5ECM Amsterdam 2008
© 2010 European Mathematical Society

Feynman integrals and motives

Matilde Marcolli

... and Beyond the Infinite
(Stanley Kubrick, *2001 – A Space Odyssey*)

Abstract. This article gives an overview of recent results on the relation between quantum field theory and motives, with an emphasis on two different approaches: a "bottom-up" approach based on the algebraic geometry of varieties associated to Feynman graphs, and a "top-down" approach based on the comparison of the properties of associated categorical structures. This survey is mostly based on joint work of the author with Paolo Aluffi, along the lines of the first approach, and on previous work of the author with Alain Connes on the second approach.

Mathematics Subject Classification (2000). 81T15, 19E15.

Keywords. Feynman integrals, mixed Tate motives, periods, Grothendieck ring, graph hypersurfaces, motivic Galois groups, Riemann–Hilbert correspondence.

Contents

1 Introduction: quantum fields and motives, an unlikely match 293
2 A bottom-up approach to Feynman integrals and motives 306
3 The Connes–Kreimer theory 316
4 A top-down approach via Galois theory 319
5 The geometry of Dim Reg 325

1. Introduction: quantum fields and motives, an unlikely match

This paper, based on the plenary lecture delivered by the author at the 5th European Congress of Mathematics in Amsterdam, aims at giving an overview of the current approaches to understanding the role of motives and periods of motives in perturbative quantum field theory. It is a priori surprising that there should be any relation at all between such distant fields. In fact, motives are a very abstract and sophisticated branch of algebraic and arithmetic geometry, introduced by Grothendieck as a universal cohomology theory for algebraic varieties. On the other hand, perturbative quantum field theory is a procedure for computing, by successive approximations in powers of the relevant coupling constants, values of physical observables in a quantum field theory. Perturbative quantum field theory is not entirely mathematically rigorous, though as we will see later in this paper, a lot of interesting mathematical structures arise when one tries to understand conceptually the procedure of extraction of finite values from divergent Feynman integrals known as renormalization.

The theory of motives itself has its mysteries, which make it a very active area of research in contemporary mathematics. The categorical structure of motives is still a problem very much under investigation. While one has a good abelian category of pure motives (with numerical equivalence), that is, of motives arising from smooth projective varieties, the "standard conjectures" of Grothendieck are still unsolved. Moreover, when it comes to the much more complicated setting of mixed motives, which no longer correspond to smooth projective varieties, one knows that they form a triangulated category, but in general one cannot improve that to the level of an abelian category with the same nice properties one has in the case of pure motives. See [14], [45] for an overview of the theory of mixed motives.

The unlikely interplay between motives and quantum field theory has recently become an area of growing interest at the interface of algebraic geometry, number theory, and theoretical physics. The first substantial indications of a relation between these two subjects came from extensive computations of Feynman diagrams carried out by Broadhurst and Kreimer [22], which showed the presence of multiple zeta values as results of Feynman integral calculations. From the number theoretic viewpoint, multiple zeta values are a prototype case of those very interesting classes of numbers which, although not themselves algebraic, can be realized by integrating algebraic differential forms on algebraic cycles in arithmetic varieties. Such numbers are called periods, *cf.* [43], and there are precise conjectures on the kind of operations (changes of variables, Stokes formula) one can perform at the level of the algebraic data that will correspond to relations in the algebra of periods. As one can consider periods of algebraic varieties, one can also consider periods of motives. In fact, the nature of the numbers one obtains is very much related to the motivic complexity of the part of the cohomology of the variety that is involved in the evaluation of the period.

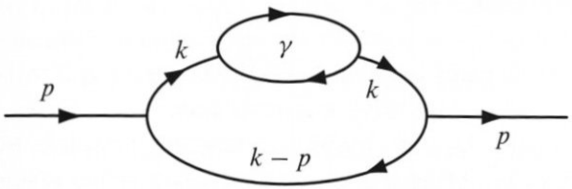
There is a special class of motives that are better understood and better behaved with respect to their categorical properties: the *mixed Tate motives*. They are also the kind of motives that are expected (see [36], [55]) to be supporting the type of periods like multiple zeta values that appear in Feynman integral computations.

At the level of pure motives the Tate motives $\mathbb{Q}(n)$ are simply motives of projective spaces and their formal inverses, but in the mixed case there are very nontrivial extensions of these objects possible. In terms of algebraic varieties, for instance, varieties that have stratifications where the successive strata are obtained by adding copies of affine spaces provide examples of mixed Tate motives. There are various conjectural geometric descriptions of such extensions (see *e.g.* [8] for one possible description in terms of hyperplane arrangements). Understanding when certain geometric objects determine motives that are or are not mixed Tate is in general a difficult question and, it turns out, one that is very much central to the relation to quantum field theory.

In fact, the main conjecture we describe here, along with an overview of some of the current approaches being developed to answer it, is whether, after a suitable subtraction of infinities, the Feynman integrals of a perturbative scalar quantum field theory always produce values that are periods of mixed Tate motives.

1.1. Feynman diagrams: graphs and integrals. We briefly introduce the main characters of our story, starting with Feynman diagrams. By these one usually means the data of a finite graph together with a prescription for assigning variables to the edges with linear relations at the vertices and a formal integral in the resulting number of independent variables.

For instance, consider a graph of the following form.



The corresponding integral gives

$$(2\pi)^{-2D} \int \frac{1}{k^4} \frac{1}{(k-p)^2} \frac{1}{(k+\ell)^2} \frac{1}{\ell^2} \, d^D k \, d^D \ell.$$

As is often the case, the resulting integral is divergent. We will explain below the regularization procedure that expresses such divergent integrals in terms of meromorphic functions. In this case one obtains

$$(4\pi)^{-D} \frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)^3\Gamma(5-D)\Gamma(D-4)}{\Gamma(D-2)\Gamma\left(4-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-5\right)} \left(p^2\right)^{D-5}$$

and one identifies the divergences with poles of the resulting function.

The renormalization problem in perturbative quantum field theory consists of removing the divergent part of such expressions by a redefinition of the running parameters (masses, coupling constants) in the Lagrangian of the theory. To avoid non-local expressions in the divergences, which cannot be canceled using the local terms in the Lagrangian, one needs a method to remove divergences from Feynman integrals that accounts for the nested structure of subdivergences inside a given Feynman graphs. Thus, the process of extracting finite values from divergent Feynman integrals is organized in two steps: *regularization*, by which one denotes a procedure that replaces a divergent integral by a function of some new regularization parameters, which is meromorphic in these parameters, and happens to have a pole at the value of the parameters that recovers the original expression; and *renormalization*, which denotes the procedure by which the polar part of the Laurent series obtained as a result of the regularization process is extracted *consistently* with the hierarchy of divergent subgraphs inside larger graphs.

1.2. Perturbative quantum field theory in a nutshell. We recall very briefly here a few notions of perturbative quantum field theory we need in the following. A detailed introduction for the use of mathematicians is given in Chapter 1 of [30].

To specify a quantum field theory, which we denote by $\mathcal{T}$ in the following, one needs to assign the Lagrangian of the theory. We restrict ourselves to the case of *scalar theories*, though it is possible that similar conjectures on number theoretic aspects of values of Feynman integrals may be formulated more generally.

A scalar field theory $\mathcal{T}$ in spacetime dimension D is determined by a classical Lagrangian density of the form

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{L}_{\mathrm{int}}(\phi), \tag{1}$$

in a single scalar field ϕ, with the interaction term $\mathcal{L}_{\mathrm{int}}(\phi)$ given by a polynomial in ϕ of degree at least three. This determines the corresponding classical action as

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{\mathrm{int}}(\phi).$$

While the variational problem for the classical action gives the classical field equations, the quantum corrections are implemented by passing to the *effective action* $S_{\mathrm{eff}}(\phi)$. The latter is not given in closed form, but in the form of an asymptotic series, the perturbative expansion parameterized by the "one-particle irreducible" (1PI) Feynman graphs. The resulting expression for the effective action is then of the form

$$S_{\mathrm{eff}}(\phi) = S_0(\phi) + \sum_{\Gamma} \frac{\Gamma(\phi)}{\#\mathrm{Aut}(\Gamma)} \tag{2}$$

where the contribution of a single graph is an integral on external momenta assigned to the "external edges" of the graph,

$$\Gamma(\phi) = \frac{1}{N!} \int_{\sum_i p_i = 0} \hat{\phi}(p_1) \ldots \hat{\phi}(p_N) U_\mu^z(\Gamma(p_1, \ldots, p_N)) dp_1 \ldots dp_N.$$

In turn, the function of the external momenta that one integrates to obtain the coefficient $\Gamma(\phi)$ is an integral in momentum variables assigned to the "internal edges" of the graph Γ, with momentum conservation at each vertex. Thus, it can be expressed as an integral in a number of variables equal to the number $b_1(\Gamma)$ of loops in the graph, of the form

$$U(\Gamma(p_1,\ldots,p_N)) = \int I_\Gamma(k_1,\ldots,k_\ell,p_1,\ldots,p_N)d^Dk_1\ldots d^Dk_\ell. \quad (3)$$

The graphs involved in the expansion (2) are the 1PI Feynman graphs of the theory $\mathcal{T}$, *i.e.* those graphs that cannot be disconnected by the removal of a single edge. As Feynman graphs of a given theory, they are also subject to certain combinatorial constraints: each vertex in the graph has valence equal to the degree of one of the monomials in the Lagrangian. The edges are subdivided into internal edges connecting two vertices and external edges (or half edges) connected to a single vertex. The Feynman rules of the theory $\mathcal{T}$ specify how to assign an integral (3) to a Feynman graph, namely it specifies the form of the function $I_\Gamma(k_1,\ldots,k_\ell,p_1,\ldots,p_N)$ of the internal momenta. This is a product of "propagators" associated to the internal lines. These are typically of the form $1/q(k)$, where q is a quadratic form in the momentum variable of a given internal edge, which is obtained from the fundamental (distributional) solution of the associated classical field equation for the free field theory coming from the $S_0(\phi)$ part of the Lagrangian, such as the Klein–Gordon equations for the scalar case. Momentum conservations are then imposed at each vertex, and multiplied by a power of the coupling constant (the coefficient of the corresponding monomial in the Lagrangian) and a power of 2π.

As we mentioned above, the resulting integrals (3) are very often divergent. Thus, a regularization and renormalization method is used to extract a finite value. There are different regularization and renormalization schemes used in the physics literature. We concentrate here on *dimensional regularization and minimal subtraction*, which is a widely used regularization method in particle physics computations, and on the recursive procedure of Bogolyubov–Parasiuk–Hepp–Zimmermann for renormalization [20], [39], [60], see also [48]. Regularization and renormalization are two distinct steps in the process of extracting finite values from divergent Feynman integrals. The first replaces the integrals with meromorphic functions with poles that account for the divergences, while the latter organizes subdivergences in such a way that the divergent parts can be eliminated (in the case of a renormalizable theory) by readjusting finitely many parameters in the Lagrangian.

The procedure of dimensional regularization is based on the curious idea of making sense of the integrals (3) in "complexified dimension" $D - z$, with $z \in \mathbb{C}^*$, instead of working in the original dimension $D \in \mathbb{N}$. It would seem at first that, to make sense of such a procedure, one would need to make sense of geometric spaces in dimension $D - z$ and of a corresponding theory of measure and integration in such spaces. However, due to the special form of the Feynman integrals (3), a lot less is needed. In fact, it turns out that it suffices to have a *formal procedure* to define the

Gaussian integral

$$\int e^{-\lambda t^2} d^D t := \pi^{D/2} \lambda^{-D/2} \tag{4}$$

in the case where D is no longer a positive integer but a complex number. Clearly, since the right hand side of (4) continues to make sense for $D \in \mathbb{C}^*$, one can use that as the definition of the left hand side and set:

$$\int e^{-\lambda t^2} d^z t := \pi^{z/2} \lambda^{-z/2} \quad \text{for all } z \in \mathbb{C}^*. \tag{5}$$

The computations of Feynman integrals can be reformulated in terms of Gaussian integrations using the method of Schwinger parameters we return to in more detail below, hence one obtains a well defined notion of integrals in dimension $D - z$:

$$\begin{aligned} &U_\mu^z(\Gamma(p_1, \dots, p_N)) \\ &\quad = \int \mu^{z\ell} d^{D-z} k_1 \dots d^{D-z} k_\ell I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N). \end{aligned} \tag{6}$$

The variable μ has the physical units of a mass and appears in these integrals for dimensional reasons. It will play an important role later on, as it sets the dependence on the energy scale of the renormalized values of the Feynman integrals, hence the renormalization group flow.

It is not an easy result to show that the dimensionally regularized integrals give meromorphic functions in the variable z, with a Laurent series expansion at $z = 0$. See a detailed discussion of this point in Chapter 1 of [30]. We will not enter in details here and talk loosely about (6) as a meromorphic function of z depending on the additional parameter μ.

We return to a discussion of a possible geometric meaning of the dimensional regularization procedure in the last section of this paper.

1.3. The Feynman rules. The integrand $I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N)$ in the Feynman integrals (3) is determined by the *Feynman rules* of the given quantum field theory, see [40], [12]. These can be summarized as follows:

- A Feynman graph Γ of a scalar quantum field theory with Lagrangian (1) has vertices of valences equal to the degrees of the monomials in the Lagrangian, internal edges connecting pairs of vertices, and external edges connecting to a single vertex.
- To each internal edge of a Feynman graph Γ one assigns a momentum variable $k_e \in \mathbb{R}^D$ and a propagator, which is a quadratic form q_e in the variable k_e, which (in Euclidean signature) is of the form

$$q_e(k_e) = k_e^2 + m^2. \tag{7}$$

- The integrand is obtained by taking a product over all internal edges of the inverse propagators

$$\frac{1}{q_1 \dots q_n}$$

and imposing a linear relation at each vertex, which expresses the conservation law

$$\sum_{e_i \in E(\Gamma): s(e_i)=v} k_i = 0$$

for the momenta flowing through that vertex. One obtains in this way the integrand

$$I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N) = \frac{\delta\big(\sum_{i \in E_{\text{int}}(\Gamma)} \epsilon_{v,i} k_i + \sum_{j \in E_{\text{ext}}(\Gamma)} \epsilon_{v,j} p_j\big)}{q_1(k_1) \dots q_n(k_n)}, \tag{8}$$

where $\epsilon_{e,v}$ denotes the incidence matrix of the graph

$$\epsilon_{e,v} = \begin{cases} +1, & t(e) = v, \\ -1, & s(e) = v, \\ 0, & \text{otherwise.} \end{cases}$$

- For each vertex of Γ one also multiplies the above by a constant factor involving the coupling constants of the terms in the Lagrangian of power corresponding to the valence of the vertex and by a power of (2π), which we omit for simplicity.

There are two properties of Feynman rules that it is useful to recall for comparison with algebro-geometric settings:

(1) Reduction from graphs to connected graphs: the Feynman rules are multiplicative over disjoint unions of graphs

$$U(\Gamma, p) = U(\Gamma_1, p_1)\, U(\Gamma_2, p_2) \quad \text{for } \Gamma = \Gamma_1 \amalg \Gamma_2. \tag{9}$$

(2) Reduction from connected graphs to 1PI graphs. An arbitrary connected finite graph can be written as a tree T where some of the vertices are replaced by 1PI graphs with a number of external edges matching the valence of the vertex, $\Gamma = \bigcup_{v \in V(T)} \Gamma_v$. For these graphs the Feynman rules satisfy

$$U(\Gamma) = \prod_{v \in V(T)} U(\Gamma_v) \prod_{e \in E_{\text{ext}}(\Gamma_v), e' \in E_{\text{ext}}(\Gamma_{v'}), e=e' \in E_{\text{int}}(\Gamma)} \frac{\delta(p_e - p_{e'})}{q_e(p_e)}. \tag{10}$$

These properties reduce the combinatorics of Feynman graphs to the 1PI case. Notice that in the particular case where $m \neq 0$ (massive theories) and the external momenta are set to zero, $p = 0$, the case (10) reduces to the simpler form

$$U(\Gamma) = U(L)^{\#E(T)} \prod_{v \in V(T)} U(\Gamma_v), \tag{11}$$

where $U(L)$ is the inverse propagator for a single edge, in this case just equal to the constant factor m^{-2}.

1.4. Parametric representation of Feynman integrals. The Feynman parameterization (also known as α-parameterization), see [12], [40], [53], reformulates the Feynman integrals (3) in such a way that they become manifestly (modulo divergences) written as the integral of an algebraic differential form on an algebraic variety, integrated over a cycle with boundary on a divisor in the variety, see [16].

One starts with the Feynman integral, written as above in the form

$$U(\Gamma) = \int \frac{\delta\big(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j\big)}{q_1 \dots q_n} \, d^D k_1 \dots d^D k_n$$

with $n = \#E_{\rm int}(\Gamma)$ and $N = \#E_{\rm ext}(\Gamma)$ and with $\epsilon_{e,v}$ the incidence matrix.

Then, one introduces the *Schwinger parameters*. These are variables $s_i \in \mathbb{R}_+$ defined by the identity

$$q_1^{-k_1} \dots q_n^{-k_n} = \frac{1}{\Gamma(k_1) \dots \Gamma(k_n)} \int_0^\infty \dots \int_0^\infty e^{-(s_1 q_1 + \dots + s_n q_n)} \, s_1^{k_1 - 1} \dots s_n^{k_n - 1} \, ds_1 \dots ds_n.$$

The *Feynman trick*, which consists of writing

$$\frac{1}{q_1 \dots q_n} = (n-1)! \int \frac{\delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \dots + t_n q_n)^n} \, dt_1 \dots dt_n,$$

is obtained from a particular case of the identity defining the Schwinger parameters, after a simple change of variables.

One then further introduces a change of variables $k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k$, where η_{ik} is the matrix

$$\eta_{ik} = \begin{cases} rl+1, & \text{edge } e_i \in \text{loop } l_k, \text{ same orientation}, \\ -1, & \text{edge } e_i \in \text{loop } l_k, \text{ reverse orientation}, \\ 0, & \text{otherwise}. \end{cases}$$

This depends on the choice of an orientation of the edges and of a basis of loops, *i.e.* a basis of $H_1(\Gamma)$. The equations imposing the conservation laws for momenta at each vertex, together with the constraint $\sum_i t_i u_i \eta_{ir} = 0$ determine uniquely u_i as functions of the external momenta p and give

$$\sum_i t_i u_i^2 = p^\dagger R_\Gamma(t) p,$$

where $R_\Gamma(t)$ is a function defined in terms of the combinatorics of the graph. Thus, one rewrites the Feynman integral after this change of coordinates in the form

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n-D\ell/2}}, \tag{12}$$

where ω_n is the volume form and the domain of integration is the simplex $\sigma_n = \{t \in \mathbb{R}_+^n | \sum_i t_i = 1\}$. In the *massless case* (with $m = 0$) the term $V_\Gamma(t,p) = p^\dagger R_\Gamma(t) p + m^2$ is of the form

$$V_\Gamma(t,p)|_{m=0} = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)},$$

where $P_\Gamma(t,p)$ is a homogeneous polynomial of degree $b_1(\Gamma) + 1$ in t, defined in terms of the cut-sets of the graph (complements of spanning tree plus one edge),

$$P_\Gamma(t,p) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e,$$

with $s_C = \left(\sum_{v \in V(\Gamma_1)} P_v\right)^2$ and $P_v = \sum_{e \in E_{\text{ext}}(\Gamma), t(e)=v} p_e$, where the momenta satisfy the conservation law $\sum_{e \in E_{\text{ext}}(\Gamma)} p_e = 0$. The *graph polynomial* $\Psi_\Gamma(t)$ is a homogeneous polynomial of degree $b_1(\Gamma)$ given by

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e \notin T} t_e,$$

with the sum over spanning trees of Γ, and the matrix

$$(M_\Gamma)_{kr}(t) = \sum_{i=0}^{n} t_i \eta_{ik} \eta_{ir}.$$

Notice how the determinant of this matrix is independent both of the choice of an orientation of the edges and of a basis of $H_1(\Gamma)$. Similarly, in the case where $m \neq 0$ but with external momenta $p = 0$ one has

$$V_\Gamma(t,p)|_{m \neq 0, p=0} = \frac{m^2}{\Psi_\Gamma(t)}.$$

After dimensional regularization the parametric Feynman integral can be rewritten as

$$U_\mu(\Gamma)(z) = \mu^{-z\ell} \frac{\Gamma(n - \frac{(D+z)\ell}{2})}{(4\pi)^{\frac{\ell(D+z)}{2}}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{\frac{(D+z)}{2}} V_\Gamma(t,p)^{n-\frac{(D+z)\ell}{2}}} .$$

Assume for simplicity that we work in the "stable range" of dimensions D such that $n \leq D\ell/2$, so that we write the integral $U(\Gamma, p)$, up to a divergent Γ-factor, in the form

$$\int_{\sigma_n} \frac{P_\Gamma(p,t)^{-n+D\ell/2}}{\Psi_\Gamma(t)^{-n+(\ell+1)D/2}} \omega_n. \tag{13}$$

The integrand is an algebraic differential form on the complement of the hypersurface

$$\hat{X}_\Gamma = \{t = (t_1, \dots, t_n) \in \mathbb{A}^n \mid \Psi_\Gamma(t) = 0\}. \tag{14}$$

Since the polynomial is homogeneous, one can also consider the projective hypersurface

$$X_\Gamma = \{t = (t_1 : \dots : t_n) \in \mathbb{P}^{n-1} \mid \Psi_\Gamma(t) = 0\}. \tag{15}$$

Moreover, the domain of integration is the simplex σ_n with boundary $\partial\sigma_n$ contained in the normal crossings divisor $\hat{\Sigma}_n = \left\{t \in \mathbb{A}^n \mid \prod_i t_i = 0\right\}$. Thus, as we discuss briefly below, if the integral converges, it defines a period of the hypersurface complement. The integral in general is still divergent, even if we have already removed a divergent Γ-factor (hence we are considering the residue of the Feynman graph $U(\Gamma)$). The divergences of (13) come from the intersections $\hat{\Sigma}_n \cap \hat{X}_\Gamma \neq \emptyset$. We discuss later how one can treat these divergences.

It is worth pointing out here that the varieties X_Γ are in general *singular* hypersurfaces, with a singularity locus that is often of low codimension. This can be seen easily by observing that the varieties defined by the derivatives of the graph polynomial are in turn cones over graph hypersurfaces of smaller graphs and that these cones do not intersect transversely. Techniques from singularity theory can be employed to estimate how singular these varieties typically are. Notice how, from the motivic viewpoint, the fact that they are highly singular is what makes it possible for many of these varieties (and possibly always for a certain part of their cohomology), to be sufficiently "simple" as motives, *i.e.* mixed Tate. This would certainly not be the case if we were dealing with smooth hypersurfaces. So the understanding of the singularities of these varieties may play a useful role in the conjectures on Feynman integrals and motives.

The parametric representation of Feynman integrals and its relation to the algebraic geometry of the graph hypersurfaces was generalized to theories with bosonic and fermionic fields in [51] where the analogous result is obtained in the form of an integration of a Berezinian on a supermanifold.

1.5. Algebraic varieties and motives. The other main objects involved in the conjecture on Feynman integrals and periods are *motives*. These are the focus of a deep chapter of arithmetic algebraic geometry, still in itself very much at the center of recent investigations in the field. Roughly speaking, motives are a universal cohomology theory for algebraic varieties, or, to say it differently, a way to embed the category of varieties into a better (triangulated, abelian, Tannakian) category.

Let $\mathcal{V}_{\mathbb{K}}$ denote the category of smooth projective algebraic varieties over a field $\mathbb{K}$. For our purposes, we may assume that $\mathbb{K}$ is $\mathbb{Q}$ or a number field. The category $\mathcal{M}_{\mathbb{K}}$ of pure motives (with the numerical equivalence relation on algebraic cycles) is defined as having objects given by triples (X, p, m) of a smooth projective variety X, a projector $p = p^2 \in \mathrm{End}(X)$, and an integer $m \in \mathbb{Z}$. The morphisms extend the usual notion of morphism of varieties, by allowing also *correspondences*, that is, algebraic cycles in the product $X \times Y$. A morphism in the usual sense is represented by the cycle given by its graph in $X \times Y$. More precisely, one has

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q\mathrm{Corr}^{m-n}_{/\sim}(X, Y)\, p,$$

for projectors $p^2 = p$, $q^2 = q$, and where $\mathrm{Corr}^{m-n}(X, Y)$ means the abelian group or vector space of cycles in $X \times Y$ of codimension equal to $\dim(X) - m + n$ and $\sim$ is the numerical equivalence relation on cycles (two cycles are the same if they have the same intersection numbers with any cycle of complementary dimension).

One defines the Tate motives $\mathbb{Q}(m)$ by formally setting $\mathbb{Q}(1) = \mathbb{L}^{-1}$, the inverse of the Lefschetz motive (the motive of an affine line) and $\mathbb{Q}(m) = \mathbb{Q}(1)^m$, with $\mathbb{Q}(0)$ the motive of a point, so that $(X, p, m) = (X, p) \otimes \mathbb{Q}(m)$. The reason for introducing these new objects in the category of motives is to allow for cycles of varying codimension: this makes it possible to have a duality $(X, p, m)^\vee = (X, p^t, -m)$ and a rigid tensor structure on the category $\mathcal{M}_{\mathbb{K}}$. It is known that, with the numerical equivalence on cycles, $\mathcal{M}_{\mathbb{K}}$ is an abelian category and it is in fact Tannakian. Since it is a semisimple category, its Tannakian Galois group (the motivic Galois group) is reductive. The subcategory generated by the $\mathbb{Q}(m)$ is the category of pure Tate motives, whose motivic Galois group is $\mathbb{G}_m$. (See [5], [41], [47].)

The situation becomes considerably more complicated when the varieties considered are not smooth projective, for instance, when one wants to include singular varieties, as is necessarily the case in relation to quantum field theory, since we have seen that the X_Γ are usually singular varieties. In this case, the theory of motives is not as well understood as in the pure case. Mixed motives, the theory of motives that accounts for these more general types of varieties, are known to form a triangulated category $\mathcal{DM}_{\mathbb{K}}$, by work of Voevodsky, Levine, Hanamura [45], [59]. Distinguished triangles in this triangulated category of motives correspond to long exact sequences in cohomology of the form

$$\mathfrak{m}(Y) \longrightarrow \mathfrak{m}(X) \longrightarrow \mathfrak{m}(X \smallsetminus Y) \longrightarrow \mathfrak{m}(Y)[1]$$

in the case of closed embeddings $Y \subset X$. Moreover, one has a homotopy invariance property expressed by the identity

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(1)[2].$$

However, in general one does not have an abelian category. The subcategory $\mathcal{DMT}_{\mathbb{K}} \subset \mathcal{DM}_{\mathbb{K}}$ of mixed Tate motives is the triangulated subcategory generated by the $\mathbb{Q}(m)$. In the case where $\mathbb{K}$ is a number field, it is known (see [45]) that one has a t-structure on $\mathcal{DMT}_{\mathbb{K}}$ whose heart defines an abelian category $\mathcal{MT}_{\mathbb{K}}$ of mixed Tate motives. It is in fact a Tannakian category (see [32]), whose Galois group is of the form $U \rtimes \mathbb{G}_m$, where the reductive part $\mathbb{G}_m$ accounts for the presence of the pure Tate motives among the mixed ones, while U is a pro-unipotent affine group scheme which accounts for the nontrivial extensions between pure Tate motives.

More concretely, examples of mixed Tate motives are given for instance by algebraic varieties that admit a stratification where all the strata are built out of locally trivial fibrations of affine spaces. We will discuss some explicit examples of this sort below, in the context of quantum field theory.

While explicitly constructing objects in $\mathcal{MT}_{\mathbb{K}}$ or checking whether given varieties that define objects in $\mathcal{DM}_{\mathbb{K}}$ are actually mixed Tate, *i.e.* whether they give objects in $\mathcal{DMT}_{\mathbb{K}}$ or $\mathcal{MT}_{\mathbb{K}}$, may in general be very difficult, there is an easier way to check the motivic nature of a variety X by looking at its class in the Grothendieck ring of varieties $K_0(\mathcal{V}_{\mathbb{K}})$. This is generated by isomorphism classes $[X]$, subject to the inclusion-exclusion relation $[X] = [Y] + [X \smallsetminus Y]$ for closed embeddings $Y \subset X$ and with the product given by $[X][Y] = [X \times Y]$.

The class in the Grothendieck ring can be thought of as a *universal Euler characteristic* for algebraic varieties, [11]. In fact, additive invariants of varieties, *i.e.* invariants with values in a commutative ring R which satisfy $\chi(X) = \chi(Y)$ if $X \cong Y$ are isomorphic varieties, $\chi(X) = \chi(Y) + \chi(X \smallsetminus Y)$, for closed embeddings $Y \subset X$, and are compatible with products, $\chi(X \times Y) = \chi(X)\chi(Y)$, correspond to ring homomorphisms $\chi\colon K_0(\mathcal{V}) \to R$. Examples of additive invariants are the usual Euler characteristic, or the motivic Euler characteristic of Gillet–Soulé [35], $\chi\colon K_0(\mathcal{V}_{\mathbb{K}}) \to K_0(\mathcal{M}_{\mathbb{K}})$ with values in the Grothendieck ring of the category of motives, defined on projective varieties by $\chi(X) = [(X, \mathrm{id}, 0)]$ and on more general varieties in terms of a complex in the category of complexes over $\mathcal{M}_{\mathbb{K}}$.

If one denotes by $\mathbb{L} = [\mathbb{A}^1] \in K_0(\mathcal{V}_{\mathbb{K}})$ the Lefschetz motive, then the part of $K_0(\mathcal{V}_{\mathbb{K}})$ generated by the Tate motives is a polynomial ring $\mathbb{Z}[\mathbb{L}]$ (or $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$ after formally inverting the Lefschetz motive in $K_0(\mathcal{M}_{\mathbb{K}})$). Checking that the class $[X]$ of a variety X lies in this subring gives strong evidence for X being a mixed Tate motive. It may seem that a lot of information is lost in passing from objects in $\mathcal{DM}_{\mathbb{K}}$ to classes in $K_0(\mathcal{V}_{\mathbb{K}})$, since this ring does not retain the information on the extensions but only keeps the rough information on scissor relations. However, at least modulo standard conjectures on motives, knowing that the class $[X]$ lies in the Tate subring

$\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$ of $K_0(\mathcal{M}_{\mathbb{K}})$ should in fact suffice to know that the motive is mixed Tate. In any case, computing in $K_0(\mathcal{V}_{\mathbb{K}})$ provides a lot of useful information on the motivic nature of given varieties.

One last thing that we need to recall briefly is the notion of period, as in [43]. A period is a complex number that can be obtained by pairing via integration

$$(\omega, \sigma) \longmapsto \int_\sigma \omega$$

an algebraic differential form $\omega \in \Omega^{\dim X}(X)$ on an algebraic variety X defined over a number field $\mathbb{K}$ with a cycle σ defined by semi-algebraic relations (equalities and inequalities) also defined over the same field $\mathbb{K}$. If the domain of integration σ has boundary $\partial\sigma \neq 0$, then the period should be thought of as a pairing with a relative homology group

$$\sigma \in H_{\dim X}(X(\mathbb{C}), \Sigma(\mathbb{C})),$$

where Σ is a divisor in X containing the boundary of σ. It is conjectured in [43] that the only relations between periods arise from the change of variable and Stokes formulae for integrals.

1.6. The mixed Tate mystery: supporting evidence. The main conjecture on the relation between quantum fields and motives can be formulated as follows.

Conjecture 1.1. *Are residues of Feynman integrals in scalar field theories always periods of mixed Tate motives?*

Here "residues" refers to the removal of the divergent Gamma factor in (12). Notice that, in general, the remaining integral still contains divergences that need to be removed by a renormalization procedure. Thus, implicit in the above conjecture is also an independence of the regularization and renormalization scheme used to eliminate divergences.

The supporting evidence for this conjecture starts from extensive numerical computations of Feynman integrals collected by Broadhurst and Kreimer [22], which showed the pervasive presence of zeta and multiple zeta values. This first suggested the fact that mixed Tate motives may be involved in this computation, in view of the fact that multiple zeta values are periods of mixed Tate motives, according to [36], [55].

Modulo the serious issue of divergences, the use of Schwinger and Feynman parameters expresses Feynman integrals as integrations of an algebraic differential form on the complement of a hypersurface X_Γ in affine space defined by a homogeneous polynomial depending on the combinatorics of the graph.

Kontsevich formulated the conjecture that the graph hypersurfaces X_Γ themselves may always be mixed Tate motives, which would imply Conjecture 1.1. Although numerically this conjecture was at first verified up to a large number of loops, Belkale and Brosnan [9] later disproved the conjecture in general, showing that in fact the X_Γ can be arbitrarily complicated as motives: they proved that the X_Γ generate the Grothendieck ring of varieties. This, however, does not disprove Conjecture 1.1. In fact, even though the varieties themselves may be more complicated as motives, the part of the cohomology that is involved in the computation of the period may still be a realization of a mixed Tate motive.

More evidence for the fact that the cohomology involved, that is the relative cohomology $H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma))$, where Σ_n denotes the union of the coordinate hyperplanes, is a realization of a mixed Tate motive was collected by Bloch–Esnault–Kreimer, [16], [13].

More recently, the question has been reformulated by Aluffi–Marcolli [4] in terms of a different relative cohomology involving determinant hypersurfaces and the motives of *varieties of frames*, which gives further evidence for the conjecture, as we explain below. A different kind of evidence comes from the approach followed in the work of Connes–Marcolli [27], where instead of constructing motives for specific Feynman graphs, one compares the "global" properties of the Tannakian category $\mathcal{MT}_{\mathbb{K}}$ with a similar category constructed out of the data of perturbative renormalization, the Tannakian category of *flat equisingular vector bundles*. Although one obtains in this way only a non-canonical identification between these Tannakian categories, it adds evidence to the conjectured relation between perturbative renormalization and mixed Tate motives.

We give in the following a general overview of these different methods and results.

2. A bottom-up approach to Feynman integrals and motives

With these preliminaries in place, we are now ready to discuss more closely the two different approaches to the relation of quantum field theory and motives. We first introduce what we refer to as a "bottom-up" approach, in the sense that it deals with the problem on a graph-by-graph basis and tries, for individual graphs or families of graphs sharing similar combinatorial properties, to construct explicit associated motives and periods computing the Feynman integrals. This approach was pioneered by the work of Bloch–Esnault–Kreimer [16] and further developed in [13], [17]. Here I will concentrate mostly on my recent joint work with Aluffi [2], [3], [4].

As we have mentioned above, the parametric formulation of Feynman integrals shows that, modulo divergences, they can be written as periods on the hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$, with $n = \#E_{\text{int}}(\Gamma)$. One can reformulate the integral in the projective setting. Then the question of whether the period so computed is a period of a mixed Tate motive can be reformulated as in [16] as the question of whether the relative cohomology

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n) \tag{16}$$

is the realization of a mixed Tate motive

$$\mathfrak{m}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n), \tag{17}$$

where $\Sigma_n = \{t \in \mathbb{P}^{n-1} \mid \prod_i t_i = 0\}$ is a normal crossings divisor containing $\partial\sigma_n$, the boundary of the domain of integration.

This leads to the question of how complex, in motivic terms, the graph hypersurfaces X_Γ can be. Clearly, if it were to be the case that these would always be mixed Tate as motives, then the conjecture on the nature of the period would follow easily. However, this is known not to be the case, as we already mentioned above: it is known by [9] that the classes $[X_\Gamma]$ generate the Grothendieck ring of varieties, hence they cannot all be contained in the Tate subring $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$. The question remains, however, on whether the particular piece (16) may nonetheless be always mixed Tate even when the variety X_Γ itself may turn out to be more complicated.

One can exhibit explicit examples of computations of classes $[X_\Gamma]$ in the Grothendieck ring. A useful method to obtain information on these classes is the observation, made in [13] and used extensively in [2], [15], that the classical Cremona transformation relates the graph hypersurfaces of a planar graph and its dual graph.

In fact, if Γ is a planar graph and $\Gamma^\vee$ denotes the dual graph in a chosen embedding of Γ, the graph polynomials are related by

$$\Psi_\Gamma(t_1, \dots, t_n) = \Big(\prod_e t_e\Big) \Psi_{\Gamma^\vee}(t_1^{-1}, \dots, t_n^{-1}).$$

This means that the graph hypersurfaces have the property that

$$\mathcal{C}(X_\Gamma \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n),$$

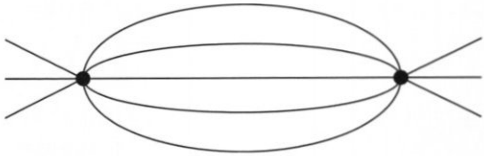
under the Cremona transformation. The latter is defined as

$$\mathcal{C} \colon (t_1 : \cdots : t_n) \longmapsto \Big(\frac{1}{t_1} : \cdots : \frac{1}{t_n}\Big),$$

which is well defined outside the singularity locus $\mathcal{S}_n$ of Σ_n defined by the ideal $I_{\mathcal{S}_n} = (t_1 \dots t_{n-1}, t_1 \dots t_{n-2} t_n, \dots, t_1 t_3 \dots t_n)$. Notice that this relation only gives an isomorphism of the parts of X_Γ and $X_{\Gamma^\vee}$ that lie outside of Σ_n.

For example, using this method, an explicit formula for the classes $[X_{\Gamma_n}]$ of the hypersurfaces of the infinite family of so called "banana graphs" were computed in [2]. The banana graphs have graph polynomial

$$\Psi_\Gamma(t) = t_1 \dots t_n \Big(\frac{1}{t_1} + \cdots + \frac{1}{t_n}\Big).$$



The parametric integral in this case is

$$\int_{\sigma_n} \frac{(t_1 \dots t_n)^{(\frac{D}{2}-1)(n-1)-1}\, \omega_n}{\Psi_\Gamma(t)^{(\frac{D}{2}-1)n}}.$$

One has in this case ([2]) that the class in the Grothendieck ring is of the form

$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L}-1)^{n-2},$$

so it is manifestly mixed Tate. In fact, in this case the dual graph $\Gamma^\vee$ is just a polygon, so that $X_{\Gamma^\vee} = \mathcal{L}$ is a hyperplane in $\mathbb{P}^{n-1}$. One has

$$[\mathcal{L} \smallsetminus \Sigma_n] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T}+1}$$

where $\mathbb{T} = [\mathbb{G}_m] = [\mathbb{A}^1] - [\mathbb{A}^0]$ is the class of the multiplicative group. Moreover, one finds that $X_{\Gamma_n} \cap \Sigma_n = \mathcal{S}_n$ and the scheme of singularities of Σ_n has class

$$[\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^{n-2}.$$

This then gives

$$[X_{\Gamma_n}] = [X_{\Gamma_n} \cap \Sigma_n] + [X_{\Gamma_n} \smallsetminus \Sigma_n],$$

where one uses the Cremona transformation to identify $[X_{\Gamma_n}] = [\mathcal{S}_n] + [\mathcal{L} \smallsetminus \Sigma_n]$.

In particular this calculation yields a value for the Euler characteristic of X_{Γ_n}, of the form $\chi(X_{\Gamma_n}) = n + (-1)^n$. A different computation of the Euler characteristic based on characteristic classes of singular varieties is also given in [2].

A very interesting observation recently made in [15] is that, although individually the varieties of Feynman graphs may not be mixed Tate, as the result of [9] shows, cancellations happen when one sums over graphs and one ends up with a class in $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V}_{\mathbb{K}})$. More precisely, it is shown in [15] that the class

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\mathrm{Aut}(\Gamma)}$$

is in $\mathbb{Z}[\mathbb{L}]$. This is in agreement with the fact that in quantum field theory individual Feynman graphs do not represent observable physical processes and only sums over graphs, usually with fixed external edges and external momenta, can be physically meaningful. This result suggests that a more appropriate formulation of the conjecture on Feyman integrals and motives may perhaps be given directly in terms that involve the full expansion of perturbative quantum field theory, with sums over graphs, rather than in terms of individual graphs. As we are going to see below, this also fits in naturally with the other, "top-down" approach to relating Feynman integrals to motives that we discuss in the second half of this paper.

2.1. Feynman rules in algebraic geometry. The graph hypersurfaces have another interesting property, namely the hypersurface complements behave like Feynman rules. This was first observed and described in detail in the work [3], but we summarize it here briefly.

As we recalled above, Feynman rules have certain multiplicative properties that makes it possible to reduce the combinatorics of graphs from arbitrary finite graphs to connected and then 1PI graphs, namely the properties listed in (9) and (11). When working in affine space, one has

$$\mathbb{A}^{n_1+n_2} \smallsetminus \hat{X}_\Gamma = (\mathbb{A}^{n_1} \smallsetminus \hat{X}_{\Gamma_1}) \times (\mathbb{A}^{n_2} \smallsetminus \hat{X}_{\Gamma_2}),$$

for a graph Γ that is a disjoint union $\Gamma = \Gamma_1 \amalg \Gamma_2$. This follows immediately from the fact that the graph polynomial factors as

$$\Psi_\Gamma(t_1, \dots, t_n) = \Psi_{\Gamma_1}(t_1, \dots, t_{n_1}) \Psi_{\Gamma_2}(t_{n_1+1}, \dots, t_{n_1+n_2}).$$

In projective space, this would no longer be the case and one has a more complicated relation in terms of *joins* instead of products of varieties, which gives a fibration

$$\mathbb{P}^{n_1+n_2-1} \smallsetminus X_\Gamma \longrightarrow (\mathbb{P}^{n_1-1} \smallsetminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \smallsetminus X_{\Gamma_2})$$

which is a $\mathbb{G}_m$-bundle (assuming that Γ_i not a forest, else the above map in projective spaces would not be well defined). Notice that the classes of the affine and the projective hypersurface complements are related by ([3])

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = (\mathbb{L} - 1)[\mathbb{P}^{n-1} \smallsetminus X_\Gamma],$$

when Γ is not a forest, since $[\hat{X}_\Gamma] = (\mathbb{L} - 1)[X_\Gamma] + 1$ is the class of the affine cone $\hat{X}_\Gamma$ over X_Γ.

One can then work either with the Grothendieck ring $K_0(\mathcal{V}_\mathbb{K})$ (in which case one can talk of *motivic Feynman rules*), or with a more refined version where one does not identify varieties up to isomorphisms but only up to linear coordinate changes coming from embeddings in some ambient affine space $\mathbb{A}^N$. This version of Grothendieck ring was introduced in [3] under the name of *ring of immersed conical varieties* $\mathcal{F}_\mathbb{K}$. It is generated by classes $[V]$ of equivalence under linear coordinate changes of varieties

$V \subset \mathbb{A}^N$ (for some arbitrarily large N) defined by homogeneous ideals (hence the name "conical"), with the usual inclusion-exclusion and product relations

$$[V \cup W] = [V] + [W] - [V \cap W],$$
$$[V] \cdot [W] = [V \times W].$$

By imposing equivalence under isomorphisms one falls back on the usual Grothendieck ring $K_0(\mathcal{V})$. The reason for working with $\mathcal{F}_{\mathbb{K}}$ instead is that it allowed us in [3] to construct invariants of the graph hypersurfaces that behave like algebro-geometric Feynman rules and that measure to some extent how singular these varieties are, and which do not factor through the Grothendieck ring, since they contain specific information on how the $\hat{X}_\Gamma$ are embedded in the ambient affine space $\mathbb{A}^{\# E_{\rm int}(\Gamma)}$.

In general, one defines an $\mathcal{R}$-valued algebro-geometric Feynman rule, for a given commutative ring $\mathcal{R}$, as in [3] in terms of a ring homomorphism $I : \mathcal{F} \to \mathcal{R}$ by setting

$$\mathbb{U}(\Gamma) := I([\mathbb{A}^n]) - I([\hat{X}_\Gamma])$$

and by taking as value of the *inverse propagator*

$$\mathbb{U}(L) = I([\mathbb{A}^1]).$$

This then satisfies both (9) and (11). The ring $\mathcal{F}$ then is the receptacle of the universal algebro-geometric Feynman rule given by

$$\mathbb{U}(\Gamma) = [\mathbb{A}^n \setminus \hat{X}_\Gamma] \in \mathcal{F}.$$

A Feynman rule defined in this way is *motivic* if the homomorphism $I : \mathcal{F} \to \mathcal{R}$ factors through the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$.

An example of algebro-geometric Feynman rule that does not factor through $K_0(\mathcal{V}_{\mathbb{K}})$ was constructed in [3] using the theory of characteristic classes of singular varieties.

In the case of smooth varieties, one knows that the Chern classes of the tangent bundle can be written as a class $c(V) = c(TV) \cap [V]$ in homology whose degree of the zero dimensional component satisfies the Poincaré–Hopf theorem $\int c(TV) \cap [V] = \chi(V)$, which gives the topological Euler characteristic of the smooth variety. This was generalized to singular varieties, following two different approaches that then turned out to be equivalent, by Marie-Hélène Schwartz [54] and Robert MacPherson [46]. The approach followed by Schwartz generalized the definition of Chern classes as the homology classes of the loci where a family of $k+1$-vector fields become linearly dependent (for the lowest degree case one reads the Poincaré–Hopf theorem as saying that the Euler characteristic measures where a single vector field has zeros). In the case of singular varieties a generalization is obtained, provided that one assigns some radial conditions on the vector fields with respect to a stratification with good properties. The approach of MacPherson was instead based on functoriality: a conjecture of Grothendieck–Deligne stated that there should be a unique natural transformation c_* between the functor $\mathbb{F}(V)$ of constructible functions on a variety V, whose objects are linear combinations of characteristic classes 1_W of subvarieties $W \subset V$ and where morphisms are defined by the prescription $f_*(1_W) = \chi(W \cap f^{-1}(p))$, with χ the Euler characteristic, to the homology (or Chow group) functor, which in the smooth case agrees with $c_*(1_V) = c(TV) \cap [V]$. MacPherson constructed this natural transformation in terms of data of Mather classes and local Euler obstructions. The results of Aluffi [1] show that, in fact, it is possible to compute these classes without having to use the original definition and the local data that are usually very difficult to compute. Most notably, the resulting characteristic classes (denoted $c_{\mathrm{CSM}}(X)$ for Chern–Schwartz–MacPherson) satisfy an inclusion–exclusion formula

$$c_{\mathrm{CSM}}(X) = c_{\mathrm{CSM}}(Y) + c_{\mathrm{CSM}}(X \smallsetminus Y),$$

but are not invariant under isomorphism, hence they are naturally defined on classes in $\mathcal{F}_{\mathbb{K}}$ but not on $K_0(\mathcal{V}_{\mathbb{K}})$. This classes give a good information on the singularities of a variety: for example, in the case of hypersurfaces with isolated singularities, they can be expressed in terms of Milnor numbers, while more generally for non-isolated singularities, as observed by Aluffi, they can be expressed in terms of Euler characteristics of varieties obtained by repeatedly taking hyperplane sections.

To construct a Feynman rule out of these Chern classes, one uses the following procedure. Given a variety $\widehat{X} \subset \mathbb{A}^N$, one can view it as a locally closed locus in $\mathbb{P}^N$, hence one can apply to its characteristic function $1_{\widehat{X}}$ the natural transformation c_* that gives an element in the Chow group $A(\mathbb{P}^N)$ or in the homology $H_*(\mathbb{P}^N)$. This gives as a result a class of the form

$$c_*(1_{\widehat{X}}) = a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + \cdots + a_N[\mathbb{P}^N].$$

One then defines an associated polynomial given by ([3])

$$G_{\widehat{X}}(T) := a_0 + a_1 T + \cdots + a_N T^N.$$

It is in fact independent of N as it stops in degree equal to $\dim \widehat{X}$. It is by construction invariant under linear changes of coordinates. It also satisfies an inclusion-exclusion property coming from the fact that the classes c_{CSM} satisfy inclusion-exclusion, namely

$$G_{\widehat{X} \cup \widehat{Y}}(T) = G_{\widehat{X}}(T) + G_{\widehat{Y}}(T) - G_{\widehat{X} \cap \widehat{Y}}(T).$$

It is a more delicate result to show that it is multiplicative,

$$G_{\widehat{X} \times \widehat{Y}}(T) = G_{\widehat{X}}(T) \cdot G_{\widehat{Y}}(T).$$

The proof of this fact is obtained in [3] using an explicit formula for the CSM classes of joins in projective spaces, where the join $J(X,Y) \subset \mathbb{P}^{m+n-1}$ of two $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ is defined as the set of

$$(sx_1 : \cdots : sx_m : ty_1 : \cdots : ty_n), \quad \text{with } (s:t) \in \mathbb{P}^1,$$

and is related to product in affine spaces by the property that the product $\hat{X} \times \hat{Y}$ of the affine cones over X and Y is the affine cone over $J(X,Y)$. The resulting multiplicative property of the polynomials $G_{\hat{X}}(T)$ shows that one has a ring homomorphism $I_{\mathrm{CSM}} \colon \mathcal{F} \to \mathbb{Z}[T]$ defined by

$$I_{\mathrm{CSM}}([\hat{X}]) = G_{\hat{X}}(T)$$

and an associated Feynman rule

$$\mathbb{U}_{\mathrm{CSM}}(\Gamma) = C_\Gamma(T) = I_{\mathrm{CSM}}([\mathbb{A}^n]) - I_{\mathrm{CSM}}([\hat{X}_\Gamma]).$$

This is not motivic, *i.e.* it does not factor through the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$, as can be seen by the example given in [3] of two graphs (see the figure below) that have different $\mathbb{U}_{\mathrm{CSM}}(\Gamma)$,

$$C_{\Gamma_1}(T) = T(T+1)^2 \quad C_{\Gamma_2}(T) = T(T^2+T+1)$$

but the same hypersurface complement class in the Grothendieck ring,

$$[\mathbb{A}^n \smallsetminus \hat{X}_{\Gamma_i}] = [\mathbb{A}^3] - [\mathbb{A}^2] \in K_0(\mathcal{V}).$$

2.2. Determinant hypersurfaces and manifolds of frames. As our excursion into the algebraic geometry of graph hypersurfaces up to this point shows, it seems very difficult to control the complexity of the motive

$$\mathfrak{m}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n)$$

that governs the computation of the parametric Feynman integral as a period.

One way to try to estimate whether the period remains mixed Tate, as the complexity of the X_Γ grows, is to use the properties of periods, in particular the change of variable formula, which allows one to recast the computation of the same integral $\int_\sigma \omega$ associated to the data (X, D, ω, σ) of a variety X, a divisor D, a differential form ω on X, and an integration domain σ with boundary $\partial\sigma \subset D$, by mapping it via a morphism f of varieties to another set of data $(X', D', \omega', \sigma')$, with the same resulting period whenever $\omega = f^*(\omega')$ and $\sigma' = f_*(\sigma)$. In other words, we try to map the variety X_Γ inside a larger ambient variety in such a way that the part of the cohomology that is involved in the period computation will not disappear, but the motivic complexity of the new ambient space will be easier to control. This is the strategy that we followed in [4], which I will briefly describe here.

The matrix $M_\Gamma(t)$ associated to a Feynman graph Γ determines a linear map of affine spaces

$$\Upsilon\colon \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik} \eta_{ir}$$

such that the affine graph hypersurface is obtained as the preimage

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

under this map of the determinant hypersurface

$$\hat{\mathcal{D}}_\ell = \{x = (x_{ij}) \in \mathbb{A}^{\ell^2} \mid \det(x_{ij}) = 0\}.$$

The advantage of moving the period computation via the map $\Upsilon = \Upsilon_\Gamma$ from the hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$ to the complement of the determinant hypersurface $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is that, unlike what happens with the graph hypersurfaces, it is well known that the determinant hypersurface $\hat{\mathcal{D}}_\ell$ is a mixed Tate motive.

One can give explicit combinatorial conditions on the graph that ensure that the map Υ is an embedding. As shown in [4], for any 3-edge-connected graph with at least 3 vertices and no looping edges, which admit a closed 2-cell embedding of face width at least 3, the map Υ is injective. These combinatorial conditions are natural from a physical viewpoint. In fact, 2-edge-connected is just the usual 1PI condition, while 3-edge-connected or 2PI is the next strengthening of this condition (the 2PI effective action is often considered in quantum field theory), and the face width condition is also the next strengthening of face width 2, which a well known combinatorial conjecture on graphs [52] expects should simply follow for graphs that are 2-vertex-connected. (The latter condition is a bit more than 1PI: for graphs with at least two vertices and no looping edges it is equivalent to all the splittings of the graph at vertices also being 1PI.) The conditions that the graph has no looping edges is only a technical device for the proof. In fact, it is then easy to show (see [4]) that adding looping edge does not affect the injectivity of the map Υ.

One can then rewrite the Feynman integral (as usual up to a divergent Γ-factor) in the form

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}},$$

for a polynomial $\mathcal{P}_\Gamma(x,p)$ on $\mathbb{A}^{\ell^2}$ that restricts to $P_\Gamma(t,p)$, and with $\omega_\Gamma(x)$ the image of the volume form. Let then $\hat{\Sigma}_\Gamma$ be a normal crossings divisor in $\mathbb{A}^{\ell^2}$, which contains the boundary of the domain of integration, $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$. The question on the motivic nature of the resulting period can then be reformulated (again modulo divergences) in this case as the question of whether the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)) \tag{18}$$

is mixed Tate. One sees immediately that, in this reformulation of the question, the difficulty has been moved from understanding the motivic nature of the hypersurface complement to having some control on the other term of the relative cohomology, namely the normal crossings divisor $\hat{\Sigma}_\Gamma$ and the way it intersects the determinant hypersurface. One would like to have an argument showing that the motive of $\hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)$ is always mixed Tate. In that case, knowing that $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is always mixed Tate, the fact that mixed Tate motives form a triangulated subcategory of the triangulated category of mixed motives would show that the motive (18) whose realization is the relative cohomology would also be mixed Tate. A first observation in [4] is that one can use the same normal crossings divisor $\hat{\Sigma}_{\ell,g}$ for all graphs Γ with a fixed number of loops and a fixed genus (that is, the minimal genus of an orientable surface in which the graph can be embedded). This divisor is given by a union of linear spaces

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}$$

defined by a set of equations

$$\begin{cases} x_{ij} = 0, & 1 \le i < j \le f-1, \\ x_{i1} + \cdots + x_{i,f-1} = 0, & 1 \le i \le f-1, \end{cases}$$

where $f = \ell - 2g + 1$ is the number of faces of an embedding of the graph Γ on a surface of genus g. A second observation of [4] is then that, using inclusion-exclusion, it suffices to show that arbitrary intersections of the components L_i of $\hat{\Sigma}_{\ell,g}$ have the property that $\left(\bigcap_{i\in I} L_i\right) \smallsetminus \hat{\mathcal{D}}_\ell$ is mixed Tate. A sufficient condition is given in [4] in terms of *manifolds of frames*. These are defined as

$$\mathbb{F}(V_1,\ldots,V_\ell) := \{(v_1,\ldots,v_\ell) \in \mathbb{A}^{\ell^2} \mid v_k \in V_k\}$$

for an assigned collection of linear subspaces V_i of a given vector space $V = \mathbb{A}^{\ell^2}$. If the manifolds of frames are mixed Tate motives for arbitrary choices of the subspaces, then the desired result would follow. One can check explicitly the cases of two and three subspaces, for which one has explicit formulae for the classes $[\mathbb{F}(V_1,\ldots,V_\ell)]$ in the Grothendieck ring:

$$[\mathbb{F}(V_1,V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L},$$

with $d_i = \dim(V_i)$ and $d_{ij} = \dim(V_i \cap V_j)$, and

$$\begin{aligned}[\mathbb{F}(V_1, V_2, V_3)] = {} & (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1) - (\mathbb{L} - 1)((\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) \\ & + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1) \\ & + (\mathbb{L} - 1)^2(\mathbb{L}^{d_1+d_2+d_3-D} - \mathbb{L}^{d_{123}+1}) + (\mathbb{L} - 1)^3\end{aligned}$$

which also depends on $d_{ijk} = \dim(V_i \cap V_j \cap V_k)$ and $D = D_{ijk} = \dim(V_i + V_j + V_k)$. However, it is difficult to establish an induction argument that would take care of the cases of more subspaces, and the combinatorics of the possible subspace arrangements quickly becomes difficult to control.

A reformulation of this problem given in [4] in terms of intersections of unions of Schubert cells in flag varieties suggests a possible connection to Kazhdan–Lusztig theory [42].

2.3. Handling divergences. So far we did not discuss how one takes care of the divergences caused by the intersections of the graph hypersurface X_Γ with the domain of integration σ_n. The poles of the integrand that fall inside the integration domain happen necessarily along the boundary $\partial\sigma_n$, as in the interior the graph polynomial Ψ_Γ takes strictly positive real values. Thus, one needs to modify the integrals suitably in such a way as to eliminate, by a regularization procedure, the intersections $X_\Gamma \cap \partial\sigma_n$, or (to work in algebro-geometric terms) the intersections $X_\Gamma \cap \Sigma_n$ which contains the former. There are different possible ways to achieve such a regularization procedure. We mention here three possible approaches.

One method was developed by Belkale and Brosnan in [10] in the logarithmically divergent case where $n = D\ell/2$, that is, when the polynomial $P_\Gamma(t, p)$ is not present and only the denominator $\Psi_\Gamma(t)^{D/2}$ appears in the parametric Feynman integral. Using dimensional regularization, one can, in this case, rewrite the Feynman integral in the form of a local Igusa L-function

$$I(s) = \int_\sigma f(t)^s \omega,$$

for $f = \Psi_\Gamma$. They prove that this L-function has a Laurent series expansion where all the coefficients are periods. In this setting, the issue of eliminating divergences becomes similar to the techniques used, for instance, in the context of log canonical thresholds. The result was more recently extended to the non-log-divergent case by Bogner and Weinzierl [18], [19].

Another method, used in [16], consists of eliminating the divergences by separating Σ_n and X_Γ performing a series of blowups. This method based on iterated blowups was investigated in great detail in [17]. Yet another method was proposed in [49], based on deformations instead of resolutions. By considering the graph hypersurface X_Γ as the special fiber X_0 of a family X_s of varieties defined by the level sets

$f^{-1}(s)$, for $f = \Psi_\Gamma : \mathbb{A}^n \to \mathbb{A}^1$, one can form a tubular neighborhood

$$D_\epsilon(X) = \bigcup_{s \in \Delta_\epsilon^*} X_s,$$

for Δ_ϵ^* a punctured disk of radius ϵ, and a circle bundle $\pi_\epsilon : \partial D_\epsilon(X) \to X_\epsilon$. One can then regularize the Feynman integral by integrating "around the singularities" in the fiber $\pi_\epsilon^{-1}(\sigma \cap X_\epsilon)$. The regularized integral has a Laurent series expansion in the parameter ϵ.

In general, as we discuss at length below, a regularization procedure for Feynman integrals replaces a divergent integral with a function of some regularization parameters (such as the complexified dimension of DimReg, or the deformation parameter ϵ in the example here above) in which the resulting function has a Laurent series expansion around the pole that corresponds to the divergent integral originally considered. One then uses a procedure of extraction of finite values to eliminate the polar parts of these Laurent series in a way that is *consistent over graphs*, that is, a renormalization procedure. We therefore turn now to recalling how renormalization can be formulated geometrically, using the results of Connes–Kreimer, as this will be the step relating the "bottom-up" approach to Feynman integrals and motives discussed so far, to the top down approach developed in [27], [28], [29], [30].

3. The Connes–Kreimer theory

We give here a very brief overview of the main results of the Connes–Kreimer theory, as they form the basis upon which the "top-down" approach to understanding the relation between quantum field theory and motives rests. As we see more in detail in the next section, in this context "top-down" means that the relation between quantum fields and motives will appear in this second approach from the comparison of the formal properties of associated abstract categorical structures rather than from a direct comparison of individual objects, as in the approach we have described in the previous sections.

3.1. The BPHZ renormalization procedure. The main steps of what is known in the physics literature as the Bogolyubov–Parashchuk–Hepp–Zimmermann procedure (BPHZ) are summarized as follows. (For more details the reader is invited to look at Chapter 1 on [30]).

Step 1: *Preparation.* One replaces the Feynman integral $U(\Gamma)$ of (6) by the expression

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma). \tag{19}$$

Here we suppress the dependence on z, μ and the external momenta p for simplicity of notation. The expression (19) is to be understood as a sum of Laurent series in z, depending on the extra parameter μ. The sum is over the set $\mathcal{V}(\Gamma)$ all proper subgraphs $\gamma \subset \Gamma$ with the property that the quotient graph Γ/γ, where each component of γ is shrunk to a vertex, is still a Feynman graph of the theory. The main result of BPHZ is that the coefficient of pole of $\bar{R}(\Gamma)$ is local.

Step 2: *Counterterms*. These are the expressions by which the Lagrangian needs to be modified to cancel the divergence produced by the graph Γ. They are defined as the polar part of the Laurent series $\bar{R}(\Gamma)$,

$$C(\Gamma) = -T(\bar{R}(\Gamma)).$$

Here T denotes the operator of projection onto the polar part of a Laurent series.

Step 3: *Renormalized value*. One then extracts a finite value from the integral $U(\Gamma)$ by removing the polar part, not of $U(\Gamma)$ itself but of its preparation:

$$\begin{aligned} R(\Gamma) &= \bar{R}(\Gamma) + C(\Gamma) \\ &= U(\Gamma) + C(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma). \end{aligned}$$

A very nice conceptual understanding of the BPHZ renormalization procedure with the DimReg + MS regularization was obtained by Connes and Kreimer [25], [26], based on a reformulation of the BPHZ procedure in geometric terms.

3.2. Renormalization, Hopf algebras, Birkhoff factorization. The first step in the geometric theory of renormalization is the understanding that the combinatorics of Feynman graphs of a given theory is governed by an algebraic structure, which accounts for the bookkeeping of the hierarchy of subdivergences that occur in multi-loop Feynman integrals. The right mathematical structure that describes their interactions is a Hopf algebra. This was first formulated by Kreimer [44] as a Hopf algebra of rooted trees decorated by Feynman diagrams, and then by Connes–Kreimer [25], [26] more directly in the form of a Hopf algebra of Feynman diagrams.

The Connes–Kreimer Hopf algebra ([25]) $\mathcal{H} = \mathcal{H}(\mathcal{T})$ depends on the choice of the physical theory, in the sense that it involves only graphs that are Feynman graphs for the specified Lagrangian $\mathcal{L}(\phi)$. As an algebra it is the free commutative algebra with generators the 1PI Feynman graphs Γ of the theory. It is graded, by loop number, or by the number of internal lines,

$$\deg(\Gamma_1 \dots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0.$$

This grading corresponds to the order in the perturbative expansion.

The coproduct already reveals a close relation to the BPHZ formulae. It is given on generators by

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma,$$

where the sum is over proper subgraphs $\gamma \subset \Gamma$ in a specific class $\mathcal{V}(\Gamma)$ determined by the property that the quotient graph Γ/γ is still a 1PI Feynman graph of the theory and that γ itself is a disjoint union of 1PI Feynman graphs of the theory. Unlike Γ which is assumed connected, the subgraphs γ can have multiple connected components, in which case the quotient graph Γ/γ is the one obtained by shrinking each component to a single vertex.

The antipode is defined inductively by

$$S(X) = -X - \sum S(X')X'',$$

where X is an element with coproduct $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$, where all the X' and X'' have lower degrees.

We only recalled how the Connes–Kreimer Hopf algebra is constructed for scalar field theories. Recently, van Suijlekom showed [56], [57], [58] how to extend it to gauge theories, incorporating Ward identities as Hopf ideals.

A commutative Hopf algebra $\mathcal{H}$ is dual to an *affine group scheme* G, defined by algebra homomorphisms

$$G(A) = \mathrm{Hom}(\mathcal{H}, A),$$

for any commutative unital algebra A. In the case of the Connes–Kreimer Hopf algebra this G is called the group of *diffeographisms* of the physical theory $\mathcal{T}$ and it was proved in [25] that it acts by local diffeomorphisms on the coupling constants of the theory.

The complex Lie group $G(\mathbb{C})$ of complex points of the affine group scheme G, defined as $G(\mathbb{C}) = \mathrm{Hom}(\mathcal{H}, \mathbb{C})$, is a pro-unipotent Lie group. For such groups, which are dual to graded connected Hopf algebras that are finite dimensional in each degree, Connes and Kreimer proved by a recursive formula that it is always possible to have a multiplicative Birkhoff factorization

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z)$$

of loops $\gamma\colon \Delta^* \to G$, defined on an infinitesimal disk Δ^* around the origin in $\mathbb{C}^*$, in terms of two holomorphic functions $\gamma_\pm(z)$ respectively defined on Δ and on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0\}$. The factorization is unique upon fixing a normalization condition $\gamma_-(\infty) = 1$. Notice that such Birkhoff factorizations do not always exist for other kinds of complex Lie groups, as one can see in the example of $\mathrm{GL}_n(\mathbb{C})$ where the existence of holomorphic vector bundles on the Riemann sphere is an obstruction.

In Hopf algebra terms, one can describe a loop $\gamma\colon \Delta^* \to G(\mathbb{C})$ on an infinitesimal punctured disk Δ^* as an algebra homomorphism $\phi \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}(\{z\}))$ with values in the field of germs of meromorphic functions (covergent Laurent series). The two terms γ_+ and γ_- of the Birkhoff factorization are, respectively, algebra homomorphisms $\phi_+ \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}\{z\})$ to convergent power series, and $\phi_- \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}[z^{-1}])$. The BPHZ recursive formula is then reformulated in [25] [26] as the Birkhoff factorization applied to the loop $\phi(\Gamma) = U(\Gamma)$ given by the dimensionally regularized unrenormalized Feynman integrals. In fact, the recursive formula of Connes and Kreimer for the Birkhoff factorization can be written as

$$\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X'')),$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$, and with T the projection onto the polar part of the Laurent series, and $\phi_+(X) = \phi(X) + \phi_-(X)$. The fact that the $\phi_\pm$ obtained in this way are still algebra homomorphism depends on the fact that the projection onto the polar part of Laurent series is a Rota–Baxter operator. In fact, this renormalization procedure by Birkhoff factorization was easily generalized in [33], [34] to arbitrary algebra homomorphisms $\phi \in \mathrm{Hom}(\mathcal{H}, \mathcal{A})$ from a commutative graded connected Hopf algebra to a Rota–Baxter algebra. When one applies this formula to $\phi(\Gamma) = U(\Gamma)$ one finds the BPHZ formula with $\phi_-(\Gamma) = C(\Gamma)$ the counterterms and $\phi_+(\Gamma)|_{z=0} = R(\Gamma)$ the renormalized values.

Notice how, from this point of view, the algebro-geometric Feynman rules discussed above, correspond to the data of a Hopf algebra homomorphism $\phi \in \mathrm{Hom}(\mathcal{H}, \mathcal{F}_{\mathbb{K}})$ or, in the motivic case, $\phi \in \mathrm{Hom}(\mathcal{H}, K_0(\mathcal{V}_{\mathbb{K}}))$, together with the assignment of the propagator $\mathbb{U}(L) = \mathbb{L}$. It would therefore be interesting to know if the rings $\mathcal{F}_{\mathbb{K}}$ and $K_0(\mathcal{V}_{\mathbb{K}})$ have a non-trivial Rota–Baxter structure.

4. A top-down approach via Galois theory

As we mentioned earlier, the "top-down" approach to the question of Feynman integrals and periods of mixed Tate motives consists of comparing categorical structures, instead of looking at varieties and motives associated to individual Feynman graphs. The main idea, developed in my joint work with Connes in [27], [28], [29], [30], is to show that the data of perturbative renormalization can be reformulated in terms of a Tannakian category of equivalence classes of differential systems with irregular singularities.

A neutral Tannakian category $\mathcal{C}$ is an abelian category, which is k-linear for some field k, has a rigid tensor structure and a fiber functor $\omega\colon \mathcal{C} \to \mathrm{Vect}_k$, which is a faithful exact tensor functor to the category of vector spaces over the same field k.

Tannakian categories are extremely rigid structures, namely, such a category is equivalent to a category of finite dimensional linear representations of an affine group scheme,

$$\mathcal{C} \simeq \mathrm{Rep}_G.$$

The affine group scheme G is reconstructed from the category as the invertible natural transformations of the fiber functor.

Thus, in order to relate two sets of objects of a seemingly very different nature, of which one is known (as is the case for mixed Tate motives over a number field) to form a Tannakian category, it suffices to show that the other set of objects can also be organized in a similar way, and check that the resulting affine group schemes are isomorphic: this gives then an equivalence of categories. This is precisely what is done in the results of [27].

The reason why this does not yet give an answer to the conjecture lies in the fact that one only obtains in this way a non-canonical identification, which cannot therefore be used to explicitly match Feynman integrals to mixed Tate motives. There are other mysterious aspects, for instance the category of mixed Tate motives involved in the result of [27] is not over $\mathbb{Q}$ or $\mathbb{Z}$, but over the ring $\mathbb{Z}[i][1/2]$, while all the varieties X_Γ involved in the parametric formulation of Feynman integrals are defined over $\mathbb{Z}$. Relating explicitly the top-down approach described below to the bottom-up approach is still an important missing ingredient in the geometric theory of renormalization, which may possibly provide the key to completing a proof of the main conjecture.

The main results of [27], [28], [29] are summarized as follows.

- *Step* 1: *Counterterms as iterated integrals.* One writes the negative piece $\gamma_-(z)$ of the Birkhoff factorization as an iterated integral depending on a single element β in the Lie algebra Lie(G) of the affine group scheme dual to the Connes–Kreimer Hopf algebra. This is a way of formulating what is known in physics as the 't Hooft–Gross relations [38], that is, the fact that counterterms only depend on the beta function of the theory (the infinitesimal generator of the renormalization group flow).
- *Step* 2: *From iterated integrals to solutions of irregular singular differential equations.* The iterated integrals obtained in the first step are uniquely solutions to certain differential equations. This makes it possible to classify the divergences of quantum field theories in terms of families of differential systems with singularities. The fact that, by dimensional analysis, counterterms are independent of the energy scale corresponds in these geometric terms to the flat singular connections describing the differential systems satisfying a certain *equisingularity* condition.
- *Step* 3: *Equisingular vector bundles.* Instead of working with equisingular connections in the context of principal G-bundles, one can formulate things equivalently in terms of linear representations and of flat connections on vector bundles. These data can then be organized in a neutral Tannakian category $\mathcal{E}$ which is independent of G and therefore universal for all physical theories.

- *Step* 4: *The Galois group.* The Tannakian category of flat equisingular connections is equivalent to a category of representations $\mathcal{E} \simeq \mathrm{Rep}_{\mathbb{U}^*}$ of an affine groups scheme $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$, where $\mathbb{U}$ is the prounipotent affine group scheme dual to the Hopf algebra $\mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^\vee$, where $\mathcal{L} = \mathcal{F}(e_{-n}; n \in \mathbb{N})$ is the free graded Lie algebra with one generator in each degree.
- *Step* 5: *Motivic Galois group.* The same group $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$ is known to arise (up to a non-canonical identification) as the motivic Galois group of the category of mixed Tate motives over the scheme $S = \mathrm{Spec}(\mathbb{Z}[i][1/2])$, by a result of Deligne–Goncharov [32].

We describe briefly each of these steps below.

4.1. Counterterms as iterated integrals. In the Birkhoff factorization, there is in fact a dependence on a mass scale μ, inherited from the same dependence of the dimensionally regularized Feynman integrals $U_\mu(\Gamma)$, so that we have

$$\gamma_\mu(z) = \gamma_-(z)^{-1}\gamma_{\mu,+}(z),$$

where one knows by reasons of dimensional analysis that the negative part is independent of μ. This part is written as a time ordered exponential

$$\gamma_-(z) = Te^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)dt} = 1 + \sum_{n=1}^{\infty} \frac{d_n(\beta)}{z^n},$$

where

$$d_n(\beta) = \int_{s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta) \dots \theta_{-s_n}(\beta) ds_1 \dots ds_n,$$

and where $\beta \in \mathrm{Lie}(G)$ is the beta function, that is, the infinitesimal generator of renormalization group flow, and the action θ_t is induced by the grading of the Hopf algebra by

$$\theta_u(X) = u^n X, \ \text{for } u \in \mathbb{G}_m, \quad \text{and} \quad X \in \mathcal{H}, \ \text{with } \deg(X) = n,$$

with generator the grading operator $Y(X) = nX$. This result follows from the analysis of the renormalization group in the Connes–Kreimer theory given in [25] [26], with the recursive formula for the coefficients d_n explicitly solved to give the time ordered exponential above.

The loop $\gamma_\mu(z)$ that collects all the unrenormalized values $U_\mu(\Gamma)$ of the Feynman integrals satisfies the scaling property

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)) \tag{20}$$

in addition to the property that its negative part is independent of μ,

$$\frac{\partial}{\partial \mu}\gamma_-(z) = 0. \tag{21}$$

The Birkhoff factorization is then written in [27] in terms of iterated integrals as

$$\gamma_{\mu,+}(z) = Te^{-\frac{1}{z}\int_0^{-z\log\mu}\theta_{-t}(\beta)dt}\ \theta_{z\log\mu}(\gamma_{\rm reg}(z)).$$

Thus $\gamma_\mu(z)$ is specified by β up to an equivalence given by the regular term $\gamma_{\rm reg}(z)$. The equivalence corresponds to "having the same negative part of the Birkhoff factorization".

4.2. From iterated integrals to differential systems. The second step of the argument of [27] goes as follows. An iterated integral (or time-ordered exponential) $g(b) = Te^{\int_a^b \alpha(t)dt}$ is the unique solution of a differential equation $dg(t) = g(t)\alpha(t)dt$ with initial condition $g(a) = 1$. In particular, given the differential field $(K = \mathbb{C}(\{z\}), \delta)$ and an affine group scheme G, and the logarithmic derivative

$$G(K) \ni f \mapsto D(f) = f^{-1}\delta(f) \in \mathrm{Lie}\, G(K),$$

one can consider differential equations of the form $D(f) = \omega$, for a flat Lie $G(\mathbb{C})$-valued connection ω, singular at $z = 0 \in \Delta^*$. The existence of solutions is ensured by the condition of trivial monodromy on Δ^*

$$M(\omega)(\ell) = Te^{\int_0^1 \ell^*\omega} = 1, \quad \ell \in \pi_1(\Delta^*).$$

These differential systems can be considered up to the gauge equivalence relation of $D(fh) = Dh + h^{-1}Df\, h$, for a regular $h \in \mathbb{C}\{z\}$. The gauge equivalence is the same thing as the requirement considered above that the solutions have the same negative piece of the Birkhoff factorization,

$$\omega' = Dh + h^{-1}\omega h \iff f_-^{\omega} = f_-^{\omega'},$$

where $D(f^\omega) = \omega$ and $D(f^{\omega'}) = \omega'$.

4.3. Flat equisingular connections. The third step of [27] consists of reformulating the data of the loops $\gamma_\mu(z)$ up to the equivalence of having the same negative piece of the Birkhoff factorization in terms of gauge equivalence classes of differential systems as above. The point here is that one keeps track of the μ-dependence and of the way $\gamma_\mu(z)$ scales with μ and the fact that the negative part of the Birkhoff factorization is independent of μ, as in (20), (21). In geometric terms these conditions are reformulated in [27] as properties of connections on a principal G-bundle $P =$

$B \times G$ over a fibration $\mathbb{G}_b \to B \to \Delta$, where $z \in \Delta$ is the complexified dimension of DimReg and the fiber $\mu^z \in \mathbb{G}_m$ over z corresponds to the changing mass scale. The multiplicative group acts by

$$u(b, g) = (u(b), u^Y(g)) \quad \text{for all } u \in \mathbb{G}_m.$$

The two conditions (20) and (21) correspond to the properties that the flat connection ϖ on P^* is *equisingular*, that is, it satisfies the following:

- Under the action of $u \in \mathbb{G}_m$ the connection transforms like
$$\varpi(z, u(v)) = u^Y(\varpi(z, u)).$$

- If γ is a solution in $G(\mathbb{C}(\{z\}))$ of the equation $D\gamma = \varpi$, then the restrictions along different sections σ_1, σ_2 of B with $\sigma_1(0) = \sigma_2(0)$ have "the same type of singularities", namely
$$\sigma_1^*(\gamma) \sim \sigma_2^*(\gamma),$$
where $f_1 \sim f_2$ means that $f_1^{-1} f_2 \in G(\mathbb{C}\{z\})$, regular at zero.

4.4. Flat equisingular vector bundles. The fourth step of [27] consists of transforming the information obtained above from equivalence classes of flat equisingular connections on the principal G-bundle P to a category $\mathcal{E}$ of flat equisingular vector bundles. This is possible without losing any amount of information, since the affine group scheme G dual to the Connes–Kreimer Hopf algebra of Feynman graphs of a given physical theory is completely determined by its category Rep_G of finite dimensional linear representations. Thus, considering all possible flat equisingular vector bundles gives rise to a category that in particular contains as a subcategory the vector bundles that come from finite dimensional representations of G, for any G associated to a particular physical theory, while in itself the category $\mathcal{E}$ does not depend on any particular G, so it is therefore universal for different physical theories.

The category $\mathcal{E}$ of flat equisingular vector bundles is defined in [27] as follows.

The objects $\mathrm{Obj}(\mathcal{E})$ are pairs $\Theta = (V, [\nabla])$, where V is a finite dimensional $\mathbb{Z}$-graded vector space, out of which one forms a bundle $E = B \times V$. The vector space has a filtration $W^{-n}(V) = \oplus_{m \geq n} V_m$ induced by the grading and a $\mathbb{G}_m$ action also coming from the grading. The class $[\nabla]$ is an equivalence class of equisingular connections, which are compatible with the filtration, trivial on the induced graded spaces $\mathrm{Gr}^W_{-n}(V)$, up to the equivalence relation of W-equivalence. This is defined by $T \circ \nabla_1 = \nabla_2 \circ T$ for some $T \in \mathrm{Aut}(E)$ which is compatible with filtration and trivial on $\mathrm{Gr}^W_{-n}(V)$. Here the condition that the connections ∇ are equisingular means that they are $\mathbb{G}_m$-invariant and that restrictions of solutions to sections of B with the same $\sigma(0)$ are W-equivalent. The morphisms $\mathrm{Hom}_{\mathcal{E}}(\Theta, \Theta')$ are linear maps $T \colon V \to V'$

that are compatible with grading, and such that on $E \oplus E'$ the following connections are W-equivalent:

$$\begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix} \overset{W\text{-equiv}}{\simeq} \begin{pmatrix} \nabla' & T\nabla - \nabla' T \\ 0 & \nabla \end{pmatrix}.$$

4.5. The Riemann–Hilbert correspondence. Finally, we proved in [27] that the category $\mathcal{E}$ is a Tannakian category,

$$\mathcal{E} \simeq \mathrm{Rep}_{\mathbb{U}^*}, \quad \text{with } \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m,$$

where $\mathbb{U}$ is dual, under the relation $\mathbb{U}(A) = \mathrm{Hom}(\mathcal{H}_{\mathbb{U}}, A)$, to the Hopf algebra $\mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^\vee$ dual (as Hopf algebra) to the universal enveloping algebra of the free graded Lie algebra $\mathcal{L} = \mathcal{F}(e_{-1}, e_{-2}, e_{-3}, \dots)$. The renormalization group $\mathbf{rg}\colon \mathbb{G}_a \to \mathbb{U}$ is a 1-parameter subgroup with generator $e = \sum_{n=1}^\infty e_{-n}$. In particular, the morphism $\mathbb{U} \to G$ that realizes the finite dimensional linear representations of G with equisingular connections as a subcategory of $\mathcal{E}$ is given by mapping the generators $e_{-n} \mapsto \beta_n$ to the n-th graded piece of the beta function of the theory, seen as an element $\beta = \sum_n \beta_n$ in the Lie algebra $\mathrm{Lie}(G)$. There are universal counterterms in $\mathbb{U}^*$ given in terms of a *universal singular frame*

$$\gamma_{\mathbb{U}}(z, v) = Te^{-\frac{1}{z}\int_0^v u^Y(e)\frac{du}{u}}.$$

For $\Theta = (V, [\nabla])$ in $\mathcal{E}$ there exists a unique $\rho \in \mathrm{Rep}_{\mathbb{U}^*}$ such that

$$D\rho(\gamma_{\mathbb{U}}) \overset{W\text{-equiv}}{\simeq} \nabla.$$

This same affine group scheme $\mathbb{U}^*$ appears in the work of Deligne–Goncharov as the motivic Galois group of the category of mixed Tate motives $\mathcal{M}_S \simeq \mathrm{Rep}_{\mathbb{U}^*}$, with $S = \mathrm{Spec}(\mathbb{Z}[i][1/2])$, albeit up to a non-canonical identification. This leads to an identification (non-canonically) of the category $\mathcal{E}$, which by the previous steps classifies the data of the counterterms in perturbative renormalization, with the category $\mathcal{M}_S$ of mixed Tate motives.

Cartier conjectured [23] the existence of a Galois group acting on the coupling constants of the physical theories and related both to the groups of diffeographisms of the Connes–Kreimer theory and to the symmetries of multiple zeta values, and he referred to it as a *cosmic Galois group*. In this sense the result of [27] is a positive answer to Cartier's conjecture, which identifies his cosmic Galois group with the affine group scheme $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$.

5. The geometry of Dim Reg

We end this exposition with a brief discussion on the subject of dimensional regularization. In physics this is taken to mean a *formal* extension of the rules of integration of Gaussians by setting

$$\int e^{-\lambda t^2} d^z t := \pi^{z/2} \lambda^{-z/2},$$

for $z \in \mathbb{C}^*$. This prescription can then be used to make sense of a larger set of integrations in complexified dimension z, which can be reduced to this Gaussian form by the use of Schwinger parameters. However, no attempt is made to make sense of an actual geometry in complexified dimension $z \in \mathbb{C}^*$. We argue here that there are (at least) two possible approaches that can be used to make sense of spaces in dimension z compatibly with the prescription for the Gaussian integration. One is based on noncommutative geometry and it was proposed first in the unpublished work [31] and later included in our book [30], while the second approach is based on motives and was proposed in [49]. The noncommutative geometry approach is based on the idea of taking a product, in the sense of metric noncommutative spaces (spectral triples) of the spacetime manifold over which the quantum field theory is constructed by a noncommutative space X_z whose dimension spectrum (the most sophisticated notion of dimension in noncommutative geometry) is given by a single point $z \in \mathbb{C}^*$. The motivic approach is based also on taking a product, but this time of the motive associated to an individual Feynman graph by a projective limit of logarithmic motives Log^{∞}.

In both cases the main idea is to deform the geometry by taking a product of the original geometry on which the computation of the un-regularized Feynman integral was performed by a new space, either noncommutative or motivic, which accounts for the shift of z in dimension. Recently there has been a considerable amount of activity in relating noncommutative geometry and motives (see [24] and [30]). It would be interesting to see if, in this context, there is a way to combine these two approaches to the geometry of dimensional regularization.

5.1. The noncommutative geometry of DimReg. The notion of metric space in noncommutative geometry is provided by *spectral triples*. These consist of data of the form $X = (\mathcal{A}, \mathcal{H}, \mathcal{D})$, with $\mathcal{A}$ an associative involutive algebra represented as an algebra of bounded operators on a Hilbert space $\mathcal{H}$, together with a self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$, with compact resolvent, and with the property that the commutators $[a, \mathcal{D}]$ are bounded operators on $\mathcal{H}$, for all $a \in \mathcal{A}$. This structure generalizes the data of a compact Riemannian spin manifold, with the (commutative) algebra of smooth functions, the Hilbert space of square integrable spinors and the Dirac operator. It makes sense, however, for a wide range of examples that are not ordinary manifolds, such as quantum groups, fractals, noncommutative tori, etc. For such spectral triples there are various different notions of dimension. The most sophisticated one is the *dimension spectrum* which is not a single number but a subset of the complex plane consisting of all poles of the family of zeta functions associated to the spectral triple,

$$\mathrm{Dim} = \{s \in \mathbb{C} \mid \zeta_a(s) = \mathrm{Tr}(a|D|^{-s}) \text{ have poles}\}.$$

These are points where one has a well defined integration theory on the noncommutative space, the analog of a volume form, given in terms of a residue for the zeta functions. It is shown in [31], [30] that there exists a (type II) spectral triple X_z with the properties that the dimension spectrum is $\mathrm{Dim} = \{z\}$ and that one recovers the DimReg prescription for the Gaussian integration in the form

$$\mathrm{Tr}(e^{-\lambda D_z^2}) = \pi^{z/2}\lambda^{-z/2}.$$

The operator D_z is of the form $D_z = \rho(z)F|Z|^{1/z}$, where $Z = F|Z|$ is a self-adjoint operator affiliated to a type II_∞ von Neumann algebra $\mathcal{N}$ and $\rho(z) = \pi^{-1/2}(\Gamma(1 + z/2))^{1/z}$, with the spectral measure $\mathrm{Tr}(\chi_{[a,b]}(Z)) = \frac{1}{2}\int_{[a,b]} dt$, for the type II trace. The ordinary spacetime over which the quantum field theory is constructed can itself be modeled as a (commutative) spectral triple

$$X = (\mathcal{A}, \mathcal{H}, \mathcal{D}) = (\mathcal{C}^\infty(X), L^2(X, S), \not{D}_X)$$

and one can take a product $X \times X_z$ given by the cup product of spectral triples (adapted to type II case)

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}) \cup (\mathcal{A}_z, \mathcal{H}_z, D_z) = (\mathcal{A} \otimes \mathcal{A}_z, \mathcal{H} \otimes \mathcal{H}_z, \mathcal{D} \otimes 1 + \gamma \otimes D_z).$$

This agrees with what is usually described in physics as the Breitenlohner–Maison prescription to resolve the problem of the compatibility of the chirality γ_5 operator with the DimReg procedure, [21]. The Breitenlohner–Maison prescription consists of changing the usual Dirac operator to a product, which is indeed of the form as in the cup product of spectral triples,

$$\mathcal{D} \otimes 1 + \gamma \otimes D_z.$$

It is shown in [31] and [30] that an explicit example of a space X_z that can be used to perform dimensional regularization geometrically can be constructed from the adèle class space, the noncommutative space underlying the spectral realization of the Riemann zeta function in noncommutative geometry (see *e.g.* [24]), by taking the crossed product of the partially defined action

$$\mathcal{N} = L^\infty(\hat{\mathbb{Z}} \times \mathbb{R}^*) \rtimes \mathrm{GL}_1(\mathbb{Q})$$

and the trace

$$\mathrm{Tr}(f) = \int_{\hat{\mathbb{Z}}\times\mathbb{R}^*} f(1,a)\,da,$$

with the operator

$$Z(1,\rho,\lambda) = \lambda, \quad Z(r,\rho,\lambda) = 0,\ r \neq 1 \in \mathbb{Q}^*.$$

5.2. The motivic geometry of DimReg. We now explain briefly the motivic approach to dimensional regularization proposed in [49]. The Kummer motives are simple examples of mixed Tate motives, given by the extensions

$$M = [u\colon \mathbb{Z} \to \mathbb{G}_m] \in \mathrm{Ext}^1_{\mathcal{DM}(\mathbb{K})}(\mathbb{Q}(0), \mathbb{Q}(1))$$

with $u(1) = q \in \mathbb{K}^*$ and the period matrix

$$\begin{pmatrix} 1 & 0 \\ \log q & 2\pi i \end{pmatrix}.$$

These can be combined in the form of the Kummer extension of Tate sheaves

$$\mathcal{K} \in \mathrm{Ext}^1_{\mathcal{DM}(\mathbb{G}_m)}(\mathbb{Q}_{\mathbb{G}_m}(0), \mathbb{Q}_{\mathbb{G}_m}(1)),$$

$$\mathbb{Q}_{\mathbb{G}_m}(1) \to \mathcal{K} \to \mathbb{Q}_{\mathbb{G}_m}(0) \to \mathbb{Q}_{\mathbb{G}_m}(1)[1].$$

The *logarithmic motives* $\mathrm{Log}^n = \mathrm{Sym}^n(\mathcal{K})$ are defined as symmetric products of this extension, [7] [37]. They form a projective system and one can take the limit as a pro-motive

$$\mathrm{Log}^\infty = \varprojlim_n \mathrm{Log}^n.$$

This corresponds to the period matrix

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & \cdots \\
\log(s) & (2\pi i) & 0 & \cdots & 0 & \cdots \\
\frac{\log^2(s)}{2!} & (2\pi i)\log(s) & (2\pi i)^2 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\
\frac{\log^n(s)}{n!} & (2\pi i)\frac{\log^{n-1}(s)}{(n-1)!} & (2\pi i)^2\frac{\log^{n-2}(s)}{(n-2)!} & \cdots & (2\pi i)^{n-1} & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots
\end{pmatrix}$$

The graph polynomials Ψ_Γ associated to Feynman graphs define motivic sheaves

$$M_\Gamma = (\Psi_\Gamma\colon \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \to \mathbb{G}_m, \hat{\Sigma}_n \smallsetminus (\hat{X}_\Gamma \cap \hat{\Sigma}_n), n-1, n-1),$$

viewed as objects $(f\colon X \to S, Y, i, w)$ in Arapura's category of motivic sheaves, [6].

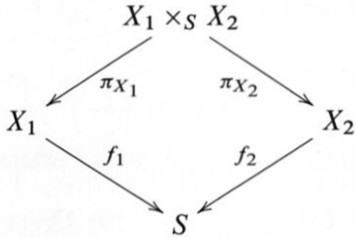
Then the procedure of dimensional regularization can be see as taking a product $M_\Gamma \times \mathrm{Log}^\infty$ in the Arapura category of the motivic sheaf M_Γ by the logarithmic pro-motive. The product in the Arapura category is given by the fibered product

$$(X_1 \times_S X_2 \to S, Y_1 \times_S X_2 \cup X_1 \times_S Y_2, i_1 + i_2, w_1 + w_2).$$

The reason for this identification is that period computations on a fibered products satisfy

$$\int \pi_{X_1}^*(\omega) \wedge \pi_{X_2}^*(\eta) = \int \omega \wedge f_1^*(f_2)_*(\eta),$$

where the integration takes place on $\sigma_1 \times_S \sigma_2$ with $\sigma_i \subset X_i$ with boundary $\partial\sigma_i \subset Y_i$, according to the diagram



This leads to writing the dimensionally regularized parametric Feynman integrals (at least in the log-divergent case where the term $P_\Gamma(t, p)$ is absent) in the Igusa L-function form $\int_\sigma \Psi_\Gamma^z \alpha$ as a period computation on $M_\Gamma \times \mathrm{Log}^\infty$.

Acknowledgement. A large part of what I described in this paper is based on recent joint work with Paolo Aluffi and on previous joint work with Alain Connes, whom I thank for their essential contributions to the development of these ideas and results. Although parts of this article reflect the content of the lecture I delivered at the ECM in the summer of 2008, new material developed in the intervening time was included for completeness and the actual writing was carried out in the winter 2009, during a stay at MSRI: I thank the institute for the hospitality and for support. An informal discussion with Nikolai Reshetikhin and a talk I gave in Yongbin Ruan's seminar helped finalizing the form of this exposition. The topics covered in this survey will appear in extended form in the book [50] based on a course given at Caltech in the fall 2008.

References

[1] P. Aluffi, Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes. *Pure Appl. Math. Q.* **2** (2006), no. 4, part 2, 915–941.

[2] P. Aluffi and M. Marcolli, Feynman motives of banana graphs. *Commun. Number Theory Phys.* **3** (2009), no. 1, 1–57.

[3] P. Aluffi and M. Marcolli, Algebro-geometric Feynman rules. arXiv:0811.2514v1 [hep-th]

[4] P. Aluffi and M. Marcolli, Parametric Feynman integrals and determinant hypersurfaces. arXiv:0901.2107v1 [math.AG]

[5] Y. André, *Une introduction aux motifs*. Panor. Syntheses 17, Soc. Math. France, Paris, 2004.

[6] D. Arapura, A category of motivic sheaves. arXiv:0801.0261v1 [math.AG]

[7] J. Ayoub, The motivic vanishing cycles and the conservation conjecture. In *Algebraic cycles and motives*, London Math. Soc. Lecture Note Ser. 343, Cambridge University Press, Cambridge, 2007, 3–54.

[8] A. A.Beilinson, A. B. Goncharov, V. V. Schechtman, and A. N. Varchenko, Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of pairs of triangles on the plane. In *The Grothendieck Festschrift*, Vol. I, Progr. Math. 86, Birkhäuser, Boston, MA, 1990, 135–172.

[9] P. Belkale and P. Brosnan, Matroids, motives, and a conjecture of Kontsevich. *Duke Math. J.* **116** (2003), 147–188.

[10] P. Belkale and P. Brosnan, Periods and Igusa local zeta functions. *Internat. Math. Res. Notices* **2003** (2003), no. 49, 2655–2670.

[11] F. Bittner, The universal Euler characteristic for varieties of characteristic zero. *Compositio Math.* **140** (2004), no. 4, 1011–1032.

[12] J. Bjorken and S. Drell, *Relativistic quantum wechanics*. McGraw-Hill, New York, 1964. *Relativistic quantum fields*. McGraw-Hill, New York, 1965.

[13] S. Bloch, Motives associated to graphs. *Japan J. Math.* **2** (2007), 165–196.

[14] S. Bloch, Lectures on mixed motives. In *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math. 62, Part 1, Amer. Math. Soc., Providence, RI, 1997, 329–359.

[15] S. Bloch, Motives associated to sums of graphs. arXiv:0810.1313v1 [math.AG]

[16] S. Bloch, E. Esnault, and D. Kreimer, On motives associated to graph polynomials. *Comm. Math. Phys.* **267** (2006), 181–225.

[17] S. Bloch and D. Kreimer, Mixed Hodge structures and renormalization in physics. *Number Theory Phys.* **2** (2008), no. 4, 637–718.

[18] C. Bogner and S. Weinzierl, Periods and Feynman integrals. *J. Math. Phys.* **50** (2009), no. 4, 042302.

[19] C. Bogner and S. Weinzierl, Resolution of singularities for multi-loop integrals. *Comput. Phys. Commun.* **178** (2008), 596–610.

[20] N. N. Bogoliubov and O. Parasiuk, Über die Multiplikation der Kausal Funktionen in der Quantentheorie der Felder. *Acta Math.* **97** (1957), 227–266.

[21] P. Breitenlohner and D. Maison, Dimensional renormalization and the action principle. *Comm. Math. Phys.* **52** (1977), no. 1, 11–38.

[22] D. Broadhurst and D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. *Phys. Lett. B* **393** (1997), 403–412.

[23] P. Cartier, A mad day's work: from Grothendieck to Connes and Kontsevich. The evolution of concepts of space and symmetry. *Bull. Amer. Math. Soc. (N.S.)* **38** (2001), no. 4, 389–408.

[24] A. Connes, C. Consani, and M. Marcolli, Noncommutative geometry and motives: the thermodynamics of endomotives. *Adv. Math.* **214** (2007), no. 2, 761–831.

[25] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem I. The Hopf algebra structure of graphs and the main theorem. *Comm. Math. Phys.* **210** (2000), no. 1, 249–273.

[26] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The β-function, diffeomorphisms and the renormalization group. *Comm. Math. Phys.* **216** (2001), no. 1, 215–241.

[27] A. Connes and M. Marcolli, Renormalization and motivic Galois theory. *Internat. Math. Res. Notices* **2004** (2004), no. 76, 4073–4091.

[28] A. Connes and M. Marcolli, From physics to number theory via noncommutative geometry. Part II: Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory. In *Frontiers in number theory, physics, and geometry, II*, Springer-Verlag, Berlin, 2006, 617–713.

[29] A. Connes and M. Marcolli, Quantum fields and motives. *J. Geom. Phys.* **56** (2005), no. 1, 55–85.

[30] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields, and motives*. Amer. Math. Soc. Colloq. Publ. 55, Amer. Math. Soc., Providence, RI , 2008.

[31] A. Connes and M. Marcolli, Anomalies, dimensional regularization and noncommutative geometry. Unpublished manuscript, 2005. www.its.caltech.edu/~matilde/work.html

[32] P. Deligne and A. B. Goncharov, Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup.* (4) **38** (2005), no. 1, 1–56.

[33] K. Ebrahimi-Fard, L. Guo, and D. Kreimer, Spitzer's identity and the algebraic Birkhoff decomposition in pQFT. *J. Phys. A* **37** (2004), no. 45, 11037–11052.

[34] K. Ebrahimi-Fard, J. M. Gracia-Bondia, and F. Patras, A Lie theoretic approach to renormalization. *Comm. Math. Phys.* **276** (2007), no. 2, 519–549.

[35] H. Gillet and C. Soulé, Descent, motives and K-theory. *J. Reine Angew. Math.* **478** (1996), 127–176.

[36] A. B. Goncharov, Multiple polylogarithms and mixed Tate motives. arXiv:math/0103059v4 [math.AG]

[37] A. B. Goncharov, Explicit regulator maps on polylogarithmic motivic complexes. In *Motives, polylogarithms and Hodge theory* (Irvine, CA, 1998), Part I, Int. Press Lect. Ser. 3, I, International Press, Somerville, MA, 2002, 245–276.

[38] D. Gross, Applications of the renormalization group to high energy physics. In *Méthodes en théorie des champs/Methods in field theory* (Les Houches 1975), North-Holland Publishing Co., Amsterdam, 1976, 141–250.

[39] K. Hepp, Proof of the Bogoliubov-Parasiuk theorem on renormalization. *Comm. Math. Phys.* **2** (1966), 301–326.

[40] C. Itzykson and J. B. Zuber, *Quantum field theory*. Dover Publications, 2006.

[41] U. Jannsen, Motives, numerical equivalence, and semi-simplicity. *Invent. Math.* **107** (1992), no. 3, 447–452.

[42] D. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality. In *Geometry of the Laplace operator*, Proc. Sympos. Pure Math. 36, Amer. Math. Soc., Providence, RI, 1980, 185–203.

[43] M. Kontsevich and D. Zagier, Periods. In *Mathematics unlimited—2001 and beyond*, Part II, Springer-Verlag, Berlin, 2001, 771–808.

[44] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories. *Adv. Theor. Math. Phys.* **2** (1998), no. 2, 303–334.

[45] M. Levine, *Mixed motives*. Math. Surveys Monogr. 57, Amer. Math. Soc., Providence, RI, 1998.

[46] R. D. MacPherson, Chern classes for singular algebraic varieties. *Ann. of Math.* (2) **100** (1974), 423–432.

[47] Yu. I. Manin, Correspondences, motifs and monoidal transformations. *Mat. Sb. (N.S.)* **77** (119) (1968), 475–507; English transl. *Math. USSR-Sb.* **6** (1968), 439–470.

[48] E. B. Manoukian, *Renormalization*. Pure Appl. Math. 106, Academic Press, New York, 1983.

[49] M. Marcolli, Motivic renormalization and singularities. arXiv:0804.4824v3 [math-ph]

[50] M. Marcolli, *Feynman motives*. Book in preparation for Imperial College Press/World Scientific.

[51] M. Marcolli and A. Rej, Supermanifolds from Feynman graphs. *J. Phys. A* **41** (2008), 315402.

[52] B. Mohar and C. Thomassen, *Graphs on surfaces*. Johns Hopkins Stud. Math. Sci., Johns Hopkins University Press, Baltimore, MD, 2001.

[53] N. Nakanishi, *Graph theory and Feynman integrals*. Math. Appl. 11, Gordon and Breach, New York, 1971.

[54] M. H. Schwartz, Classes caractéristiques définies par une stratification d'une variété analytique complexe. I. *C. R. Acad. Sci. Paris* **260** (1965), 3262–3264.

[55] T. Terasoma, Mixed Tate motives and multiple zeta values. *Invent. Math.* **149** (2002), no. 2, 339–369.

[56] W. van Suijlekom, Renormalization of gauge fields: a Hopf algebra approach. *Comm. Math. Phys.* **276** (2007), no. 3, 773–798.

[57] W. van Suijlekom, The Hopf algebra of Feynman graphs in quantum electrodynamics. *Lett. Math. Phys.* **77** (2006), N.3, 265–281.

[58] W. van Suijlekom, Representing Feynman graphs on BV-algebras. *Comm. Math. Phys.* **290** (2009), 291–319.

[59] V. Voevodsky, Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud. 143, Princeton University Press, Princeton, NJ, 2000, 188–238.

[60] W. Zimmermann, Convergence of Bogoliubov's method of renormalization in momentum space. *Comm. Math. Phys.* **15** (1969), 208–234.

Matilde Marcolli, Mathematics Department, California Institute of Technology,
1200 E. California Blvd., Pasadena, CA 91125, U.S.A.

E-mail: matilde@caltech.edu