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Parametric Feynman integrals and determinant hypersurfaces

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Abstract

The purpose of this paper is to show that, under certain combinatorial conditions on the graph, parametric Feynman integrals can be realized as periods on the complement of the determinant hypersurface $\hat{\mathcal{D}}_\ell$ in affine space $\mathbb{A}^{\ell^2}$, with ℓ the number of loops of the Feynman graph. The question of whether these are periods of mixed Tate motives can then be reformulated as a question on a relative cohomology of the pair $(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\Sigma}_{\ell,g} \cap \hat{\mathcal{D}}_\ell))$ being a realization of a mixed Tate motive, where $\hat{\Sigma}_{\ell,g}$ is a normal crossing divisor depending only on the number of loops and the genus of the graph. We show explicitly that the relative cohomology is a realization of a mixed Tate motive in the case of three loops and we give alternative formulations of the main question in the general case, by describing the locus $\hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\Sigma}_\ell \cap \hat{\mathcal{D}}_\ell)$ in terms of intersections of unions of Schubert cells in flag varieties. We also discuss different methods of regularization aimed at removing the divergences of the Feynman integral.

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1 Introduction

The question of whether Feynman integrals arising in perturbative scalar quantum field theory are periods of mixed Tate motives can be seen (see [9, 10]) as a question on whether certain relative cohomologies associated to algebraic varieties defined by the data of the parametric representation of the Feynman integral are realizations of mixed Tate motives. In this paper we investigate another possible viewpoint on the problem, which leads us to consider a different relative cohomology, defined in terms of the complement of the affine determinant hypersurface and the locus where the hypersurface intersects the image of a simplex under a linear map defined by the Feynman graph. For all graphs with a given number of loops ℓ, admitting a minimal embedding in an orientable surface of genus g, and satisfying a natural combinatorial condition, we relate the question mentioned above to a problem in the geometry of coordinate subspaces of an ℓ-dimensional vector space, which only depends on the genus g.

More precisely, we consider for each graph Γ as above and satisfying a transparent combinatorial condition (summarized at the beginning of §5) a normal crossing divisor $\hat{\Sigma}_\Gamma$ in the affine space $\mathbb{A}^{\ell^2}$ of $\ell \times \ell$ matrices. We observe that, modulo the issue of divergences, the parametric Feynman integral is a period of the pair $(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_\Gamma))$, where $\hat{\mathcal{D}}_\ell$ is the determinant hypersurface. We then observe that all these normal crossing divisors $\hat{\Sigma}_\Gamma$ may be immersed into a fixed normal crossing divisor $\hat{\Sigma}_{\ell,g}$, determined by the number of loops ℓ and the embedding genus g; therefore, the question of whether Feynman integrals are periods of mixed Tate motives may be decided by verifying that the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_{\ell,g})),$$

whose realization is the relative cohomology of the corresponding pair, is mixed Tate. In fact, we show that verifying this assertion for $g = 0$ would suffice to deal with all graphs Γ with $b_1(\Gamma) = \ell$ (and satisfying our combinatorial condition), simultaneously for all genera.

We approach this question by an inclusion–exclusion argument, reducing it to verifying that specific loci in $\mathbb{A}^{\ell^2}$ are mixed Tate (see §5.3). We carry out this verification for $\ell \leq 3$ loops (§6), showing that the motive $\mathfrak{m}(\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3, \hat{\Sigma}_{3,0} \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_{3,0})$ is mixed Tate. In doing so, we obtain explicit formulae for the class $[\hat{\Sigma}_{3,0} \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_{3,0})]$ (corresponding to the 'wheel with three spokes') and for the classes of strata of the same locus, in the Grothendieck group of varieties. These classes may be assembled to construct the corresponding class for any graph with three loops (satisfying our combinatorial condition). This illustrates a simple case of our strategy: it follows that, modulo the issue of divergences, Feynman integrals of graphs with three or fewer loops are indeed periods of mixed Tate motives. Carrying out the same strategy for a larger number of loops is a worthwhile project.

At present, the restriction to $\ell \leq 3$ is dictated by the fact that only in this case we are able to provide an explicit description as mixed Tate motives of the manifolds of frames $\mathbb{F}(V_1, \dots, V_\ell)$ that we introduce in Section 6 as a way to control the motivic nature of the locus $\hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_{\ell,g})$.

Finally, in Section 7 we discuss the problem of regularization of divergent Feynman integrals, and how different possible regularizations can be made compatible with the approach via determinant hypersurfaces described here.

We recall the basic notation and terminology we use in the following. The notion of a mixed Tate motives over a field $\mathbb{K}$ that we adopt for the purpose of this paper is objects of the triangulated subcategory $\mathcal{DMT}_{\mathbb{K}}$ of the Voevodsky triangulated category $\mathcal{MD}_{\mathbb{K}}$ of mixed motives [33]. All the graph hypersurfaces and the other algebraic varieties we consider in this paper are defined over $\mathbb{K} = \mathbb{Q}$. We will not need the detailed technical construction of the triangulated category of mixed motives, as for our purposes it will suffice to use some of the formal properties that elements of this category satisfy. The relevant properties are recalled in Section 4.1.

Moreover, throughout the paper we use the following terminology.

Definition 1.1. Consider a scalar field theory with Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{\text{int}}(\phi), \tag{1.1}$$

where $\mathcal{L}_{int}(\phi)$ is a polynomial in ϕ of degree at least three. Then a one particle irreducible (1PI) Feynman graph Γ of the theory is a finite connected graph with the following properties.

- The valence of each vertex is equal to the degree of one of the monomials in the Lagrangian (1.1).
- The set $E(\Gamma)$ of edges of the graph is divided into *internal* and *external* edges, $E(\Gamma) = E_{\text{int}}(\Gamma) \cup E_{\text{ext}}(\Gamma)$. Each internal edge connects two vertices of the graph, while the external edges have only one vertex. (One thinks of an internal edges as being a union of two half-edges and an external one as being a single half-edge.)
- The graph cannot be disconnected by removing a single internal edge. This is the 1PI condition.

In the following we denote by $n = \#E_{\rm int}(\Gamma)$ the number of internal edges, by $N = \#E_{\rm ext}(\Gamma)$ the number of external edges, and by $\ell = b_1(\Gamma)$ the number of loops.

In their parametric form, the Feynman integrals of *massless* perturbative scalar quantum field theories (cf. [22, §6-2-3, 7, §18 and 27, §6]) are integrals of the form

$$U(\Gamma, p) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\, \omega_n}{\Psi_\Gamma(t)^{-n+(\ell+1)D/2}}, \tag{1.2}$$

where $\Gamma(n - D\ell/2)$ is a possibly divergent Γ-factor, σ_n is the simplex

$$\sigma_n = \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n_+ \,\middle|\, \sum_i t_i = 1 \right\} \tag{1.3}$$

and the polynomials $\Psi_\Gamma(t)$ and $P_\Gamma(t,p)$ are obtained from the combinatorics of the graph, respectively, as

$$\Psi_\Gamma(t) = \sum_{T\subset\Gamma} \prod_{e\notin E(T)} t_e, \tag{1.4}$$

where the sum is over all the spanning trees T of Γ and

$$P_\Gamma(p,t) = \sum_{C\subset\Gamma} s_C \prod_{e\in C} t_e, \tag{1.5}$$

where the sum is over the cut-sets $C \subset \Gamma$, i.e., the collections of $b_1(\Gamma) + 1$ internal edges that divide the graph Γ in exactly two connected components $\Gamma_1 \cup \Gamma_2$. The coefficient s_C is a function of the external momenta attached to the vertices in either one of the two components

$$s_C = \left(\sum_{v\in V(\Gamma_1)} P_v \right)^2 = \left(\sum_{v\in V(\Gamma_2)} P_v \right)^2. \tag{1.6}$$

Here the P_v are defined as

$$P_v = \sum_{e\in E_{\rm ext}(\Gamma),\, t(e)=v} p_e, \tag{1.7}$$

where the p_e are incoming external momenta attached to the external edges of Γ and satisfying the conservation law

$$\sum_{e\in E_{\mathrm{ext}}(\Gamma)} p_e = 0. \tag{1.8}$$

In order to work with algebraic differential forms defined over $\mathbb{Q}$, we assume that the external momenta are also taking rational values $p_e \in \mathbb{Q}^D$.

Ignoring the Γ-function factor in (1.2), one is interested in understanding what kind of period is the integral

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\,\omega_n}{\Psi_\Gamma(t)^{-n+(\ell+1)D/2}}. \tag{1.9}$$

In quantum field theory one can consider the same physical theory (with specified Lagrangian) in different spacetime dimensions $D \in \mathbb{N}$. In fact, one should think of the dimension D as one of the variable parameters in the problem. For the purposes of this paper, we work in the range where D is sufficiently large, so that $n \le D\ell/2$. The case $n = D\ell/2$ is the *log divergent* case, where the integral (1.9) simplifies to the form

$$\int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}. \tag{1.10}$$

Another case where the Feynman integral has the simpler form (1.10), even for graphs that do not necessarily satisfy the log divergent condition, i.e., for $n \ne D\ell/2$, is where one considers the case with non-zero mass $m \ne 0$, but with external momenta set equal to zero. In such cases, the parametric Feynman integral becomes of the form

$$\int_{\sigma_n} \frac{V_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{D/2}}|_{p=0} = m^{-2n+D\ell} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}, \tag{1.11}$$

where $V_\Gamma(t,p)$ is of the form

$$V_\Gamma(t,p) = p^\dagger R_\Gamma(t) p + m^2,$$

with

$$V_\Gamma(t,p)|_{m=0} = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)}.$$

In the following we assume that we are either in the massless case (1.9) and in the range of dimensions D satisfying $n \leq D\ell/2$, or in the massive case with zero external momenta (1.11) and arbitrary dimension.

A first issue one needs to clarify in addressing the question of Feynman integrals and periods is the fact that the integral (1.9) is often divergent. Divergences are contributed by the intersection $\sigma_n \cap \hat{X}_\Gamma$, with $\hat{X}_\Gamma = \{t \in \mathbb{A}^n \mid \Psi_\Gamma(t) = 0\}$, which is often non-empty. Although there are cases where a non-empty intersection $\sigma_n \cap \hat{X}_\Gamma$ may still give rise to an absolutely convergent integral, hence a period, these are relatively rare cases and usually some regularization and renormalization procedure is needed to eliminate the divergences over the locus where the domain of integration meets the graph hypersurface. Notice that these intersections only occur on the boundary $\partial\sigma_n$, since in the interior of σ_n the polynomial $\Psi_\Gamma(t)$ is strictly positive (see (1.4)).

Our results will apply directly to all cases where the integral is convergent, while we discuss in Section 7 the case where a regularization procedure is required to treat divergences in the Feynman integrals. The main question is then, more precisely formulated, whether it is true that the numbers obtained by computing such integrals (after removing a possibly divergent Gamma factor, and after regularization and renormalization when needed) are always periods of mixed Tate motives.

The main contribution of this paper is the reformulation of the problem, where instead of working with the graph hypersurfaces X_Γ defined by the vanishing of the graph polynomial Ψ_Γ, one works with the complement of a fixed determinant hypersurface in an affine space of matrices. This allows us to reduce the problem to one that only depends on the number of loops of the graph, at least for the class of graphs satisfying the combinatorial condition discussed in Section 2 (for example, 3-vertex connected planar graphs with ℓ loops). We propose specific questions in terms of ℓ alone, in Section 5.3; these questions may be appreciated independently of our motivation, as they do not refer directly to Feynman graphs. We hope that these reformulations might help to connect the problem to other interesting questions, such as the geometry of intersections of Schubert cells and Kazhdan–Lusztig theory.

2 Feynman parameters and determinants

With the notation as above, for a given Feynman graph Γ, the graph hypersurface X_Γ is defined as the locus of zeros

$$X_\Gamma = \{t = (t_1 : \ldots : t_n) \in \mathbb{P}^{n-1} \mid \Psi_\Gamma(t) = 0\}. \tag{2.1}$$

Indeed, Ψ_Γ is homogeneous of degree ℓ, hence it defines a hypersurface of degree ℓ in the projective space $\mathbb{P}^{n-1}$. We will also consider the affine cone on X_Γ, namely the affine hypersurface

$$\hat{X}_\Gamma = \{t \in \mathbb{A}^n \mid \Psi_\Gamma(t) = 0\}. \tag{2.2}$$

The question of whether the Feynman integral is a period of a mixed Tate motive can be approached (modulo the divergence problem) as a question on whether the relative cohomology

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma)) \tag{2.3}$$

is a realization of a mixed Tate motive, where Σ_n is the algebraic simplex

$$\Sigma_n = \left\{ t \in \mathbb{P}^{n-1} \,\middle|\, \prod_i t_i = 0 \right\}, \tag{2.4}$$

i.e., the union of the coordinate hyperplanes containing the boundary of the domain of integration $\partial\sigma_n \subset \Sigma_n$. See for instance [9, 10].

Although working in the projective setting is very natural (see [10]), there are several reasons why it may be preferable to consider affine hypersurfaces:

- Only in the limit cases of a massless theory or of zero external momenta in the massive case does the parameteric Feynman integral involve the quotient of two homogeneous polynomial ([7, §18]).
- The deformation of the ϕ^4 quantum field theory to non-commutative spacetime, which has been the focus of much recent research (see, e.g., [20]), shows that, even in the massless case, the graph polynomials Ψ_Γ and P_Γ are no longer homogeneous in the non-commutative setting and only in the limit commutative case they recover this property (see [21, 23]).
- As shown in [2], in the affine setting the graph hypersurface complement satisfies a multiplicative property over disjoint unions of graphs that makes it possible to define algebro-geometric and motivic Feynman rules.

For these various reasons, in this paper we primarily work in the affine rather than in the projective setting.

In the present paper, we approach the problem in a different way, where instead of working with the hypersurface $\hat{X}_\Gamma$, we map the Feynman integral computation and the graph hypersurface in a larger hypersurface $\hat{\mathcal{D}}_\ell$ inside a larger affine space, so that we will be dealing with a relative cohomology replacing (2.3) where the ambient space (the hypersurface complement) only depends on the number of loops in the graph.

2.1 Determinant hypersurfaces and graph polynomials

We now show that all the affine varieties $\hat{X}_\Gamma$, for fixed number of loops ℓ, map naturally to a larger hypersurface in a larger affine space, by realizing the polynomial Ψ_Γ for the given graph as a pullback of a fixed polynomial Ψ_ℓ in ℓ^2-variables.

Recall that the determinant hypersurface $\mathcal{D}_\ell$ is defined in the following way. Let $k[x_{kr}, k, r = 1, \ldots, \ell]$ be the polynomial ring in ℓ^2 variables and set

$$\mathcal{D}_\ell = \{x = (x_{kr}) \mid \det(x) = 0\}. \tag{2.5}$$

Since the determinant is a homogeneous polynomial Ψ_ℓ, this in particular also defines a projective hypersurface in $\mathbb{P}^{\ell^2-1}$. We will however mostly concentrate on the affine hypersurface $\hat{\mathcal{D}}_\ell \subset \mathbb{A}^{\ell^2}$ defined by the vanishing of the determinant, i.e., the cone in $\mathbb{A}^{\ell^2}$ of the projective hypersurface $\mathcal{D}_\ell$.

Suppose given any Feynman graph Γ with $b_1(\Gamma) = \ell$, and with $\#E_{\rm int}(\Gamma) = n$. It is well known (see, e.g., [7, §18]) that the graph polynomial $\Psi_\Gamma(t)$ can be equivalently written in the form of a determinant

$$\Psi_\Gamma(t) = \det\, M_\Gamma(t) \tag{2.6}$$

of an $\ell \times \ell$-matrix

$$(M_\Gamma)_{kr}(t) = \sum_{i=1}^{n} t_i \eta_{ik} \eta_{ir}, \tag{2.7}$$

where the $n \times \ell$-matrix η_{ik} is defined in terms of the edges $e_i \in E(\Gamma)$ and a choice of a basis for the first homology group, $l_k \in H_1(\Gamma, \mathbb{Z})$, with $k = 1, \ldots, \ell = b_1(\Gamma)$, by setting

$$\eta_{ik} = \begin{cases} +1 & \text{edge } e_i \in \text{loop } l_k, \text{ same orientation} \\ -1 & \text{edge } e_i \in \text{loop } l_k, \text{ reverse orientation} \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

The determinant $\det\, M_\Gamma(t)$ is independent both of the choice of orientation on the edges of the graph and of the choice of generators for $H_1(\Gamma, \mathbb{Z})$.

The expression of the matrix $M_\Gamma(t)$ defines a linear map $\tau : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ of the form

$$\tau = \tau_\Gamma : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \tau(t_1, \ldots, t_n) = \sum_i t_i \eta_{ki} \eta_{ir}. \tag{2.9}$$

We can write this equivalently in the shorter form

$$\tau = \eta^\dagger \Lambda \eta, \tag{2.10}$$

where Λ is the diagonal $n \times n$-matrix with $t_1, \ldots, t_n$ as diagonal entries, and $\eta = \eta_\Gamma$ is the matrix (2.8).

Then by construction we have that $\hat{X}_\Gamma = \tau^{-1}(\hat{\mathcal{D}}_\ell)$, from (2.6). We formalize this as follows:

Lemma 2.1. *Let Γ be a Feynman graph with n internal edges and ℓ loops. Let $\hat{X}_\Gamma \subset \mathbb{A}^n$ denote the affine cone on the projective hypersurface $X_\Gamma \subset \mathbb{P}^{n-1}$. Then*

$$\hat{X}_\Gamma = \tau^{-1}(\hat{\mathcal{D}}_\ell), \tag{2.11}$$

where $\tau : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ is a linear map depending on Γ.

The next lemma, which follows directly from the definitions, details some of the properties of the map τ introduced above that we will be using in the following.

Lemma 2.2. *The matrix of τ, $M_\Gamma(t) = \eta^\dagger \Lambda \eta$, has the following properties.*

- *For $i \neq j$, the corresponding entry is the sum of $\pm t_k$, where the t_k correspond to the edges common to the ith and jth loop, and the sign is $+1$ if the orientations of the edges both agree or both disagree with the loop orientations, and -1 otherwise.*
- *For $i = j$, the entry is the sum of the variables t_k corresponding to the edges in the ith loop (all taken with sign $+$).*

Now consider a specific edge e, and let t_e be the corresponding variable. Then

- *The variable t_e appears in $\eta^\dagger \Lambda \eta$ if and only if e is part of at least one loop.*
- *If e belongs to a single loop ℓ_i, then t_e only appears in the diagonal entry (i, i), added to the variables corresponding to the other edges forming the loop ℓ_i.*
- *If there are two loops ℓ_i, ℓ_j containing e, and not having any other edge in common, then the $\pm t_e$ appears by itself at the entries (i, j) and (j, i) in the matrix $\eta^\dagger \Lambda \eta$.*

When the map τ constructed above is injective, it is possible to rephrase the computation of the parametric Feynman integral (1.9) as a period of the complement of the determinant hypersurface $\hat{\mathcal{D}}_\ell \subset \mathbb{A}^{\ell^2}$.

Lemma 2.3. *Assume that the map $\tau : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ of* (2.10) *is injective. Then the integral* (1.9) *can be rewritten in the form*

$$\int_{\tau(\sigma_n)} \frac{\mathcal{P}_\Gamma(p,x)^{-n+D\ell/2}\,\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}, \tag{2.12}$$

where $\mathcal{P}_\Gamma(p,x)$ is a homogeneous polynomial on $\mathbb{A}^{\ell^2}$ whose restriction to the image of $\mathbb{A}^n$ under the map τ agrees with $P_\Gamma(p,t)$, and ω_Γ is the induced volume form.

Proof. It is possible to regard the polynomial $P_\Gamma(p,t)$ as the restriction to $\mathbb{A}^n$ of a homogeneous polynomial $\mathcal{P}_\Gamma(p,x)$ defined on all of $\mathbb{A}^{\ell^2}$. Clearly, such $\mathcal{P}_\Gamma(p,x)$ will not be unique, but different choices of $\mathcal{P}_\Gamma(p,x)$ will not affect the integral calculation, which all happens inside the linear subspace $\mathbb{A}^n$. The simplex σ_n is also linearly embedded inside $\mathbb{A}^{\ell^2}$, and we denote its image by $\tau(\sigma_n)$. The volume form ω_n can also be identified, under such a choice of coordinates in $\mathbb{A}^{\ell^2}$ with a form $\omega_\Gamma(x)$ such that

$$\omega_\Gamma(x) \wedge \langle \xi_\Gamma, dx\rangle = \omega_{\ell^2},$$

with ξ_Γ the $(\ell^2 - n)$-frame associated to the linear subspace $\tau(\mathbb{A}^n) \subset \mathbb{A}^{\ell^2}$ and

$$\langle \xi_\Gamma, dx\rangle = \langle \xi_1, dx\rangle \wedge \cdots \wedge \langle \xi_{\ell^2-n}, dx\rangle.$$

□

Notice in particular that if the map τ is injective then one has a well defined map $\mathbb{P}^{n-1} \to \mathbb{P}^{\ell^2-1}$, which is otherwise not everywhere defined.

We are interested in the following, heuristically formulated, consequence of Lemma 2.3.

Claim 2.1. *Assume that the map $\tau : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ of* (2.10) *is injective. Then the complexity of Feynman integrals corresponding to the graph Γ is controlled by the motive $\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_\Gamma))$, where $\hat{\Sigma}_\Gamma$ is a normal crossings divisor in $\mathbb{A}^{\ell^2}$ such that $\tau(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$.*

The explicit construction of the normal crossings divisor $\hat{\Sigma}_\Gamma$ is given in Lemma 5.1 below. We will further improve on this observation by reformulating it in a way that will only depend on the number of loops ℓ of Γ and on its genus, and not on the specific graph Γ. To this purpose, we will determine subsets of $\mathbb{A}^{\ell^2}$ which will contain the components of the image $\tau(\partial\sigma_n)$ of the boundary of the simplex in $\mathbb{A}^n$, independently of Γ (see Section 3.4).

In any case, this type of results motivates us to determine conditions on the Feynman graph Γ which ensure that the corresponding map $\tau : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ is injective.

3 Graph theoretic conditions for embeddings

3.1 Injectivity of τ

In the following, we denote by τ_i the composition of the map τ of (2.10) with the projection to the ith row of the matrix $\eta^\dagger \Lambda \eta$, viewed as a map of the variables corresponding only to the edges that belong to the ith loop in the chosen bases of the first homology of the graph Γ.

We first make the following simple observation.

Lemma 3.1. *If τ_i is injective for i ranging over a set of loops such that every edge of Γ is part of a loop in that set, then τ is itself injective.*

Proof. Let $(t_1, \dots, t_n) = (c_1, \dots, c_n)$ be in the kernel of τ. Since each (i, j) entry in the target matrix is a combination of edges in the ith loop, the map τ_i must send to zero the tuple of c_j's corresponding to the edges in the ith loop. Since we are assuming τ_i to be injective, that tuple is the zero-tuple. Since every edge is in some loop for which τ_i is injective, it follows that every c_j is zero, as needed. □

The properties detailed in Lemma 2.2 immediately provide a sufficient condition for the maps τ_i to be injective.

Lemma 3.2. *The map τ_i is injective if the following conditions are satisfied:*

- *For every edge e of the ith loop, there is another loop having only e in common with the ith loop, and*
- *The ith loop has at most one edge not in common with any other loop.*

Proof. In this situation, all but at most one edge variable appear by themselves as an entry of the ith row, and the possible last remaining variable appears summed together with the other variables. More explicitly, if $t_{i_1}, \dots, t_{i_v}$ are the variables corresponding to the edges of a loop ℓ_i, up to rearranging the entries in the corresponding row of $\eta^\dagger \Lambda \eta$ and neglecting other entries, the map τ_i is given by

$$(t_{i_1}, \dots, t_{i_v}) \mapsto (t_{i_1} + \cdots + t_{i_v}, \pm t_{i_1}, \dots, \pm t_{i_v})$$

if ℓ_i has no edge not in common with any other loop, and

$$(t_{i_1}, \dots, t_{i_v}) \mapsto (t_{i_1} + \cdots + t_{i_v}, \pm t_{i_1}, \dots, \pm t_{i_{v-1}})$$

if ℓ_i has a single edge t_v not in common with any other loop. In either case the map τ_i is injective, as claimed. □

Now we need a sufficiently natural combinatorial condition on the graph Γ that ensures that the conditions of Lemma 3.2 and Lemma 3.1 are fulfilled. We first recall some useful facts about graphs and embeddings of graphs on surfaces which we need in the following.

Every (finite) graph Γ may be embedded in a compact orientable surface of finite genus. The minimum genus of an orientable surface in which Γ may be embedded is the *genus* of Γ. Thus, Γ is planar if and only if it may be embedded in a sphere, if and only if its genus is 0.

Definition 3.1. An embedding of a graph Γ in an orientable surface S is a *2-cell embedding* if the complement of Γ in S is homeomorphic to a union of open 2-cells (the *faces*, or *regions* determined by the embedding). An embedding of Γ in S is a *closed 2-cell embedding* if the closure of every face is a disk.

It is known that an embedding of a connected graph is minimal genus if and only if it is a 2-cell embedding [26, Proposition 3.4.1 and Theorem 3.2.4]. We discuss below conditions on the existence of *closed* 2-cell embeddings, cf. [26, §5.5].

For our purposes, the advantage of having a closed 2-cell embedding for a graph Γ is that the faces of such an embedding determine a choice of loops of Γ, by taking the boundaries of the 2-cells of the embedding together with a basis of generators for the homology of the Riemann surface in which the graph is embedded.

Lemma 3.3. *A closed 2-cell embedding $\iota : \Gamma \to S$ of a connected graph Γ on a surface of (minimal) genus g, together with the choice of a face of the embedding and a basis for the homology $H_1(S, \mathbb{Z})$ determine a basis of $H_1(\Gamma, \mathbb{Z})$ given by $2g + f - 1$ loops, where f is the number of faces of the embedding.*

Proof. Orient (arbitrarily) the edges of Γ and the faces, and then add the edges on the boundary of each face with sign determined by the orientations. The fact that the closure of each face is a 2-disk guarantees that the boundary is null-homotopic. This produces a number of loops equal to the number f of faces. It is clear that these f loops are *not* independent: the sum of any $f-1$ of them must equal the remaining one, up to sign. Any $f-1$ loops, however, will be independent of $H_1(\Gamma)$. Indeed, these $f-1$ loops, together with $2g$ generators of the homology of S, generate $H_1(\Gamma)$. The homology group $H_1(\Gamma)$ has rank $2g+f-1$, as one can see from the Euler characteristic formula

$$\begin{aligned} b_0(S) - b_1(S) + b_2(S) &= 2 - 2g = \chi(S) = v - e + f \\ &= b_0(\Gamma) - b_1(\Gamma) + f = 1 - \ell + f, \end{aligned}$$

so there will be no other relations. □

One refers to the chosen one among the f faces as the "external face" and the remaining $f-1$ faces as the "internal faces".

Thus, given a closed 2-cell embedding $\iota : \Gamma \to S$, we can use a basis of $H_1(\Gamma, \mathbb{Z})$ costructed as in Lemma 3.3 to compute the map τ of (2.10) and the maps τ_i of (2.2). We then have the following result.

Lemma 3.4. *Assume that Γ is closed-2-cell embedded in a surface. With notation as above, assume that*

- *any two of the f faces have at most one edge in common.*

Then the $f-1$ maps τ_i, defined with respect to a choice of basis for $H_1(\Gamma)$ as in Lemma 3.3, are all injective. If further

- *every edge of Γ is in the boundary of two of the f faces,*

then τ is injective.

Proof. The injectivity of the $f-1$ maps τ_i follows from Lemma 3.2. If ℓ is a loop determined by an internal face, the variables corresponding to edges in common between ℓ and any other internal loop will appear as $(\pm)$ individual entries on the row corresponding to ℓ. Since ℓ has at most one edge in common with the external region, this accounts for all but at most one of the edges in ℓ. By Lemma 3.2, the injectivity of τ_i follows.

Finally, as shown in Lemma 3.1, the map τ is injective if every edge is in one of the $f-1$ loops and the $f-1$ maps τ_i are injective. The stated

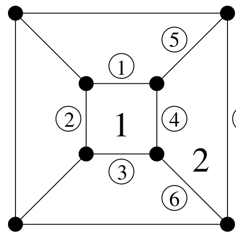

Figure 1: An example satisfying Lemma 3.4.

condition guarantees that the edge appears in the loops corresponding to the faces separated by that edge. At least one of them is internal, so that every edge is accounted for. □

Example 3.1. Consider the example of the planar graph in figure 1. The conditions stated in Lemma 3.4 are evidently satisfied. Edges are marked by circled numbers. The loop corresponding to region 1 consists of edges 1, 2, 3, 4. The corresponding row of $\eta^t T\eta$ is

$$(t_1 + t_2 + t_3 + t_4, \pm t_4, \pm t_3, \pm t_2, \pm t_1).$$

Region 2 consists of edges 4, 5, 6, 7. Edge 7 is not in any other internal region. The corresponding row of $\eta^\dagger \Lambda\eta$ is

$$(t_4 + t_5 + t_6 + t_7, \pm t_4, \pm t_5, \pm t_6).$$

These maps are injective, as claimed. Given the symmetry of the situation, it is clear that all maps τ_i (and hence τ as well) are injective for this graph, as guaranteed by Lemma 3.4.

The considerations that follow will allow us to improve on Lemma 3.4, by showing that in natural situations the second condition listed in Lemma 3.4 is automatically satisfied.

3.2 Connectivity of graphs

In this section we review some notions on connectivity for graphs, both for contextual reasons, since these notions relate well with conditions that are natural from the physical point of view, and in order to improve the results obtained above.

Given a graph Γ and a vertex $v \in V(\Gamma)$, the graph $\Gamma \smallsetminus v$ is the graph with vertex set $V(\Gamma) \smallsetminus \{v\}$ and edge set $E(\Gamma) \smallsetminus \{e : v \in \partial(e)\}$, i.e., the graph obtained by removing from Γ the star of the vertex v. It is customary to refer to $\Gamma \smallsetminus v$ simply as "the graph obtained by removing the vertex v", even though one in fact removes also all the edges adjacent to v.

There are two different notions of connectivity for graphs. To avoid confusion, we refer to them here as k-edge-connectivity and k-vertex-connectivity. For the notion of k-vertex-connectivity we follow [26, p. 11], though in our notation graphs include the case of multigraphs.

Definition 3.2. The notions of k-edge-connectivity and k-vertex-connectivity are defined as follows:

- A graph is k-edge-connected if it cannot be disconnected by removal of any set of $k-1$ (or fewer) edges.
- A graph is 2-vertex-connected if it has no looping edges, it has at least 3 vertices, and it cannot be disconnected by removal of a single vertex, where vertex removal is defined as above.
- For $k \geq 3$, a graph is k-vertex-connected if it has no looping edges and no multiple edges, it has at least $k+1$ vertices, and it cannot be disconnected by removal of any set of $k-1$ vertices.

Thus, 1-vertex-connected and 1-edge-connected simply mean connected, while 2-edge-connected is the one-particle-irreducible (1PI) condition recalled in Definition 1.1. To see how the condition of 2-vertex-connectivity relates to the physical 1PI condition, we first recall the notion of *splitting of a vertex* in a graph Γ (cf. [26, §4.2]).

Definition 3.3. A graph Γ' is a splitting of Γ at a vertex $v \in V(\Gamma)$ if it is obtained by partitioning the set $E \subset E(\Gamma)$ of edges adjacent to v into two disjoint non-empty subsets, $E = E_1 \cup E_2$ and inserting a new edge e to whose end vertices v_1 and v_2 the edges in the two sets E_1 and E_2 are respectively attached (see figure 2).

We have the following relation between 2-vertex-connectivity and two-edge-connectivity (1PI). The first observation will be needed in the proof of Proposition 3.1; the second is offered mostly for contextual reasons.

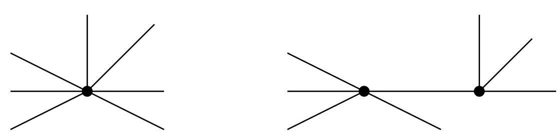

Figure 2: A splitting of a graph Γ at a vertex v.

Lemma 3.5. *Let Γ be a graph with at least 3 vertices and no looping edges.*

(1) *If Γ is 2-vertex-connected then it is also 2-edge-connected (1PI).*
(2) *Γ is 2-vertex-connected if and only if all the graphs Γ' obtained as splittings of Γ at any $v \in V(\Gamma)$ are 2-edge-connected (1PI).*

Proof. (1) We have to show that, for a graph Γ with at least 3 vertices and no looping edges, 2-vertex-connectivity implies 2-edge-connectivity. Assume that Γ is not 1PI. Then there exists an edge e such that $\Gamma \smallsetminus e$ has two connected components Γ_1 and Γ_2. Since Γ has no looping edges, e has two distinct endpoints v_1 and v_2, which belong to the two different components after the edge removal. Since Γ has at least 3 vertices, at least one of the two components contains at least two vertices. Assume then that there exists $v \neq v_1 \in V(\Gamma_1)$. Then, after the removal of the vertex v_1 from Γ, the vertices v and v_2 belong to different connected components, so that Γ is not 2-vertex-connected.

(2) We need to show that 2-vertex-connectivity is equivalent to all splittings Γ' being 1PI. Suppose first that Γ is not 2-vertex-connected. Since Γ has at least 3 vertices and no looping edges, the failure of 2-vertex-connectivity means that there exists a vertex v whose removal disconnects the graph. Let $V \subset V(\Gamma)$ be the set of vertices other than v that are endpoints of the edges adjacent to v. This set is a union $V = V_1 \cup V_2$ where the vertices in the two subsets V_i are contained in at least two different connected components of $\Gamma \smallsetminus v$. Then the splitting Γ' of Γ at v obtained by inserting an edge e such that the endpoints v_1 and v_2 are connected by edges, respectively, to the vertices in V_1 and V_2 is not 1PI.

Conversely, assume that there exists a splitting Γ' of Γ at a vertex v that is not 1PI. There exists an edge e of Γ' whose removal disconnects the graph. If e already belonged to Γ, then Γ would not be 1PI (and hence not 2-vertex connected, by (1)), as removal of e would disconnect it. So e must be the edge added in the splitting of Γ at the vertex v.

Let v_1 and v_2 be the endpoints of e. None of the other edges adjacent to v_1 or v_2 is a looping edge, by hypothesis; therefore, there exist at least another vertex $v_1' \neq v_2$ adjacent to v_1, and a vertex $v_2' \neq v_1$ adjacent to v_2. Since $\Gamma' \smallsetminus e$ is disconnected, v_1' and v_2' are in distinct connected components of $\Gamma' \smallsetminus e$. Since v_1' and v_2' are in $\Gamma \smallsetminus v$, and $\Gamma \smallsetminus v$ is contained in $\Gamma' \smallsetminus e$, it follows that removing v from Γ would also disconnect the graph. Thus Γ is not 2-vertex-connected. □

The first statement in Lemma 3.5 admits the following analog for 3-connectivity.

Lemma 3.6. *Let Γ be a graph with at least 4 vertices, with no looping edges and no multiple edges. Then 3-vertex-connectivity implies 3-edge-connectivity.*

Proof. We argue by contradiction. Assume that Γ is 3-vertex-connected but not 2PI. We know it is 1PI because of the previous lemma. Thus, there exist two edges e_1 and e_2 such that the removal of both edges is needed to disconnect the graph. Since we are assuming that Γ has no multiple or looping edges, the two edges have at most one end in common.

Suppose first that they have a common endpoint v. Let v_1 and v_2 denote the remaining two endpoints, $v_i \in \partial e_i$, $v_1 \neq v_2$. If the vertices v_1 and v_2 belong to different connected components after removing e_1 and e_2, then the removal of the vertex v disconnects the graph, so that Γ is not 3-vertex-connected (in fact not even 2-vertex-connected). If v_1 and v_2 belong to the same connected component, then v must be in a different component. Since the graph has at least 4 vertices and no multiple or looping edges, there exists at least another edge attached to either v_1, v_2, or v, with the other endpoint $w \notin \{v, v_1, v_2\}$. If w is adjacent to v, then removing v and v_1 leaves v_2 and w in different connected components. Similarly, if w is adjacent to (say) v_1, then the removal of the two vertices v_1 and v_2 leave v and w in two different connected components. Hence Γ is not 3-vertex-connected.

Next, suppose that e_1 and e_2 have no endpoint in common. Let v_1 and w_1 be the endpoints of e_1 and v_2 and w_2 be the endpoints of e_2. At least one pair $\{v_i, w_i\}$ belongs to two separate components after the removal of the two edges, though not all four points can belong to different connected components, else the graph would not be 1PI. Suppose then that v_1 and w_1 are in different components. It also cannot happen that v_2 and w_2 belong to the same component, else the removal of e_1 alone would disconnect the graph. We can assume then that, say, v_2 belongs to the same component as v_1 while w_2 belongs to a different component (which may or may not be the same as that of w_1). Then the removal of v_1 and w_2 leaves v_2 and w_1 in two different components so that the graph is not 3-vertex-connected. $\square$

Remark 3.1. While the 2-edge-connected hypothesis on Feynman graphs is very natural from the physical point of view, since it is just the 1PI condition that arises when one considers the perturbative expansion of the effective action of the quantum field theory (cf. [22]), conditions of 3-connectivity (3-vertex-connected or 3-edge-connected) arise in a more subtle manner in the theory of Feynman integrals, in the analysis of Laundau singularities (see for instance [29]). In particular, the 2PI effective action is often considered in quantum field theory in relation to non-equilibrium phenomena, see, e.g., [28, §10.5.1].

3.3 Connectivity and embeddings

We now recall another property of graphs on surfaces, namely the *face width* of an embedding $\iota : \Gamma \hookrightarrow S$. The face width $fw(\Gamma, \iota)$ is the largest number $k \in \mathbb{N}$ such that every non-contractible simple closed curve in S intersects Γ at least k times. When S is a sphere, hence $\iota : \Gamma \hookrightarrow S$ is a planar embedding, one sets $fw(\Gamma, \iota) = \infty$.

Remark 3.2. For a graph Γ with at least 3 vertices and with no looping edges, the condition that an embedding $\iota : \Gamma \hookrightarrow S$ is a *closed* 2-cell embedding is equivalent to the properties that Γ is 2-vertex-connected and that the embedding has face width $fw(\Gamma, \iota) \geq 2$, see [26, Proposition 5.5.11].

In particular, this implies that a planar graph with at least three vertices and no looping edges admits a closed 2-cell embedding in the sphere if and only if it is 2-vertex-connected. Notice that the condition that Γ has at least 3 vertices and no looping edges is necessary for this statement to be true. For example, the graph with two vertices, one edge between them, and one looping edge attached to each vertex cannot be disconnected by removal of a single vertex, but does not have a closed 2-cell embedding in the sphere. Similarly, the graph consisting of two vertices, one edge between them and one looping edge attached to one of the vertices admits a closed 2-cell embedding in the sphere, but is not 2-vertex-connected. (see figure 3).

It is not known whether every 2-vertex-connected graph Γ admits a closed 2-cell embedding. The "strong orientable embedding conjecture" states that this is the case, namely, that every 2-vertex-connected graph Γ admits a closed 2-cell embedding in some orientable surface S, of face width at least two (see [26, Conjecture 5.5.16]).

We are now ready for the promised improvement of Lemma 3.4.

Proposition 3.1. *Let Γ be a graph with at least 3 vertices and with no looping edges, which is closed-2-cell embedded in an orientable surface S. Then, if any two of the faces have at most one edge in common, the map τ is injective.*

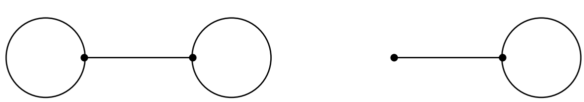

Figure 3: Vertex conditions and 2-cell embeddings.

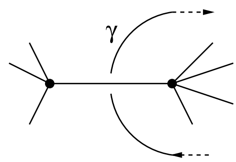

Figure 4: An edge not in the boundary of two faces.

Proof. It suffices to show that, under these conditions on the graph Γ, the second condition of Lemma 3.4 is automatically satisfied, so that only the first condition remains to be checked. That is, we show that every edge of Γ is in the boundary of two faces.

Assume an edge is not in the boundary of two faces. Then that edge must bound the same face on both of its sides, as in figure 4. The closure of the face is a cell, by assumption. Let γ be a path from one side of the edge to the other. Since γ splits the cell into two connected components, it follows that removing the edge splits Γ into two connected components, hence Γ is not 2-edge-connected. However, as recalled in Remark 3.2, the fact that Γ has at least 3 vertices and no looping edges and it admits a closed 2-cell embedding implies that Γ is 2-vertex-connected, hence in particular it is 1PI by the first part of Lemma 3.5, and this gives a contradiction. □

The condition that Γ has at least 3 vertices and no looping edges is necessary for Proposition 3.1. For example, the second graph shown in figure 3 does not satisfy the property that each edge is in the boundary of two faces; in the case of this graph, clearly the map τ is not injective.

Here is another direct consequence of the previous embedding results.

Proposition 3.2. *Let Γ be a 3-edge-connected graph, with at least 3 vertices and no looping edges, admitting a closed-2-cell embedding $\iota : \Gamma \hookrightarrow S$ with face width $fw(\Gamma, \iota) \geq 3$. Then the maps τ_i, τ are all injective.*

Proof. The result of Proposition 3.1 shows that the second condition stated in Lemma 3.4 is automatically satisfied, so the only thing left to check is that the first condition stated in Lemma 3.4 holds. Assume that two faces F_1, F_2 have more than one edge in common, see figure 5. Since F_1, F_2 are (path-)connected, there are paths γ_i in F_i connecting corresponding sides of the edges. With suitable care, it can be arranged that $\gamma_1 \cup \gamma_2$ is a closed path γ meeting Γ in two points, see figure 5. Since the embedding has face width ≥ 3, γ must be null-homotopic in the surface, and in particular it splits it into two connected components. This implies that Γ is split into

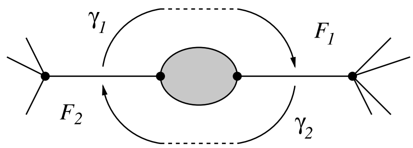

Figure 5: Two faces with more than one edge in common.

two connected components by removing the two edges, hence Γ cannot be 3-edge-connected. □

The 3-edge-connectivity hypothesis in Proposition 3.2 can be viewed as the next step strengthening of the 1PI condition, cf. Remark 3.1. Similarly, the condition of the face width of the embedding $fw(\Gamma, \iota) \geq 3$ is the next step strengthening of the condition $fw(\Gamma, \iota) \geq 2$ conjecturally implied by 2-vertex-connectivity.

In fact, if we enhance in Proposition 3.2 the 3-edge-connected hypothesis with 3-vertex-connectivity (see Lemma 3.6), we can refer to a result of graph theory ([26, Proposition 5.5.12]) which shows that for a 3-vertex-connected graph it is equivalent to admit an embedding with $fw(\Gamma, \iota) \geq 3$ and to have the *wheel neighborhood property*, that is, every vertex of Γ has a wheel neighborhood. Another equivalent condition to $fw(\Gamma, \iota) \geq 3$ for a 3-vertex-connected graph is that the loops determined by the faces of the embedding as in Lemma 3.3 are either disjoint or their intersection is just a vertex or a single edge ([26, Proposition 5.5.12]). For example, we can formulate an analog of Proposition 3.2 in the following way.

Corollary 3.1. *Let Γ be a 3-vertex-connected graph such that each vertex has a wheel neighborhood. Then the maps τ_i and τ of* (2.10), (2.2) *are all injective.*

The results derived in this section thus identify classes of graphs that satisfy simple geometric properties for which the injectivity of the map τ holds.

3.4 Dependence on Γ

The preceding results refer to the injectivity of the maps τ_i, τ determined by a given graph Γ, where τ maps an affine space $\mathbb{A}^n$ (where n is the number of internal edges of Γ) to $\mathbb{A}^{\ell^2}$ (where ℓ is the number of loops of Γ), by means of the matrix $M_\Gamma(t)$. The whole matrix $M_\Gamma(t)$ depends on course on the graph Γ. However, the injectivity of τ may be detected by a suitable submatrix.

In the following statement, choose a basis for $H_1(\Gamma, \mathbb{Z})$ as prescribed in Lemma 3.3; thus, $f - 1 = \ell - 2g$ rows of $M_\Gamma(t)$ correspond to the "internal" faces in an embedding of Γ.

Lemma 3.7. *For a graph Γ with at least 3 vertices and no looping edges that is closed-2-cell embedded by $\iota : \Gamma \hookrightarrow S$ in an orientable surface S of genus g, the map defined by the $(\ell - 2g) \times (\ell - 2g)$ minor in the matrix $M_\Gamma(t)$, which corresponds to the loops that are boundaries of faces on S is injective if and only if the map τ is injective.*

Proof. Indeed, under the given assumptions, every edge appears in the loop corresponding to some internal face of the embedding. The argument proving Lemma 3.4 shows that the given minor determines the injectivity of τ. □

A further refinement of the foregoing considerations will allow us to obtain statements that will be to some extent independent of Γ, and only hinge on $\ell = b_1(\Gamma)$ and the genus g of Γ.

We have pointed out earlier (Section 2.1) that det $M_\Gamma(t)$ does not depend on the choice of orientation for the loops of Γ. It is however advantageous to make a coherent choice for these orientations. We are now assuming that we have chosen a closed 2-cell embedding of Γ into an orientable surface of genus g; such an embedding has f faces, where $\ell = 2g + f - 1$; we can arrange $M_\Gamma(t)$ so that the first $f - 1$ rows correspond to the $f - 1$ loops determined by the "internal" faces of the embedding.

On each face, choose the positive orientation (counterclockwise with respect to an outgoing normal vector). Then each edge-variable in common between two faces i, j will appear with a minus sign in the entries (i, j) and (j, i) of $M_\Gamma(t)$. These entries are both in the $(\ell - 2g) \times (\ell - 2g)$ upper-left minor, which is the minor singled out in Lemma 3.7.

The upshot is that in the cases covered by the above results (such as Proposition 3.1), the edge variables t_e can all be obtained by either pulling-back entries $-x_{ij}$ with $1 \le i < j \le \ell - 2g$, or a sum

$$x_{i1} + x_{i2} + \cdots + x_{i,\ell-2g}$$

with $1 \le i \le \ell - 2g$. Note that these expressions only depend on ℓ and g; it follows that all components of the image $\tau(\partial\sigma_n)$ in $\mathbb{A}^{\ell^2}$ of the boundary of the simplex σ_n can be realized as pull-backs of subspaces of $\mathbb{A}^{\ell^2}$ from a list which *only depends on the number* $\ell - 2g$ $(= f - 1$, where f is the number of faces in a closed 2-cell embedding of $\Gamma)$. This observation essentially emancipates the domain of integration in the integral appearing in the statement of Lemma 2.3 from the specific graph Γ.

We will return to this point in Section 5, cf. Proposition 5.1.

3.5 More general graphs

The previous combinatorial statements were obtained under the assumption that the graphs have no looping edges. However, the statements can then be generalized easily to the case with looping edges using the following observation.

Lemma 3.8. *Let Γ be a graph obtained by attaching a looping edge at a vertex of a given graph Γ'. Then the map τ_Γ of* (2.10) *is injective if and only if $\tau_{\Gamma'}$ is.*

Proof. Let t be the variable assigned to the looping edge and t_e the variables assigned to the edges of Γ'. The matrix $M_\Gamma(t,t_e)$ is of the block diagonal form

$$M_\Gamma(t,t_e) = \begin{pmatrix} t & 0 \\ 0 & M_{\Gamma'}(t_e) \end{pmatrix}.$$

This proves the statement. □

This allows us to extend the results of Proposition 3.2 and Corollary 3.1 to all graphs obtained by attaching an arbitrary number of looping edges at the vertices of a graph satisfying the hypothesis of Proposition 3.2 or Corollary 3.1.

Corollary 3.2. *Let Γ be a graph such that, after removing all the looping edges, the remaining graph is 3-vertex-connected with a wheel neighborhood at each vertex. Then the maps τ_i, τ are all injective.*

We can further extend the class of graphs to which the results of this section apply by including those graphs that are obtained from graphs satisfying the hypotheses of Proposition 3.1, Proposition 3.2, Corollary 3.1, or Corollary 3.2 by subdividing edges.

Let e_n be the edge of Γ that is subdivided in two edges e'_n and e''_n to form the graph Γ'. The effect on the graph polynomial is

$$\Psi_{\Gamma'}(t_1,\dots,t_{n-1},t'_n,t''_n) = \Psi_\Gamma(t_1,\dots,t_{n-1},t'_n+t''_n),$$

since the spanning trees of Γ' are obtained by adding either e'_n or e''_n to those spanning trees of Γ that do not contain e_n and by replacing e_n with $e'_n \cup e''_n$ in the spanning trees of Γ that contain e_n. Thus, notice that in this case the injectivity of the map τ is not preserved by the operation of splitting edges. However, one can check directly that this operation does not affect the nature of the period computed by the Feynman integral, as the following result shows, so that any result that will show that the Feynman integral is a period of a mixed Tate motive for a class of graphs with no valence two vertices will automatically extend to graphs obtained by splitting edges.

Proposition 3.3. *Let Γ' be a graph obtained from a given graph Γ by subdividing one of the edges by inserting a valence two vertex. Then the parametric Feynman integral for Γ' will be of the form*

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-(n+1)+D\ell/2} t_n \omega_n}{\Psi_\Gamma(t)^{-(n+1)+(\ell+1)D/2}}, \tag{3.1}$$

with $n = \#E_{int}(\Gamma)$.

Proof. When one subdivides an edge as above, the Feynman rules imply that one finds as corresponding Feynman integral an expression of the form

$$\int \frac{\delta(\sum_i \epsilon_{v,i} k_i + \sum_j \epsilon_{v,j} p_j)}{q_1 \cdots q_{n-1} q_n^2} \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_n}{(2\pi)^D},$$

where the $q_i(k_i) = k_i^2 + m^2$ are the quadratic forms that give the propagators associated to the internal edges of the graph. We have used the constraint $\delta(k_n - k_{n+1})$ for the two momentum variables associated to the two parts of the split edge, so that we find q_n^2 in the denominator. One then uses the usual formula

$$\frac{1}{q_1^{a_1} \cdots q_n^{a_n}} = \frac{\Gamma(a_1 + \cdots + a_n)}{\Gamma(a_1) \cdots \Gamma(a_n)} \int_{\mathbb{R}_+^n} \frac{t_1^{a_1-1} \cdots t_n^{a_n-1} \delta(1 - \sum_i t_i)}{(t_1 q_1 + \cdots t_n q_n)^{a_1 + \cdots + a_n}}$$

to obtain the parametric form of the Feynman integral. In our case this gives

$$\frac{1}{q_1 \cdots q_{n-1} q_n^2} = n! \int_{\sigma_n} \frac{t_n \, dt_1 \cdots dt_n}{(t_1 q_1 + \cdots t_n q_n)^{n+1}}.$$

Thus, one obtains the parametric form of the Feynman integral as

$$\int \frac{d^D x_1 \cdots d^D x_\ell}{(\sum_i t_i q_i)^{n+1}} = C_{\ell,n+1} \det(M_\Gamma(t))^{-D/2} V_\Gamma(t,p)^{-(n+1)+D\ell/2},$$

where $V_\Gamma(t,p) = P_\Gamma(t,p)/\Psi_\Gamma(t)$ and with

$$C_{\ell,n+1} = \int \frac{d^D x_1 \cdots d^D x_\ell}{(1 + \sum_k x_k^2)^{n+1}}.$$

This gives (3.1). □

In particular, Proposition 3.3 shows that the parametric Feynman integral for the graph Γ' is still a period of the same type as that of the graph Γ, since it is still a period associated to the complement of the graph hypersurface $\hat{X}_\Gamma$ and evaluated over the same simplex σ_n. Only the algebraic differential form changes from $\Psi_\Gamma^{-D/2} V_\Gamma(t,p)^{-n+D\ell/2}\omega_n$ to $\Psi_\Gamma^{-D/2} V_\Gamma(t,p)^{-(n+1)+D\ell/2} t_n \omega_n$, but this does not affect the nature of the period, at least in the "stable range" where D is sufficiently large ($D\ell/2 > n$).

4 The motive of the determinant hypersurface complement

Our work in Sections 2 and 3 relates the complexity of a Feynman integral over a graph satisfying suitable combinatorial conditions to the complexity of the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell))$$

whose realizations give the relative cohomology of the pair of the complement of the determinant hypersurface and a normal crossing divisor $\hat{\Sigma}_\Gamma$ containing the image of the boundary $\tau_\Gamma(\partial\sigma_n)$, as in Lemma 5.1 below (see Corollary 2.1, Proposition 3.1 and ff.).

In this section we exhibit an explicit filtration of the complement of the determinant hypersurface, from which we can directly prove that the motive of $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\Gamma$ is mixed Tate. We use this filtration to compute explicitly the class of $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\Gamma$ in the Grothendieck group of varieties, as well as the class of the projective version $\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$.

Notice that the mixed Tate nature of the motive of the determinant hypersurface also follows directly from the results of Belkale–Brosnan [3], or from those of Biglari [5, 6], but we prefer to give here a very explicit computation, which will be useful as a preliminary for the similar but more involved analysis of the loci that contain the boundary of the domain of integration that we discuss in the following sections.

4.1 The motive

As we already argued, it is more natural to consider the graph hypersurfaces $\hat{X}_\Gamma$ in the affine space $\mathbb{A}^n$, instead of the projective X_Γ in $\mathbb{P}^{n-1}$. Thus, here also we work with the affine space $\mathbb{A}^{\ell^2}$ parametrizing $\ell \times \ell$ matrices. The cone $\hat{\mathcal{D}}_\ell$ over the determinant hypersurface consists of matrices of rank $< \ell$. Realizing the complement of $\hat{\mathcal{D}}_\ell$ in $\mathbb{A}^{\ell^2}$ amounts then to 'parametrizing' matrices M of rank exactly ℓ.

It is clear how this should be done:

— The first row of M must be a non-zero vector v_1.
— The second row of M must be a vector v_2 that is non-zero modulo v_1.
— The third row of M must be a vector v_3 that is non-zero modulo v_1 and v_2.
— And so on.

To formalize this construction, let E be a fixed ℓ-dimensional vector space, and work inductively. The first steps of the construction are as follows.

— Denote by $\mathcal{W}_1$ the variety $E \smallsetminus \{0\}$.
— Note that $\mathcal{W}_1$ is equipped with a trivial vector bundle $E_1 = E \times \mathcal{W}_1$, and with a line bundle $S_1 := L_1 \subseteq E_1$ whose fiber over $v_1 \in \mathcal{W}_1$ consists of the line spanned by v_1.
— Let $\mathcal{W}_2 \subseteq E_1$ be the complement $E_1 \smallsetminus L_1$.
— Note that $\mathcal{W}_2$ is equipped with a trivial vector bundle $E_2 = E \times \mathcal{W}_2$, and *two* line subbundles of E_2: the pull-back of L_1 (still denoted L_1) and the line-bundle L_2 whose fiber over $v_2 \in \mathcal{W}_2$ consists of the line spanned by v_2.
— By construction, L_1 and L_2 span a rank-2 subbundle S_2 of E_2.
— Let $\mathcal{W}_3 \subseteq E_2$ be the complement $E_2 \smallsetminus S_2$.
— And so on.

Inductively: at the kth step, this procedure produces a variety $\mathcal{W}_k$, endowed with k line bundles $L_1, \ldots, L_k$ spanning a rank-k subbundle S_k of the trivial vector bundle $E_k := E \times \mathcal{W}_k$. If $S_k \subsetneq E_k$, define $\mathcal{W}_{k+1} := E_k \smallsetminus S_k$. Let $E_{k+1} = E \times \mathcal{W}_{k+1}$, and define line subbundles $L_1, \ldots, L_k$ to be the pull-backs of the like-named line bundles on $\mathcal{W}_k$; and let L_{k+1} be the line bundle whose fiber over v_{k+1} is the line spanned by v_{k+1}. The line bundles $L_1, \ldots, L_{k+1}$ span a rank-$k+1$ subbundle S_{k+1} of E_{k+1}, and the construction can continue. The sequence stops at the ℓth step, where S_ℓ has rank ℓ, equal to the rank of E_ℓ, so that $E_\ell \smallsetminus S_\ell = \emptyset$.

Lemma 4.1. *The variety $\mathcal{W}_\ell$ constructed as above is isomorphic to $\mathbb{A}^{\ell^2} \setminus \hat{D}_\ell$.*

Proof. Each variety $\mathcal{W}_k$ maps to $\mathbb{A}^{\ell^2}$ as follows: a point of $\mathcal{W}_k$ determines k vectors $v_1, \ldots, v_k$, and can be mapped to the matrix whose first k rows are $v_1, \ldots, v_k$ resp. (and the remaining rows are 0). By construction, this matrix has rank exactly k. Conversely, any such rank k matrix is the image of a point of $\mathcal{W}_k$, by construction. □

In particular, we have the following result on the bundles S_k involved in the construction described above.

Lemma 4.2. *The bundle S_k over the variety $\mathcal{W}_k$ is trivial for all $1 \leq k \leq \ell$.*

Proof. Points of $\mathcal{W}_k$ are parameterized by k-tuples of vectors $v_1, \dots, v_k$ spanning $S_k \subseteq \mathbb{K}^\ell \times \mathcal{W}_k = E_k$. This means precisely that the map

$$\mathbb{K}^k \times \mathcal{W}_k \xrightarrow{\alpha} S_k$$

defined by

$$\alpha : ((c_1, \dots, c_r), (v_1, \dots, v_r)) \mapsto c_1 v_1 + \cdots + c_r v_r$$

is an isomorphism. □

Recall that, given a triangulated category $\mathcal{D}$, a full subcategory $\mathcal{D}'$ is a triangulated subcategory if and only if it is invariant under the shift T of $\mathcal{D}$ and for any distinguished triangle

$$A \to B \to C \to A[1]$$

for $\mathcal{D}$ where A and B are in $\mathcal{D}'$ there is an isomorphism $C \simeq C'$ with C' also in $\mathcal{D}'$. A full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is thick if it is closed under direct sums.

Let $\mathcal{MD}_{\mathbb{K}}$ be the Voevodsky triangulated category of mixed motives over a field $\mathbb{K}$, [33]. The triangulated category $\mathcal{DMT}_{\mathbb{K}}$ of mixed Tate motives is the full triangulated thick subcategory of $\mathcal{MD}_{\mathbb{K}}$ generated by the Tate objects $\mathbb{Q}(n)$. It is known that, over a number field $\mathbb{K}$, there is a canonical t-structure on $\mathcal{DMT}_{\mathbb{K}}$ and one can therefore construct an abelian category $\mathcal{MT}_{\mathbb{K}}$ of mixed Tate motives (see [24]).

We then have the following result on the nature of the motive of the determinant hypersurface complement.

Theorem 4.1. *The determinant hypersurface complement $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ defines an object in the category $\mathcal{DMT}_{\mathbb{K}}$ of mixed Tate motives.*

Proof. First recall that by Proposition 4.1.4 of [33], over a field $\mathbb{K}$ of characteristic zero a closed embedding $Y \subset X$ determines a distinguished triangle

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1]$$

in $\mathcal{MD}_{\mathbb{K}}$. Here we use the notation $\mathfrak{m}(X)$ for the motivic complex with compact support denoted by $\underline{C^c_*}(X)$ in [33]. In particular, if $\mathfrak{m}(Y)$ and $\mathfrak{m}(X)$ are in $\mathcal{DMT}_{\mathbb{K}}$ then $\mathfrak{m}(X \smallsetminus Y)$ is isomorphic to an object in $\mathcal{DMT}_{\mathbb{K}}$, by the property of full triangulated subcategories recalled above. Similarly, using the invariance of $\mathcal{DMT}_{\mathbb{K}}$ under the shift, if $\mathfrak{m}(Y)$ and $\mathfrak{m}(X \smallsetminus Y)$ are in $\mathcal{DMT}_{\mathbb{K}}$ then $\mathfrak{m}(X)$ is isomorphic to an object in $\mathcal{DMT}_{\mathbb{K}}$.

We also know (see [8, §1.2.3]) that in the Voevodsky category $\mathcal{MD}_{\mathbb{K}}$ one inverts the morphism $X \times \mathbb{A}^1 \to X$ induced by the projection, so that taking the product with an affine space $\mathbb{A}^k$ is an isomorphism at the level of the corresponding motives and for the motivic complexes with compact support this gives $\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2]$, see [33, Corollary 4.1.8]. Thus, for any given $\mathfrak{m}(X)$ in $\mathcal{DMT}_{\mathbb{K}}$, the motive $\mathfrak{m}(X \times \mathbb{A}^k)$ is obtained from $\mathfrak{m}(X)$ by Tate twists and shifts, hence it is also in $\mathcal{DMT}_{\mathbb{K}}$.

These two properties of the derived category $\mathcal{DMT}_{\mathbb{K}}$ of mixed Tate motives suffice to show that the motive of the affine hypersurface complement $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is mixed Tate,

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell) \in Obj(\mathcal{DMT}_{\mathbb{Q}}). \tag{4.1}$$

In fact, one sees from the inductive construction of $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ described above that at each step we are dealing with varieties defines over $\mathbb{K} = \mathbb{Q}$ and we now show that, at each step, the corresponding motives are mixed Tate.

Single points obviously belong to the category of mixed Tate motives. At the first step, one takes the complement $\mathcal{W}_1$ of a point in an affine space, which gives a mixed Tate motive by the first observation above on distinguished triangles associated to closed embeddings. At the next step one considers the complement of the line bundle S_1 inside the trivial vector bundle E_1 over $\mathcal{W}_1$. Again, both $\mathfrak{m}(S_1)$ and $\mathfrak{m}(E_1)$ are mixed Tate motives, since both are products by affine spaces by Lemma 4.2 above, hence $\mathfrak{m}(E_1 \smallsetminus S_1)$ is also mixed Tate. The same argument shows that, for all $1 \leq k \leq \ell$, the motive $\mathfrak{m}(E_k \smallsetminus S_k)$ is mixed Tate, by repeatedly using Lemma 4.2 and the two properties of $\mathcal{DMT}_{\mathbb{Q}}$ recalled above. □

4.2 The class in the Grothendieck ring

Lemma 4.1 suffices to obtain an explicit formula for the class in the Grothendieck ring of varieties of the complement of the determinant hypersurface. This is of course well-known: see for example [3, §3.3].

Proposition 4.1. *In the affine case the class in the Grothendieck ring of varieties is*

$$[\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1) \tag{4.2}$$

where $\mathbb{L}$ is the class of $\mathbb{A}^1$. In the projective case, the class is

$$[\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=2}^{\ell} (\mathbb{L}^i - 1). \tag{4.3}$$

Proof. Using Lemma 4.1 one sees inductively that the class of $\mathcal{W}_k$ is given by

$$\begin{aligned}[\mathcal{W}_k] &= (\mathbb{L}^\ell - 1)(\mathbb{L}^\ell - \mathbb{L})(\mathbb{L}^\ell - \mathbb{L}^2)\cdots(\mathbb{L}^\ell - \mathbb{L}^{k-1}) \\ &= \mathbb{L}^{\binom{k}{2}}(\mathbb{L}^\ell - 1)(\mathbb{L}^{\ell-1} - 1)\cdots(\mathbb{L}^{\ell-k+1} - 1). \end{aligned} \tag{4.4}$$

This completes the proof. □

The class (4.3) can be written equivalently in the form

$$[\mathbb{P}^{\ell^2-1} \smallsetminus D_\ell] = (\mathbb{L}[\mathbb{P}^1]\mathbb{T}) \cdot (\mathbb{L}^2[\mathbb{P}^2]\mathbb{T}) \cdot (\mathbb{L}^3[\mathbb{P}^3]\mathbb{T}) \cdots (\mathbb{L}^{\ell-1}[\mathbb{P}^{\ell-1}]\mathbb{T}), \tag{4.5}$$

where $\mathbb{L} = [\mathbb{A}^1]$ and $\mathbb{T} = [\mathbb{G}_m]$ is the class of the multiplicative group. Here the motive $\mathbb{L}^\ell[\mathbb{P}^1]\cdots[\mathbb{P}^{\ell-1}]$ can be thought of as the motive of the "variety of frames".

Example 4.1. In the cases $\ell = 2$ and $\ell = 3$, the class of $\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$ is given, respectively, by

$$\mathbb{L}^3 - \mathbb{L} \quad \text{and} \quad \mathbb{L}^8 - \mathbb{L}^5 - \mathbb{L}^6 + \mathbb{L}^3.$$

(Note however that, for $\ell \geq 5$, coefficients other than 0, ± 1 appear in the class.) Thus, the class $[\mathcal{D}_\ell]$ is given, for $\ell = 2$ and $\ell = 3$ by the expressions

$$\begin{aligned} &[\mathcal{D}_2] = \mathbb{L}^2 + 2\mathbb{L} + 1 = (\mathbb{L}+1)^2 \\ &[\mathcal{D}_3] = \mathbb{L}^7 + 2\mathbb{L}^6 + 2\mathbb{L}^5 + \mathbb{L}^4 + \mathbb{L}^2 + \mathbb{L} + 1 = (\mathbb{L}^3 - \mathbb{L} + 1)(\mathbb{L}^2 + \mathbb{L} + 1)^2. \end{aligned}$$

The $\ell = 2$ case is otherwise evident: $\mathcal{D}_2$ is the set of rank-1, 2×2–matrices, and as such it may be realized as $\mathbb{P}^1 \times \mathbb{P}^1$, with the indicated class. The $\ell = 3$ case can also be easily verified independently.

5 Relative cohomology and mixed Tate motives

We now assume that Γ is a graph satisfying the condition studied in Sections 2 and 3: the map τ is injective. By Proposition 3.1, this is the case if Γ has at least three vertices, no looping edges, and is closed-2-cell embedded in an orientable surface in such a way that any two of the faces determined by the embedding have at most one edge in common. Proposition 3.2 and Corollary 3.1 provide us with specific combinatorial conditions ensuring that this is the case. For instance, all 3-edge-connected planar graphs are included in this class.

Also note that by the considerations in Section 3.5 (especially Lemma 3.8 and Proposition 3.3), any estimate for the complexity of Feynman integrals for graphs satisfying these conditions generalizes automatically to the larger class of graphs obtained from those considered here by adding arbitrarily many looping edges, and by arbitrarily subdividing edges.

5.1 Algebraic simplexes and normal crossing divisors

In our setting and under the injectivity assumption, the property that the Feynman integral (1.9) is a period of a mixed Tate motive (modulo divergences) would follow from showing that a certain relative cohomology is a realization of a mixed Tate motive. Instead of the relative cohomology

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma))$$

considered in [9,10], we consider here a different relative cohomology, where the hypersurface complement $\mathbb{P}^{n-1} \smallsetminus X_\Gamma$ is replaced by the complement $\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$ of the determinant hypersurface, or better its affine counterpart $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$, and instead of the algebraic simplex $\Sigma_n = \{t : t_1 \cdots t_n = 0\}$, we consider a locus $\hat{\Sigma}_\Gamma$ in $\mathbb{A}^{\ell^2}$ that pulls back to the algebraic simplex Σ_n under the map τ of (2.10) and that consists of a union of n linear subspaces of codimension one in $\mathbb{A}^{\ell^2}$ that meet the image of $\mathbb{A}^n$ under τ along divisors with normal crossings. The following observation is a direct consequence of the construction of the matrix $M_\Gamma(t)$ (cf. Section 2.1).

Lemma 5.1. *Suppose given a graph Γ such that the corresponding maps τ and τ_i are injective. Then the n coordinates t_i associated to the internal edges of Γ can be written as preimages via the (injective) map $\tau : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ of n linear subspaces X_i of codimension 1 in $\mathbb{A}^{\ell^2}$. These n subspaces form a divisor $\hat{\Sigma}_\Gamma$ with normal crossings in $\mathbb{A}^{\ell^2}$.*

Proof. Consider the various possible cases for a specific edge listed in Lemma 2.2. In the third case listed there, where there are two loops ℓ_i, ℓ_j containing e, and not having any other edge in common, the variable t_e is immediately expressed as the pullback to $\mathbb{A}^n$ of a coordinate in $\mathbb{A}^{\ell^2}$. Consider then the second case listed in Lemma 2.2, where an edge e belongs to a single loop ℓ_i. Under the assumption that the map τ_i is injective, then any linear combination of the variables corresponding to the edges in the ith loop may be written as a linear combination of coordinates of the ith row. ☐

The considerations in Section 3.4 allow us to improve this observation, by passing to a larger normal crossing divisor, so that one can generate all the $\hat{\Sigma}_\Gamma$ from the components of a single normal crossings divisor $\hat{\Sigma}_{\ell,g}$ that only depends on the number of loops of the graph and on the minimal genus of the embedding of the graph on a Riemann surface. We formalize this remark as follows.

Proposition 5.1. *There exists a normal crossings divisor $\hat{\Sigma}_{\ell,g} \subset \mathbb{A}^{\ell^2}$, which is a union of $N = \binom{f}{2}$ linear spaces*

$$\hat{\Sigma}_{\ell,g} := X_1 \cup \cdots \cup X_N, \tag{5.1}$$

such that, for all graphs Γ with ℓ loops and genus g closed 2-cell embedding, the preimage under $\tau = \tau_\Gamma$ of the union $\hat{\Sigma}_\Gamma$ of a subset of components of $\hat{\Sigma}_{\ell,g}$ is the algebraic simplex Σ_n in $\mathbb{A}^n$. More explicitly, the components of the divisor $\hat{\Sigma}_{\ell,g}$ can be described by the $N = \binom{f}{2}$ equations

$$\begin{cases} x_{ij} = 0, & 1 \le i < j \le f-1, \\ x_{i1} + \cdots + x_{i,f-1} = 0, & 1 \le i \le f-1, \end{cases} \tag{5.2}$$

where $f = \ell - 2g + 1$ is the number of faces of the embedding.

Proof. Using Lemma 3.7, we know that the injectivity of an $(\ell - 2g) \times (\ell - 2g)$ minor of the matrix M_Γ suffices to control the injectivity of the map τ. We can in fact arrange so that the minor is the upper-left part of the $\ell \times \ell$ ambient matrix. Then, as in Lemma 5.1, the hyperplanes in $\mathbb{A}^n$ associated to the coordinates t_i can be obtained by pulling back linear spaces along this minor. On the diagonal of the $(f-1) \times (f-1)$ submatrix we find all edges making up each face, with a positive sign. It follows that the pull-backs of the equations (5.2) produce a list of all the edge variables, possibly with redundancies. The components of $\hat{\Sigma}_{\ell,g}$ that form the divisor $\hat{\Sigma}_\Gamma$ are selected by eliminating those components of $\hat{\Sigma}_{\ell,g}$ that contain the image of the graph hypersurface (i.e., coming from the zero entries of the matrix $M_\Gamma(t)$). ☐

Thus, for every Γ satisfying the conditions recalled at the beginning of the section (for example, every 3-edge-connected planar graph, or every graph obtained from one of these by adding looping edges or subdividing edges), *the nature of period appearing as a Feynman integral over* Γ *in the sense explained in Section 2 is controlled by the motive*

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_\Gamma)), \tag{5.3}$$

for a normal crossing divisor $\hat{\Sigma}_\Gamma \subset \mathbb{A}^{\ell^2}$ *consisting of a subset of components of the fixed* (for given ℓ and g) *normal crossing divisor* $\hat{\Sigma}_{\ell,g} \subset \mathbb{A}^{\ell^2}$ *introduced above.*

More explicitly, the boundary of the topological simplex σ_n, that is, the domain of integration of the Feynman integral in Lemma 2.3, satisfies

$$\tau(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma \subset \hat{\Sigma}_{\ell,g}. \tag{5.4}$$

Thus, the main goal here will be to understand the motivic nature of the complement

$$\hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_\Gamma). \tag{5.5}$$

Since $\hat{\Sigma}_\Gamma$ consists of components from the fixed normal crossing divisor $\hat{\Sigma}_{\ell,g}$, this question will be recast in terms that only depend on ℓ and g: we show in Corollary 5.1 below that, using the inclusion–exclusion principle applied to the components of $\hat{\Sigma}_{\ell,g}$, it is possible to answer these questions simultaneously for all the divisors $\hat{\Sigma}_\Gamma$, for all graphs with ℓ loops and genus g, by investigating the nature of a motive constructed out of the intersections of the components of the divisor $\hat{\Sigma}_{\ell,g}$.

Notice in fact that one can derive the case of $\hat{\Sigma}_{\ell,g}$ from the case of $g = 0$, since $\hat{\Sigma}_{\ell,g} \subseteq \hat{\Sigma}_{\ell,0}$, corresponding to an $(\ell - 2g) \times (\ell - 2g)$ minor of the matrix $M_\Gamma(t)$.

There are general and explicit conditions (see [18, Proposition 3.6]) implying that the relative cohomology of a pair (X, Y) comes from a mixed Tate motive $\mathfrak{m}(X, Y)$ (see also [19] for a concrete application to the geometric case of moduli spaces of curves). In general, these rely on assumptions on the divisors involved and their associated stratification, which may not directly apply to the cases considered here. We discuss here a direct approach to constructing stratifications of our loci $\hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_{\ell,g})$ that can be used to investigate the nature of the motive (5.3).

5.2 Inclusion–exclusion

The procedure we follow will be the one outlined above, based on the divisors $\hat{\Sigma}_{\ell,g}$ and the inclusion–exclusion principle. Since we already know by the results of Section 4 that the complement $X = \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is a mixed Tate motive, we aim at providing a direct argument showing that $Y = \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)$ also is a mixed Tate motive. The same argument used in Section 4 based on the distinguished triangles in the Voevodsky triangulated category of mixed Tate motives [33] would then show that the relative cohomology of the pair (X, Y) comes from an object $\mathfrak{m}(X, Y) \in \mathrm{Obj}(\mathcal{DMT}_{\mathbb{Q}})$.

As a first step we transform the problem of a complement in a union of linear spaces into an equivalent formulation in terms of intersections of linear spaces, using inclusion–exclusion. For a collection $\{Z_i\}_{i\in I}$ of varieties Z_i we set

$$Z_I^\circ := (\cap_{i\in I} Z_i) \smallsetminus (\cup_{j\notin I} Z_j). \tag{5.6}$$

Notice that, for all I,

$$\cap_{i\in I} Z_i = \amalg_{J\supseteq I} Z_J^\circ.$$

This is a *disjoint* union. We then have the following result.

Lemma 5.2. *Let $Z_1, \dots, Z_m$ be varieties; assume that the intersections $\cap_{i\in I} Z_i$ are mixed Tate, for all non-empty $I \subseteq \{1, \dots, m\}$. Then $Z_1 \cup \cdots \cup Z_m$ is mixed Tate.*

Proof. We want to show that Z_I° is mixed Tate for all non-empty $I \subseteq \{1, \dots, m\}$. To see this, notice that it is true by hypothesis for $I = \{1, \dots, m\}$, since in this case $Z_I^\circ = \cap_{i\in I} Z_i$. Thus, it suffices to prove that if it is true for all I with $|I| > k$, then it is true for all I with $|I| = k$ (provided $k \geq 1$). Recall that, as we already used in Section 4 above, the distinguished triangles in the Voevodsky category of mixed Tate motives imply that, if $X \hookrightarrow Y$ is a closed embedding, and $U = Y \smallsetminus X$ the complement, then if any two of X, Y, U are mixed Tate so is the third as well. The result then follows from the combined use of this property, the hypothesis, and the identity

$$Z_I^\circ = (\cap_{i\in I} Z_i) \smallsetminus (\amalg_{J\supsetneq I} Z_J^\circ).$$

Since we have

$$Z_1 \cup \cdots \cup Z_m = \amalg_{I\neq\emptyset} Z_I^\circ,$$

we conclude that the union $Z_1 \cup \cdots \cup Z_m$ is mixed Tate, again by the property of mixed Tate motives mentioned above. □

Now, we have observed that for every graph Γ with ℓ loops and genus g (and satisfying the condition specified at the beginning of the section) the divisor $\hat{\Sigma}_\Gamma$ consists of components of the divisor $\hat{\Sigma}_{\ell,g}$. Therefore, the strata of $\hat{\Sigma}_\Gamma$ are unions of strata from $\hat{\Sigma}_{\ell,g}$. We can then reformulate our main problem as follows.

Corollary 5.1. *Let, as above, $\hat{\Sigma}_{\ell,g} = X_1 \cup \cdots \cup X_N$ and let $\hat{\Sigma}_\Gamma$ be the divisors constructed out of subsets of components of $\hat{\Sigma}_{\ell,g}$, associated to the individual graphs. Then, for all graphs Γ with ℓ loops and genus g, the complement $\hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_\Gamma)$ is mixed Tate if the locus*

$$(\cap_{i\in I} X_i) \smallsetminus \hat{\mathcal{D}}_\ell \tag{5.7}$$

is mixed Tate for all $I \subseteq \{1, \ldots, N\}$, $I \neq \emptyset$.

Proof. This is a direct consequence of Lemma 5.2. □

Corollary 5.1 encapsulates the main reformulation of our problem, mentioned at the end of Section 1: the target becomes that of proving that the loci $(\cap_{i\in I} X_i) \smallsetminus \hat{\mathcal{D}}_\ell$ determined by the normal crossing divisor $\hat{\Sigma}_{\ell,g}$ are mixed Tate. This result shows that, although in principle one is working with a different divisor $\hat{\Sigma}_\Gamma$ for each graph Γ, in fact it suffices to consider the divisor $\hat{\Sigma}_{\ell,g}$, for fixed number of loops ℓ and genus g. It is conceivable that the loci associated to a specific graph (that is, to a specific choice of components of $\hat{\Sigma}_{\ell,g}$) may be mixed Tate while the loci corresponding to the whole divisor $\hat{\Sigma}_{\ell,g}$ is not. As we are seeking an explanation that would imply that *all* periods arising from Feynman integrals are periods of mixed Tate motives, we will optimistically venture that *all loci $(\cap_{i\in I} X_i) \smallsetminus \hat{\mathcal{D}}_\ell$ may in fact turn out to be mixed Tate,* for all ℓ and for $g = 0$: by Corollary 5.1, it would follow that all complements $\hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_\ell \cap \hat{\Sigma}_\Gamma)$ are mixed Tate, for *all* graphs Γ (satisfying our running combinatorial hypothesis).

Our task is now to formulate this working hypothesis as a more concrete problem. The intersection $\cap_{i\in I} X_i$ is a linear subspace of codimension $|I|$ in $\mathbb{A}^{\ell^2}$; in general, the intersection of a linear subspace with the determinant is *not* mixed Tate (for example, the intersection of a general $\mathbb{A}^3$ with $\hat{D}_3$ is a cone over a genus-1 curve). Thus, we have to understand in what sense the intersections $\cap_{i\in I} X_i$ appearing in Corollary 5.1 are special; the following lemma determines some key features of these subspaces.

Lemma 5.3. *Let E be a fixed ℓ-dimensional vector space, as in Section 4.1 above. Every $I \subseteq \{1, \ldots, N\}$ as above determines a choice of linear subspaces $V_1, \ldots, V_\ell$ of E, such that*

$$\cap_{k\in I} X_k = \{(v_1, \ldots, v_\ell) \in \mathbb{A}^{\ell^2} \mid \forall i, v_i \in V_i\}. \tag{5.8}$$

(*Here, we denote an $\ell \times \ell$ matrix in $\mathbb{A}^{\ell^2}$ by its ℓ row-vectors $v_i \in E$.*)

Further, $\dim V_i \geq i-1$. *Further still, there exists a basis* $(e_1, \dots, e_\ell)$ *of* E *such that each space* V_i *is the span of a subset (of cardinality* $\geq i-1$*) of the vectors* e_j.

Proof. Recall (Proposition 5.1) that the components X_k of $\hat{\Sigma}_{\ell,g}$ consist of matrices for which either the (i,j) entry x_{ij} equals 0, for $1 \leq i < j \leq \ell - 2g$, or

$$x_{i1} + \cdots + x_{i,\ell-2g} = 0$$

for $1 \leq i \leq \ell - 2g$. Thus, each X_k consists of ℓ-tuples $(v_1, \dots, v_\ell)$ for which exactly one row v_i belongs to a fixed hyperplane of E, and more precisely to one of the hyperplanes

$$x_1 + \cdots + x_{\ell-2g} = 0, \quad x_2 = 0, \quad \cdots, \quad x_{\ell-2g} = 0 \tag{5.9}$$

(with evident notation). The statement follows by choosing V_i to be the intersection of the hyperplanes corresponding to the X_k in row i, among those listed in (5.9). Since there are at most $\ell - 2g - i + 1$ hyperplanes X_k in the ith row,

$$\dim V_i \geq \ell - (\ell - 2g - i + 1) = 2g + i - 1 \geq i - 1.$$

Finally, to obtain the basis $(e_1, \dots, e_\ell)$ mentioned in the statement, simply choose the basis dual to the basis $(x_1 + \cdots + x_{\ell-2g}, x_2, \dots, x_\ell)$ of the dual space to E. □

5.3 The main questions

In view of Lemma 5.3, for any choice $V_1, \dots, V_\ell$ of subspaces of an ℓ-dimensional space E, let

$$\mathbb{F}(V_1, \dots, V_\ell) := \{(v_1, \dots, v_\ell) \in \mathbb{A}^{\ell^2} \mid \forall k, v_k \in V_k\} \smallsetminus \hat{\mathcal{D}}_\ell \tag{5.10}$$

denote the complement of the determinant hypersurface in the set of matrices determined by $V_1, \dots, V_\ell$. An optimistic version of the question we are led to is:

Question I_ℓ. *Let* $V_1, \dots, V_\ell$ *be subspaces of an* ℓ*-dimensional vector space. Is the locus* $\mathbb{F}(V_1, \dots, V_\ell)$ *mixed Tate?*

By Corollary 5.1 and Lemma 5.3, an affirmative answer to Question I_ℓ implies that the complement $\hat{\Sigma}_\Gamma \smallsetminus (\hat{D}_\ell \cap \hat{\Sigma}_\Gamma)$ is mixed Tate for all graphs Γ with ℓ loops and satisfying the combinatorial condition given at the beginning of this section. Modulo divergence issues, this would imply that all Feynman integrals corresponding to these graphs are periods of mixed Tate motives. We will give an affirmative answer to Question I_ℓ for $\ell \leq 3$, in Section 6.

As Lemma 5.3 is in fact more precise, the same conclusion would be reached by answering affirmatively the following weak version of Question I_ℓ:

Question $\mathbf{II}_\ell$. *Let $(e_1, \dots, e_\ell)$ be a basis of an ℓ-dimensional vector space. For $i = 1, \dots, \ell$, let V_i be a subspace spanned by a choice of $\geq i-1$ basis vectors. Is $\mathbb{F}(V_1, \dots, V_\ell)$ mixed Tate?*

Notice that, when $V_k = E$ for all k, both questions reproduce the statement about the hypersurface complement $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ proved in Section 4.1. One might expect that a similar inductive procedure would provide a simple approach to these questions. It is natural to consider the following apparent refinement of Question I_ℓ for $1 \leq r \leq \ell$ (and we could similarly consider an analogous refinement Question $\mathrm{II}'_{\ell,r}$ of Question II_ℓ):

Question $\mathbf{I}'_{\ell,r}$. *In a vector space E of dimension ℓ, and for any choice of subspaces $V_1, \dots, V_r$ of E, let*

$$\mathbb{F}_\ell(V_1, \dots, V_r) = \{(v_1, \dots, v_r) \,|\, v_i \in V_i \text{ and } \dim\langle v_1, \dots v_r\rangle = r\}.$$

Is the locus $\mathbb{F}_\ell(V_1, \dots, V_r)$ mixed Tate?

Question I_ℓ is then the same as Question $\mathrm{I}'_{\ell,\ell}$; and Question $\mathrm{I}'_{\ell,r}$ is obtained by taking $V_{r+1} = \cdots = V_\ell = E$ in Question I_ℓ: thus, answering Question I_ℓ is equivalent to answering Question $\mathrm{I}'_{\ell,r}$ for all $r \leq \ell$.

Now, for all ℓ, the case $r = 1$ is immediate: $\mathbb{F}_\ell(V_1)$ consists of all non-zero vectors in V_1, which is trivially mixed Tate. One could then hope that an inductive procedure may yield a method for increasing r. This is carried out in Section 6 for $r = 2$ and $r = 3$ (in particular, we give an affirmative answer to Question I_ℓ for $\ell \leq 3$); but this approach quickly leads to the analysis of several different cases, with an increase in complexity that makes further progress along these lines seem unlikely. The main problem is that once all tuples $(v_1, \dots, v_k)$ of linearly independent vectors such that $v_i \in V_i$ have been constructed, controlling

$$\dim(V_{k+1} \cap \langle v_1, \dots, v_k\rangle)$$

requires consideration of a range of possibilities that depend on the position of the vectors v_i and their spans vis-a-vis the position of the next space V_{k+1}. The number of these possibilities increases rapidly. A similar approach to the simpler (but sufficient for our purposes) Question II does not appear to circumvent this problem.

There are special cases where an inductive argument works nicely. We mention two here.

- Suppose that all the V_k in (5.10) are hyperplanes in E. Then $\mathbb{F}(V_1, \dots, V_\ell)$ is mixed Tate.

In this case, following the inductive argument mentioned above, the only possibilities for $V_{k+1} \cap \langle v_1, \dots, v_k \rangle$ are $\langle v_1, \dots, v_k \rangle$, and a hyperplane in $\langle v_1, \dots, v_k \rangle$. The first occurs when

$$\langle v_1, \dots, v_k \rangle \subseteq V_k.$$

This locus is under control, since it amounts to doing the whole construction in V_k rather than E, i.e., one can argue by induction on the dimension of E. Thus, this locus is mixed Tate. The other case gives a locus that is the complement of this mixed Tate variety in another mixed Tate variety, hence, by the same argument about closed embeddings and distinguished triangles used in Section 4, it is also mixed Tate.

- Suppose $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r$; then $\mathbb{F}_\ell(V_1, \dots, V_r)$ is mixed Tate.

Indeed, in this case $\langle v_1, \dots, v_k \rangle \subseteq V_{k+1}$ for all k. The condition on v_{k+1} is simply $v_{k+1} \in V_{k+1} \smallsetminus \langle v_1, \dots, v_k \rangle$, and these conditions clearly produce a mixed Tate locus. Arguing as in Section 4.1, the class of $\mathbb{F}_\ell(V_1, \dots, V_r)$ is immediately seen to equal

$$(\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_3} - \mathbb{L}^2) \cdots (\mathbb{L}^{d_r} - \mathbb{L}^{r-1})$$

in this case, where $d_k = \dim V_k$.

5.4 A reformulation

For given subspaces $V_i \subset E$, the inductive approach suggested by Question $\mathrm{I}'_{\ell,r}$ aims at constructing the set of ℓ-uples $(v_1, \dots, v_\ell)$ with the two properties

(1) $v_i \in V_i$;
(2) $\dim\langle v_1, \dots, v_r \rangle = r$, for all r,

and proving inductively that these loci are mixed Tate, in order to show that the loci (5.10) are mixed Tate. By (2), the sets

$$0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \dots, v_\ell \rangle = E$$

form a complete flag in E; let $E_r = \langle v_1, \dots, v_r \rangle$. Our main question can then be phrased in terms of these moving complete flags:

Question III$_\ell$. *Let $V_1, \dots, V_\ell$ be subspaces of an ℓ-dimensional vector space E, and let d_i, e_i be integers. Is the locus Flag$_{\ell,\{d_i,e_i\}}(\{V_i\})$ of complete flags*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell = E$$

such that

- dim $E_i \cap V_i = d_i$
- dim $E_i \cap V_{i+1} = e_i$

mixed Tate?

An affirmative answer to this question (for all choices of d_i, e_i) would give an affirmative answer to our main Question I$_\ell$. Indeed, the locus $\mathbb{F}(V_1, \dots, V_\ell)$ is a fibration on the locus $\mathrm{Flag}_{\ell,\{d_i,e_i\}}(\{V_i\})$ determined in Question III$_\ell$. Concretely, the procedure constructing the tuples $(v_1, \dots, v_\ell)$ in $\mathbb{F}(V_1, \dots, V_\ell)$ over a flag $E_\bullet$ in this locus is

- choose $v_1 \in (E_1 \cap V_1) \smallsetminus \{0\}$;
- choose $v_2 \in (E_2 \cap V_2) \smallsetminus (E_1 \cap V_2)$;
- choose $v_3 \in (E_3 \cap V_3) \smallsetminus (E_2 \cap V_3)$;
- etc.

The class of $\mathbb{F}(V_1, \dots, V_\ell)$ in the Grothendieck group would then be computed as a sum of terms

$$[\mathrm{Flag}_{\ell,\{d_i,e_i\}}(\{V_i\})] \cdot (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - \mathbb{L}^{e_1})(\mathbb{L}^{d_3} - \mathbb{L}^{e_2}) \cdots (\mathbb{L}^{d_r} - \mathbb{L}^{e_{r-1}}).$$

The set of flags $E_\bullet$ satisfying conditions analogous to those specified in Question III$_\ell$ with respect to all terms of a fixed flat $E'_\bullet$ (that is: with prescribed $\dim(E_i \cap E'_j)$ for all i and j) is a cell of the corresponding Schubert variety in the flag manifold.

It follows that $\mathrm{Flag}_{\ell,\{d_i,e_i\}}(\{V_i\})$ is a disjoint union of cells, and thus certainly mixed Tate, if the V_i's form a complete flag. This gives a high-brow alternative viewpoint for the last case mentioned in Section 5.3.

By the same token, the set of flags $E_\bullet$ for which dim $E_i \cap F$ is a fixed constant is a union of Schubert cells in the flag manifold, for all subspaces F. It follows that the locus $\mathrm{Flag}_{\ell,\{d_i,e_i\}}(\{V_i\})$ of Question III_ℓ is an *intersection of unions of Schubert cells* in the flag manifold. Such loci were studied, e.g., in [16, 17, 30, 31].

6 Motives and manifolds of frames

The manifolds of r-frames in a given vector space are defined as follows.

Definition 6.1. Let $\mathbb{F}(V_1, \ldots, V_r) \subset V_1 \times \cdots \times V_r$ denote the locus of r-tuples of linearly independent vectors in a vector space, where each v_i is constrained to belong to the given subspace V_i.

These are the loci appearing in Question $\mathrm{I}'_{\ell,r}$; we now omit the explicit mention of the dimension ℓ of the ambient space. The question we consider here is the one formulated in Section 5.3, namely to establish when the motive of the manifold of frames $\mathbb{F}(V_1, \ldots, V_r)$ is mixed Tate. A possible strategy to answering this question is based on the following simple observations.

Lemma 6.1. *Let $V_1, \ldots, V_r$ be subspaces of a given vector space V. Let $v_r \in V_r$, and let $\pi : V \to V' := V/\langle v_r\rangle$ be the natural projection. Let $v_1, \ldots, v_{r-1}$ be vectors such that $v_i \in V_i$, and $\pi(v_1), \ldots, \pi(v_{r-1})$ are linearly independent. Then $v_1, \ldots, v_r$ are linearly independent.*

Proof. The dimension of $\pi(\langle v_1, \ldots, v_{r-1}\rangle) = \langle \pi(v_1), \ldots, \pi(v_{r-1})\rangle$ is $r-1$ by hypothesis, therefore dim $\pi^{-1}(\pi(\langle v_1, \ldots, v_{r-1}\rangle)) = r$. Since $\pi^{-1}(\pi(\langle v_1, \ldots, v_{r-1}\rangle)) \subseteq \langle v_1, \ldots, v_r\rangle$, it follows that $\dim\langle v_1, \ldots, v_r\rangle = r$, as needed. □

A second equally elementary remark is that for a given $v' \neq 0$ in the quotient $V/\langle v_r\rangle$, and letting as above π denote the projection $V \to V/\langle v_r\rangle$, $\pi^{-1}(v') \cap V_i$ consists of either a single vector, if $v_r \notin V_i$, or a copy of the field k, if $v_r \in V_i$.

This implies the following.

Lemma 6.2. *Suppose given a stratification $\{S_\alpha\}$ of V_r with the properties that*

- *$\{S_\alpha\}$ is finer than the stratification induced on V_r by the subspace arrangement $V_1 \cap V_r, \ldots, V_{r-1} \cap V_r$, hence the number s_α of spaces*

V_i $(1 \le i < r)$ containing a vector $v_r \in S_\alpha$ is independent of the vector and only depends on α.

- *For $v_r \in S_\alpha$, the class $\mathbb{F}_\alpha := [\mathbb{F}(\pi(V_1), \ldots, \pi(V_{r-1}))]$ also depends only on α, and not on the chosen vector $v_r \in S_\alpha$.*

Then the class in the Grothendieck group satisfies

$$[\mathbb{F}(V_1, \ldots, V_r)] = \sum_\alpha \mathbb{L}^{s_\alpha} \cdot [\mathbb{F}_\alpha] \cdot [S_\alpha]. \tag{6.1}$$

Proof. Indeed, by Lemma 6.1 every frame in the quotient will determine frames in V, and by the observation following the Lemma, there is a whole k^{s_α} of frames over a given one in the quotient. □

In an inductive argument, the loci $[\mathbb{F}_\alpha]$ could be assumed to be mixed Tate, and (6.1) would provide a strong indication that $[\mathbb{F}(V_1, \ldots, V_r)]$ is then mixed Tate as well. We focus here on giving statements at the level of classes in the Grothendieck ring, for simplicity, though these same arguments, based on constructing explicit stratifications, can be also used to derive conclusion on the motives at the level of the derived category of mixed motives in a way similar to what we did in the case of the complement of the determinant hypersurface in Section 4 above.

The main question is then reduced to finding conditions under which a stratification of the type described here exists. We see explicitly how the argument goes in the simplest cases of two and three subspaces. As we discuss below, the case of three subspaces is already more involved and exhibits some of the features one is bound to encounter, with a more complicated combinatorics, in the more general cases.

6.1 The case of two subspaces

Let V_1, V_2 be subspaces of a vector space V. We want to parametrize all pairs of vectors

$$(v_1, v_2)$$

such that $v_1 \in V_1$, $v_2 \in V_2$, and $\dim\langle v_1, v_2\rangle = 2$. This locus can be decomposed into two pieces (which may be empty), defined by the following prescriptions:

(1) choose $v_1 \in V_1 \smallsetminus (V_1 \cap V_2)$, and $v_2 \in V_2 \smallsetminus \{0\}$;
(2) choose $v_1 \in (V_1 \cap V_2) \smallsetminus \{0\}$, and $v_2 \in V_2 \smallsetminus \langle v_1\rangle$.

It is clear that each of these two recipes produces linearly independent vectors, and that (1) and (2) exhaust the ways in which this can be done. So $\mathbb{F}(V_1, V_2)$ is the union of the corresponding loci. Pairs (v_1, v_2) as in (1) range over the locus $(V_1 \smallsetminus (V_1 \cap V_2)) \times (V_2 \smallsetminus \{0\})$, which is clearly mixed Tate. As for (2), realize it as follows:

- Consider the projective space $\mathbb{P}(V_1 \cap V_2)$, and the trivial bundles $\mathcal{V}_{12} \subseteq \mathcal{V}_2$ with fiber $V_1 \cap V_2 \subseteq V_2$.
- $\mathcal{V}_{12}$ contains the tautological line bundle $\mathcal{O}_{12}(-1)$ over $\mathbb{P}(V_1 \cap V_2)$, hence this line bundle is naturally contained in $\mathcal{V}_2$ as well.
- Then the pairs (v_1, v_2) as in (2) are obtained by choosing a point $p \in \mathbb{P}(V_1 \cap V_2)$, a vector $v_1 \neq 0$ in the fiber of $\mathcal{O}_{12}(-1)$ over p, and a vector v_2 in the fiber of $\mathcal{V}_2 \smallsetminus \mathcal{O}_{12}(-1)$ over p.

It is clear that this description also produces a mixed Tate motive.

Note that the prescriptions given as (1) and (2) suffice to compute the class in the Grothendieck group.

Lemma 6.3. *The class in the Grothendieck group of the manifold of frames* $\mathbb{F}(V_1, V_2)$ *is of the form*

$$[\mathbb{F}(V_1, V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}, \tag{6.2}$$

where $d_i = \dim V_i$ *and* $d_{12} = \dim(V_1 \cap V_2)$.

Proof. The two loci (1) and (2), respectively, have classes

(1) $(\mathbb{L}^{d_1} - \mathbb{L}^{d_{12}})(\mathbb{L}^{d_2} - 1)$;
(2) $(\mathbb{L}^{d_{12}} - 1)(\mathbb{L}^{d_2} - \mathbb{L})$.

The class of $\mathbb{F}(V_1, V_2)$ is then given by the sum

$$\begin{aligned}[\mathbb{F}(V_1, V_2)] &= (\mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2+d_{12}} + \mathbb{L}^{d_{12}}) + (\mathbb{L}^{d_2+d_{12}} - \mathbb{L}^{d_{12}+1} - \mathbb{L}^{d_2} + \mathbb{L}) \\ &= \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}.\end{aligned}$$

This gives (6.2). □

Notice that the expression for $[\mathbb{F}(V_1, V_2)]$ is symmetric in V_1 and V_2, though the two individual contributions (1) and (2) are not. Of course, a more symmetric description of the locus can be obtained by subdividing it into four cases according to whether v_1, v_2 are or are not in $V_1 \cap V_2$.

6.2 The case of three subspaces

We are given three subspaces V_1, V_2, V_3 of a vector space, and we want to parametrize all triples of linearly independent vectors (v_1, v_2, v_3) with $v_i \in V_i$. As above, d_i will stand for the dimension of V_i, and d_{ij} for $\dim(V_i \cap V_j)$. Further, let $d_{123} = \dim(V_1 \cap V_2 \cap V_3)$, and $D = \dim(V_1 + V_2 + V_3)$.

Notice that now the information on the dimension D is also needed and does not follow from the other data. This can be seen easily by thinking of the cases of three distinct lines spanning a three-dimensional vector space or of three distinct coplanar lines. These configurations only differ in the number D, yet the set of linearly independent triples is nonempty in the first case, empty in the second.

We proceed as follows. Given a choice of $v_3 \in V_3$, consider the projection $\pi : V \to V' := V/\langle v_3\rangle$; in V' we have the images $\pi(V_1), \pi(V_2)$, to which we can apply the case $r = 2$ analyzed above. As we have seen, $\mathbb{F}(V_1', V_2')$ is determined by the dimensions of V_1', V_2', and $V_1' \cap V_2'$. Thus, we need a stratification of V_3 such that, for $v_3 \in V_3$ and denoting as above by π the projection $V \to V/\langle v_3\rangle$, the dimensions of the spaces

$$\pi(V_1), \quad \pi(V_2), \quad \pi(V_1) \cap \pi(V_2)$$

are constant along strata.

Lemma 6.4. *The following five loci give a stratification of $V_3 \smallsetminus \{0\}$ with the properties of Lemma 6.2.*

(1) $S_{123} := (V_1 \cap V_2 \cap V_3) \smallsetminus \{0\}$;
(2) $S_{13} := (V_1 \cap V_3) \smallsetminus (V_1 \cap V_2 \cap V_3)$;
(3) $S_{23} := (V_2 \cap V_3) \smallsetminus (V_1 \cap V_2 \cap V_3)$;
(4) $S_{(12)3} := ((V_1 + V_2) \cap V_3) \smallsetminus ((V_1 \cup V_2) \cap V_3)$;
(5) $S_3 := V_3 \smallsetminus ((V_1 + V_2) \cap V_3)$.

Proof. First observe that

$$\dim \pi(V_i) = \begin{cases} d_i, & \text{if } v_3 \notin V_i, \\ d_i - 1, & \text{if } v_3 \in V_i. \end{cases}$$

As for $\dim(\pi(V_1) \cap \pi(V_2))$, note that

$$\dim(\pi(V_1) \cap \pi(V_2)) = \dim(\pi(V_1)) + \dim(\pi(V_2)) - \dim(\pi(V_1) + \pi(V_2))$$

and

$$\begin{aligned}\dim(\pi(V_1)+\pi(V_2)) &= \dim(\pi(V_1+V_2))\\ &= \begin{cases}\dim(V_1+V_2), & \text{if } v_3 \notin V_1+V_2,\\ \dim(V_1+V_2)-1, & \text{if } v_3 \in V_1+V_2.\end{cases}\end{aligned}$$

It follows easily that the three numbers $\dim \pi(V_1)$, $\dim \pi(V_2)$, $\dim(\pi(V_1 \cap V_2))$ are constant along the strata. More explicitly one has the following data:

	$\dim \pi(V_1)$	$\dim \pi(V_2)$	$\dim(\pi(V_1)\cap\pi(V_2))$
S_{123}	d_1-1	d_2-1	$d_{12}-1$
S_{13}	d_1-1	d_2	d_{12}
S_{23}	d_1	d_2-1	d_{12}
$S_{(12)3}$	d_1	d_2	$d_{12}+1$
S_3	d_1	d_2	d_{12}

For example, in the fourth (and most interesting) case, $\dim \pi(V_1) = d_1$ and $\dim \pi(V_2) = d_2$ since $v_3 \notin V_i$ if $v_3 \in S_{(12)3}$; $\dim \pi(V_1+V_2) = \dim(V_1 + V_2) - 1$ since $v_3 \in V_1 + V_2$; and hence

$$\begin{aligned}\dim \pi(V_1)\cap\pi(V_2) &= \dim V_1 + \dim V_2 - \dim(V_1+V_2)+1\\ &= \dim(V_1\cap V_2)+1 = d_{12}+1.\end{aligned}$$

Lemma 6.3 converts this information into the list of the classes $[\mathbb{F}_\alpha]$ and one obtains the following list of cases:

	$[\mathbb{F}_\alpha]$
S_{123}	$\mathbb{L}^{d_1+d_2-2}-\mathbb{L}^{d_1-1}-\mathbb{L}^{d_2-1}-\mathbb{L}^{d_{12}}+\mathbb{L}^{d_{12}-1}+\mathbb{L}$
S_{13}	$\mathbb{L}^{d_1+d_2-1}-\mathbb{L}^{d_1-1}-\mathbb{L}^{d_2}-\mathbb{L}^{d_{12}+1}+\mathbb{L}^{d_{12}}+\mathbb{L}$
S_{23}	$\mathbb{L}^{d_1+d_2-1}-\mathbb{L}^{d_1}-\mathbb{L}^{d_2-1}-\mathbb{L}^{d_{12}+1}+\mathbb{L}^{d_{12}}+\mathbb{L}$
$S_{(12)3}$	$\mathbb{L}^{d_1+d_2}-\mathbb{L}^{d_1}-\mathbb{L}^{d_2}-\mathbb{L}^{d_{12}+2}+\mathbb{L}^{d_{12}+1}+\mathbb{L}$
S_3	$\mathbb{L}^{d_1+d_2}-\mathbb{L}^{d_1}-\mathbb{L}^{d_2}-\mathbb{L}^{d_{12}+1}+\mathbb{L}^{d_{12}}+\mathbb{L}$

The number s_α is immediately read off the geometry. The last ingredient consists of the class $[S_\alpha]$, which is also essentially immediate. The only item that deserves attention is the dimension of $(V_1+V_2)\cap V_3$. This is

$$\dim(V_1+V_2)+\dim V_3-\dim(V_1+V_2+V_3)=\dim(V_1+V_2)+d_3-D$$

and as

$$\dim(V_1+V_2)=\dim V_1+\dim V_2-\dim(V_1\cap V_2)=d_1+d_2-d_{12},$$

we have

$$\dim((V_1+V_2)\cap V_3)=d_1+d_2+d_3-D-d_{12}.$$

With this understanding one obtains the following list of cases:

	$[S_\alpha]$	s_α
S_{123}	$\mathbb{L}^{d_{123}}-1$	2
S_{13}	$\mathbb{L}^{d_{13}}-\mathbb{L}^{d_{123}}$	1
S_{23}	$\mathbb{L}^{d_{23}}-\mathbb{L}^{d_{123}}$	1
$S_{(12)3}$	$\mathbb{L}^{d_1+d_2+d_3-D-d_{12}}-\mathbb{L}^{d_{13}}-\mathbb{L}^{d_{23}}+\mathbb{L}^{d_{123}}$	0
S_3	$\mathbb{L}^{d_3}-\mathbb{L}^{d_1+d_2+d_3-D-d_{12}}$	0

This completes the proof. $\square$

We can now apply equation (6.1), and this gives the following result.

Lemma 6.5. *The class of* $\mathbb{F}(V_1,V_2,V_3)$ *in the Grothendieck group is of the form*

$$\begin{aligned}[\mathbb{F}(V_1,V_2,V_3)]=&(\mathbb{L}^{d_1}-1)(\mathbb{L}^{d_2}-1)(\mathbb{L}^{d_3}-1)-(\mathbb{L}-1)((\mathbb{L}^{d_1}-\mathbb{L})(\mathbb{L}^{d_{23}}-1)\\&+(\mathbb{L}^{d_2}-\mathbb{L})(\mathbb{L}^{d_{13}}-1)+(\mathbb{L}^{d_3}-\mathbb{L})(\mathbb{L}^{d_{12}}-1))\\&+(\mathbb{L}-1)^2(\mathbb{L}^{d_1+d_2+d_3-D}-\mathbb{L}^{d_{123}+1})+(\mathbb{L}-1)^3.\end{aligned}\tag{6.3}$$

Notice once again that the expression (6.3) is symmetric in V_1, V_2, V_3, unlike the contributions of the individual strata. Slightly more refined considerations, in the style of those sketched in Section 6.1, prove that $[\mathbb{F}(V_1,V_2,V_3)]$ is in fact mixed Tate.

In principle, the procedure applied here should work for a larger number of subspaces: the main task amounts to the determination of a stratification of the last subspace satisfying the properties given in Lemma 6.2. This is bound to be rather challenging for $r\geq 4$: already for $r=4$ one can produce examples for which the closures of the strata are not linear subspaces. This is in fact the case already for $V_1, \ldots, V_3$ planes in general position in a four-dimensional ambient space E: the unique quadric cone containing V_1, V_2, V_3 is the closure of a stratum in a stratification of $V_4 = E$ satisfying the properties listed in Lemma 6.2.

6.3 Graphs with three loops

One can apply the formula of Lemma 6.5 to compute explicitly the motive (as a class in the Grothendieck group) for the locus

$$\hat{\Sigma}_{3,0} \smallsetminus (\hat{\Sigma}_{3,0} \cap \hat{\mathcal{D}}_3) \tag{6.4}$$

of intersection of the divisor with normal crossings $\hat{\Sigma}_{\ell,g}$ of (5.1) with the complement of the determinant hypersurface, in the case of (planar) graphs with three loops.

As pointed out in the discussion following Corollary 5.1, studying $\hat{\Sigma}_{3,0}$ suffices in order to get analogous information for $\hat{\Sigma}_\Gamma$ for every graph with three loops and satisfying the condition specified at the beginning of Section 5 (guaranteeing that the corresponding map τ is injective). The divisor $\hat{\Sigma}_{3,0}$ is the divisor corresponding to the "wheel with three spokes" graph (the skeleton of the tetrahedron).

This graph has matrix $M_\Gamma(t)$ given by

$$\begin{pmatrix} t_1 + t_2 + t_5 & -t_1 & -t_2 \\ -t_1 & t_1 + t_3 + t_4 & -t_3 \\ -t_2 & -t_3 & t_2 + t_3 + t_6 \end{pmatrix}.$$

Here, $t_1, \ldots, t_6$ are variables associated with the six edges of the graph, labeled as in figure 6.

Choose the internal faces with counterclockwise orientation as the basis of loops. Then any orientation for the edges leads to the matrix displayed

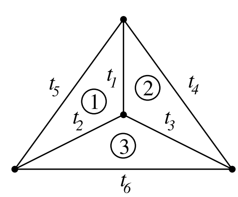

Figure 6: The wheel with three spokes graph.

above. Labeling entries of the matrix as x_{ij}, we can obtain $t_1, \dots, t_6$ as pullbacks of the following:

$$\begin{cases} t_1 = -x_{12}, \\ t_2 = -x_{13}, \\ t_3 = -x_{23}, \\ t_4 = x_{21} + x_{22} + x_{23}, \\ t_5 = x_{11} + x_{12} + x_{13}, \\ t_6 = x_{31} + x_{32} + x_{33}. \end{cases}$$

Thus, we are considering the divisor $\hat{\Sigma}_{3,0}$ with normal crossings given by the equation

$$x_{12}x_{13}x_{23}(x_{11} + x_{12} + x_{13})(x_{21} + x_{22} + x_{23})(x_{31} + x_{32} + x_{33}) = 0.$$

We want to obtain an explicit description, as a class in the Grothendieck group, of the intersection of this locus with the complement of determinant hypersurface $\hat{\mathcal{D}}_3$ in $\mathbb{A}^9$. By inclusion–exclusion (cf. Section 5.2) this can be done by carrying out the computation for all intersections of subsets of the components of this divisor. Since there are six components, there are $2^6 = 64$ such intersections.

Each of these possibilities determines a triple of subspaces V_1, V_2, V_3 inside the ambient $\mathbb{A}^9$ (cf. Lemma 5.3), corresponding to linearly independent vectors v_1, v_2, v_3, i.e., the rows of the matrix x_{ij}, parameterizing points in the complement of the determinant.

Thus, to begin with, one computes for each of these cases the corresponding class $[\mathbb{F}(V_1, V_2, V_3)]$ using Lemma 6.5.

Note that each of these classes is necessarily a multiple of $(\mathbb{L} - 1)^3$: indeed, once the directions of v_1, v_2, v_3 are specified, the set of vectors with those directions forms a $(\mathbb{C}^*)^3$. We list the classes here, divided by this constant factor $(\mathbb{L} - 1)^3$. Each class is marked according to the components of $\hat{\Sigma}_{3,0}$ containing the corresponding locus: for example, $\bullet\bullet\circ\circ\circ\bullet$ corresponds to the complement of $\hat{\mathcal{D}}_3$ in the intersection of $X_1 \cap X_2 \cap X_6$, where X_i pulls back to t_i via τ as above (thus, $X_1 \cap X_2 \cap X_6$ has equations $x_{12} = x_{13} = x_{31} + x_{32} + x_{33} = 0$). See the first of the following tables.

Next, one applies inclusion–exclusion to go from the class $[\mathbb{F}(V_1, V_2, V_3)]$ as above, which corresponds to the complement of the determinant in subspaces obtained as intersections of the six divisors, to classes corresponding to the

●●●●●●	0	●●○○●●	0	●○○○●●	$\mathbb{L}^3$	○●○○○●	$\mathbb{L}^3(\mathbb{L}+2)$
○●●●●●	0	●●○●○●	$\mathbb{L}(\mathbb{L}+1)$	●○○●○●	$\mathbb{L}^2(\mathbb{L}+1)$	○●○○●○	$\mathbb{L}^3(\mathbb{L}+1)$
●○●●●●	0	●●○●●○	0	●○○●●○	$\mathbb{L}^3$	○●○●○○	$\mathbb{L}^3(\mathbb{L}+2)$
●●○●●●	0	●●●○○●	$\mathbb{L}^2$	●○●○○●	$\mathbb{L}(\mathbb{L}^2+2\mathbb{L}-1)$	○●●○○○	$\mathbb{L}^3(\mathbb{L}+1)$
●●●○●●	0	●●●○●○	0	●○●○●○	$\mathbb{L}^2(\mathbb{L}+1)$	●○○○○●	$\mathbb{L}^3(\mathbb{L}+2)$
●●●●○●	$\mathbb{L}$	●●●●○○	$\mathbb{L}^2$	●○●●○○	$\mathbb{L}^2(\mathbb{L}+1)$	●○○○●○	$\mathbb{L}^3(\mathbb{L}+1)$
●●●●●○	0	○○○●●●	0	●●○○○●	$\mathbb{L}^2(\mathbb{L}+1)$	●○○●○○	$\mathbb{L}^3(\mathbb{L}+2)$
○○●●●●	0	○○●○●●	$\mathbb{L}^2(\mathbb{L}+1)$	●●○○●○	0	●○●○○○	$\mathbb{L}^3(\mathbb{L}+2)$
○●○●●●	0	○○●●○●	$\mathbb{L}^3$	●●○●○○	$\mathbb{L}^2(\mathbb{L}+1)$	●●○○○○	$\mathbb{L}^3(\mathbb{L}+1)$
○●●○●●	$\mathbb{L}^2$	○○●●●○	$\mathbb{L}^3$	●●●○○○	$\mathbb{L}^3$	○○○○○●	$\mathbb{L}^3(\mathbb{L}+1)^2$
○●●●○●	$\mathbb{L}^2$	○●○○●●	$\mathbb{L}^3$	○○○○●●	$\mathbb{L}^3(\mathbb{L}+1)$	○○○○●○	$\mathbb{L}^3(\mathbb{L}+1)^2$
○●●●●○	0	○●○●○●	$\mathbb{L}^2(\mathbb{L}+1)$	○○○●○●	$\mathbb{L}^3(\mathbb{L}+1)$	○○○●○○	$\mathbb{L}^3(\mathbb{L}+1)^2$
●○○●●●	0	○●○●●○	$\mathbb{L}^3$	○○○●●○	$\mathbb{L}^3(\mathbb{L}+1)$	○○●○○○	$\mathbb{L}^3(\mathbb{L}+1)^2$
●○●○●●	$\mathbb{L}^2$	○●●○○●	$\mathbb{L}^2(\mathbb{L}+1)$	○○●○○●	$\mathbb{L}^3(\mathbb{L}+2)$	○●○○○○	$\mathbb{L}^3(\mathbb{L}+1)^2$
●○●●○●	$\mathbb{L}^2$	○●●○●○	$\mathbb{L}^3$	○○●○●○	$\mathbb{L}^3(\mathbb{L}+2)$	●○○○○○	$\mathbb{L}^3(\mathbb{L}+1)^2$
●○●●●○	$\mathbb{L}^2$	○●●●○○	$\mathbb{L}^3$	○○●●○○	$\mathbb{L}^3(\mathbb{L}+1)$	○○○○○○	$\mathbb{L}^3(\mathbb{L}+1)(\mathbb{L}^2+\mathbb{L}+1)$

●●●●●●	0	●●○○●●	0	●○○○●●	$\mathbb{L}^2(\mathbb{L}-1)^4$	○●○○○●	$\mathbb{L}^2(\mathbb{L}-1)^5$
○●●●●●	0	●●○●○●	$\mathbb{L}^2(\mathbb{L}-1)^3$	●○○●○●	$\mathbb{L}^2(\mathbb{L}-1)^4$	○●○○●○	$\mathbb{L}^2(\mathbb{L}-1)^5$
●○●●●●	0	●●○●●○	0	●○○●●○	$\mathbb{L}^2(\mathbb{L}-1)^4$	○●○●○○	$\mathbb{L}^2(\mathbb{L}-1)^5$
●●○●●●	0	●●●○○●	$\mathbb{L}(\mathbb{L}-1)^4$	●○●○○●	$\mathbb{L}^2(\mathbb{L}-1)^4$	○●●○○○	$\mathbb{L}(\mathbb{L}-1)^6$
●●●○●●	0	●●●○●○	0	●○●○●○	$\mathbb{L}^2(\mathbb{L}-1)^4$	●○○○○●	$\mathbb{L}(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^4$
●●●●○●	$\mathbb{L}(\mathbb{L}-1)^3$	●●●●○○	$\mathbb{L}(\mathbb{L}-1)^4$	●○●●○○	$\mathbb{L}(\mathbb{L}-1)^5$	●○○○●○	$\mathbb{L}^2(\mathbb{L}-1)^5$
●●●●●○	0	○○○●●●	0	●●○○○●	$\mathbb{L}^2(\mathbb{L}-1)^4$	●○○●○○	$\mathbb{L}^2(\mathbb{L}-1)^5$
○○●●●●	0	○○●○●●	$\mathbb{L}^2(\mathbb{L}-1)^4$	●●○○●○	0	●○●○○○	$\mathbb{L}^2(\mathbb{L}-1)^5$
○●○●●●	0	○○●●○●	$\mathbb{L}(\mathbb{L}-1)^5$	●●○●○○	$\mathbb{L}^2(\mathbb{L}-1)^4$	●●○○○○	$\mathbb{L}^2(\mathbb{L}-1)^5$
○●●○●●	$\mathbb{L}^2(\mathbb{L}-1)^3$	○○●●●○	$\mathbb{L}^2(\mathbb{L}-1)^4$	●●●○○○	$\mathbb{L}(\mathbb{L}-1)^5$	○○○○○●	$\mathbb{L}(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^5$
○●●●○●	$\mathbb{L}(\mathbb{L}-1)^4$	○●○○●●	$\mathbb{L}^2(\mathbb{L}-1)^4$	○○○○●●	$\mathbb{L}^2(\mathbb{L}-1)^5$	○○○○●○	$\mathbb{L}^2(\mathbb{L}-1)^6$
○●●●●○	0	○●○●○●	$\mathbb{L}^2(\mathbb{L}-1)^4$	○○○●○●	$\mathbb{L}^2(\mathbb{L}-1)^5$	○○○●○○	$\mathbb{L}^2(\mathbb{L}-1)^6$
●○○●●●	0	○●○●●○	$\mathbb{L}^3(\mathbb{L}-1)^3$	○○○●●○	$\mathbb{L}^2(\mathbb{L}-1)^5$	○○●○○○	$\mathbb{L}^2(\mathbb{L}-1)^6$
●○●○●●	$\mathbb{L}^2(\mathbb{L}-1)^3$	○●●○○●	$\mathbb{L}(\mathbb{L}-1)^5$	○○●○○●	$\mathbb{L}^2(\mathbb{L}-1)^5$	○●○○○○	$\mathbb{L}^2(\mathbb{L}-1)^6$
●○●●○●	$\mathbb{L}(\mathbb{L}-1)^4$	○●●○●○	$\mathbb{L}^2(\mathbb{L}-1)^4$	○○●○●○	$\mathbb{L}^2(\mathbb{L}-1)^5$	●○○○○○	$\mathbb{L}(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^5$
●○●●●○	$\mathbb{L}^2(\mathbb{L}-1)^3$	○●●●○○	$\mathbb{L}(\mathbb{L}-1)^5$	○○●●○○	$\mathbb{L}(\mathbb{L}-1)^6$	○○○○○○	$\mathbb{L}(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^6$

complement of the determinant in *the complement of smaller subspaces in a given subspace.* This produces the list of classes in the Grothendieck group in the second table: here the classes do include the common factor $(\mathbb{L}-1)^3$.

These are the classes of the individual strata of the stratification of $\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3$ determined by $\hat{\Sigma}_{3,0}$ (including several empty strata). The sum of the classes in this table is the class $[\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3]$, that is $\mathbb{L}^3(\mathbb{L}+1)(\mathbb{L}^2+\mathbb{L}+1)(\mathbb{L}-1)^3 = (\mathbb{L}-1)(\mathbb{L}^8-\mathbb{L}^5-\mathbb{L}^6+\mathbb{L}^3)$ (cf. Example 4.1).

It is interesting to notice that the expressions simplify when one takes inclusion–exclusion into account. The cancellations due to inclusion–exclusion mostly lead to classes of the form $\mathbb{L}^a(\mathbb{L}-1)^b$.

In terms of Feynman integrals, in the case of the wheel with three spokes, we are interested in the relative cohomology

$$H^*(\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3, \hat{\Sigma}_{3,0} \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_{3,0})).$$

The hypersurface complement $\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3$ has class

$$[\mathbb{A}^9 \smallsetminus \hat{\mathcal{D}}_3] = \mathbb{L}^3(\mathbb{L}+1)(\mathbb{L}^2+\mathbb{L}+1)(\mathbb{L}-1)^3, \tag{6.5}$$

while the class of $\hat{\Sigma}_{3,0} \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_{3,0})$ may be obtained as the sum of all the classes listed above except $\circ\circ\circ\ \circ\circ\circ$, which corresponds to the choice of subspaces where $V_1 = V_2 = V_3$ is the whole space (these are all the strata of $\hat{\Sigma}_{3,0} \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_{3,0})$) or, equivalently, the difference of (6.5) and the last item $\circ\circ\circ\ \circ\circ\circ$. This gives

$$\begin{aligned}[\hat{\Sigma}_{3,0} \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_{3,0})] &= \mathbb{L}^3(\mathbb{L}+1)(\mathbb{L}^2+\mathbb{L}+1)(\mathbb{L}-1)^3 \\ &\quad - \mathbb{L}(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^6 \\ &= \mathbb{L}(6\mathbb{L}^4 - 3\mathbb{L}^3 + 2\mathbb{L}^2 + 2\mathbb{L} - 1)(\mathbb{L}-1)^3\end{aligned}$$

The main information is carried by the class $\circ\circ\circ\ \circ\circ\circ$,

$$\mathbb{L}(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^6. \tag{6.6}$$

In the case of other three-loop graphs Γ, such as the one illustrated in figure 7, the divisor $\hat{\Sigma}_\Gamma$ is a union of components of $\hat{\Sigma}_{3,0}$ (cf. Proposition 5.1). The class of the locus $\hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_{3,0})$ may be obtained by adding up all contributions listed above, for the strata contained in $\hat{\Sigma}_\Gamma$. For the example given in figure 7, these are the strata contained in the divisors $X_1, \dots, X_5$; the corresponding classes are those marked by $*\,*\,*\,*\,*\,*$, where at least one of the first five $*$ is $\bullet$; or, equivalently, the difference of (6.5) and the classes marked $\circ\circ\circ\ \circ\circ\bullet$ and $\circ\circ\circ\ \circ\circ\circ$. The sum of these two classes is

$$\mathbb{L}(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^5 + \mathbb{L}(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^6 = \mathbb{L}^2(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^5$$

(cf. (6.6)), and hence

$$\begin{aligned}[\hat{\Sigma}_\Gamma \smallsetminus (\hat{\mathcal{D}}_3 \cap \hat{\Sigma}_{3,0})] &= \mathbb{L}^3(\mathbb{L}+1)(\mathbb{L}^2+\mathbb{L}+1)(\mathbb{L}-1)^3 \\ &\quad - \mathbb{L}^2(\mathbb{L}^2-\mathbb{L}-1)(\mathbb{L}-1)^5 = \mathbb{L}^2(5\mathbb{L}^3+1)(\mathbb{L}-1)^3.\end{aligned}$$

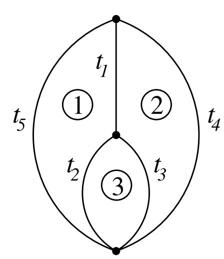

Figure 7: Another graph with three loops and injective τ.

7 Divergences and renormalization

Our analysis in the previous sections of this paper concentrated on the task of showing that a certain relative cohomology is a realization of a mixed Tate motive $\mathfrak{m}(X,Y)$, where the loci X and Y are constructed, respectively, as the complement of the determinant hypersurface and the intersection with this complement of a normal crossing divisor that contains the image of the boundary of the domain of integration σ_n under the map τ_Γ, for any graph Γ with fixed number of loops and fixed genus. Knowing that $\mathfrak{m}(X,Y)$ is a mixed Tate motive implies that, when convergent, the parametric Feynman integral for all such graphs is a period of a mixed Tate motive. This, however, does not take into account the presence of divergences in the Feynman integrals.

There are several different approaches to regularize and renormalize the divergent integrals. We outline here some of the possibilities and comment on how they can be made compatible with our approach.

7.1 Blowups

One possible approach to dealing with divergences coming from the intersections of the divisor Σ_n with the graph hypersurface X_Γ is the one proposed by Bloch–Esnault–Kreimer in [10], namely one can proceed to perform a series of blowups of strata of this intersection until one has separated the domain of integration from the hypersurface and in this way regularized the integral.

In our setting, a similar approach should be reformulated in the ambient $\mathbb{A}^{\ell^2}$ and in terms of the intersection of the determinant hypersurface $\hat{\mathcal{D}}_\ell$ with the divisor $\hat{\Sigma}_{\ell,g}$. If the main question posed in Section 5.3 has an affirmative answer, then this intersection admits a stratification by mixed Tate non-singular loci. It seems likely that a suitable sequence of blowups would then have the effect of regularizing the integral, while at the same time maintaining the motivic nature of the relevant loci unaltered. We intend to return to a more detailed analysis of this approach in future work.

7.2 Dimensional regularization and L-functions

Belkale and Brosnan showed in [4] that dimensionally regularized Feynman integrals can be written in the form of a local Igusa L-function, where the coefficients of the Laurent series expansion are periods, provided the integrals describing them are convergent. Such periods have an explicit description in terms of integrals on simplices σ_n and cubes $[0,1]^r$ of algebraic differental forms

$$f(t)^{s_0}\omega_n \wedge \frac{f(t)-1}{(f(t)-1)t_1+1}dt_1 \wedge \cdots \wedge \frac{f(t)-1}{(f(t)-1)t_r+1}dt_r,$$

for $f(t) = \Psi_\Gamma(t)$ the graph polynomial. The nature of such integrals as periods would still be controlled by the same motivic loci that are involved in the original parametric Feynman integral before dimensional regularization. The result of [4] is formulated only for the case of log-divergent integrals where only the graph polynomial $\Psi_\Gamma(t)$ is present in the Feynman parametric form and not the polynomial $P_\Gamma(t,p)$. The result was extended to the more general non-log-divergent case by Bogner and Weinzierl in [12]. For the renormalization of Feynman integrals via blow-ups in cases more general than log-divergent, see also [11].

In this approach, if there are singularities in the integrals that compute the coefficients of the Laurent series expansion of the local Igusa L-function giving the dimensionally regularized Feynman integral, these can be treated by an algorithmic procedure developed by Bogner and Weinzierl in [13] (see also the short survey [14]). The algorithm is designed to split the divergent integral into sectors where a change of variable that introduces a blowup at the origin isolates the divergence as a pole in a parameter $1/\epsilon$. One can then do a subtraction of this polar part in the Laurent series expansion in the variable ϵ and eliminate the divergence. The iteration part of the algorithm is based on Hironaka's polyhedral game and it is shown in [13] that the resulting algorithm terminates in finite time.

If one uses this approach in our context one will have to show that the changes of variables introduced in the process of evaluating the integrals in sectors do not alter the motivic nature of the loci involved.

7.3 Deformations

An alternative to the use of blowups is the use of deformations. We discuss here the simplest possible procedure one can think of that uses deformations of the graph hypersurface (or of the determinant hypersurface). It is not the most satisfactory deformation method, because it does not lead immediately to a "minimal subtraction" procedure, but it suffices here to illustrate the idea.

Consider the original parametric Feynman integral of the form

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^\beta \omega_n}{\Psi_\Gamma(t)^\alpha}, \tag{7.1}$$

with exponents α and β as in (1.2),

$$\alpha = -n + (\ell+1)D/2, \quad \beta = -n + D\ell/2.$$

Again, for our purposes, we can assume to work in the "stable range" where D is sufficiently large so that both α and β are positive. The case of small D, which is of direct physics interest, leads one to the different problem of considering the hypersurfaces defined by $P_\Gamma(t,p)$, as a function of the external momenta p and the singularities produced by the intersections of these with the domain of integration. This type of analysis can be found in the physics literature, for instance in [32]. See also [7, §18].

In the range where α and β are positive, one can choose to regularize the integral (7.1) by introducing a deformation parameter $\epsilon \in \mathbb{C} \smallsetminus \mathbb{R}_+$ and replaing (7.1) with the deformed

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^\beta \omega_n}{(\Psi_\Gamma(t)-\epsilon)^\alpha}. \tag{7.2}$$

This has the effect of replacing, as locus of the singularities of the integrand, the graph hypersurface $\hat{X}_\Gamma = \{\Psi_\Gamma(t) = 0\}$, with the level set $\hat{X}_{\Gamma,\epsilon} = \{\Psi_\Gamma(t) = \epsilon\}$ of the map $\Psi_\Gamma : \mathbb{A}^n \to \mathbb{A}$. For a choice of ϵ in the cut plane $\mathbb{C} \smallsetminus \mathbb{R}_+$, the hypersurface $\hat{X}_{\Gamma,\epsilon}$ does not intersect the domain of integration σ_n. In fact, for $t_i \geq 0$ one has $\Psi_\Gamma(t) \geq 0$. This choice has therefore the effect of desingularizing the integral. The resulting function of ϵ extends holomorphically to a function on $\mathbb{C} \smallsetminus I$, where $I \subset \mathbb{R}_+$ is the bounded interval of values of Ψ_Γ on σ_n.

When we transform the parametric integral using the map τ_Γ into an integral of a form defined on the complement of the determinant hypersurface $\hat{\mathcal{D}}_\ell$ in $\mathbb{A}^{\ell^2}$ on a domain of integration $\tau_\Gamma(\sigma_n)$ with boundary on the divisor $\hat{\Sigma}_{\ell,g}$, we can similarly separate the divisor from the hypersurface by the same deformation, where instead of the locus $\hat{\mathcal{D}}_\ell = \{\det(x) = 0\}$ one considers the level set $\hat{\mathcal{D}}_{\ell,\epsilon} = \{\det(x) = \epsilon\}$, so that $\hat{\mathcal{D}}_{\ell,\epsilon}$ does not intersect $\tau_\Gamma(\sigma_n)$. The nature of the period described by the deformed integral is then controlled by the motive $\mathfrak{m}(X_\epsilon, Y_\epsilon)$ for $X_\epsilon = \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_{\ell,\epsilon}$ and $Y_\epsilon = \hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\mathcal{D}}_{\ell,\epsilon} \cap \hat{\Sigma}_{\ell,g})$. The question becomes then whether the motivic nature of $\mathfrak{m}(X,Y)$ with $X = X_0$ and $Y = Y_0$ and $\mathfrak{m}(X_\epsilon, Y_\epsilon)$ is the same. This in general is not the case, as one can easily construct examples of fibrations where the generic fiber is not a mixed Tate motive while the special one is. However, in this setting one is dealing with a very special case, where the deformed variety $\hat{\mathcal{D}}_{\ell,\epsilon}$ is given by matrices of fixed determinant. Up to a rescaling, one can check that the fiber $\hat{\mathcal{D}}_{\ell,1} = \mathrm{SL}_n$ is indeed a mixed Tate motive, from the general results of

Biglari [5,6] on reductive groups. Thus, over a set of algebraic values of ϵ one does not leave the world of mixed Tate motives. This will give a statement on the nature of the regularized Feynman integrals as a period of a mixed Tate motive $\mathfrak{m}(X_\epsilon, Y_\epsilon)$ and reduces then the problem to that of removing the divergence as $\epsilon \to 0$, in such way that what remains is a convergent integral whose nature as a period is controlled by the original motive $\mathfrak{m}(X, Y)$.

A different approach to the regularization of parametric Feynman integrals using deformations was discussed in [25] in terms of Leray cocycles and a related regularization procedure.

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