Feynman integrals in configuration space and mixed Tate motives

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based on

Question: Are Feynman integrals periods of mixed Tate motives? (multiple zeta values: extensive example collection Broadhurst–Kreimer)

- Two methods of computing Feynman integrals (scalar massless Euclidean quantum field theory): momentum space or configuration space (Fourier transform)

\[
G_m^R(x_s - x_t) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} dp \, \frac{e^{ip \cdot (x_s - x_t)}}{p^2 + m^2 + i\epsilon}
\]
General setting: Motives of algebraic varieties (Grothendieck)
Universal cohomology theory for algebraic varieties (with realizations)

- **Pure motives**: smooth projective varieties with correspondences

\[
\text{Hom}((X, p, m), (Y, q, n)) = q \text{Corr}_{\sim, \mathbb{Q}}^{m-n}(X, Y) p
\]

Algebraic cycles mod equivalence (rational, homological, numerical), composition

\[
\text{Corr}(X, Y) \times \text{Corr}(Y, Z) \rightarrow \text{Corr}(X, Z)
\]

\[
(\pi_{X,Z})_*(\pi^*_{X,Y}(\alpha) \bullet \pi^*_{Y,Z}(\beta))
\]

intersection product in \( X \times Y \times Z \); with projectors \( p^2 = p \) and \( q^2 = q \)
and Tate twists \( \mathbb{Q}(m) \) with \( \mathbb{Q}(1) = \mathbb{L}^{-1} \)

Numerical pure motives: \( \mathcal{M}_{\text{num}, \mathbb{Q}}(k) \) semi-simple abelian category (Jannsen)
• **Mixed motives**: varieties that are possibly singular or not projective (much more complicated theory!) Triangulated category $\mathcal{D}m$ (Voevodsky, Levine, Hanamura)

$$m(Y) \to m(X) \to m(X \setminus Y) \to m(Y)[1]$$

$$m(X \times \mathbb{A}^1) = m(X)(-1)[2]$$

• **Mixed Tate motives** $\mathcal{D}mT \subset \mathcal{D}m$ generated by the $\mathbb{Q}(m)$

Over a number field: $t$-structure, abelian category of mixed Tate motives (vanishing result, M. Levine)
Periods and motives: $\int_{\sigma} \omega$ numbers obtained integrating an algebraic differential form over a cycle defined by algebraic equations
Constraints on numbers obtained as periods from the motive of the variety!

• Periods of mixed Tate motives are Multiple Zeta Values

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_1 > n_2 > \ldots > n_r \geq 1} n_1^{-k_1} n_2^{-k_2} \cdots n_r^{-k_r}$$

Conjecture proved recently:

Feynman integrals and periods: MZVs as typical outcome:
• D. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, arXiv:hep-th/9609128
General setting: scalar perturbative QFTs

\[ S(\phi) = \int \mathcal{L}(\phi) d^Dx = S_0(\phi) + S_{\text{int}}(\phi) \]

in \( D \) dimensions, with Lagrangian density

\[ \mathcal{L}(\phi) = \frac{1}{2} (\partial \phi)^2 - \frac{m^2}{2} \phi^2 - \mathcal{L}_{\text{int}}(\phi) \]

Perturbative expansion: Feynman rules and Feynman diagrams

\[ S_{\text{eff}}(\phi) = S_0(\phi) + \sum \frac{U(\Gamma, \phi)}{\# \text{Aut}(\Gamma)} \quad (1\text{PI graphs}) \]

Amplitudes \( U(\Gamma) \) for fixed external edges of the graph are integral (generally divergent) on:

- momenta associated to internal edges of the graph with momentum conservation rules at vertices
- configurations associated to vertices of the graph with divergences where coordinates collide (diagonals)
• Momentum space: parametric Feynman integrals, graph hypersurfaces, motives of graph hypersurfaces not mixed Tate in general (Belkale–Brosnan, Doryn, Schnetz), period can still be mixed Tate (Brown, Brown–Schnetz); various results on classes in the Grothendieck ring (Aluffi-M.)

\[ U(\Gamma) = \frac{\Gamma(n-D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t, p)^{-n+D\ell/2} \omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}} \]

\( \sigma_n = \{ t \in \mathbb{R}^n_+ | \sum_i t_i = 1 \} \), volume form \( \omega_n \)

\[ \Psi_\Gamma(t) = \sum_T \prod_{e \notin T} t_e \]

\[ P_\Gamma(p, t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e \]

\( s_C \) quadratic function of external momenta \( p_e \)

\[ \chi_\Gamma = \{ t \in \mathbb{P}^{n-1} : \Psi_\Gamma(t) = 0 \} \]

(projective) graph hypersurfaces
• **Configuration space**: wonderful compactifications of graph configuration spaces; mixed Tate motives; Feynman amplitude and Laplacian Green functions; explicit results using Gegenbauer polynomial expansion; pullback to wonderful compactification, cohomologous to algebraic form with logarithmic poles; deformation and renormalization.
Feynman amplitude in configuration space \((\text{dim } D = 2\lambda + 2)\)

Version N.1:

\[
\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_s(e) - x_t(e)\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v
\]

defines a \(C^\infty\)-differential form on \(X^{V_\Gamma}\) with singularities along diagonals \(x_s(e) = x_t(e)\)

- not closed form
- chain of integration:

\[
\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}
\]
Version N.2: (complexification)

\[ Z = X \times X \text{ with projection } p : Z \to X, \; p : z = (x, y) \mapsto x \]

\[ \omega^{(Z)}_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\left\| x_{s(e)} - x_{t(e)} \right\|^{2D-2}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v \]

where \[ \left\| x_{s(e)} - x_{t(e)} \right\| = \left\| p(z)_{s(e)} - p(z)_{t(e)} \right\| \]

- closed form
- chain of integration:

\[ \sigma^{(Z,y)} = X^{V_\Gamma} \times \{ y = (y_v) \} \subset Z^{V_\Gamma} = X^{V_\Gamma} \times X^{V_\Gamma} \]

for a fixed \( y = (y_v \mid v \in V_\Gamma) \)
Relation to Green functions:

- Green function of real Laplacian on $\mathbb{A}^D(\mathbb{R})$, with $D = 2\lambda + 2$:

$$G_\mathbb{R}(x, y) = \frac{1}{\|x - y\|^{2\lambda}}$$

- On $\mathbb{A}^D(\mathbb{C})$ complex Laplacian

$$\Delta = \sum_{k=1}^{D} \frac{\partial^2}{\partial x_k \partial \bar{x}_k}$$

has Green form

$$G_\mathbb{C}(x, y) = \frac{-(D - 2)!}{(2\pi i)^D \|x - y\|^{2D-2}}$$

Feynman amplitudes modeled on the two cases.
Different method:

- **Version N.1**: explicit computation of regularized integral
  \[ \int_{\sigma \Gamma} \omega_{\Gamma} \]
  using expansion of Green function in Gegenbauer polynomials:
  explicit occurrence of multiple zeta values

- **Version N.2**: cohomological method, pullback \( \omega_{\Gamma}^{(Z)} \) to a compactification of configuration space where cohomologous to algebraic form with log poles; regularize to separate poles from chain of integration; show explicitly motive is mixed Tate

- Discuss first second case (geometric method)
Graph configuration spaces

$X$ a smooth projective algebraic variety that contains a dense $\mathbb{A}^D$: for instance $X = \mathbb{P}^D$, with $D$ spacetime dimension.

Feynman amplitude $\omega_\Gamma$ on $X^{V_\Gamma}$

Singularities of Feynman amplitude along diagonals

$$\Delta_e = \{(x_v)_{v \in V_\Gamma} \mid x_{v_1} = x_{v_2} \text{ for } \partial_\Gamma(e) = \{v_1, v_2\}\}$$

Graph configuration space:

$$Conf_\Gamma(X) = X^{V_\Gamma} \setminus \bigcup_{e \in E_\Gamma} \Delta_e$$

Goal N.1: compactify $Conf_\Gamma(X)$ to a smooth projective algebraic variety $\overline{Conf_\Gamma(X)}$ so that

$$\overline{Conf_\Gamma(X)} \setminus Conf_\Gamma(X)$$

is a normal crossings divisor
Variants: Version N.2 of configuration space for amplitude $\omega^{(Z)}_{\Gamma}$

$$F(X, \Gamma) = Z^{V_{\Gamma}} \setminus \bigcup_{e \in E_{\Gamma}} \Delta^{(Z)}_{e} \cong (X \times X)^{V_{\Gamma}} \setminus \bigcup_{e \in E_{\Gamma}} \Delta^{(Z)}_{e}$$

with diagonals

$$\Delta^{(Z)}_{e} \cong \{(z_{v} \mid v \in V_{\Gamma}) \in Z^{V_{\Gamma}} \mid p(z_{s(e)}) = p(z_{t(e)})\}$$

Relation to previous:

$$F(X, \Gamma) \cong \text{Conf}_{\Gamma}(X) \times X^{V_{\Gamma}}$$

$$\Delta^{(Z)}_{e} \cong \Delta_{e} \times X^{V_{\Gamma}}$$

Compactify to $F(X, \Gamma)$ in same way
Wonderful compactifications

- Fulton–MacPherson configuration spaces (= complete graph case of $Conf_{\Gamma}(X)$)
- More general setting for arrangements of subvarieties: DeConcini–Procesi, Li Li
- General method: realize $Conf_{\Gamma}(X)$ or $F(X, \Gamma)$ as a sequence of blowups of $X^{\nu_{\Gamma}}$ (or $Z^{\nu_{\Gamma}}$) along a collection of dominant transforms of diagonals
- Equivalent description: closure in

$$Conf_{\Gamma}(X) \hookrightarrow \prod_{\gamma \in \mathcal{G}_{\Gamma}} \text{Bl}_{\Delta_{\gamma}} X^{\nu_{\Gamma}}$$

with $\mathcal{G}_{\Gamma}$ subgraphs induced (all edges of $\Gamma$ between subset of vertices) and 2-vertex-connected
Blowup construction of wonderful compactifications

- Connected induced subgraphs: \( \mathcal{SG}_k(\Gamma) = \{ \gamma \in \mathcal{SG}(\Gamma) \mid |V_\gamma| = k \} \)
  and polydiagonals \( \hat{\Delta}_\gamma = \{ x_{s(e)} = x_{t(e)} : e \in E_\gamma \} \) (same as diagonal \( \Delta_\gamma \) if \( \gamma \) connected)

- Arrangement of subvarieties in \( X^{V_\Gamma} \): \( S_\Gamma \) polydiagonals of disjoint unions of connected induced subgraphs

- Building set for the arrangement:

\[
\mathcal{G}_\Gamma = \{ \Delta_\gamma : \gamma \text{ induced, biconnected} \}
\]

\[
\text{Conf}_\Gamma(X) = X^{V_\Gamma} \setminus \bigcup_{\gamma \subset \mathcal{G}_\Gamma} \Delta_\gamma
\]

- Start with \( Y_0 = X^{V_\Gamma} \); obtain \( Y_k \) from \( Y_{k-1} \) by blowup along the iterated dominant transforms (=proper transform or inverse image of exceptional divisor) of

\[
\bigcup_{\gamma \in \mathcal{G}_{n-k+1, \Gamma}} \Delta_\gamma
\]

with \( \mathcal{G}_{k, \Gamma} = \mathcal{G}_\Gamma \cap \mathcal{SG}_k(\Gamma) \) then

\[
Y_{n-1} = \overline{\text{Conf}_\Gamma(X)}
\]
Boundary structure

- $G_\Gamma$-nests: sets of biconnected induced subgraphs with $\gamma \cap \gamma' = \emptyset$ or $\gamma \cap \gamma' = \{v\}$ single vertex or $\gamma \subseteq \gamma'$ or $\gamma' \subseteq \gamma$

\[
\overline{\text{Conf}}_\Gamma(X) \setminus \text{Conf}_\Gamma(X) = \bigcup_{\Delta_\gamma \in G_\Gamma} D_\gamma
\]

- divisors $D_\gamma$ (iterated dominant transform of $\Delta_\gamma$) with

\[
D_{\gamma_1} \cap \cdots \cap D_{\gamma_\ell} \neq \emptyset \iff \{\gamma_1, \ldots, \gamma_\ell\} \text{ is a } G_\Gamma\text{-nest}
\]

and transverse intersections

- strata parameterized by forests of nested subgraphs (as in Fulton–MacPherson case)

- case of $\overline{F}(X, \Gamma)$ completely analogous
Motives of configuration spaces – Key ingredient: Blowup formulae

- For mixed motives (Voevodsky category):

\[
m(\text{Bl}_V(Y)) \cong m(Y) \oplus \bigoplus_{k=1}^{\text{codim}_Y(V) - 1} m(V)(k)[2k]
\]

- For Grothendieck classes Bittner relation

\[
[\text{Bl}_V(Y)] = [Y] - [V] + [E] = [Y] + [V](\mathbb{P}^{\text{codim}_Y(V) - 1} - 1)
\]

exceptional divisor \(E\)

- **Conclusion**: the motive of \(\text{Conf}_\Gamma(X)\) and of \(F(X, \Gamma)\) is mixed Tate if \(X\) is mixed Tate.
Voevodsky motive: (quasi-projective smooth $X$)

$$m(\overline{\text{Conf}}_{\Gamma}(X)) = m(X)^{V_{\Gamma}} \bigoplus \bigoplus_{\mathcal{N} \in \mathcal{G}_{\Gamma}\text{-nests}, \mu \in M_{\mathcal{N}}} m(X)^{V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}(\|\mu\|)[2\|\mu\|]}$$

where $M_{\mathcal{N}} := \{(\mu_{\gamma})_{\Delta_{\gamma} \in \mathcal{G}_{\Gamma}} : 1 \leq \mu_{\gamma} \leq r_{\gamma} - 1, \mu_{\gamma} \in \mathbb{Z}\}$ with $r_{\gamma} = r_{\gamma}, \mathcal{N} := \dim(\bigcap_{\gamma' \in \mathcal{N} : \gamma' \subset \gamma} \Delta_{\gamma'}) - \dim \Delta_{\gamma}$ and $\|\mu\| := \sum_{\Delta_{\gamma} \in \mathcal{G}_{\Gamma}} \mu_{\gamma}$

$$\Gamma/\delta_{\mathcal{N}}(\Gamma) = \Gamma//((\gamma_1 \cup \cdots \cup \gamma_r))$$

for $\mathcal{N} = \{\gamma_1, \ldots, \gamma_r\}$

Class in the Grothendieck ring:

$$[\overline{\text{Conf}}_{\Gamma}(X)] = [X]^{V_{\Gamma}} + \sum_{\mathcal{N} \in \mathcal{G}_{\Gamma}\text{-nests}} [X]^{V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}} \sum_{\mu \in M_{\mathcal{N}}} \|\mu\|$$

Chow motive: (smooth projective $X$): from result of Li Li on wonderful compactifications of arrangements of subvarieties
Pullback and forms with logarithmic poles

- $\pi^*_\gamma(\omega^{(Z)}_\Gamma)$ pullback to iterated blowup $\overline{F(X, \Gamma)}$ of $Z^{V_\Gamma}$ along dominant transforms of $\Delta^{(Z)}_\gamma$ of biconnected induced subgraphs
- Divergence locus union of divisors (dominant transforms of $\Delta^{(Z)}_\gamma$)
  \[
  \bigcup_{\Delta^{(Z)}_\gamma \in \mathcal{G}_\Gamma} D^{(Z)}_\gamma
  \]
- Chain of integration $\tilde{\sigma}^{(Z, y)}_\Gamma = \overline{\text{Conf}_\Gamma(X) \times \{y\} \subset \overline{F(X, \Gamma)}}$ intersects divergence locus in
  \[
  \mathcal{D}_\Gamma = \bigcup_{\Delta^{(Z)}_\gamma \in \mathcal{G}_\Gamma} D_\gamma \times \{y\} \subset \overline{\text{Conf}_\Gamma(X) \times \{y\}}
  \]
- Pullback $\pi^*_\gamma(\omega^{(Z)}_\Gamma)$ on $\tilde{\sigma}^{(Z, y)}_\Gamma$ smooth closed form on
  \[
  \overline{\text{Conf}_\Gamma(X)} \setminus \left( \bigcup_{\gamma \in \mathcal{G}_\Gamma} D_\gamma \right)
  \]
Smooth and algebraic forms

- de Rham cohomology of a smooth quasi-projective varieties computed using algebraic differential forms (Grothendieck)
- if complement of normal crossings divisor can use forms with log poles (Deligne)

\[ H^*(\mathcal{U}) \cong H^*_\mathbb{H}(\mathcal{X}, \Omega^*_\mathcal{X}(\log(D))) \]

- \( \mathcal{X} \) smooth projective variety \( \dim_{\mathbb{C}} m \); \( D \) normal crossings divisor;
- \( \mathcal{U} = \mathcal{X} \setminus D \); \( \omega \) smooth closed differential form \( \deg m \) on \( \mathcal{U} \);
- \( \Rightarrow \exists \) algebraic differential form \( \eta \) log poles along \( D \), with
  \( [\eta] = [\omega] \in H^m_{dR}(\mathcal{U}) \)

- Conclusion: \( \exists \) algebraic form \( \eta_{\Gamma}^{(Z)} \) with log poles along union of \( D_{\gamma} \), cohomologous to \( \pi^*_\gamma(\omega_{\Gamma}^{(Z)}) \) on \( \tilde{\sigma}_{\Gamma}^{(Z,y)} \)
Regularization problem

- $\eta_\Gamma^{(Z)}$ algebraic differential form; $\tilde{\sigma}_\Gamma^{(Z,y)}$ algebraic cycle: Feynman integral becomes

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)} \setminus \mathcal{D}_\Gamma} \eta_\Gamma^{(Z)}$$

would be a period... but divergent!! (because of intersection $\mathcal{D}_\Gamma$ of chain with divisors)

- need a regularization procedure: separate chain of integration from divergence locus

Two regularization methods

- Principal value current regularization and iterated Poincaré residues
- Deformation to the normal cone
Current regularization

- Regularized Feynman amplitude:

\[
\langle PV(\eta^{(Z)}_\Gamma), \varphi \rangle = \lim_{\lambda \to 0} \int_{\tilde{\sigma}^{(Z,y)}_\Gamma} |f_n|^{2\lambda n} \cdots |f_1|^{2\lambda_1} \eta^{(Z,y)}_\Gamma \varphi
\]

where \( \varphi \) test functions; \( n = n_\Gamma = \# \mathcal{G}_\Gamma \); and \( f_k \) equation of \( D^{(Z)}_{\gamma_k} \)

- Ambiguities of regularization:

\[
\tilde{\sigma}^{(Z,y)}_{\Gamma,\mathcal{N},\epsilon} := \tilde{\sigma}^{(Z,y)}_\Gamma \cap T_{\mathcal{N},\epsilon}(f) \cap N_{\mathcal{N},\epsilon}(f)
\]

\[
T_{\mathcal{N},\epsilon}(f) = \{|f_k| = \epsilon_k, \ k = 1, \ldots, r\}
\]

\[
N_{\mathcal{N},\epsilon}(f) = \{|f_k| > \epsilon, \ k = r + 1, \ldots, n\}
\]

\( n \) graphs in \( \mathcal{G}_\Gamma \) ordered so that first \( r \) in the nest \( \mathcal{N} \)

\[
\lim_{\epsilon \to 0} \int_{\tilde{\sigma}^{(Z,y)}_{\Gamma,\mathcal{N},\epsilon}} \varphi \eta^{(Z,y)}_\Gamma
\]

has a residue (iterated Poincaré residue) supported on

\[
V^{(Z)}_{\mathcal{N}} = D^{(Z)}_{\gamma_1} \cap \cdots \cap D^{(Z)}_{\gamma_r}
\]
Iterated Poincaré residue

\[ \int_{\Sigma_N} R_N(\eta_\Gamma) = \frac{1}{(2\pi i)^r} \int_{\mathcal{L}_N(\Sigma_N)} \eta_\Gamma \]

\((2D|V_\Gamma| - r)\)-cycle \(\Sigma_N\) in \(V^{(Z)}_N\); iterated Leray coboundary \(\mathcal{L}_N(\Sigma_N)\) in \(F(X, \Gamma)\) is a \(T^r\)-torus bundle over \(\Sigma_N\)

- If the variety \(X\) is a mixed Tate motive, these residues are all periods of mixed Tate motives
- On intersections of chain of integration and divergence loci

\[ \langle R_N(\eta_\Gamma), V_N \rangle = \int_{V_N \times \{y\}} R_N(\eta_\Gamma) \]
Deformation to the normal cone

- extend integral

\[ \int_{\tilde{\sigma}_{\Gamma}^{(Z,y)}} \pi^{*}(\omega_{\Gamma}^{(Z)}) \]

to a larger ambient deformation space where can separate \( \tilde{\sigma}_{\Gamma}^{(Z,y)} \) from the divergence locus

- start with \( Z^{\nu_{\Gamma}} \times \mathbb{P}^{1} \), deformation coordinate \( \zeta \in \mathbb{P}^{1} \), and

\[
\tilde{\omega}_{\Gamma}^{(Z)} = \prod_{e \in E_{\Gamma}} \frac{1}{(\|x(s(e)) - x(t(e))\|^2 + |\zeta|^2)^{D-1}} \bigwedge_{v \in V_{\Gamma}} dx_v \wedge d\bar{x}_v \wedge d\zeta \wedge d\bar{\zeta}
\]

- divergence locus in the central fiber \( \zeta = 0 \)

\[
\bigcup_{e \in E_{\Gamma}} \Delta_{e}^{(Z)} \subset Z^{\nu_{\Gamma}} \times \{0\}
\]
• starting with $Z^\nu \times \mathbb{P}^1$ blowups along $\Delta^{(Z)}_{\gamma} \times \{0\}$, induced biconnected subgraphs

• obtain smooth projective variety $\mathcal{D}(Z[\Gamma])$ fibered over $\mathbb{P}^1$: fiber over $\zeta \neq 0 \in \mathbb{P}^1$ equal to $Z^\nu$; fiber over $\zeta = 0$ has a component $F(X, \Gamma)$ plus other components projectivizations $\mathbb{P}(C \oplus 1)$ of normal cones of blowups

• in $\mathcal{D}(Z[\Gamma])$ the chain of integration $\tilde{\sigma}_{\Gamma}^{(Z,y)}$ becomes separated from the locus of divergence
Deformation: motive and period

• if the motive of $X$ is mixed Tate, then the motive of $\mathcal{D}(Z[\Gamma])$ is also mixed Tate (again blowup formulae)

• pullback $\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$ of form to the deformation along blowup $\tilde{\pi}_\Gamma : \mathcal{D}(Z[\Gamma]) \to Z^\vee_\Gamma \times \mathbb{P}^1$

• locus of divergence union of divisors in the central fiber of projection $\pi : \mathcal{D}(Z[\Gamma]) \to \mathbb{P}^1$

$$\bigcup_{\gamma \in \mathcal{G}_\Gamma} D^{(Z)}_\gamma \subset \pi^{-1}(0)$$

• chain $\sigma_{\Gamma}^{(Z,y)} \times \mathbb{P}^1$ with proper transform $\overline{\sigma_{\Gamma}^{(Z,y)} \times \mathbb{P}^1}$ deformed inside normal cone away from union of divisors (as in figure) to $\Sigma_{\Gamma}^{(Z,y)}$

• Regularized Feynman amplitude

$$\int_{\Sigma_{\Gamma}^{(Z,y)}} \delta(\pi^{-1}(0)) \tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$$

is a period of a mixed Tate motive
Explicit computations of Feynman amplitudes:

Step 1: explicit chains in $X^{V_{\Gamma}}$

- Acyclic orientations: $\Gamma$ no looping edges, $\Omega(\Gamma)$ set of acyclic orientations; Stanley: $(-1)^{V_{\Gamma}} P_{\Gamma}(-1)$ acyclic orientations where $P_{\Gamma}(t)$ chromatic polynomial

- orientation $o \in \Omega(\Gamma) \Rightarrow$ partial ordering of vertices $w \geq_o v$

- chain with boundary $\partial \Sigma_o \subset \bigcup_{e \in E_{\Gamma}} \Delta_e$

$$\Sigma_o := \{(x_v) \in X^{V_{\Gamma}}(\mathbb{R}) : r_w \geq r_v \text{ whenever } w \geq_o v\}$$

middle dimensional relative homology class

$$[\Sigma_o] \in H_{|V_{\Gamma}|}(X^{V_{\Gamma}}, \bigcup_{e \in E_{\Gamma}} \Delta_e)$$

- $\Sigma_o \setminus \bigcup_v \{r_v = 0\}$ bundle fiber $(S^{D-1})^{V_{\Gamma}}$ base

$$\overline{\Sigma}_o = \{(r_v) \in (\mathbb{R}^*_+)^{V_{\Gamma}} : r_w \geq r_v \text{ whenever } w \geq_o v\}$$
Step 2: Gegenbauer polynomials

- Generating function and orthogonality (|t| < 1 and \( \lambda > -1/2 \))

\[
\frac{1}{(1 - 2tx + t^2)^\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x)t^n
\]

\[
\int_{-1}^{1} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x) (1 - x^2)^{\lambda-1/2} dx = \delta_{n,m} \frac{\pi 2^{1-2\lambda} \Gamma(n + 2\lambda)}{n!(n + \lambda)\Gamma(\lambda)^2}
\]

- \( D = 2\lambda + 2 \) Newton potential expansion in Gegenbauer polynomials:

\[
\frac{1}{\| x_{s(e)} - x_{t(e)} \|^2\lambda} = \frac{1}{\rho_e^{2\lambda}(1 + (\frac{r_e}{\rho_e})^2 - 2\frac{r_e}{\rho_e} \omega_{s(e)} \cdot \omega_{t(e)})^\lambda}
\]

\[
= \rho_e^{-2\lambda} \sum_{n=0}^{\infty} \left( \frac{r_e}{\rho_e} \right)^n C_n^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}),
\]

with \( \rho_e = \max\{\| x_{s(e)} \|, \| x_{t(e)} \|\} \) and \( r_e = \min\{\| x_{s(e)} \|, \| x_{t(e)} \|\} \) and with \( \omega \in S^{D-1} \)
Step 3: angular and radial integrals

• on chain of integration $\sigma_\Gamma = X(\mathbb{R})^V$ Feynman integral becomes (Version N.1)

$$\sum_{o \in \Omega(\Gamma)} m_o \int_{\Sigma_o} \prod_{e \in E_\Gamma} r_{t_0(e)}^{-2\lambda} \left( \sum_n \left( \frac{r_{s_0(e)}}{r_{t_0(e)}} \right)^n C_n^{(\lambda)} (\omega_{s_0(e)} \cdot \omega_{t_0(e)}) \right) dV$$

with positive integers $m_o$ (multiplicities) and volume form $dV = \prod_v d^D x_v = \prod_v r_v^{D-1} dr_v d\omega_v$

• angular integrals:

$$A(e)_{e \in E_\Gamma} = \int_{(S^{D-1})^V} \prod_e C_n^{(\lambda)} (\omega_{s(e)} \cdot \omega_{t(e)}) \prod_v d\omega_v$$

• radial integrals:

$$\sum_{o \in \Omega(\Gamma)} m_o \int_{\Sigma_o} \prod_{e \in E_\Gamma} \mathcal{F}(r_{s_0(e)}, r_{t_0(e)}) \prod_v r_v^{D-1} dr_v$$

$$\mathcal{F}(r_{s_0(e)}, r_{t_0(e)}) = r_{t_0(e)}^{-2\lambda} \sum_{n_e} A_{n_e} \left( \frac{r_{s_0(e)}}{r_{t_0(e)}} \right)^{n_e}$$
Example: polygons and polylogarithms

- Γ polygon with \( k \) edges, \( D = 2\lambda + 2 \):

\[
\mathcal{A}_n = \left( \frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)(n+\lambda)} \right)^k \cdot \dim \mathcal{H}_n(S^{2\lambda+1})
\]

\( \mathcal{H}_n(S^{2\lambda+1}) \) space of harmonic functions deg \( n \) on \( S^{2\lambda+1} \) (Gegenbauer polynomial and zonal spherical harmonics)

- when \( D = 4 \), Feynman amplitude:

\[
(2\pi^2)^k \sum_{\mathcal{O}} m_{\mathcal{O}} \int_{\Sigma_{\mathcal{O}}} \text{Li}_{k-2} \left( \prod_i r_{w_i}^2 \right) \prod_v r_v \, dr_v
\]

polylogarithm functions

\[
\text{Li}_s(z) = \sum_{n=1}^\infty \frac{z^n}{n^s}
\]

vertices \( v_i, w_i \) sources and tails of oriented paths of \( \mathcal{O} \).
Step 4: stars of vertices and isoscalars

- star (corolla) of a vertex, with unpaired half-edges: angular integral

\[ \mathcal{A}_n(\omega) = \int_{S^{D-1}} \prod_j C_{n_j}^{(\lambda)}(\omega_j \cdot \omega) \, d\omega \]

with \( n = (n_j)_{e_j \in E_R} \) and \( \omega = (\omega_j)_{e_j \in E_R} \)

- integrals of products of spherical harmonics:

\[ \mathcal{A}_{(n_j)}(\omega_{v_j}) = c_{D,n_1} \cdots c_{D,n_k} \tilde{\mathcal{A}}_{(n_j)}(\omega_{v_j}) \]

\[ \tilde{\mathcal{A}}_{(n_j)}(\omega_{v_j}) = \sum_{\ell_1, \ldots, \ell_k} Y_{\ell_1}^{(n_1)}(\omega_1) \cdots Y_{\ell_k}^{(n_k)}(\omega_k) \int_{S^{D-1}} Y_{\ell_1}^{(n_1)}(\omega) \cdots Y_{\ell_k}^{(n_k)}(\omega) \, d\omega \]

\( \{ Y_{\ell}^{(n)} \}_{\ell=1,\ldots,d_n} \) orthonormal basis of \( \mathcal{H}_n(S^{D-1}) \); \( d_n = \dim \mathcal{H}_n(S^{D-1}) \)

and

\[ c_{D,n} = \frac{\text{Vol}(S^{D-1})(D-2)}{2n + D - 2} \]
isoscalar factors

- reduce to trivalent vertices: Gaunt coefficients \( \langle Y_{\ell_1}^{(n_1)}, Y_{\ell_2}^{(n_2)} Y_{\ell_3}^{(n_3)} \rangle_D \)

Racah’s factorization in terms of isoscalar factors

\[
\langle Y_{\ell_1}^{(n_1)}, Y_{\ell_2}^{(n_2)}, Y_{\ell_3}^{(n_3)} \rangle_D = \left( \begin{array}{ccc} n_1 & n_2 & n_3 \\ n_1' & n_2' & n_3' \end{array} \right)_{D: D-1} \langle Y_{\ell_1'}^{(n_1')}, Y_{\ell_2'}^{(n_2')}, Y_{\ell_3'}^{(n_3')} \rangle_{D-1}
\]

\( \ell_i = (n'_i, \ell'_i) \) with \( n'_i = m_{D-2,i} \) and \( \ell'_i = (m_{D-3,i}, \ldots, m_{1,i}) \)

there are general explicit expressions for the isoscalar factors
Step 5: gluing trivalent stars by matching half edges

- integrate on variables of matched half-edges:

\[ \mathcal{A}(n_i)_{i=1,\ldots,4}((\omega_i)_{i=1,\ldots,4}) = \sum \prod_{i=1}^{4} c_{D,n_i} Y_{\ell_i}^{(n_i)}(\omega_i) \mathcal{K}_{n_i,\ell_i}(n) \]

\[ \mathcal{K}_{n_i,\ell_i}(n) = c_{D,n}^2 \sum_{\ell=1}^{d_n} \langle Y_{\ell}^{(n)}, Y_{\ell_1}^{(n_1)}, Y_{\ell_2}^{(n_2)} \rangle_D \cdot \langle Y_{\ell}^{(n)}, Y_{\ell_3}^{(n_3)}, Y_{\ell_4}^{(n_4)} \rangle_D \]

- when \( D = 4 \) and \( \ell_i = 0 \)

\[ \mathcal{K}^{(D=4)}_{n,0}(n) = \left( \prod_{i=1}^{4} \frac{1}{(n_i + 1)^{1/2}} \right) \frac{4\pi^4}{(n + 1)^3}, \]

in range \( n + n_1 + n_2 \) and \( n + n_3 + n_4 \) even and

\(|n_j - n_k| \leq n_i \leq n_j + n_k\) for \((n_i, n_j, n_k)\) equal to \((n, n_1, n_2)\) or \((n, n_3, n_4)\) and transpositions; zero otherwise
• radial integral for matched half-edges:

\[
r^9 \prod_{i=1}^{3} t_i^{\alpha_i} \sum_{n_1,n_2,n_3} \mathcal{A}(n_1,n_2,n_3) (\omega_1,\omega_2,\omega_3) t_1^{\epsilon_1 n_1} t_2^{\epsilon_2 n_2} t_3^{\epsilon_3 n_3} dr \prod_{i=1}^{3} dt_i
\]

\(\alpha_i = 1\) and \(\epsilon_i = 1\) outgoing; \(\alpha_i = 3\) and \(\epsilon_i = -1\) incoming

• leading term of integral for matched half-edges \((D = 4)\):

\[
\sum_n \left( \prod_{i=1}^{4} c_{D,n_i} Y_0^{(n_i)} (\omega_i) \right) t_i^{\alpha_i + \epsilon_i n_i} dt_i \int_\Sigma t^4 dt \sum_n \frac{4\pi^2}{(n+1)^3} t^{\epsilon n}
\]

sum with constraints \(n + n_1 + n_2\) and \(n + n_3 + n_4\) even and

\[|n_j - n_k| \leq n_i \leq n_j + n_k\] for \((n_i, n_j, n_k)\) equal to \((n, n_1, n_2)\) or \((n, n_3, n_4)\) and transpositions
Step 6: gluing all half edges and nested sums

- $\mathcal{R}$ a domain of summation for integers $(n_1, \ldots, n_k)$

\[
\mathcal{R} = \mathcal{R}_{P}^{(k)} := \{ (n_1, \ldots, n_k) \mid n_i > 0, \ i = 1, \ldots, k \}
\]

\[
\mathcal{R} = \mathcal{R}_{MP}^{(k)} := \{ (n_1, \ldots, n_k) \mid n_k > \cdots > n_2 > n_1 > 0 \}
\]

\[
\mathcal{R} = \mathcal{R}_{T}^{(3)} := \{ (n_1, n_2, n_3) \mid n_2 > n_1, \ n_2 - n_1 < n_3 < n_2 + n_1 \}.
\]

associated series

\[
\text{Li}_{s_1, \ldots, s_k}^{\mathcal{R}} (z_1, \ldots, z_k) = \sum_{(n_1, \ldots, n_k) \in \mathcal{R}} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}
\]

includes products of polylogs, multiple polylogs, etc.

- even/odd: $\text{Li}_{s_1, \ldots, s_k}^{\mathcal{R}} (z_1, \ldots, z_k)$ and $\text{Li}_{s_1, \ldots, s_k}^{\mathcal{R}} (z_1, \ldots, z_k)$ respectively:

\[
\frac{1}{2} \left( \text{Li}_{s_1, \ldots, s_k}^{\mathcal{R}} (z_1, \ldots, z_k) + \text{Li}_{s_1, \ldots, s_k}^{\mathcal{R}} (-z_1, \ldots, -z_k) \right)
\]

\[
\frac{1}{2} \left( \text{Li}_{s_1, \ldots, s_k}^{\mathcal{R}} (z_1, \ldots, z_k) - \text{Li}_{s_1, \ldots, s_k}^{\mathcal{R}} (-z_1, \ldots, -z_k) \right).
\]
more general odd/even summations ($\mathcal{E}_i = 2\mathbb{N}$ or $\mathcal{E}_i = \mathbb{N} \setminus 2\mathbb{N}$)

$$\text{Li}^{\mathcal{R},\mathcal{E}_1,\ldots,\mathcal{E}_k}_{s_1,\ldots,s_k}(z_1, \ldots, z_k) = \sum_{(n_1,\ldots,n_k) \in \mathcal{R}, n_i \in \mathcal{E}_i} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

Example: matching all half-edges

$$\int_0^1 t^9 \left( 2^6 \text{Li}^{\mathcal{R}_{MP,\text{odd,even}}}_{6,3}(t, t) + 2 \text{Li}^{\mathcal{R}_{T,\text{even}}}_{3,3,3}(t, t, t) \right) dt$$

then relate $\text{Li}^{\mathcal{T}}_{s_1,s_2,s_3}(z_1, z_2, z_3)$ to well known generalizations of multiple zeta values and multiple polylogarithms.
• **Mordell–Tornheim multiple series**

\[ \zeta_{MT,k}(s_1, \ldots, s_k; s_{k+1}) = \sum_{(n_1,\ldots,n_k) \in \mathcal{R}_P^{(k)}} n_1^{-s_1} \cdots n_k^{-s_k} (n_1 + \cdots + n_k)^{-s_{k+1}} \]

and function \( \text{Li}_{s_1,\ldots,s_k; s_{k+1}}^{MT} (z_1, \ldots, z_k; z_{k+1}) \)

\[ \sum_{(n_1,\ldots,n_k) \in \mathcal{R}_P^{(k)}} \frac{z_1^{n_1} \cdots z_k^{n_k} z_{k+1}^{(n_1 + \cdots + n_k)}}{n_1^{s_1} \cdots n_k^{s_k} (n_1 + \cdots + n_k)^{s_{k+1}}} \]

• **Apostol–Vu multiple series**

\[ \zeta_{AV,k}(s_1, \ldots, s_k; s_{k+1}) = \sum_{(n_1,\ldots,n_k) \in \mathcal{R}_MP^{(k)}} n_1^{-s_1} \cdots n_k^{-s_k} (n_1 + \cdots + n_k)^{-s_{k+1}} \]

and function \( \text{Li}_{s_1,\ldots,s_k; s_{k+1}}^{AV} (z_1, \ldots, z_k; z_{k+1}) \)

\[ \sum_{(n_1,\ldots,n_k) \in \mathcal{R}_MP^{(k)}} \frac{z_1^{n_1} \cdots z_k^{n_k} z_{k+1}^{(n_1 + \cdots + n_k)}}{n_1^{s_1} \cdots n_k^{s_k} (n_1 + \cdots + n_k)^{s_{k+1}}} \]
Euler–Maclaurin summation formula for $f(t) = x^t t^{-s}$

\[
f^{(k)}(t) = \sum_{j=0}^{k} (-1)^{k-j} \left( \begin{array}{c} k \\ j \end{array} \right) \left( \begin{array}{c} s + k - j - 1 \\ k - j \end{array} \right) (k-j)! t^{-(s+k-j)} x^t \log(x)^j
\]

gives

\[
\sum_{n=a}^{b} f(n) = \int_{a}^{b} f(t) dt + \frac{1}{2} (f(b) + f(a)) \\
+ \sum_{k=2}^{N} \frac{b_k}{k!} (f^{(k-1)}(b) - f^{(k-1)}(a)) \\
- \int_{a}^{b} \frac{B_N(t - [t])}{N!} f^{(N)}(t) dt,
\]

$b_k$ Bernoulli numbers and $B_k$ Bernoulli polynomials
applied to \( \text{Li}_{s_1, s_2, s_3}^{R}(z_1, z_2, z_3) \) with \( R = R_T^{(3)} \) summation terms

\[
\pm F_{j,k}(s_3, z_3) \, \text{Li}_{s_1, s_2; s_3+k-j}^{AV}(z_1, z_2; z_3)
\]

\[
\pm F_{j,k}(s_3, z_3) \, \text{Li}_{s_1, s_3+k-j; s_2}^{MT}(z_1, z_2; z_3)
\]

with

\[
F_{j,k}(s, z) = \frac{b_k}{k!} \binom{k}{j} \binom{s + k - j - 1}{k - j} (k - j)! \log(z)^j
\]

Conclusion: by this method can see explicit integrals leading to multiple zeta values, but computations become easily extremely complicated even for simple graphs!