Feynman integrals in configuration space and mixed Tate motives

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based on

- O.Ceyhan, M.Marcolli, *Feynman integrals and motives of configuration spaces*, Communications in Mathematical Physics: Vol.313, N.1 (2012), 35–70, arXiv:1012.5485
- O.Ceyhan, M.Marcolli, *Feynman integrals and periods in configuration spaces*, arXiv:1207.3544

Question: Are Feynman integrals periods of mixed Tate motives? (multiple zeta values: extensive example collection Broadhurst–Kreimer)

• Two methods of computing Feynman integrals (scalar massless Euclidean quantum field theory): momentum space or configuration space (Fourier transform)

$$G_m^{\mathbb{R}}(x_s - x_t) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} dp \, \frac{e^{ip\cdot(x_s - x_t)}}{p^2 + m^2 + i\epsilon}$$

General setting: Motives of algebraic varieties (Grothendieck)
Universal cohomology theory for algebraic varieties (with realizations)

- *Pure motives*: smooth projective varieties with correspondences

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{m-n}_{/\sim,\mathbb{Q}}(X,Y)\,p$$

Algebraic cycles mod equivalence (rational, homological, numerical), composition

$$\mathrm{Corr}(X,Y)\times \mathrm{Corr}(Y,Z)\to \mathrm{Corr}(X,Z)$$

$$(\pi_{X,Z})_*(\pi^*_{X,Y}(\alpha)\bullet \pi^*_{Y,Z}(\beta))$$

intersection product in $X\times Y\times Z$; with projectors $p^2=p$ and $q^2=q$ and Tate twists $\mathbb{Q}(m)$ with $\mathbb{Q}(1)=\mathbb{L}^{-1}$

Numerical pure motives: $\mathscr{M}_{num,\mathbb{Q}}(k)$ semi-simple abelian category (Jannsen)

• *Mixed motives*: varieties that are possibly singular or not projective (much more complicated theory!) Triangulated category $\mathcal{DM}$ (Voevodsky , Levine, Hanamura)

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1]$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2]$$

• *Mixed Tate motives* $\mathcal{DMT} \subset \mathcal{DM}$ generated by the $\mathbb{Q}(m)$

Over a number field: t-structure, abelian category of mixed Tate motives (vanishing result, M.Levine)

Periods and motives: $\int_\sigma \omega$ numbers obtained integrating an algebraic differential form over a cycle defined by algebraic equations
Constraints on numbers obtained as periods from the motive of the variety!

- Periods of mixed Tate motives are Multiple Zeta Values

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} n_1^{-k_1} n_2^{-k_2} \cdots n_r^{-k_r}$$

Conjecture proved recently:
- Francis Brown, *Mixed Tate motives over* $\mathbb{Z}$, Annals of Math 2012, arXiv:1102.1312.

Feynman integrals and periods: MZVs as *typical* outcome:
- D. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, arXiv:hep-th/9609128

General setting: scalar perturbative QFTs

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

in D dimensions, with Lagrangian density

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi)$$

Perturbative expansion: Feynman rules and Feynman diagrams

$$S_{eff}(\phi) = S_0(\phi) + \sum_\Gamma \frac{U(\Gamma, \phi)}{\#\mathrm{Aut}(\Gamma)} \quad \text{(1PI graphs)}$$

Amplitudes $U(\Gamma)$ for fixed external edges of the graph are integral (generally divergent) on:

• momenta associated to internal edges of the graph with momentum conservation rules at vertices

• configurations associated to vertices of the graph with divergences where coordinates collide (diagonals)

• Momentum space: parametric Feynman integrals, graph hypersurfaces, motives of graph hypersurfaces *not* mixed Tate in general (Belkale–Brosnan, Doryn, Schnetz), period can still be mixed Tate (Brown, Brown–Schnetz); various results on classes in the Grothendieck ring (Aluffi-M.)

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

$\sigma_n = \{t \in \mathbb{R}_+^n \,|\, \sum_i t_i = 1\}$, volume form ω_n

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin T} t_e$$

$$P_\Gamma(p,t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

s_C quadratic function of external momenta p_e

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} \,:\, \Psi_\Gamma(t) = 0\}$$

(projective) graph hypersurfaces

• Configuration space: wonderful compactifications of graph configuration spaces; mixed Tate motives; Feynman amplitude and Laplacian Green functions; explicit results using Gegenbauer polynomial expansion; pullback to wonderful compactification, cohomologous to algebraic form with logarithmic poles; deformation and renormalization.

Feynman amplitude in configuration space (dim $D = 2\lambda + 2$)

Version N.1:

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v$$

defines a $\mathscr{C}^\infty$-differential form on X^{V_Γ} with singularities along diagonals $x_{s(e)} = x_{t(e)}$

- not closed form
- chain of integration:

$$\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}$$

Version N.2: (complexification)

$Z = X \times X$ with projection $p : Z \to X$, $p : z = (x, y) \mapsto x$

$$\omega_\Gamma^{(Z)} = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v$$

where $\|x_{s(e)} - x_{t(e)}\| = \|p(z)_{s(e)} - p(z)_{t(e)}\|$

- closed form
- chain of integration:

$$\sigma^{(Z,y)} = X^{V_\Gamma} \times \{y = (y_v)\} \subset Z^{V_\Gamma} = X^{V_\Gamma} \times X^{V_\Gamma}$$

for a fixed $y = (y_v \mid v \in V_\Gamma)$

Relation to Green functions:

- Green function of real Laplacian on $\mathbb{A}^D(\mathbb{R})$, with $D = 2\lambda + 2$:

$$G_{\mathbb{R}}(x,y) = \frac{1}{\|x-y\|^{2\lambda}}$$

- On $\mathbb{A}^D(\mathbb{C})$ complex Laplacian

$$\Delta = \sum_{k=1}^{D} \frac{\partial^2}{\partial x_k \partial \bar{x}_k}$$

has Green form

$$G_{\mathbb{C}}(x,y) = \frac{-(D-2)!}{(2\pi i)^D \|x-y\|^{2D-2}}$$

Feynman amplitudes modeled on the two cases

Different method:

• Version N.1: explicit computation of regularized integral

$$\int_{\sigma_\Gamma} \omega_\Gamma$$

using expansion of Green function in Gegenbauer polynomials: explicit occurrence of multiple zeta values

• Version N.2: cohomological method, pullback $\omega_\Gamma^{(Z)}$ to a compactification of configuration space where cohomologous to algebraic form with log poles; regularize to separate poles from chain of integration; show explicitly motive is mixed Tate

• Discuss first second case (geometric method)

Graph configuration spaces

X a smooth projective algebraic variety that contains a dense $\mathbb{A}^D$: for instance $X = \mathbb{P}^D$, with D spacetime dimension.

Feynman amplitude ω_Γ on X^{V_Γ}

Singularities of Feynman amplitude along diagonals

$$\Delta_e = \{(x_v)_{v \in V_\Gamma} \,|\, x_{v_1} = x_{v_2} \text{ for } \partial_\Gamma(e) = \{v_1, v_2\}\}$$

Graph configuration space:

$$Conf_\Gamma(X) = X^{V_\Gamma} \smallsetminus \bigcup_{e \in E_\Gamma} \Delta_e$$

Goal N.1: compactify $Conf_\Gamma(X)$ to a smooth projective algebraic variety $\overline{Conf}_\Gamma(X)$ so that

$$\overline{Conf}_\Gamma(X) \smallsetminus Conf_\Gamma(X)$$

is a normal crossings divisor

Variants: Version N.2 of configuration space for amplitude $\omega_\Gamma^{(Z)}$

$$F(X,\Gamma) = Z^{V_\Gamma} \setminus \bigcup_{e \in E_\Gamma} \Delta_e^{(Z)} \cong (X \times X)^{V_\Gamma} \setminus \bigcup_{e \in E_\Gamma} \Delta_e^{(Z)}$$

with diagonals

$$\Delta_e^{(Z)} \cong \{(z_v \mid v \in V_\Gamma) \in Z^{V_\Gamma} \mid p(z_{s(e)}) = p(z_{t(e)})\}$$

Relation to previous:

$$F(X,\Gamma) \simeq \mathrm{Conf}_\Gamma(X) \times X^{V_\Gamma}$$

$$\Delta_e^{(Z)} \cong \Delta_e \times X^{V_\Gamma}$$

Compactify to $\overline{F(X,\Gamma)}$ in same way

Wonderful compactifications

• Fulton–MacPherson configuration spaces (= complete graph case of $Conf_\Gamma(X)$)

• More general setting for arrangements of subvarieties: DeConcini–Procesi, Li Li

• General method: realize $\overline{Conf_\Gamma}(X)$ or $\overline{F(X,\Gamma)}$ as a sequence of blowups of X^{V_Γ} (or Z^{V_Γ}) along a collection of dominant transforms of diagonals

• Equivalent description: closure in

$$Conf_\Gamma(X) \hookrightarrow \prod_{\gamma \in \mathscr{G}_\Gamma} \mathrm{Bl}_{\Delta_\gamma} X^{V_\Gamma}$$

with $\mathscr{G}_\Gamma$ subgraphs induced (all edges of Γ between subset of vertices) and 2-vertex-connected

Blowup construction of wonderful compactifications

• Connected induced subgraphs: $\mathbb{SG}_k(\Gamma) = \{\gamma \in \mathbb{SG}(\Gamma) \mid |V_\gamma| = k\}$ and polydiagonals $\hat{\Delta}_\gamma = \{x_{s(e)} = x_{t(e)} \,:\, e \in E_\gamma\}$ (same as diagonal Δ_γ if γ connected)

• Arrangement of subvarieties in X^{V_Γ}: $\mathscr{S}_\Gamma$ polydiagonals of disjoint unions of connected induced subgraphs

• Building set for the arrangement:

$$\mathscr{G}_\Gamma = \{\Delta_\gamma \,:\, \gamma \text{ induced, biconnected }\}$$

$$Conf_\Gamma(X) = X^{V_\Gamma} \smallsetminus \cup_{\gamma \subset \mathscr{G}_\Gamma} \Delta_\gamma$$

• Start with $Y_0 = X^{V_\Gamma}$; obtain Y_k from Y_{k-1} by blowup along the iterated *dominant transforms* (=proper transform or inverse image of exceptional divisor) of

$$\cup_{\gamma \in \mathscr{G}_{n-k+1,\Gamma}} \Delta_\gamma$$

with $\mathscr{G}_{k,\Gamma} = \mathscr{G}_\Gamma \cap \mathbb{SG}_k(\Gamma)$ then

$$Y_{n-1} = \overline{Conf}_\Gamma(X)$$

Boundary structure

• $\mathcal{G}_\Gamma$-nests: sets of biconnected induced subgraphs with $\gamma \cap \gamma' = \emptyset$ or $\gamma \cap \gamma' = \{v\}$ single vertex or $\gamma \subseteq \gamma'$ or $\gamma' \subseteq \gamma$

$$\overline{Conf}_\Gamma(X) \smallsetminus Conf_\Gamma(X) = \bigcup_{\Delta_\gamma \in \mathcal{G}_\Gamma} D_\gamma$$

• divisors D_γ (iterated dominant transform of Δ_γ) with

$$D_{\gamma_1} \cap \cdots \cap D_{\gamma_\ell} \neq \emptyset \Leftrightarrow \{\gamma_1, \ldots, \gamma_\ell\} \text{ is a } \mathcal{G}_\Gamma\text{-nest}$$

and transverse intersections

• strata parameterized by forests of nested subgraphs (as in Fulton–MacPherson case)

• case of $\overline{F(X,\Gamma)}$ completely analogous

Motives of configuration spaces – Key ingredient: Blowup formulae

- For mixed motives (Voevodsky category):

$$\mathfrak{m}(\mathrm{Bl}_V(Y)) \cong \mathfrak{m}(Y) \oplus \bigoplus_{k=1}^{\mathrm{codim}_Y(V)-1} \mathfrak{m}(V)(k)[2k]$$

- For Grothendieck classes Bittner relation

$$[\mathrm{Bl}_V(Y)] = [Y] - [V] + [E] = [Y] + [V]([\mathbb{P}^{\mathrm{codim}_Y(V)-1}] - 1)$$

exceptional divisor E

- Conclusion: the motive of $\overline{Conf_\Gamma}(X)$ and of $\overline{F(X,\Gamma)}$ is mixed Tate if X is mixed Tate.

Voevodsky motive: (quasi-projective smooth X)

$$\mathfrak{m}(\overline{Conf}_\Gamma(X)) = \mathfrak{m}(X)^{V_\Gamma} \oplus \bigoplus_{\mathcal{N} \in \mathcal{G}_\Gamma\text{-nests}, \mu \in M_{\mathcal{N}}} \mathfrak{m}(X)^{V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}}(\|\mu\|)[2\|\mu\|]$$

where $M_{\mathcal{N}} := \{(\mu_\gamma)_{\Delta_\gamma \in \mathcal{G}_\Gamma} : 1 \leq \mu_\gamma \leq r_\gamma - 1, \; \mu_\gamma \in \mathbb{Z}\}$ with $r_\gamma = r_{\gamma,\mathcal{N}} := \dim(\cap_{\gamma' \in \mathcal{N} : \gamma' \subset \gamma} \Delta_{\gamma'}) - \dim \Delta_\gamma$ and $\|\mu\| := \sum_{\Delta_\gamma \in \mathcal{G}_\Gamma} \mu_\gamma$

$$\Gamma/\delta_{\mathcal{N}}(\Gamma) = \Gamma//(\gamma_1 \cup \cdots \cup \gamma_r)$$

for $\mathcal{N} = \{\gamma_1, \ldots, \gamma_r\}$

Class in the Grothendieck ring:

$$[\overline{Conf}_\Gamma(X)] = [X]^{V_\Gamma} + \sum_{\mathcal{N} \in \mathcal{G}_\Gamma\text{-nests}} [X]^{V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}} \sum_{\mu \in M_{\mathcal{N}}} \mathbb{L}^{\|\mu\|}$$

Chow motive: (smooth projective X): from result of Li Li on wonderful compactifications of arrangements of subvarieties

Pullback and forms with logarithmic poles

- $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ pullback to iterated blowup $\overline{F(X,\Gamma)}$ of Z^{V_Γ} along dominant transforms of $\Delta_\gamma^{(Z)}$ of biconnected induced subgraphs
- Divergence locus union of divisors (dominant transforms of $\Delta_\gamma^{(Z)}$)

$$\bigcup_{\Delta_\gamma^{(Z)}\in\mathscr{G}_\Gamma} D_\gamma^{(Z)}$$

- Chain of integration $\tilde{\sigma}_\Gamma^{(Z,y)} = \overline{\mathrm{Conf}}_\Gamma(X)\times\{y\}\subset\overline{F(X,\Gamma)}$ intersects divergence locus in

$$\mathscr{D}_\Gamma = \bigcup_{\Delta_\gamma^{(Z)}\in\mathscr{G}_\Gamma} D_\gamma\times\{y\}\subset\overline{\mathrm{Conf}}_\Gamma(X)\times\{y\}$$

- pullback $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ on $\tilde{\sigma}_\Gamma^{(Z,y)}$ smooth closed form on

$$\overline{\mathrm{Conf}}_\Gamma(X)\smallsetminus\left(\bigcup_{\gamma\in\mathscr{G}_\Gamma} D_\gamma\right)$$

Smooth and algebraic forms

• de Rham cohomology of a smooth quasi-projective varieties computed using algebraic differential forms (Grothendieck)

• if complement of normal crossings divisor can use forms with log poles (Deligne)

$$H^*(\mathscr{U}) \simeq \mathbb{H}^*(\mathscr{X}, \Omega^*_{\mathscr{X}}(\log(\mathscr{D})))$$

• $\mathscr{X}$ smooth projective variety $\dim_{\mathbb{C}} m$; $\mathscr{D}$ normal crossings divisor; $\mathscr{U} = \mathscr{X} \smallsetminus \mathscr{D}$; ω smooth closed differential form deg$m$ on $\mathscr{U}$; $\Rightarrow \exists$ algebraic differential form η log poles along $\mathscr{D}$, with $[\eta] = [\omega] \in H^m_{dR}(\mathscr{U})$

• Conclusion: $\exists$ algebraic form $\eta_\Gamma^{(Z)}$ with log poles along union of D_γ, cohomologous to $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ on $\tilde{\sigma}_\Gamma^{(Z,y)}$

Regularization problem

• $\eta_\Gamma^{(Z)}$ algebraic differential form; $\tilde{\sigma}_\Gamma^{(Z,y)}$ algebraic cycle: Feynman integral becomes

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)} \smallsetminus \mathscr{D}_\Gamma} \eta_\Gamma^{(Z)}$$

would be a period... but divergent!! (because of intersection $\mathscr{D}_\Gamma$ of chain with divisors)

• need a regularization procedure: separate chain of integration from divergence locus

Two regularization methods

• Principal value current regularization and iterated Poincaré residues

• Deformation to the normal cone

Current regularization

- Regularized Feynman amplitude:

$$\langle PV(\eta_\Gamma^{(Z)}), \varphi\rangle = \lim_{\lambda\to 0} \int_{\tilde{\sigma}_\Gamma^{(Z,y)}} |f_n|^{2\lambda_n} \cdots |f_1|^{2\lambda_1} \eta_\Gamma^{(Z,y)} \, \varphi$$

where φ test functions; $n = n_\Gamma = \#\mathscr{G}_\Gamma$; and f_k equation of $D_{\gamma_k}^{(Z)}$

- Ambiguities of regularization:

$$\tilde{\sigma}_{\Gamma,\mathcal{N},\epsilon}^{(Z,y)} := \tilde{\sigma}_\Gamma^{(Z,y)} \cap T_{\mathcal{N},\epsilon}(f) \cap N_{\mathcal{N},\epsilon}(f)$$

$$T_{\mathcal{N},\epsilon}(f) = \{|f_k| = \epsilon_k, \, k = 1, \ldots, r\}$$

$$N_{\mathcal{N},\epsilon}(f) = \{|f_k| > \epsilon, \, k = r+1, \ldots, n\}$$

n graphs in $\mathscr{G}_\Gamma$ ordered so that first r in the nest $\mathcal{N}$

$$\lim_{\epsilon\to 0} \int_{\tilde{\sigma}_{\Gamma,\mathcal{N},\epsilon}^{(Z,y)}} \varphi \, \eta_\Gamma^{(Z,y)}$$

has a residue (iterated Poincaré residue) supported on

$$V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$$

Iterated Poincaré residue

$$\int_{\Sigma_{\mathcal{N}}} \mathcal{R}_{\mathcal{N}}(\eta_\Gamma) = \frac{1}{(2\pi i)^r} \int_{\mathcal{L}_{\mathcal{N}}(\Sigma_{\mathcal{N}})} \eta_\Gamma$$

$(2D|V_\Gamma| - r)$-cycle $\Sigma_{\mathcal{N}}$ in $V_{\mathcal{N}}^{(Z)}$; iterated Leray coboundary $\mathcal{L}_{\mathcal{N}}(\Sigma_{\mathcal{N}})$ in $\overline{F(X,\Gamma)}$ is a T^r-torus bundle over $\Sigma_{\mathcal{N}}$

- If the variety X is a mixed Tate motive, these residues are all periods of mixed Tate motives

- On intersections of chain of integration and divergence loci

$$\langle \mathcal{R}_{\mathcal{N}}(\eta_\Gamma), V_{\mathcal{N}} \rangle = \int_{V_{\mathcal{N}} \times \{y\}} \mathcal{R}_{\mathcal{N}}(\eta_\Gamma)$$

Deformation to the normal cone

• extend integral

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)}} \pi_\Gamma^*(\omega_\Gamma^{(Z)})$$

to a larger ambient deformation space where can separate $\tilde{\sigma}_\Gamma^{(Z,y)}$ from the divergence locus

• start with $Z^{V_\Gamma} \times \mathbb{P}^1$, deformation coordinate $\zeta \in \mathbb{P}^1$, and

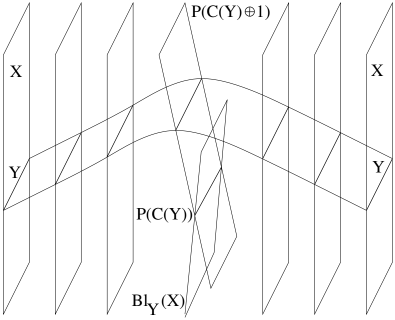
$$\tilde{\omega}_\Gamma^{(Z)} = \prod_{e \in E_\Gamma} \frac{1}{(\|x_{s(e)} - x_{t(e)}\|^2 + |\zeta|^2)^{D-1}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v \wedge d\zeta \wedge d\bar{\zeta}$$

• divergence locus in the central fiber $\zeta = 0$

$$\cup_{e \in E_\Gamma} \Delta_e^{(Z)} \subset Z^{V_\Gamma} \times \{0\}$$

• starting with $Z^{V_\Gamma} \times \mathbb{P}^1$ blowups along $\Delta_\gamma^{(Z)} \times \{0\}$, induced biconnected subgraphs

• obtain smooth projective variety $\mathscr{D}(Z[\Gamma])$ fibered over $\mathbb{P}^1$: fiber over $\zeta \neq 0 \in \mathbb{P}^1$ equal to Z^{V_Γ}; fiber over $\zeta = 0$ has a component $\overline{F(X,\Gamma)}$ plus other components projectivizations $\mathbb{P}(C \oplus 1)$ of normal cones of blowups



• in $\mathscr{D}(Z[\Gamma])$ the chain of integration $\tilde{\sigma}_\Gamma^{(Z,y)}$ becomes separated from the locus of divergence

Deformation: motive and period

• if the motive of X is mixed Tate, then the motive of $\mathscr{D}(Z[\Gamma])$ is also mixed Tate (again blowup formulae)

• pullback $\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$ of form to the deformation along blowup $\tilde{\pi}_\Gamma : \mathscr{D}(Z[\Gamma]) \to Z^{V_\Gamma} \times \mathbb{P}^1$

• locus of divergence union of divisors in the central fiber of projection $\pi : \mathscr{D}(Z[\Gamma]) \to \mathbb{P}^1$

$$\bigcup_{\gamma \in \mathscr{G}_\Gamma} D_\gamma^{(Z)} \subset \pi^{-1}(0)$$

• chain $\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1$ with proper transform $\overline{\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1}$ deformed inside normal cone away from union of divisors (as in figure) to $\Sigma_\Gamma^{(Z,y)}$

• Regularized Feynman amplitude

$$\int_{\Sigma_\Gamma^{(Z,y)}} \delta(\pi^{-1}(0)) \; \tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$$

is a period of a mixed Tate motive

Explicit computations of Feynman amplitudes:

Step 1: explicit chains in X^{V_Γ}

- Acyclic orientations: Γ no looping edges, $\Omega(\Gamma)$ set of acyclic orientations; Stanley: $(-1)^{V_\Gamma} P_\Gamma(-1)$ acyclic orientations where $P_\Gamma(t)$ chromatic polynomial
- orientation $\mathbf{o} \in \Omega(\Gamma) \Rightarrow$ partial ordering of vertices $w \geq_{\mathbf{o}} v$
- chain with boundary $\partial\Sigma_{\mathbf{o}} \subset \cup_{e\in E_\Gamma} \Delta_e$

$$\Sigma_{\mathbf{o}} := \{(x_v) \in X^{V_\Gamma}(\mathbb{R}) \, : \, r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}$$

middle dimensional relative homology class

$$[\Sigma_{\mathbf{o}}] \in H_{|V_\Gamma|}(X^{V_\Gamma}, \cup_{e\in E_\Gamma} \Delta_e)$$

- $\Sigma_{\mathbf{o}} \smallsetminus \cup_v \{r_v = 0\}$ bundle fiber $(S^{D-1})^{V_\Gamma}$ base

$$\overline{\Sigma}_{\mathbf{o}} = \{(r_v) \in (\mathbb{R}^*_+)^{V_\Gamma} \, : \, r_w \geq r_v \text{ whenever } w \geq_{\mathbf{o}} v\}$$

Step 2: Gegenbauer polynomials

- Generating function and orthogonality ($|t| < 1$ and $\lambda > -1/2$)

$$\frac{1}{(1-2tx+t^2)^\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) t^n$$

$$\int_{-1}^{1} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)\,(1-x^2)^{\lambda-1/2} dx = \delta_{n,m} \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}$$

- $D = 2\lambda + 2$ Newton potential expansion in Gegenbauer polynomials:

$$\frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} = \frac{1}{\rho_e^{2\lambda}(1 + (\frac{r_e}{\rho_e})^2 - 2\frac{r_e}{\rho_e}\omega_{s(e)} \cdot \omega_{t(e)})^\lambda}$$
$$= \rho_e^{-2\lambda} \sum_{n=0}^{\infty} (\frac{r_e}{\rho_e})^n C_n^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}),$$

with $\rho_e = \max\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and $r_e = \min\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and with $\omega_v \in S^{D-1}$

Step 3: angular and radial integrals

- on chain of integration $\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}$ Feynman integral becomes (Version N.1)

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\Sigma_{\mathbf{o}}} \prod_{e\in E_\Gamma} r_{t_{\mathbf{o}}(e)}^{-2\lambda} \left(\sum_n (\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}})^n C_n^{(\lambda)}(\omega_{s_{\mathbf{o}}(e)} \cdot \omega_{t_{\mathbf{o}}(e)}) \right) dV$$

with positive integers $m_{\mathbf{o}}$ (multiplicities) and volume form $dV = \prod_v d^D x_v = \prod_v r_v^{D-1} dr_v \, d\omega_v$

- angular integrals:

$$\mathscr{A}_{(n_e)_{e\in E_\Gamma}} = \int_{(S^{D-1})^{V_\Gamma}} \prod_e C_{n_e}^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}) \prod_v d\omega_v$$

- radial integrals:

$$\sum_{\mathbf{o}\in\Omega(\Gamma)} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \prod_{e\in E_\Gamma} \mathscr{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) \prod_v r_v^{D-1} dr_v$$

$$\mathscr{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) = r_{t_{\mathbf{o}}(e)}^{-2\lambda} \sum_{n_e} \mathscr{A}_{n_e} (\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}})^{n_e}$$

Example: polygons and polylogarithms

- Γ polygon with k edges, $D = 2\lambda + 2$:

$$\mathscr{A}_n = \left(\frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)(n+\lambda)} \right)^k \cdot \dim \mathscr{H}_n(S^{2\lambda+1})$$

$\mathscr{H}_n(S^{2\lambda+1})$ space of harmonic functions deg n on $S^{2\lambda+1}$ (Gegenbauer polynomial and zonal spherical harmonics)

- when $D = 4$, Feynman amplitude:

$$(2\pi^2)^k \sum_{\mathbf{o}} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \mathrm{Li}_{k-2}\Big(\prod_i \frac{r_{w_i}^2}{r_{v_i}^2}\Big) \prod_v r_v \, dr_v$$

polylogarithm functions

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

vertices v_i, w_i sources and tails of oriented paths of $\mathbf{o}$

Step 4: stars of vertices and isoscalars

- star (corolla) of a vertex, with unpaired half-edges: angular integral

$$\mathscr{A}_{\underline{n}}(\underline{\omega}) = \int_{S^{D-1}} \prod_j C^{(\lambda)}_{n_j}(\omega_j \cdot \omega)\, d\omega$$

with $\underline{n} = (n_j)_{e_j \in E_\Gamma}$ and $\underline{\omega} = (\omega_j)_{e_j \in E_\Gamma}$

- integrals of products of spherical harmonics:

$$\mathscr{A}_{(n_j)}(\omega_{v_j}) = c_{D,n_1} \cdots c_{D,n_k}\ \tilde{\mathscr{A}}_{(n_j)}(\omega_{v_j})$$

$$\tilde{\mathscr{A}}_{(n_j)}(\omega_{v_j}) = \sum_{\ell_1,\ldots,\ell_k} \overline{Y^{(n_1)}_{\ell_1}(\omega_1) \cdots Y^{(n_k)}_{\ell_k}(\omega_k)} \int_{S^{D-1}} Y^{(n_1)}_{\ell_1}(\omega) \cdots Y^{(n_k)}_{\ell_k}(\omega)\ d\omega$$

$\{Y^{(n)}_\ell\}_{\ell=1,\ldots,d_n}$ orthonormal basis of $\mathscr{H}_n(S^{D-1})$; $d_n = \dim \mathscr{H}_n(S^{D-1})$ and

$$c_{D,n} = \frac{Vol(S^{D-1})\,(D-2)}{2n+D-2}$$

isoscalar factors

• reduce to trivalent vertices: Gaunt coefficients $\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} Y^{(n_3)}_{\ell_3}\rangle_D$
Racah's factorization in terms of *isoscalar factors*

$$\langle Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2}, Y^{(n_3)}_{\ell_3}\rangle_D = \begin{pmatrix} n_1 & n_2 & n_3 \\ n'_1 & n'_2 & n'_3 \end{pmatrix}_{D:D-1} \langle Y^{(n'_1)}_{\ell'_1}, Y^{(n'_2)}_{\ell'_2}, Y^{(n'_3)}_{\ell'_3}\rangle_{D-1}$$

$\ell_i = (n'_i, \ell'_i)$ with $n'_i = m_{D-2,i}$ and $\ell'_i = (m_{D-3,i}, \ldots, m_{1,i})$
there are general explicit expressions for the isoscalar factors

Step 5: gluing trivalent stars by matching half edges

- integrate on variables of matched half-edges:

$$\mathscr{A}_{(n_i)_{i=1,\ldots,4}}((\omega_i)_{i=1,\ldots,4}) = \sum_{\ell_i} \prod_{i=1}^{4} c_{D,n_i} \overline{Y^{(n_i)}_{\ell_i}(\omega_i)} \mathscr{K}_{n_i,\ell_i}(n)$$

$$\mathscr{K}_{n_i,\ell_i}(n) = c^2_{D,n} \sum_{\ell=1}^{d_n} \langle Y^{(n)}_{\ell}, Y^{(n_1)}_{\ell_1}, Y^{(n_2)}_{\ell_2} \rangle_D \cdot \langle Y^{(n)}_{\ell}, Y^{(n_3)}_{\ell_3}, Y^{(n_4)}_{\ell_4} \rangle_D$$

- when $D = 4$ and $\ell_i = 0$

$$\mathscr{K}^{(D=4)}_{\underline{n},\underline{0}}(n) = (\prod_{i=1}^{4} \frac{1}{(n_i+1)^{1/2}}) \frac{4\pi^4}{(n+1)^3},$$

in range $n + n_1 + n_2$ and $n + n_3 + n_4$ even and $|n_j - n_k| \leq n_i \leq n_j + n_k$ for (n_i, n_j, n_k) equal to (n, n_1, n_2) or (n, n_3, n_4) and transpositions; zero otherwise

• radial integral for matched half-edges:

$$r^9 \prod_{i=1}^{3} t_i^{\alpha_i} \sum_{n_1,n_2,n_3} \mathscr{A}_{(n_1,n_2,n_3)}(\omega_1,\omega_2,\omega_3) t_1^{\epsilon_1 n_1} t_2^{\epsilon_2 n_2} t_3^{\epsilon_3 n_3} \, dr \prod_{i=1}^{3} dt_i$$

$\alpha_i = 1$ and $\epsilon_i = 1$ outgoing; $\alpha_i = 3$ and $\epsilon_i = -1$ incoming

• leading term of integral for matched half-edges ($D = 4$):

$$\sum_{\underline{n}} \Big(\prod_{i=1}^{4} c_{D,n_i} \overline{Y_0^{(n_i)}(\omega_i)} \frac{t_i^{\alpha_i+\epsilon_i n_i} \, dt_i}{(n_i+1)^{1/2}}\Big) \int_{\bar{\Sigma}} t^4 \, dt \sum_n \frac{4\pi^2}{(n+1)^3} t^{\epsilon n}$$

sum with constraints $n + n_1 + n_2$ and $n + n_3 + n_4$ even and $|n_j - n_k| \leq n_i \leq n_j + n_k$ for (n_i, n_j, n_k) equal to (n, n_1, n_2) or (n, n_3, n_4) and transpositions

Step 6: gluing all half edges and nested sums

- $\mathscr{R}$ a domain of summation for integers $(n_1, \dots, n_k)$

$$\mathscr{R} = \mathscr{R}_P^{(k)} \quad := \{(n_1, \dots, n_k) \mid n_i > 0, \;\; i = 1, \dots, k\}$$

$$\mathscr{R} = \mathscr{R}_{MP}^{(k)} \quad := \{(n_1, \dots, n_k) \mid n_k > \cdots > n_2 > n_1 > 0\}$$

$$\mathscr{R} = \mathscr{R}_T^{(3)} \quad := \{(n_1, n_2, n_3) \mid n_2 > n_1, \quad n_2 - n_1 < n_3 < n_2 + n_1\}.$$

associated series

$$\mathrm{Li}^{\mathscr{R}}_{s_1,\dots,s_k}(z_1, \dots, z_k) = \sum_{(n_1,\dots,n_k)\in\mathscr{R}} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

includes products of polylogs, multiple polylogs, etc.

- even/odd: $\mathrm{Li}^{\mathscr{R},\mathrm{even}}_{s_1,\dots,s_k}(z_1, \dots, z_k)$ and $\mathrm{Li}^{\mathscr{R},\mathrm{odd}}_{s_1,\dots,s_k}(z_1, \dots, z_k)$ respectively:

$$\frac{1}{2}\left(\mathrm{Li}^{\mathscr{R}}_{s_1,\dots,s_k}(z_1, \dots, z_k) + \mathrm{Li}^{\mathscr{R}}_{s_1,\dots,s_k}(-z_1, \dots, -z_k)\right)$$

$$\frac{1}{2}\left(\mathrm{Li}^{\mathscr{R}}_{s_1,\dots,s_k}(z_1, \dots, z_k) - \mathrm{Li}^{\mathscr{R}}_{s_1,\dots,s_k}(-z_1, \dots, -z_k)\right).$$

• more general odd/even summations ($\mathscr{E}_i = 2\mathbb{N}$ or $\mathscr{E}_i = \mathbb{N} \smallsetminus 2\mathbb{N}$)

$$\mathrm{Li}^{\mathscr{R},\mathscr{E}_1,\ldots,\mathscr{E}_k}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{(n_1,\ldots,n_k)\in\mathscr{R},\, n_i\in\mathscr{E}_i} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

• Example: matching all half-edges

$$\int_0^1 t^9 \, (2^6 \mathrm{Li}^{\mathscr{R}_{MP},\text{odd},\text{even}}_{6,3}(t,t) + 2\mathrm{Li}^{\mathscr{R}_T,\text{even}}_{3,3,3}(t,t,t))\, dt$$

• then relate $\mathrm{Li}^{\mathscr{R}_T}_{s_1,s_2,s_3}(z_1,z_2,z_3)$ to well known generalizations of multiple zeta values and multiple polylogarithms

• Mordell–Tornheim multiple series

$$\zeta_{MT,k}(s_1,\ldots,s_k;s_{k+1}) = \sum_{(n_1,\ldots,n_k)\in\mathscr{R}_P^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}}$$

and function $\mathrm{Li}^{MT}_{s_1,\ldots,s_k;s_{k+1}}(z_1,\ldots,z_k;z_{k+1})$

$$\sum_{(n_1,\ldots,n_k)\in\mathscr{R}_P^{(k)}} \frac{z_1^{n_1}\cdots z_k^{n_k} z_{k+1}^{(n_1+\cdots+n_k)}}{n_1^{s_1}\cdots n_k^{s_k}(n_1+\cdots+n_k)^{s_{k+1}}}$$

• Apostol–Vu multiple series

$$\zeta_{AV,k}(s_1,\ldots,s_k;s_{k+1}) = \sum_{(n_1,\ldots,n_k)\in\mathscr{R}_{MP}^{(k)}} n_1^{-s_1}\cdots n_k^{-s_k}(n_1+\cdots+n_k)^{-s_{k+1}}$$

and function $\mathrm{Li}^{AV}_{s_1,\ldots,s_k;s_{k+1}}(z_1,\ldots,z_k;z_{k+1})$

$$\sum_{(n_1,\ldots,n_k)\in\mathscr{R}_{MP}^{(k)}} \frac{z_1^{n_1}\cdots z_k^{n_k} z_{k+1}^{(n_1+\cdots+n_k)}}{n_1^{s_1}\cdots n_k^{s_k}(n_1+\cdots+n_k)^{s_{k+1}}}$$

Euler–Maclaurin summation formula for $f(t) = x^t t^{-s}$

$$f^{(k)}(t) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{s+k-j-1}{k-j} (k-j)!\, t^{-(s+k-j)} x^t \log(x)^j$$

gives

$$\begin{aligned} \sum_{n=a}^{b} f(n) &= \int_a^b f(t)dt + \frac{1}{2}(f(b)+f(a)) \\ &+ \sum_{k=2}^{N} \frac{b_k}{k!}(f^{(k-1)}(b) - f^{(k-1)}(a)) \\ &- \int_a^b \frac{B_N(t-[t])}{N!} f^{(N)}(t)\, dt, \end{aligned}$$

b_k Bernoulli numbers and B_k Bernoulli polynomials

- applied to $\mathrm{Li}^{\mathscr{R}}_{s_1,s_2,s_3}(z_1,z_2,z_3)$ with $\mathscr{R} = \mathscr{R}^{(3)}_T$ summation terms

$$\pm F_{j,k}(s_3,z_3)\ \mathrm{Li}^{AV}_{s_1,s_2;s_{3+k-j}}(z_1,z_2;z_3)$$

$$\pm F_{j,k}(s_3,z_3)\ \mathrm{Li}^{MT}_{s_1,s_{3+k-j};s_2}(z_1,z_2;z_3)$$

with

$$F_{j,k}(s,z) = \frac{b_k}{k!}\binom{k}{j}\binom{s+k-j-1}{k-j}(k-j)!\ \log(z)^j$$

Conclusion: by this method can see explicit integrals leading to multiple zeta values, but computations become easily extremely complicated even for simple graphs!