53**], §3.2 and [**60**]):

$$\begin{aligned} \mathcal{K}_{\mathbb{C}}(x,y) &= \sum_{k=1}^{D} (-1)^{k-1} \frac{\partial G_{\mathbb{C}}}{\partial x_k} d\bar{x}_{[k]} \wedge dx \\ &= (-1)^{D-1} \partial_x G_{\mathbb{C}} \wedge \sum_{k=1}^{D} d\bar{x}_{[k]} \wedge dx_{[k]}. \end{aligned} \tag{8.11}$$

where ∂_x and $\bar{\partial}_x$ denote the operators ∂ and $\bar{\partial}$ in the variables $x = (x_k)$. For fixed y, the coefficients of $\mathcal{K}_{\mathbb{C}}(x,y)$ are harmonic functions on $\mathbb{A}^D \smallsetminus \{y\}$ and $\mathcal{K}_{\mathbb{C}}(x,y)$ is closed, $d_x \mathcal{K}(x,y) = 0$. Moreover, the Bochner–Martinelli integral formula holds: for a bounded domain Σ with piecewise smooth boundary $\partial\Sigma$, a function $f \in \mathcal{C}^2(\overline{\Sigma})$ and for $y \in \Sigma$,

$$f(y) = \int_{\partial\Sigma} f(x) \mathcal{K}_{\mathbb{C}}(x,y) + \int_{\Sigma} \Delta(f)(x)\, G_{\mathbb{C}}(x,y) d\bar{x} \wedge dx - \int_{\partial\Sigma} G_{\mathbb{C}}(x,y)\, \mu_f(x), \tag{8.12}$$

where $\mu_f(x)$ is the form

$$\mu_f(x) = \sum_{k=1}^{D} (-1)^{D+k-1} \frac{\partial f}{\partial \bar{x}_k}\, dx_{[k]} \wedge d\bar{x}. \tag{8.13}$$

The integral (8.12) vanishes when $y \notin \Sigma$. A related Bochner–Martinelli integral, which can be derived from (8.12) (see Thm 1.3 of [**60**]) is of the form

$$f(y) = \int_{\partial\Sigma} f(x)\, \mathcal{K}_{\mathbb{C}}(x,y) - \int_{\Sigma} \bar\partial f \wedge \mathcal{K}_{\mathbb{C}}(x,y), \tag{8.14}$$

for $y \in \Sigma$ and $f \in \mathcal{C}^1(\overline{\Sigma})$, with $\bar\partial f = \sum_k \partial_{\bar x_k} f \, d\bar x_k$.

Similarly, one can consider Green forms associated to the Laplacians on $\Omega^{p,q}$ forms and related Bochner–Martinelli kernels, see [**82**].

8.4. Complexified amplitudes and the Bochner–Martinelli kernels.

We now consider the Feynman amplitude $\hat\omega_\Gamma$ of (3.12) in the complexified case discussed in §3.3.3. We first introduce a Bochner–Martinelli (BM) kernel for graphs.

8.4.1. *Bochner–Martinelli kernel for graphs.* We define Bochner–Martinelli kernels for graphs in the following way.

Suppose given an oriented graph Γ and vertices $v, w \in V_\Gamma$. In the coordinates $x_v, x_w \in X^{V_\Gamma}$, consider the forms

$$\kappa^{\mathbb{C}}_{v,w} = \sum_{k=1}^{D} (-1)^{k-1} \frac{\bar x_{v,k} - \bar x_{w,k}}{\|x_v - x_w\|^{2D}}\; dx_v \wedge d\bar x_{v,[k]}, \tag{8.15}$$

$$\kappa^{\mathbb{C},*}_{v,w} = \sum_{k=1}^{D} (-1)^{k-1} \frac{x_{v,k} - x_{w,k}}{\|x_v - x_w\|^{2D}}\; dx_{v,[k]} \wedge d\bar x_{v}. \tag{8.16}$$

We construct two kinds of integral kernels, in terms of the $\kappa^{\mathbb{C}}_{v,w}$ and $\kappa^{\mathbb{C},*}_{v,w}$ and of expressions

$$S^{\mathbb{C}}_{\Gamma,v} = \prod_{\substack{e\in E_\Gamma \\ v=s(e)}} \frac{1}{\|x_v - x_{t(e)}\|^{2D-2}}, \qquad T^{\mathbb{C}}_{\Gamma,v} = \prod_{\substack{e\in E_\Gamma \\ v=t(e)}} \frac{1}{\|x_{s(e)} - x_v\|^{2D-2}}. \tag{8.17}$$

which are the analogs of (8.4) in the complexified case.

Definition 8.4. For an oriented graph Γ and a vertex $v \in V_\Gamma$, we set

$$\kappa^{\mathbb{C}}_{\Gamma,v} = \sum_{e: v=s(e)} S^{\mathbb{C}}_{\Gamma,v} T^{\mathbb{C}}_{\Gamma\smallsetminus e,v} \kappa^{\mathbb{C}}_{v,t(e)} - \sum_{e: v=t(e)} S^{\mathbb{C}}_{\Gamma\smallsetminus e,v} T^{\mathbb{C}}_{\Gamma,v} \kappa^{\mathbb{C}}_{s(e),v}. \tag{8.18}$$

$$\kappa^{\mathbb{C},*}_{\Gamma,v} = \sum_{e: v=s(e)} S^{\mathbb{C}}_{\Gamma,v} T^{\mathbb{C}}_{\Gamma\smallsetminus e,v} \kappa^{\mathbb{C},*}_{v,t(e)} - \sum_{e: v=t(e)} S^{\mathbb{C}}_{\Gamma\smallsetminus e,v} T^{\mathbb{C}}_{\Gamma,v} \kappa^{\mathbb{C},*}_{s(e),v}. \tag{8.19}$$

We also define a simpler BM kernel for graphs, of the form

$$\kappa^{BM,\mathbb{C}}_{\Gamma,v} = \sum_{e: v\in\partial(e)} \epsilon_e\, \kappa^{\mathbb{C}}_{s(e),t(e)}\, dx_v \wedge d\bar x_{v,[k]} \tag{8.20}$$

and

$$\kappa^{BM,\mathbb{C},*}_{\Gamma,v} = \sum_{e: v\in\partial(e)} \epsilon_e\, \kappa^{\mathbb{C},*}_{s(e),t(e)}\, dx_{v,[k]} \wedge d\bar x_{v}, \tag{8.21}$$

where the sign ϵ_e is ±1 depending on whether $v = s(e)$ or $v = t(e)$.

8.4.2. *Complexified amplitude and Bochner–Martinelli kernel.* The Bochner–Martinelli kernel of graphs defined above is related to the Feynman amplitude (3.12) by the following.

PROPOSITION 8.5. *Let $\hat{\omega}_\Gamma$ be the Feynman amplitude* (3.12). *Then*

$$\partial \hat{\omega}_\Gamma = \sum_{v\in V_\Gamma} \epsilon_v \ \hat{\omega}_{\Gamma\smallsetminus\{v\}} \wedge \kappa^{\mathbb{C}}_{\Gamma,v} \tag{8.22}$$

$$\bar{\partial} \hat{\omega}_\Gamma = \sum_{v\in V_\Gamma} \epsilon_v \ \hat{\omega}_{\Gamma\smallsetminus\{v\}} \wedge (-1)^{D-1} \kappa^{\mathbb{C},*}_{\Gamma,v}, \tag{8.23}$$

where the sign ϵ_v is defined by

$$\epsilon_v \left(\bigwedge_{w\neq v} \sum_k (-1)^{k-1} dx_{w,[k]} \wedge d\bar{x}_{w,[k]} \right) \wedge \left(\sum_k (-1)^{k-1} dx_{v,[k]} \wedge d\bar{x}_{v,[k]} \right) =$$

$$\bigwedge_{v'\in V_\Gamma} \sum_k (-1)^{k-1} dx_{v',[k]} \wedge d\bar{x}_{v',[k]}.$$

PROOF. The argument is analogous to Proposition 8.3. □

8.5. Bochner–Martinelli integral on graphs. There is an analog of the classical Bochner–Martinelli integral (8.14) for the kernel (8.18) of graphs.

We first recall some well known facts about the Laplacian on graphs, see e.g. [**17**]. Given a graph Γ, one defines the exterior differential δ from functions on V_Γ to functions on E_Γ by

$$(\delta h)(e) = h(s(e)) - h(t(e))$$

and the δ^* operator from functions on edges to functions on vertices by

$$(\delta^* \xi)(v) = \sum_{e:v=s(e)} \xi(e) - \sum_{e:v=t(e)} \xi(e).$$

Thus, the Laplacian $\Delta_\Gamma = \delta^*\delta$ on Γ is given by

$$(\Delta_\Gamma f)(v) = \sum_{e:v=s(e)} (h(v) - h(t(e)) - \sum_{e:v=t(e)} (h(s(e)) - h(v))$$

$$= N_v \, h(v) - \sum_{e:v\in\partial(e)} h(v_e),$$

where N_v is the number of vertices connected to v by an edge, and v_e is the other endpoint of e (we assume as usual that Γ has no looping edges). Thus, a harmonic function h on a graph is a function on V_Γ satisfying

$$h(v) = \frac{1}{N_v} \sum_{e:v\in\partial(e)} h(v_e). \tag{8.24}$$

Motivated by the usual notion of graph Laplacian Δ_Γ and the harmonic condition (8.24) for graphs recalled here above, we introduce an operator

$$(\Delta_{\Gamma,v} f)(x) = \sum_{e:v\in\partial(e)} f(x_{v_e}), \tag{8.25}$$

which assigns to a complex valued function f defined on $\mathbb{A}^D \subset X$ a complex valued function $\Delta_{\Gamma,v} f$ defined on X^{V_Γ}.

We then have the following result.

PROPOSITION 8.6. *Let f be a complex valued function defined on $\mathbb{A}^D \subset X$. Also suppose given a bounded domain Σ with piecewise smooth boundary $\partial\Sigma$ in $\mathbb{A}^D$ and assume that f is $\mathcal{C}^1$ on $\overline{\Sigma}$. For a given $v \in \mathbf{V}_\Gamma$ consider the set of $x = (x_w) \in \mathbb{A}^{D|\mathbf{V}_\Gamma|} \subset X^{\mathbf{V}_\Gamma}$, such that $x_{v_e} \in \Sigma$ for all $v_e \neq v$ endpoints of edges e with $v \in \partial(e)$. For such $x = (x_w)$ we have*

(8.26)
$$(\Delta_{\Gamma,v} f)(x) = \frac{(D-1)!}{(2\pi i)^D\, N_v} \left(\int_{\partial\Sigma} f(x_v) \kappa^{BM,\mathbb{C}}_{\Gamma,v}(x) - \int_\Sigma \bar\partial_{x_v} f(x_v) \wedge \kappa^{BM,\mathbb{C}}_{\Gamma,v}(x) \right),$$

with $\kappa^{BM,\mathbb{C}}_{\Gamma,v}$ and $\kappa^{BM,\mathbb{C}}_{\Gamma,v}$ as in (8.20) and (8.21), where the integration on Σ and $\partial\Sigma$ is in the variable x_v and $\Delta_{\Gamma,v} f$ is defined as in (8.25).

PROOF. We have

$$\int_{\partial\Sigma} f(x_v) \kappa^{BM,\mathbb{C}}_{\Gamma,v}(x) = \sum_{e: v \in \partial(e)} \epsilon_e \int_{\partial\Sigma} f(x_v) \sum_{k=1}^D (-1)^{k-1} \frac{(\bar x_{s(e),k} - \bar x_{t(e),k})}{\|x_{s(e)} - x_{t(e)}\|^{2D}}\, \eta_v$$

where

$$\eta_v = dx_v \wedge d\bar x_{v,[k]}.$$

We write the integral as

$$\sum_{e: v=s(e)} \int_{\partial\Sigma} f(x_v) \sum_{k=1}^D (-1)^{k-1} \frac{(\bar x_{v,k} - \bar x_{v_e,k})}{\|x_v - x_{v_e}\|^{2D}}\, \eta_v$$

$$- \sum_{e: v=t(e)} \int_{\partial\Sigma} f(x_v) \sum_{k=1}^D (-1)^{k-1} \frac{(\bar x_{v_e,k} - \bar x_{v,k})}{\|x_v - x_{v_e}\|^{2D}}\, \eta_v$$

$$= \sum_{e: v\in\partial(e)} \int_{\partial\Sigma} f(x_v) \sum_{k=1}^D (-1)^{k-1} \frac{(\bar x_{v,k} - \bar x_{v_e,k})}{\|x_v - x_{v_e}\|^{2D}}\, \eta_v.$$

The case of the integral on Σ is analogous. We then apply the classical result (8.14) about the Bochner–Martinelli integral and we obtain

$$\int_{\partial\Sigma} f(x) \kappa^{BM,\mathbb{C}}_{\Gamma,v}(x) - \int_\Sigma \bar\partial_{x_v} f(x) \wedge \kappa^{BM,\mathbb{C}}_{\Gamma,v}(x) = \frac{(2\pi i)^D}{(D-1)!} \sum_{e: v\in\partial(e)} f(x_{v_e}).$$

□

8.6. Bochner–Martinelli kernel and forms with logarithmic poles. In view of our discussion, later in the paper, about replacing the complexified amplitude by an algebraic form with logarithmic poles, we mention here a well known property of the Bochner–Martinelli kernel, which provides a completely explicit toy model model case for the kind of substitution we aim for in the more general case of the complexified amplitude.

Consider then the Bochner–Martinelli form, for $x, w \in \mathbb{C}^D$

$$\Psi(x,y) = \sum_{k=1}^{D} (-1)^{k-1} \frac{(\bar{x}_k - \bar{y}_k)}{\|x-y\|^{2D}}\, dx \wedge d\bar{x}_{[k]} \tag{8.27}$$

Let Ω^{D-1} be the pullback, under the projection $p : \mathbb{C}^D \smallsetminus \{0\} \to \mathbb{P}^{D-1}(\mathbb{C})$, of the Kähler volume form of the Fubini–Study metric on $\mathbb{P}^{D-1}(\mathbb{C})$, and let $\pi : \mathrm{Bl}_y(\mathbb{C}^D) \to \mathbb{C}^D$ denote the projection of the blowup of $\mathbb{C}^D$ at a point y. We then have the following result.

Lemma 8.7. *The pullback $\pi^*(\Psi)$ of the Bochner–Martinelli form* (8.27) *is*

$$\pi^*(\Psi) = C\,\theta \wedge \Omega^{D-1}, \tag{8.28}$$

where Ω is as above and $\theta = d\log(\lambda)$ on the fibers $\{\lambda x\}_{\lambda\in\mathbb{C}}$, and with the constant $C \in \mathbb{Q}(2\pi i)$.

Proof. The result follows as in [**53**], pp. 371-373. One checks with a direct computation that the constant is given by $C = (2\pi i)^{D-1}/(D-1)!$. ☐

Lemma 8.8. *The form*

$$\hat{\omega}_{v,w} = \|x_v - x_w\|^{-(2D-2)} \sum_{k=1}^{D} (-1)^{k-1} dx_{v,[k]} \wedge d\bar{x}_{v,[k]} \tag{8.29}$$

satisfies

$$\partial \hat{\omega}_{v,w} = \Psi(x_v, x_w). \tag{8.30}$$

where the ∂-operator is taken in the x_v variables.

Proof. The form $\partial\hat{\omega}_{v,w}$ is given by

$$\sum_{j=1}^{D} \frac{\partial}{\partial x_{v,j}} \left(\frac{1}{\|x_w - x_v\|^{2D-2}} \right) dx_{v,j} \wedge \sum_{k=1}^{D} (-1)^{k-1} dx_{v,[k]} \wedge d\bar{x}_{v,[k]} = \Psi(x_v, x_w).$$

☐

Note that the form $\hat{\omega}_{v,w}$ of (8.29) is the form $\hat{\omega}_\Gamma$ for the graph Γ consisting of two vertices v, w and a single edge between them. We then have the following result, which shows that the form $\hat{\omega}_{v,w}$ can be replaced by an algebraic form with logarithmic poles, up to a coboundary.

Proposition 8.9. *Let π be the projection of the blowup along $\Delta = \{x_v = x_w\}$. The form $\hat{\omega}_{v,w}$ satisfies, for some form ξ,*

$$\pi^*(\hat{\omega}_{v,w}) = C\, \log(\lambda)\, \Omega^{D-1} + d\xi, \tag{8.31}$$

with $C = (2\pi i)^{D-1}/(D-1)!$.

Proof. By (8.28) and (8.30), we have $\partial \pi^*(\hat{\omega}_{v,w}) = C\theta\wedge\Omega^{D-1}$ and $\bar{\partial}\pi^*(\hat{\omega}_{v,w}) = 0$. Thus, we have

$$d(\pi^*(\hat{\omega}_{v,w}) - C\, \log(\lambda)\, \Omega^{D-1}) = 0.$$

The cohomology of the blowup is given additively by (see [**53**], p. 605)

$$H^*(\mathrm{Bl}_\Delta(\mathbb{P}^{D-1} \times \mathbb{P}^{D-1})) = \pi^* H^*(\mathbb{P}^{D-1} \times \mathbb{P}^{D-1}) \oplus H^*(E)\,/\pi^* H^*(\Delta).$$

This has no odd dimensional cohomology, hence for the above form of degree $2D-1$ we obtain that $\pi^*(\hat{\omega}_{v,w})$ and $C\,\log(\lambda)\,\Omega^{D-1}$ differ by a coboundary, with C as in Lemma 8.7. □

In particular, as a direct consequence of the previous result, in the case of a tree we obtain the following.

Corollary 8.10. *If the graph* Γ *is a tree, then*

$$\pi^*(\hat{\omega}_\Gamma) = \wedge_{e\in E_\Gamma}\pi^*(\hat{\omega}_{s(e),t(e)})$$

is cohomologous to a differential form with logarithmic poles.

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