Feynman quadrics, Prym varieties, and the motive of the Sunset Graph

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Combinatorial Structures in Perturbative QFT
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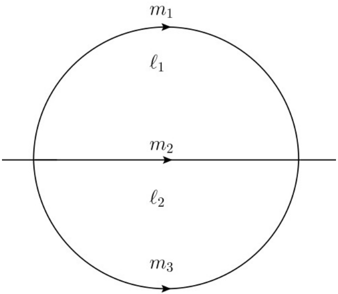
This lecture is based on:

- Matilde Marcolli, Gonçalo Tabuada, *Feynman quadrics motive of the massive sunset graph*, arXiv:1705.10307

Related work:

- M. Bernardara, G. Tabuada, *Chow groups of intersections of quadrics via homological projective duality and (Jacobians of) non-commutative motives*, Izv. Math. 80 (2016) no. 3, 463–480.
- M. Marcolli, G. Tabuada, *Jacobians of noncommutative motives*, Moscow Mathematical Journal 14 (2014) no. 3, 577–594.

Sunset Graph



perturbative scalar QFT with masses m_i (massive propagators along the edges)

Previously known (via different method)

- S. Bloch P. Vanhove, *The elliptic dilogarithm for the sunset graph*. J. Number Theory 148 (2015), 328–364
- S. Bloch, M. Kerr and P. Vanhove, *A Feynman integral via higher normal functions*. Compos. Math. 151 (2015) no. 12, 2329–2375.

The motive of the massive sunset graph (computed using graph hypersurfaces) in dimension $D = 2$ and $D = 4$ (case of equal masses and more general case) is non-mixed Tate (expressed in terms of elliptic curves).

We work in general dimension D and with Feynman quadrics instead of graph hypersurfaces

Feynman graphs

- $D > 0$ spacetime dimension (Euclidean)
- (Γ, m, κ) Feynman graph equipped with mass parameters $m = (m_e)$ and external momenta $\kappa = (\kappa_i)$
- internal edges $e_i \in E_{\rm int}(\Gamma)$ carry momentum variables $k_i = (k_{i,r}) \in \mathbb{A}^D$
- edge propagator

$$q_i(k_i) = \sum_{r=1}^{D} k_{i,r}^2 + m_i^2$$

Feynman integral $\mathcal{I}_{(\Gamma,m,\kappa)}$

$$C \int \frac{\prod_{v\in V_{\rm int}(\Gamma)} \delta(\sum_{e_i\in E_{\rm int}(\Gamma)} \epsilon_{v,i} k_i + \sum_{e_j\in E_{\rm ext}(\Gamma)} \epsilon_{v,j}\kappa_j)}{\prod_{e_i\in E_{\rm int}(\Gamma)} q_i(k_i)} \prod_{e_i\in E_{\rm int}(\Gamma)} \frac{d^D k_i}{(2\pi)^D}$$

- $C = \prod_v \lambda_v (2\pi)^{-D}$ with λ_v coupling constant at vertex v
- $\epsilon_{v,i}$ incidence matrix with entries 1, -1, or 0, for $v = s(e)$, $v = t(e)$, $v \notin \partial(e)$
- $\prod_{e_i} d^D k_i$ standard volume form in $\mathbb{A}^{nD}(\mathbb{R})$
- $n := \# E_{\rm int}(\Gamma)$ number of internal edges

Unrenormalized (usually divergent) Feynman integral

Feynman quadrics

- Notation: $v = (v_{i,r}) \in \mathbb{A}^{nD}$ and $v' = (v'_{i,r}) \in \mathbb{A}^{nD}$, let $\langle v, v' \rangle := \sum_{i=1}^{n} \sum_{r=1}^{D} v_{i,r} v'_{i,r}$ and $v^2 := \langle v, v \rangle = \sum_{i,r} v_{i,r}^2$
- Homogeneous polynomial ($nD + 1$ variables):

$$q'_i(k_i, x) := \sum_{r=1}^{D} k_{i,r}^2 + m_i^2 x^2 = k_i^2 + m_i^2 x^2$$

(identify $k_i = (k_{i,r}) \in \mathbb{A}^D$ with $v = (v_{j,r})$ of $\mathbb{A}^{nD}$ with $k_{i,r}$ for $i = j$ and 0 otherwise)

- Quadric Hypersurface: $Q'_i \subset \mathbb{P}^{nD}$

Linear relations

• delta function in Feynman integral imposes linear relations at vertices between the momentum variables

$$\sum_{\substack{e_i \in E_{\rm int}(\Gamma) \\ s(e_i)=v}} k_i + \sum_{\substack{e_j \in E_{\rm ext}(\Gamma) \\ s(e_j)=v}} \kappa_j = \sum_{\substack{e_i \in E_{\rm int}(\Gamma) \\ t(e_i)=v}} k_i + \sum_{\substack{e_j \in E_{\rm ext}(\Gamma) \\ t(j)=v}} \kappa_j$$

- N number of *independent* linear relations
- choose $n - N$ *independent* variables $\ell = \{\ell_i\}$ among $\{k_1, \dots, k_n\}$ (loop variables)
- have $N = \#V_{\rm int}(\Gamma) - 1$ so difference $n - N = \#E_{\rm int}(\Gamma) - \#V_{\rm int}(\Gamma) + 1$ is first Betti number $L = b_1(\Gamma)$

Vanishing external momenta $\kappa = 0$

- linear subspace of momentum conservation

$$H_\Gamma := \bigcap_{v \in V_{\rm int}(\Gamma)} \{ \sum_{\substack{e_i \in E_{\rm int}(\Gamma) \\ s(e_i)=v}} k_i - \sum_{\substack{e_i \in E_{\rm int}(\Gamma) \\ t(e_i)=v}} k_i = 0 \} \subset \mathbb{P}^{nD}$$

- Feynman quadrics in loop variables

$$Q_i := Q_i' \cap H_\Gamma = \{ q_i(\ell, x) = 0 \}$$

- The quadrics Q_i are usually singular (cones)
- Notation: coordinates $u = (u_0 : \cdots : u_{LD})$ on $\mathbb{P}^{LD}$ with

$$u_0 := x, \quad (u_1, \ldots, u_D) := \ell_1 \quad \cdots (u_{(L-1)D}, \ldots, u_{LD}) := \ell_L$$

Nets of quadrics

- parameterizing space of all quadric hypersurfaces in $\mathbb{P}^{LD}$ is the projective space $\mathbb{P}^{\binom{LD+2}{2}-1}$ of symmetric $(LD+1) \times (LD+1)$-matrices up to scalar multiples
- inside this parameterizing space discriminant hypersurface $\mathcal{D}$: quadratic forms with non-trivial kernel
- a *net of n quadric hypersurfaces* in $\mathbb{P}^{LD}$ consists of an embedding $\rho \colon \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{\binom{LD+2}{2}-1}$

Net of Feynman quadrics of a graph Γ:

$$\rho\colon \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{\binom{LD+2}{2}-1} \quad (0:\cdots:0:\underset{i}{1}:0:\cdots:0) \mapsto Q_i$$

- quadric hypersurfaces Q_i belong to $\mathbb{P}^{\binom{LD+2}{2}-1}(\mathbb{R})$ (the defining quadratic form q_i of the quadric Q_i is real)
- symmetric matrices A_i, defined by $q_i(u) = \langle u, A_i u\rangle$, can be written as $A_i = T_i^\dagger T_i$, with $T_i^\dagger$ adjoint of T_i with respect to the bilinear form $\langle v, v'\rangle$
- momentum conservation condition:

$$\sum_{s(e_i)=v} \bar{T}_i = \sum_{t(e_i)=v} \bar{T}_i$$

$\bar{T}_i = P T_i P$ with projection $P : (u_0, \ldots, u_{LD}) \mapsto (u_1, \ldots, u_{LD})$

Deformations of nets of quadrics

- *one-parameter deformation* of $\rho : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{\binom{LD+2}{2}-1}$ is a morphism $\tilde{\rho} : \mathbb{P}^{n-1} \times \mathbb{A}^1 \to \mathbb{P}^{\binom{LD+2}{2}-1}$ with $\rho = \tilde{\rho}|_{\mathbb{P}^{n-1}\times\{0\}}$
- given $\epsilon \in \mathbb{A}^1(\mathbb{Q})$, $\epsilon \neq 0$, write ρ_ϵ for net $\tilde{\rho}|_{\mathbb{P}^{n-1}\times\{\epsilon\}}$ (ϵ-deformation of ρ)

For any Feynman graph Γ there is always a one-parameter deformation $\tilde{\rho}$ of the net of Feynman quadrics

$$\rho_\epsilon : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{\binom{LD+2}{2}-1} \quad (0 : \cdots : 0 : \underset{i}{1} : 0 : \cdots : 0) \mapsto Q_{i,\epsilon}$$

such that for sufficiently small ϵ

- the quadrics $Q_{i,\epsilon}$ belong to $\mathbb{P}^{\binom{LD+2}{2}-1} \setminus \mathcal{D}$ (smooth)
- the quadrics $Q_{i,\epsilon}$ belong to $\mathbb{P}^{\binom{LD+2}{2}-1}(\mathbb{R})$ (real)
- the symmetric matrices $A_{i,\epsilon}$ can be written as $A_{i,\epsilon} = T^\dagger_{i,\epsilon} T_{i,\epsilon}$ (positive)
- momentum conservation condition: $\sum_{s(e_i)=v} \bar{T}_{i,\epsilon} = \sum_{t(e_i)=v} \bar{T}_{i,\epsilon}$ for $\bar{T}_{i,\epsilon} = P T_{i,\epsilon} P$

These deformations ρ_ϵ maintain physical properties (real, positive, momentum conservation) while they gain smoothness (replacing cones with smooth quadrics)

- Sketch of the argument for momentum conservation:
  - choose a spanning tree for the Feynman graph Γ
  - constructing an ϵ-deformation $q_{i,\epsilon}$ of quadratic forms q_i associated to the L edges in the complement of the spanning tree
  - show that there is a unique way to extend the deformation to the remaining quadratic forms q_i on the edges of the spanning tree so that momentum conservation holds

The Motive and the Period

Feynman integral in terms of periods of motive associated to the net of Feynman quadrics

- motive: in the category of mixed motives $\mathrm{DM}_{\mathrm{gm}}(F)_{\mathbb{Q}}$ with $F \subseteq \mathbb{C}$ algebraically closed

$$M^{Q}_{(\Gamma,m)} = M(\mathbb{P}^{LD} \setminus Q_{(\Gamma,m)})_{\mathbb{Q}}$$

$Q_{(\Gamma,m)} := \bigcup_{i=1}^{n} Q_{i,\epsilon}$ union of the quadric hypersurfaces

- algebraic differential forms: $\alpha \in \mathbb{N}$

$$\omega := \sum_{i=0}^{LD} (-1)^i u_i \, du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_{LD}$$

$$\eta_\alpha := \frac{\omega}{\prod_{i=1}^{n} q_i^\alpha} \qquad \eta_{\alpha,\epsilon} := \frac{\omega}{\prod_{i=1}^{n} q_{i,\epsilon}^\alpha}$$

- restriction of ω to the affine chart $\mathbb{A}^{LD}$ coords $(1, u_1, \ldots, u_{LD})$ is affine volume form $du_1 \wedge \cdots \wedge du_{LD}$

Properties:

- $\alpha = 1$ (divergent) Feynman integral

$$\frac{C}{(2\pi)^D} \cdot \int_{\mathbb{A}^{LD}(\mathbb{R})} \eta_1$$

- $\alpha > \frac{LD}{2n}$ convergent (regularization)

$$\int_{\mathbb{A}^{LD}(\mathbb{R})} \eta_{\alpha,\epsilon} = \int_{\mathbb{P}^{2D}(\mathbb{R})} \eta_{\alpha,\epsilon}$$

period of $\mathbb{P}^{LD} \setminus Q_{(\Gamma,m)}$

exponent α changes superficial degree of convergence of Feynman integral from $\delta(\Gamma) = LD - 2n$ to $\delta_\alpha(\Gamma) = DL - 2n\alpha$

ϵ-deformation $Q_{i,\epsilon}$ ensures differential form $\eta_{\alpha,\epsilon}$ has no poles on the hyperplane at infinity $\mathbb{P}^{LN}(\mathbb{R}) \setminus \mathbb{A}^{LN}(\mathbb{R}) = \mathbb{P}^{LN-1}(\mathbb{R})$

Igusa zeta function (interpolate $\alpha \in \mathbb{N}$ by complex variable s)

$$\mathcal{I}(s) = \int_{\mathbb{P}^{LD}(\mathbb{R})} \eta_{s,\epsilon}$$

- Laurent series expansion: for some $N \in \mathbb{Z}$, $\alpha \in \mathbb{Z}$

$$\mathcal{I}(s) = \sum_{k \geq N} \gamma_k \, (s-\alpha)^k$$

- coefficients γ_k are periods of

$$(\mathbb{P}^{LD} \setminus Q_{(\Gamma,m)}) \times \mathbb{A}^k$$

- key: Bernstein functional equation

$$\mathcal{I}_\Gamma(s) = a_1(s)\mathcal{I}(s+1) + \cdots + a_k(s)\mathcal{I}(s+k)$$

- similar to argument in
  - P. Belkale and P. Brosnan, *Periods and Igusa local zeta functions*. Int. Math. Res. Not. 49 (2003), 2655–2670.
- for $\alpha > \frac{LD}{2n}$ Laurent expansion

$$\mathcal{I}_\Gamma(s) = \sum_{k\geq 0} \frac{(-1)^k}{k!} \int_{\mathbb{P}^{LD}(\mathbb{R})} \eta_{\alpha,\epsilon} \cdot \log^k(\prod_{j=1}^n q_{j,\epsilon}) \ (s-\alpha)^k.$$

$$\log(f(u)) = \int_0^1 \theta(u,t) \quad \text{with} \quad \theta(u,t) = \frac{f(u)-1}{(f(u)-1)t+1}\, dt$$

$$\gamma_k = \frac{(-1)^k}{k!} \int_{\mathbb{P}^{LD}(\mathbb{R})\times[0,1]^k} \eta_{\alpha,\epsilon} \wedge \theta(u,t_1) \wedge \cdots \wedge \theta(u,t_k)$$

functional equation for integers $\alpha \leq \frac{LD}{2n}$

Renormalization recursive procedure (Connes-Kreimer)

- renormalized value regularized integral $\mathcal{I}_\Gamma(s)$
- $\mathcal{H}_{CK}$ Hopf algebra of Feynman graphs: commutative polynomial algebra in connected 1-edge-connected graphs; coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} (\gamma \otimes \Gamma/\gamma)$$

- $\mathcal{R}$ algebra of Laurent series centered at $s = 1$ with $\mathcal{T}$ projection onto polar part
- Rota-Baxter identity

$$\mathcal{T}(f_1)\mathcal{T}(f_2) = \mathcal{T}(f_1\mathcal{T}(f_2)) + \mathcal{T}(\mathcal{T}(f_1)f_2) - \mathcal{T}(f_1 f_2)$$

- splitting $\mathcal{R}_+ = (1 - \mathcal{T})\mathcal{R}$ and $\mathcal{R}_- = \mathcal{T}\mathcal{R}^u$ (unitization)

- morphism of commutative algebras $\phi : \mathcal{H}_{CK} \to \mathcal{R}$
- Birkhoff factorization $\phi_\pm : \mathcal{H}_{CK} \to \mathcal{R}_\pm$

$$\phi_-(X) = -\mathcal{T}(\phi(X) + \sum \phi_-(X')\phi(X''))$$

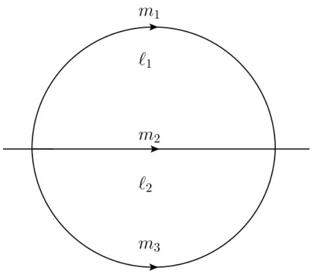
$$\phi_+(X) = (1-\mathcal{T})(\phi(X) + \sum \phi_-(X')\phi(X''))$$

  for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum (X' \otimes X'')$ with
  $\phi = (\phi_- \circ S) \star \phi_+$
- product dual to coproduct $\phi_1 \star \phi_s(X) := \langle \phi_1 \otimes \phi_2, \Delta(X) \rangle$

Feynman graph Γ: Laurent series $\phi_+(\Gamma)(s)$ regular at $s = 1$ with value $\phi_+(\Gamma)(1)$ the renormalized value

Focus here on computing leading term $(1-\mathcal{T})\mathcal{I}_\Gamma(s)|_{s=1}$ of renormalized $\phi_+(\Gamma)$, which is a period of $(\mathbb{P}^{2D} \setminus Q_{(\Gamma,m)}) \times \mathbb{A}^1$ (other terms are similar)

Sunset Graph



in the specific case of the massive sunset graph there is an explicit deformation $Q_{i,\epsilon}$ of the net of quadrics such that

- $Q_{i,\epsilon}$ smooth, real, positive, satisfying momentum conservation
- the double intersections $Q_{i,\epsilon} \cap Q_{j,\epsilon}$ and the triple intersection $Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}$ are all transversal

main idea

- usual physical perturbation

$$q_{e,\epsilon}(k,x) := k_e^2 + (m_e^2 + i\epsilon)x^2$$

- main purpose to move location of poles in the complex plane
- not good for smoothness and for transversality
- ... but use as model idea

unperturbed quadrics of the Sunset Graph

$$Q_1 = \{q_1(u) = \langle u, A_1 u\rangle = 0\} \quad \text{with} \quad A_1 = \mathrm{diag}(m_1^2, \underbrace{1,\ldots,1}_{D}, \underbrace{0,\ldots,0}_{D})$$

$$Q_2 = \{q_2(u) = \langle u, A_2 u\rangle = 0\} \quad \text{with} \quad A_2 = \mathrm{diag}(m_2^2, \underbrace{0,\ldots,0}_{D}, \underbrace{1,\ldots,1}_{D})$$

$$Q_3 = \{q_3(u) = \langle u, A_3 u\rangle = 0\} \quad \text{with} \quad A_3 = \mathrm{diag}(m_3^2, \underbrace{1,\ldots,1}_{D}, \underbrace{1,\ldots,1}_{D})$$

perturbed quadrics of the Sunset Graph

$$Q_{1,\epsilon} = \{q_{1,\epsilon}(u) = \langle u, A_{1,\epsilon} u\rangle = 0\} \quad A_{1,\epsilon} = \mathrm{diag}(m_1^2, \underbrace{1, \ldots, 1}_{D}, \underbrace{\epsilon^2, \ldots, \epsilon^{2D}}_{D})$$

$$Q_{2,\epsilon} = \{q_{2,\epsilon}(u) = \langle u, A_{2,\epsilon} u\rangle = 0\} \quad A_{2,\epsilon} = \mathrm{diag}(m_2^2, \underbrace{\epsilon^2, \ldots, \epsilon^{2D}}_{D}, \underbrace{1, \ldots, 1}_{D})$$

$$Q_{3,\epsilon} = \{q_{3,\epsilon}(u) = \langle u, A_{3,\epsilon} u\rangle = 0\}$$

$$A_{3,\epsilon} = \mathrm{diag}(m_3^2, \underbrace{(1+\epsilon)^2, \ldots, (1+\epsilon^D)^2}_{D}, \underbrace{(1+\epsilon)^2, \ldots, (1+\epsilon^D)^2}_{D}) .$$

Zariski open $W(m) \subset \mathbb{A}^1$ (depending on mass parameter $m = (m_1, m_2, m_3)$, with $m_i \neq 0$), for every $\epsilon \in W(m)$ the deformed quadrics $Q_{1,\epsilon}, Q_{2,\epsilon}, Q_{3,\epsilon} \subset \mathbb{P}^{2D}$ are smooth, real, positive, satisfying momentum conservation and transverse intersections $Q_{i,\epsilon} \cap Q_{j,\epsilon}$ and $Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}$

Motives

- $M(X)_{\mathbb{Q}}$ mixed motive and $M^c(X)_{\mathbb{Q}}$ mixed motive with compact support (isomorphic for smooth projective)
- dual motive dual $M(X)^\vee \simeq M^c(X)_{\mathbb{Q}}(-d)[-2d]$
- category of mixed-Tate motives: triangulated subcategory in $\mathrm{DM}_{\mathrm{gm}}(F)_{\mathbb{Q}}$ generated by Tate motives $\mathbb{L}^k$
- if distinguished triangle in $\mathrm{DM}_{\mathrm{gm}}(F)_{\mathbb{Q}}$ with two out of three terms mixed-Tate $\Rightarrow$ third one also mixed-Tate
- also if distinguished triangle with one term mixed-Tate and one not $\Rightarrow$ third one must be non-mixed-Tate

- for any X smooth $M(X)_{\mathbb{Q}}$ mixed-Tate iff $M^c(X)_{\mathbb{Q}}$ mixed-Tate (mixed Tate subcategory stable under duals)
- Mayer-Vietoris triangle: Zariski open cover $X = U \cup V$

$$M^c(X)_{\mathbb{Q}} \longrightarrow M^c(U)_{\mathbb{Q}} \oplus M^c(V)_{\mathbb{Q}} \longrightarrow M^c(U \cap V)_{\mathbb{Q}} \longrightarrow M^c(X)_{\mathbb{Q}}[1]$$

- Gysin triangle: Zariski closed subscheme $Z \subset X$ with open complement U

$$M^c(Z)_{\mathbb{Q}} \longrightarrow M^c(X)_{\mathbb{Q}} \longrightarrow M^c(U)_{\mathbb{Q}} \longrightarrow M^c(Z)_{\mathbb{Q}}[1]$$

Step 1: the motives $M^c(\mathbb{P}^{2D} \setminus Q_{i,\epsilon})_\mathbb{Q}$ are mixed-Tate

- $2D$ even, quadric hypersurface $Q_{i,\epsilon} \subset \mathbb{P}^{2D}$ odd-dimensional
- motivic decomposition of Chow motive (smooth quadrics)

$$\mathfrak{h}(Q_{i,\epsilon})_\mathbb{Q} \simeq 1 \oplus \mathbb{L} \oplus \mathbb{L}^{\otimes 2} \oplus \cdots \oplus \mathbb{L}^{\otimes(2D-1)}$$

  so $M^c(Q_{i,\epsilon})_\mathbb{Q} \simeq M(Q_{i,\epsilon})_\mathbb{Q}$ is mixed-Tate
- Gysin triangle with $X = \mathbb{P}^{2D}$ and $Z = Q_{i,\epsilon}$ gives $M^c(\mathbb{P}^{2D} \setminus Q_{i,\epsilon})_\mathbb{Q}$ mixed-Tate

Step 2: $M^c(\mathbb{P}^{2D} \backslash (Q_{i,\epsilon} \cup Q_{j,\epsilon}))_{\mathbb{Q}}$ mixed-Tate iff $M^c(\mathbb{P}^{2D} \backslash (Q_{i,\epsilon} \cap Q_{j,\epsilon}))_{\mathbb{Q}}$ mixed-Tate

- Mayer-Vietoris triangle with $X := \mathbb{P}^{2D} \backslash (Q_{i,\epsilon} \cap Q_{j,\epsilon})$, $U := \mathbb{P}^{2D} \backslash Q_{i,\epsilon}$ and $V := \mathbb{P}^{2D} \backslash Q_{j,\epsilon}$
- this has $U \cap V = \mathbb{P}^{2D} \backslash (Q_{i,\epsilon} \cup Q_{j,\epsilon})$
- $M^c(U)_{\mathbb{Q}}$ and $M^c(V)_{\mathbb{Q}}$ mixed-Tate by Step 1
- if two out of three terms mixed-Tate then third term in the Mayer-Vietoris triangle also mixed-Tate

Step 3: assume all $M^c(\mathbb{P}^{2D}\setminus(Q_{i,\epsilon}\cap Q_{j,\epsilon}))_{\mathbb{Q}}$ are mixed-Tate, then $M^c(\mathbb{P}^{2D}\setminus Q_{(\Gamma,m)})_{\mathbb{Q}}$ is mixed-Tate iff $M^c(\mathbb{P}^{2D}\setminus(Q_{1,\epsilon}\cap Q_{2,\epsilon}\cap Q_{3,\epsilon}))_{\mathbb{Q}}$ is mixed-Tate

- take $U=\mathbb{P}^{2D}\setminus(Q_{1,\epsilon}\cup Q_{2,\epsilon})$ and $V=\mathbb{P}^{2D}\setminus Q_{3,\epsilon}$ then $U\cap V=\mathbb{P}^{2D}\setminus Q_{(\Gamma,m)}$
- $M^c(V)_{\mathbb{Q}}$ mixed-Tate by Step 1
- $M^c(\mathbb{P}^{2D}\setminus(Q_{1,\epsilon}\cap Q_{2,\epsilon}))_{\mathbb{Q}}$ mixed-Tate by assumption
- get $M^c(U)_{\mathbb{Q}}$ mixed Tate by Step 2 and assumption
- Mayer-Vietoris triangle: $M^c(\mathbb{P}^{2D}\setminus Q_{(\Gamma,m)})_{\mathbb{Q}}$ mixed-Tate iff $M^c(U\cup V)_{\mathbb{Q}}$ mixed-Tate
- take $U_{13}:=\mathbb{P}^{2D}\setminus(Q_{1,\epsilon}\cap Q_{3,\epsilon})$ and $U_{23}:=\mathbb{P}^{2D}\setminus(Q_{2,\epsilon}\cap Q_{3,\epsilon})$

$$U_{13}\cap U_{23}=\mathbb{P}^{2D}\setminus((Q_{1,\epsilon}\cap Q_{3,\epsilon})\cup(Q_{2,\epsilon}\cap Q_{3,\epsilon}))$$

$$=\mathbb{P}^{2D}\setminus((Q_{1,\epsilon}\cup Q_{2,\epsilon})\cap Q_{3,\epsilon})=U\cup V$$

- Mayer-Vietoris triangle: $M^c(U\cup V)_{\mathbb{Q}}$ mixed-Tate iff $M^c(U_{13}\cup U_{23})_{\mathbb{Q}}$ mixed-Tate

$$U_{13}\cup U_{23}=\mathbb{P}^{2D}\setminus(Q_{1,\epsilon}\cap Q_{2,\epsilon}\cap Q_{3,\epsilon})$$

Step 4: the motives $M^c(\mathbb{P}^{2D}\setminus(Q_{i,\epsilon}\cap Q_{j,\epsilon}))_{\mathbb{Q}}$ are mixed-Tate

- by transversality the intersections $Q_{i,\epsilon}\cap Q_{j,\epsilon}$ are smooth complete intersections of two odd-dimensional quadrics
- motivic decomposition of Chow motive $\mathfrak{h}(Q_{i,\epsilon}\cap Q_{j,\epsilon})_{\mathbb{Q}}$ (Bernardara-Tabuada)

$$1\oplus\mathbb{L}\oplus\mathbb{L}^{\otimes 2}\oplus\cdots\oplus\mathbb{L}^{\otimes(D-2)}\oplus(\mathbb{L}^{\otimes(D-1)})^{\oplus(2D+2)}\oplus\mathbb{L}^{\otimes D}\oplus\cdots\oplus\mathbb{L}^{\otimes(2D-2)}$$

so $M^c(Q_{i,\epsilon}\cap Q_{j,\epsilon})_{\mathbb{Q}}\simeq M(Q_{i,\epsilon}\cap Q_{j,\epsilon})_{\mathbb{Q}}$ mixed-Tate

- Gysin triangle with with $X:=\mathbb{P}^{2D}$ and $Z:=Q_{i,\epsilon}\cap Q_{j,\epsilon}\Rightarrow$ $M^c(\mathbb{P}^{2D}\setminus(Q_{i,\epsilon}\cap Q_{j,\epsilon}))_{\mathbb{Q}}$ mixed-Tate

Step 5: the motive $M^c(\mathbb{P}^{2D} \backslash (Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}))_{\mathbb{Q}}$ is not mixed-Tate

- same Gysin triangle argument: $M^c(\mathbb{P}^{2D} \backslash (Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}))_{\mathbb{Q}}$ mixed-Tate iff $M^c(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon})_{\mathbb{Q}}$ mixed-Tate
- by transversality $Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}$ smooth complete intersection of three odd-dimensional quadrics
- motivic decomposition of Chow motive $\mathfrak{h}(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon})_{\mathbb{Q}}$ (Bernardara-Tabuada)

$$1 \oplus \mathbb{L} \oplus \mathbb{L}^{\otimes 2} \oplus \cdots \oplus \mathbb{L}^{\otimes (2D-3)} \oplus (\mathfrak{h}^1(J_a^{D-2}(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}))_{\mathbb{Q}} \otimes \mathbb{L}^{\otimes (D-1)})$$

- $J_a^{D-2}(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon})$ is the $(D-2)$-th intermediate algebraic Jacobian of $Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}$
- abelian variety $J_a^{D-2}(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon})$ isomorphic to Prym variety $\mathrm{Prym}(\widetilde{C}/C)$
- Prym variety $\mathrm{Prym}(\widetilde{C}/C)$ with C discriminant divisor of quadric fibration associated to the triple intersection and $\widetilde{C}$ étale double cover of curve C

- if motive $\mathfrak{h}(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon})_{\mathbb{Q}}$ were mixed-Tate it would be sum of powers of Lefschetz motive $\mathbb{L}$, hence only even dimensional cohomology
- but first cohomology

$$H^1(\mathfrak{h}(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon})_{\mathbb{Q}}) = H^1(\mathfrak{h}^1(\mathrm{Prym}(\tilde{C}/C))_{\mathbb{Q}})$$

$$= H^1(\mathrm{Prym}(\tilde{C}/C)) \neq 0$$

- so non-mixed-Tate because of term

$$\mathfrak{h}^1(J_a^{D-2}(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}))_{\mathbb{Q}} \otimes \mathbb{L}^{\otimes(D-1)}$$

Conclusion: the Feynman motive $M^c(\mathbb{P}^{2D} \setminus Q_{(\Gamma,m)})_{\mathbb{Q}}$ of the Sunset Graph is non-mixed-Tate (for generic non-zero mass parameters)

Prym varieties and intermediate Jacobians

- A. Beauville, *Variétés de Prym et Jacobiennes intermédiaires*. Ann. Sci. de l'ENS 10 (1977) 309–391.

Prym varieties

- $\pi : \tilde{C} \to C$ étale double covering of curves
- pullback maps on the Jacobians $\pi^* : J \to \tilde{J}$
- norm map $N : \tilde{J} \to J$ (project points of divisor)
- Prym variety is the kernel of the norm map (largest abelian subvariety of $\tilde{J}$ on which norm map is trivial)

Algebraic intermediate Jacobian

- Jacobian $J = J(C)$ of curves C extended for complex varieties (Griffiths) to family

$$J^i(X) = H^{2i+1}(X,\mathbb{C})/(F^{i+1}H_B^{2i+1}(X,\mathbb{C}) + H_B^{2i+1}(X,\mathbb{Z}))$$

  using Hodge filtration
- except for $i = 0, d-1$ (Picard, Albanese) intermediate Jacobians are not algebraic
- however they contain algebraic $J_a^i(X) \subset J^i(X)$ given by image of Abel-Jacobi map

$$AJ_i : CH^{i+1}(X)_{\mathbb{Z}}^{alg} \to J^i(X)$$

  on algebraic cycles of codimension $i+1$ trivial wrt the algebraic equivalence relation
- algebraic equivalence relation: $Z \sim Z'$ is $\exists$ curve C and cycle α in $X \times C$ with $[\alpha \cap (X \times \{c\})] - [\alpha \cap [X \times \{c'\}] = [Z] - [Z']$ for some $c, c' \in C$ (as cycles)
- abelian varieties: relation between algebraic intermediate Jacobians and Prym varieties (Beauville)

Other periods

- same differential form $\eta_{\alpha,\epsilon}$
- taking the derivative with respect to the mass parameter
- this raises powers of the edge propagators

$$\eta_{\alpha_1,\ldots,\alpha_n} = \frac{\omega}{\prod_{i=1}^{n} q_i^{\alpha_i}}$$

- these are solutions of differential system satisfied by the Feynman integral

Question: provide an upper bound estimative for the dimension of the space of these Feynman integrals, through a bound on the dimension of the space of periods on the Feynman motive

Dimension bound: upper bound given by $7 + \dim H^1(\mathrm{Prym}(\widetilde{C}/C))$

- estimate dimension of space of periods in middle cohomology
- periods are pairing between de Rham and Betti cohomology:

$$H^{2D}_{dR}(\mathbb{P}^{2D}\setminus Q_{(\Gamma,m)}) \times H^{2D}_{B}(\mathbb{P}^{2D}\setminus Q_{(\Gamma,m)}) \longrightarrow \mathbb{C}$$

- estimate dimension of middle cohomology $H^{2D}(\mathbb{P}^{2D}\setminus Q_{(\Gamma,m)})$
- take $\mathbb{P} = \mathbb{P}^{2D}$, $U = \mathbb{P}^{2D}\setminus(Q_{1,\epsilon} \cup Q_{2,\epsilon})$, $V = \mathbb{P}^{2D}\setminus Q_{3,\epsilon}$, $U_{13} = \mathbb{P}^{2D}\setminus(Q_{1,\epsilon} \cap Q_{3,\epsilon})$, $U_{23} = \mathbb{P}^{2D}\setminus(Q_{2,\epsilon} \cap Q_{3,\epsilon})$, and $Q_{123} = Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}$
- Mayer-Vietoris triangle with $U \cap V = \mathbb{P}\setminus Q_{(\Gamma,m)}$ gives long exact sequence in cohomology

- in a long exact sequence $\cdots \to V^{r-1} \to V^r \to V^{r+1} \to \cdots$ dimensions $\dim(V^r) \leq \dim(V^{r-1}) + \dim(V^{r+1})$
- long exact sequence in cohomology

$$\cdots \to H^r(U \cup V) \to H^r(U) \oplus H^r(V) \to H^r(U \cap V) \to H^{r+1}(U \cup V) \to \cdots$$

- this gives estimate on dimensions

$$\dim H^{2D}(\mathbb{P}^{2D} \setminus Q_{\Gamma,m}) \leq \dim H^{2D}(U) + \dim H^{2D}(V) + \dim H^{2D+1}(U \cup V)$$

$$\dim H^{2D+1}(U \cup V) \leq \dim H^{2D+1}(U_{13}) + \dim H^{2D+1}(U_{23}) + \dim H^{2D+2}(\mathbb{P} \setminus Q_{12}$$

- given complete intersection $Z_1 \cap \cdots \cap Z_c =: Z \subset \mathbb{P}$ codimension c, Gysin long exact sequence in cohomology:

$$\cdots \to H^{r-2c}(Z) \to H^r(\mathbb{P}) \to H^r(\mathbb{P} \setminus Z) \to H^{r-2c+1}(Z) \to H^{r+1}(\mathbb{P}) \to \cdots$$

- for U_{ij} either U_{13} or U_{23} with $Q_{i,\epsilon} \cap Q_{j,\epsilon}$ codimension 2

$$H^{r-4}(Q_{i,\epsilon} \cap Q_{j,\epsilon}) \to H^r(\mathbb{P}) \to H^r(U_{ij}) \to H^{r-3}(Q_{i,\epsilon} \cap Q_{j,\epsilon}) \to H^{r+1}(\mathbb{P})$$

- for $r = 2D + 1$ estimate:

$$\dim H^{2D+1}(U_{ij}) \leq \dim H^{2D+1}(\mathbb{P}) + \dim H^{2D-2}(Q_{i,\epsilon} \cap Q_{j,\epsilon})$$
$$\leq \dim H^{2D-2}(Q_{i,\epsilon} \cap Q_{j,\epsilon})$$

- by motivic decomposition have $\mathfrak{h}^{2D-2}(Q_{i,\epsilon} \cap Q_{j,\epsilon})_{\mathbb{Q}} \simeq \mathbb{L}^{D-1}$
- so get $\dim H^{2D-2}(Q_{i,\epsilon} \cap Q_{j,\epsilon}) = 1$
- previous estimates give

$$\dim H^{2D+1}(U \cup V) \leq 2 + \dim H^{2D+2}(\mathbb{P} \setminus Q_{123})$$

$$\dim H^{2D}(\mathbb{P}^{2D} \setminus Q_{(\Gamma,m)}) \leq 2 + \dim H^{2D}(U) + \dim H^{2D}(V) + \dim H^{2D+2}(\mathbb{P} \setminus Q_{123})$$

- estimate $\dim H^{2D+2}(\mathbb{P} \setminus Q_{123})$ via Gysin exact sequence

$$H^{r-6}(Q_{123}) \to H^r(\mathbb{P}) \to H^r(\mathbb{P} \smallsetminus Q_{123}) \to H^{r-6+1}(Q_{123})$$

- taking $r = 2D + 2$ and using
$\dim H^{2D-3}(Q_{123}) = \dim H^1(\mathrm{Prym}(\tilde{C}/C))$

$$\dim H^{2D+2}(\mathbb{P} \setminus Q_{123}) \leq \dim H^1(\mathrm{Prym}(\tilde{C}/C) + 1$$

- estimate $\dim H^{2D}(V)$ using Gysin exact sequence for $Q_{3,\epsilon}$ codimension 1

$$\cdots \to H^{i-2}(Q_{3,\epsilon}) \to H^i(\mathbb{P}) \to H^i(V) \to H^{i-2+1}(Q_{3,\epsilon}) \to \cdots$$

- get estimate

$$\dim H^{2D}(V) \leq \dim H^{2D-1}(Q_{3,\epsilon}) + \dim H^{2D}(\mathbb{P})$$

- by motivic decomposition $Q_{3,\epsilon}$ has no odd cohomology so get $\dim H^{2D}(V) \leq 1$
- estimate $\dim H^{2D}(U)$ with Mayer-Vietoris long exact sequence for open cover $\{\mathbb{P}\setminus Q_{1,\epsilon}, \mathbb{P}\setminus Q_{2,\epsilon}\}$ of $\mathbb{P}\setminus(Q_{1,\epsilon} \cap Q_{2,\epsilon})$

$$\begin{aligned}\dim H^{2D}(U) \leq \quad & \dim H^{2D+1}(\mathbb{P}\setminus(Q_{1,\epsilon} \cap Q_{2,\epsilon})) + \dim H^{2D}(\mathbb{P}\setminus Q_{1,\epsilon}) \\ & + \dim H^{2D}(\mathbb{P}\setminus Q_{2,\epsilon}).\end{aligned}$$

- as for U_{13} and U_{23} have $\dim H^{2D+1}(U_{12}) \leq 1$
- similarly to V have $\dim H^{2D}(\mathbb{P} \setminus Q_{i,\epsilon}) \leq 1$
- so get $\dim H^{2D}(U) \leq 3$
- combining all these estimates get middle cohomology bounded by $7 + \dim H^1(\mathrm{Prym}(\widetilde{C}/C))$

Massless case: motive of the sunset graph known to be mixed-Tate

- P. Aluffi, M. Marcolli, *Feynman motives of banana graphs*, Commun. Number Theory Phys. 3 (2009), no. 1, 1–57.

Why previous argument does not apply?

- quadrics Q_i in $\mathbb{P}^{LD-1}$ (no x coordinate)
- still deform so quadrics smooth and transverse
- even dimensional quadrics $Q_{i,\epsilon}$ odd dimensional $\mathbb{P}^{2D-1}$
- motivic decomposition (Bernardara-Tabuada)

$$\mathfrak{h}(Q_{i,\epsilon} \cap Q_{j,\epsilon})_{\mathbb{Q}} = \begin{cases} \mathbb{L}^{\otimes i/2} & 0 \leq i \leq 2d, \ i \text{ even} \\ \mathfrak{h}^1(J_a^{D-2}(Q_{i,\epsilon} \cap Q_{j,\epsilon}))_{\mathbb{Q}} & i = d \\ 0 & \text{otherwise} \end{cases}$$

- now find distinguished triangles with two terms non-mixed-Tate: cannot say anything about third

One-loop triangle graph: massive case known to be mixed-Tate

- S. Bloch, D. Kreimer, *Mixed Hodge structures and renormalization in physics*. Comm. Number Theory Phys. 2 (2008), no. 4, 637–718.

Why previous argument does not apply?

- if zero external momenta, momentum conservation at vertices: same momentum entering through an edge exists through the other
- three Feynman quadrics in same momentum variable $k \in \mathbb{A}^D$
- momentum conservation condition for the deformations forces identification $\bar{T}_{i,\epsilon} = \bar{T}_{j,\epsilon}$ on consecutive edges
- no deformation satisfying momentum conservation can achieve transversality