

Feynman integrals and motives in configuration spaces

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based on

- O.Ceyhan, M.Marcolli, *Feynman integrals and motives of configuration spaces*, Communications in Mathematical Physics: Vol.313, N.1 (2012), 35–70, arXiv:1012.5485
- O.Ceyhan, M.Marcolli, *Feynman integrals and periods in configuration spaces*, arXiv:1207.3544

Question: Are Feynman integrals periods of mixed Tate motives?
(multiple zeta values: extensive example collection
Broadhurst–Kreimer)

- Two methods of computing Feynman integrals (scalar massless Euclidean quantum field theory): momentum space or configuration space (Fourier transform)

$$G_m^{\mathbb{R}}(x_s - x_t) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} dp \frac{e^{ip \cdot (x_s - x_t)}}{p^2 + m^2 + i\epsilon}$$

General setting: Motives of algebraic varieties (Grothendieck)
Universal cohomology theory for algebraic varieties (with realizations)

- Pure motives: smooth projective varieties with correspondences

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q\mathrm{Corr}_{/\sim, \mathbb{Q}}^{m-n}(X, Y) p$$

Algebraic cycles mod equivalence (rational, homological, numerical), composition

$$\mathrm{Corr}(X, Y) \times \mathrm{Corr}(Y, Z) \rightarrow \mathrm{Corr}(X, Z)$$

$$(\pi_{X,Z})_*(\pi_{X,Y}^*(\alpha) \bullet \pi_{Y,Z}^*(\beta))$$

intersection product in $X \times Y \times Z$; with projectors $p^2 = p$ and $q^2 = q$ and Tate twists $\mathbb{Q}(m)$ with $\mathbb{Q}(1) = \mathbb{L}^{-1}$

Numerical pure motives: $\mathcal{M}_{num, \mathbb{Q}}(k)$ semi-simple abelian category (Jannsen)

- Mixed motives: varieties that are possibly singular or not projective (much more complicated theory!) Triangulated category \mathcal{DM} (Voevodsky , Levine, Hanamura)

$$m(Y) \rightarrow m(X) \rightarrow m(X \setminus Y) \rightarrow m(Y)[1]$$

$$m(X \times \mathbb{A}^1) = m(X)(-1)[2]$$

- Mixed Tate motives $\mathcal{DMT} \subset \mathcal{DM}$ generated by the $\mathbb{Q}(m)$
- Over a number field: t-structure, abelian category of mixed Tate motives (vanishing result, M.Levine)

Periods and motives: $\int_{\sigma} \omega$ numbers obtained integrating an algebraic differential form over a cycle defined by algebraic equations
Constraints on numbers obtained as periods from the motive of the variety!

- Periods of mixed Tate motives are Multiple Zeta Values

$$\zeta(k_1, k_2, \dots, k_r) = \sum_{n_1 > n_2 > \dots > n_r \geq 1} n_1^{-k_1} n_2^{-k_2} \dots n_r^{-k_r}$$

Conjecture **proved** recently:

- Francis Brown, *Mixed Tate motives over \mathbb{Z}* , Annals of Math 2012, arXiv:1102.1312.

Feynman integrals and periods: MZVs as *typical* outcome:

- D. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, arXiv:hep-th/9609128

General setting: scalar perturbative QFTs

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

in D dimensions, with Lagrangian density

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi)$$

Perturbative expansion: Feynman rules and Feynman diagrams

$$S_{eff}(\phi) = S_0(\phi) + \sum_{\Gamma} \frac{U(\Gamma, \phi)}{\#\text{Aut}(\Gamma)} \quad (1\text{PI graphs})$$

Amplitudes $U(\Gamma)$ for fixed external edges of the graph are integral (generally divergent) on:

- momenta associated to internal edges of the graph with momentum conservation rules at vertices
- configurations associated to vertices of the graph with divergences where coordinates collide (diagonals)

- **Momentum space**: parametric Feynman integrals, graph hypersurfaces, motives of graph hypersurfaces *not* mixed Tate in general (Belkale–Brosnan, Doryn, Schnetz), period can still be mixed Tate (Brown, Brown–Schnetz); various results on classes in the Grothendieck ring (Aluffi-M.)

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t, p)^{-n+D\ell/2} \omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

$\sigma_n = \{t \in \mathbb{R}_+^n \mid \sum_i t_i = 1\}$, volume form ω_n

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin T} t_e$$

$$P_\Gamma(p, t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

s_C quadratic function of external momenta p_e

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} : \Psi_\Gamma(t) = 0\}$$

(projective) graph hypersurfaces

- **Configuration space**: wonderful compactifications of graph configuration spaces; mixed Tate motives; Feynman amplitude and Laplacian Green functions; explicit results using Gegenbauer polynomial expansion; pullback to wonderful compactification, cohomologous to algebraic form with logarithmic poles; deformation and renormalization.

Feynman amplitude in configuration space ($\dim D = 2\lambda + 2$)

Version N.1:

$$\omega_\Gamma = \prod_{e \in E_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} \bigwedge_{v \in V_\Gamma} dx_v$$

defines a \mathcal{C}^∞ -differential form on X^{V_Γ} with singularities along diagonals $x_{s(e)} = x_{t(e)}$

- not closed form
- chain of integration:

$$\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}$$

Version N.2: (complexification)

$Z = X \times X$ with projection $p : Z \rightarrow X$, $p : z = (x, y) \mapsto x$

$$\omega_{\Gamma}^{(Z)} = \prod_{e \in E_{\Gamma}} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2D-2}} \bigwedge_{v \in V_{\Gamma}} dx_v \wedge d\bar{x}_v$$

where $\|x_{s(e)} - x_{t(e)}\| = \|p(z)_{s(e)} - p(z)_{t(e)}\|$

- closed form
- chain of integration:

$$\sigma^{(Z,y)} = X^{V_{\Gamma}} \times \{y = (y_v)\} \subset Z^{V_{\Gamma}} = X^{V_{\Gamma}} \times X^{V_{\Gamma}}$$

for a fixed $y = (y_v \mid v \in V_{\Gamma})$

Relation to Green functions:

- Green function of real Laplacian on $\mathbb{A}^D(\mathbb{R})$, with $D = 2\lambda + 2$:

$$G_{\mathbb{R}}(x, y) = \frac{1}{\|x - y\|^{2\lambda}}$$

- On $\mathbb{A}^D(\mathbb{C})$ complex Laplacian

$$\Delta = \sum_{k=1}^D \frac{\partial^2}{\partial x_k \partial \bar{x}_k}$$

has Green form

$$G_{\mathbb{C}}(x, y) = \frac{-(D-2)!}{(2\pi i)^D \|x - y\|^{2D-2}}$$

Feynman amplitudes modeled on the two cases

Different method:

- Version N.1: explicit computation of regularized integral

$$\int_{\sigma_\Gamma} \omega_\Gamma$$

using expansion of Green function in Gegenbauer polynomials:
explicit occurrence of multiple zeta values

- Version N.2: cohomological method, pullback $\omega_\Gamma^{(Z)}$ to a compactification of configuration space where cohomologous to algebraic form with log poles; regularize to separate poles from chain of integration; show explicitly motive is mixed Tate
- Discuss first second case (geometric method)

Graph configuration spaces

X a smooth projective algebraic variety that contains a dense \mathbb{A}^D : for instance $X = \mathbb{P}^D$, with D spacetime dimension.

Feynman amplitude ω_Γ on X^{V_Γ}

Singularities of Feynman amplitude along diagonals

$$\Delta_e = \{(x_v)_{v \in V_\Gamma} \mid x_{v_1} = x_{v_2} \text{ for } \partial_\Gamma(e) = \{v_1, v_2\}\}$$

Graph configuration space:

$$\text{Conf}_\Gamma(X) = X^{V_\Gamma} \setminus \bigcup_{e \in E_\Gamma} \Delta_e$$

Goal N.1: compactify $\text{Conf}_\Gamma(X)$ to a smooth projective algebraic variety $\overline{\text{Conf}}_\Gamma(X)$ so that

$$\overline{\text{Conf}}_\Gamma(X) \setminus \text{Conf}_\Gamma(X)$$

is a normal crossings divisor

Variants: Version N.2 of configuration space for amplitude $\omega_{\Gamma}^{(Z)}$

$$F(X, \Gamma) = Z^{V_{\Gamma}} \setminus \bigcup_{e \in E_{\Gamma}} \Delta_e^{(Z)} \cong (X \times X)^{V_{\Gamma}} \setminus \bigcup_{e \in E_{\Gamma}} \Delta_e^{(Z)}$$

with diagonals

$$\Delta_e^{(Z)} \cong \{(z_v \mid v \in V_{\Gamma}) \in Z^{V_{\Gamma}} \mid \rho(z_{s(e)}) = \rho(z_{t(e)})\}$$

Relation to previous:

$$F(X, \Gamma) \simeq \text{Conf}_{\Gamma}(X) \times X^{V_{\Gamma}}$$

$$\Delta_e^{(Z)} \cong \Delta_e \times X^{V_{\Gamma}}$$

Compactify to $\overline{F(X, \Gamma)}$ in same way

Wonderful compactifications

- Fulton–MacPherson configuration spaces (= complete graph case of $Conf_{\Gamma}(X)$)
- More general setting for arrangements of subvarieties: DeConcini–Procesi, Li Li
- General method: realize $\overline{Conf}_{\Gamma}(X)$ or $\overline{F}(X, \Gamma)$ as a sequence of blowups of $X^{V_{\Gamma}}$ (or $Z^{V_{\Gamma}}$) along a collection of dominant transforms of diagonals
- Equivalent description: closure in

$$Conf_{\Gamma}(X) \hookrightarrow \prod_{\gamma \in \mathcal{G}_{\Gamma}} Bl_{\Delta_{\gamma}} X^{V_{\Gamma}}$$

with \mathcal{G}_{Γ} subgraphs induced (all edges of Γ between subset of vertices) and 2-vertex-connected

Blowup construction of wonderful compactifications

- Connected induced subgraphs: $\mathcal{SG}_k(\Gamma) = \{\gamma \in \mathcal{SG}(\Gamma) \mid |V_\gamma| = k\}$ and polydiagonals $\hat{\Delta}_\gamma = \{x_{s(e)} = x_{t(e)} : e \in E_\gamma\}$ (same as diagonal Δ_γ if γ connected)
- Arrangement of subvarieties in X^{V_Γ} : \mathcal{S}_Γ polydiagonals of disjoint unions of connected induced subgraphs
- Building set for the arrangement:

$$\mathcal{G}_\Gamma = \{\Delta_\gamma : \gamma \text{ induced, biconnected}\}$$

$$\text{Conf}_\Gamma(X) = X^{V_\Gamma} \setminus \bigcup_{\gamma \in \mathcal{G}_\Gamma} \Delta_\gamma$$

- Start with $Y_0 = X^{V_\Gamma}$; obtain Y_k from Y_{k-1} by blowup along the iterated *dominant transforms* (=proper transform or inverse image of exceptional divisor) of

$$\bigcup_{\gamma \in \mathcal{G}_{n-k+1, \Gamma}} \Delta_\gamma$$

with $\mathcal{G}_{k, \Gamma} = \mathcal{G}_\Gamma \cap \mathcal{SG}_k(\Gamma)$ then

$$Y_{n-1} = \overline{\text{Conf}_\Gamma(X)}$$

Boundary structure

- \mathcal{G}_Γ -nests: sets of biconnected induced subgraphs with $\gamma \cap \gamma' = \emptyset$ or $\gamma \cap \gamma' = \{v\}$ single vertex or $\gamma \subseteq \gamma'$ or $\gamma' \subseteq \gamma$

$$\overline{\text{Conf}}_\Gamma(X) \setminus \text{Conf}_\Gamma(X) = \bigcup_{\Delta_\gamma \in \mathcal{G}_\Gamma} D_\gamma$$

- divisors D_γ (iterated dominant transform of Δ_γ) with

$$D_{\gamma_1} \cap \cdots \cap D_{\gamma_\ell} \neq \emptyset \Leftrightarrow \{\gamma_1, \dots, \gamma_\ell\} \text{ is a } \mathcal{G}_\Gamma\text{-nest}$$

and transverse intersections

- strata parameterized by forests of nested subgraphs (as in Fulton–MacPherson case)
- case of $\overline{F}(X, \Gamma)$ completely analogous

Motives of configuration spaces – Key ingredient: Blowup formulae

- For mixed motives (Voevodsky category):

$$\mathfrak{m}(\mathrm{Bl}_V(Y)) \cong \mathfrak{m}(Y) \oplus \bigoplus_{k=1}^{\mathrm{codim}_Y(V)-1} \mathfrak{m}(V)(k)[2k]$$

- For Grothendieck classes Bittner relation

$$[\mathrm{Bl}_V(Y)] = [Y] - [V] + [E] = [Y] + [V]([\mathbb{P}^{\mathrm{codim}_Y(V)-1}] - 1)$$

exceptional divisor E

- **Conclusion:** the motive of $\overline{\mathrm{Conf}}_\Gamma(X)$ and of $\overline{F}(X, \Gamma)$ is mixed Tate if X is mixed Tate.

Voevodsky motive: (quasi-projective smooth X)

$$m(\overline{\text{Conf}}_\Gamma(X)) = m(X)^{V_\Gamma} \oplus \bigoplus_{\mathcal{N} \in \mathcal{G}_\Gamma\text{-nests}, \mu \in M_\mathcal{N}} m(X)^{V_{\Gamma/\delta_\mathcal{N}(\Gamma)}}(\|\mu\|)[2\|\mu\|]$$

where $M_\mathcal{N} := \{(\mu_\gamma)_{\Delta_\gamma \in \mathcal{G}_\Gamma} : 1 \leq \mu_\gamma \leq r_\gamma - 1, \mu_\gamma \in \mathbb{Z}\}$ with
 $r_\gamma = r_{\gamma, \mathcal{N}} := \dim(\bigcap_{\gamma' \in \mathcal{N}: \gamma' \subset \gamma} \Delta_{\gamma'}) - \dim \Delta_\gamma$ and $\|\mu\| := \sum_{\Delta_\gamma \in \mathcal{G}_\Gamma} \mu_\gamma$

$$\Gamma/\delta_\mathcal{N}(\Gamma) = \Gamma // (\gamma_1 \cup \dots \cup \gamma_r)$$

for $\mathcal{N} = \{\gamma_1, \dots, \gamma_r\}$

Class in the Grothendieck ring:

$$[\overline{\text{Conf}}_\Gamma(X)] = [X]^{V_\Gamma} + \sum_{\mathcal{N} \in \mathcal{G}_\Gamma\text{-nests}} [X]^{V_{\Gamma/\delta_\mathcal{N}(\Gamma)}} \sum_{\mu \in M_\mathcal{N}} \mathbb{L}^{\|\mu\|}$$

Chow motive: (smooth projective X): from result of Li Li on wonderful compactifications of arrangements of subvarieties

Pullback and forms with logarithmic poles

- $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ pullback to iterated blowup $\overline{F(X, \Gamma)}$ of Z^{\vee_Γ} along dominant transforms of $\Delta_\gamma^{(Z)}$ of biconnected induced subgraphs
- Divergence locus union of divisors (dominant transforms of $\Delta_\gamma^{(Z)}$)

$$\bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma^{(Z)}$$

- Chain of integration $\tilde{\sigma}_\Gamma^{(Z, y)} = \overline{\text{Conf}_\Gamma(X)} \times \{y\} \subset \overline{F(X, \Gamma)}$ intersects divergence locus in

$$\mathcal{D}_\Gamma = \bigcup_{\Delta_\gamma^{(Z)} \in \mathcal{G}_\Gamma} D_\gamma \times \{y\} \subset \overline{\text{Conf}_\Gamma(X)} \times \{y\}$$

- pullback $\pi_\gamma^*(\omega_\Gamma^{(Z)})$ on $\tilde{\sigma}_\Gamma^{(Z, y)}$ smooth closed form on

$$\overline{\text{Conf}_\Gamma(X)} \setminus \left(\bigcup_{\gamma \in \mathcal{G}_\Gamma} D_\gamma \right)$$

Smooth and algebraic forms

- de Rham cohomology of a smooth quasi-projective varieties computed using algebraic differential forms (Grothendieck)
- if complement of normal crossings divisor can use forms with log poles (Deligne)

$$H^*(\mathcal{U}) \simeq \mathbb{H}^*(\mathcal{X}, \Omega_{\mathcal{X}}^*(\log(\mathcal{D})))$$

- \mathcal{X} smooth projective variety $\dim_{\mathbb{C}} m$; \mathcal{D} normal crossings divisor; $\mathcal{U} = \mathcal{X} \setminus \mathcal{D}$; ω smooth closed differential form $\deg m$ on \mathcal{U} ;
 $\Rightarrow \exists$ algebraic differential form η log poles along \mathcal{D} , with $[\eta] = [\omega] \in H_{dR}^m(\mathcal{U})$
- **Conclusion:** \exists algebraic form $\eta_{\Gamma}^{(Z)}$ with log poles along union of D_{γ} , cohomologous to $\pi_{\gamma}^*(\omega_{\Gamma}^{(Z)})$ on $\tilde{\sigma}_{\Gamma}^{(Z,Y)}$

Regularization problem

- $\eta_{\Gamma}^{(Z)}$ algebraic differential form; $\tilde{\sigma}_{\Gamma}^{(Z,y)}$ algebraic cycle: Feynman integral becomes

$$\int_{\tilde{\sigma}_{\Gamma}^{(Z,y)} \setminus \mathcal{D}_{\Gamma}} \eta_{\Gamma}^{(Z)}$$

would be a period... but divergent!! (because of intersection \mathcal{D}_{Γ} of chain with divisors)

- need a regularization procedure: separate chain of integration from divergence locus

Two regularization methods

- Principal value current regularization and iterated Poincaré residues
- Deformation to the normal cone

Current regularization

- Regularized Feynman amplitude:

$$\langle PV(\eta_{\Gamma}^{(Z)}), \varphi \rangle = \lim_{\lambda \rightarrow 0} \int_{\tilde{\sigma}_{\Gamma}^{(Z,y)}} |f_n|^{2\lambda_n} \cdots |f_1|^{2\lambda_1} \eta_{\Gamma}^{(Z,y)} \varphi$$

where φ test functions; $n = n_{\Gamma} = \#\mathcal{G}_{\Gamma}$; and f_k equation of $D_{\gamma_k}^{(Z)}$

- Ambiguities of regularization:

$$\tilde{\sigma}_{\Gamma, \mathcal{N}, \epsilon}^{(Z,y)} := \tilde{\sigma}_{\Gamma}^{(Z,y)} \cap T_{\mathcal{N}, \epsilon}(f) \cap N_{\mathcal{N}, \epsilon}(f)$$

$$T_{\mathcal{N}, \epsilon}(f) = \{|f_k| = \epsilon_k, k = 1, \dots, r\}$$

$$N_{\mathcal{N}, \epsilon}(f) = \{|f_k| > \epsilon, k = r+1, \dots, n\}$$

n graphs in \mathcal{G}_{Γ} ordered so that first r in the nest \mathcal{N}

$$\lim_{\epsilon \rightarrow 0} \int_{\tilde{\sigma}_{\Gamma, \mathcal{N}, \epsilon}^{(Z,y)}} \varphi \eta_{\Gamma}^{(Z,y)}$$

has a residue (iterated Poincaré residue) supported on

$$V_{\mathcal{N}}^{(Z)} = D_{\gamma_1}^{(Z)} \cap \cdots \cap D_{\gamma_r}^{(Z)}$$

Iterated Poincaré residue

$$\int_{\Sigma_{\mathcal{N}}} \mathcal{R}_{\mathcal{N}}(\eta_{\Gamma}) = \frac{1}{(2\pi i)^r} \int_{\mathcal{L}_{\mathcal{N}}(\Sigma_{\mathcal{N}})} \eta_{\Gamma}$$

$(2D|V_{\Gamma}| - r)$ -cycle $\Sigma_{\mathcal{N}}$ in $V_{\mathcal{N}}^{(Z)}$; iterated Leray coboundary $\mathcal{L}_{\mathcal{N}}(\Sigma_{\mathcal{N}})$ in $F(X, \Gamma)$ is a T^r -torus bundle over $\Sigma_{\mathcal{N}}$

- If the variety X is a mixed Tate motive, these residues are all periods of mixed Tate motives
- On intersections of chain of integration and divergence loci

$$\langle \mathcal{R}_{\mathcal{N}}(\eta_{\Gamma}), V_{\mathcal{N}} \rangle = \int_{V_{\mathcal{N}} \times \{y\}} \mathcal{R}_{\mathcal{N}}(\eta_{\Gamma})$$

Deformation to the normal cone

- extend integral

$$\int_{\tilde{\sigma}_\Gamma^{(Z,y)}} \pi_\Gamma^*(\omega_\Gamma^{(Z)})$$

to a larger ambient deformation space where can separate $\tilde{\sigma}_\Gamma^{(Z,y)}$ from the divergence locus

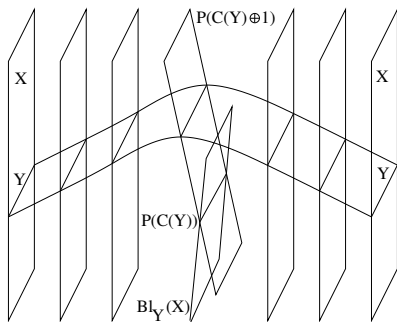
- start with $Z^{V_\Gamma} \times \mathbb{P}^1$, deformation coordinate $\zeta \in \mathbb{P}^1$, and

$$\tilde{\omega}_\Gamma^{(Z)} = \prod_{e \in E_\Gamma} \frac{1}{(\|x_{s(e)} - x_{t(e)}\|^2 + |\zeta|^2)^{D-1}} \bigwedge_{v \in V_\Gamma} dx_v \wedge d\bar{x}_v \wedge d\zeta \wedge d\bar{\zeta}$$

- divergence locus in the central fiber $\zeta = 0$

$$\cup_{e \in E_\Gamma} \Delta_e^{(Z)} \subset Z^{V_\Gamma} \times \{0\}$$

- starting with $Z^{\text{Vr}} \times \mathbb{P}^1$ blowups along $\Delta_\gamma^{(Z)} \times \{0\}$, induced biconnected subgraphs
- obtain smooth projective variety $\mathcal{D}(Z[\Gamma])$ fibered over \mathbb{P}^1 : fiber over $\zeta \neq 0 \in \mathbb{P}^1$ equal to Z^{Vr} ; fiber over $\zeta = 0$ has a component $\overline{F(X, \Gamma)}$ plus other components projectivizations $\mathbb{P}(C \oplus 1)$ of normal cones of blowups



- in $\mathcal{D}(Z[\Gamma])$ the chain of integration $\tilde{\sigma}_\Gamma^{(Z, y)}$ becomes separated from the locus of divergence

Deformation: motive and period

- if the motive of X is mixed Tate, then the motive of $\mathcal{D}(Z[\Gamma])$ is also mixed Tate (again blowup formulae)
- pullback $\tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$ of form to the deformation along blowup $\tilde{\pi}_\Gamma : \mathcal{D}(Z[\Gamma]) \rightarrow Z^{V_\Gamma} \times \mathbb{P}^1$
- locus of divergence union of divisors in the central fiber of projection $\pi : \mathcal{D}(Z[\Gamma]) \rightarrow \mathbb{P}^1$

$$\bigcup_{\gamma \in \mathcal{G}_\Gamma} D_\gamma^{(Z)} \subset \pi^{-1}(0)$$

- chain $\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1$ with proper transform $\overline{\sigma_\Gamma^{(Z,y)} \times \mathbb{P}^1}$ deformed inside normal cone away from union of divisors (as in figure) to $\Sigma_\Gamma^{(Z,y)}$
- Regularized Feynman amplitude

$$\int_{\Sigma_\Gamma^{(Z,y)}} \delta(\pi^{-1}(0)) \tilde{\pi}_\Gamma^*(\tilde{\omega}_\Gamma^{(Z)})$$

is a period of a mixed Tate motive

Explicit computations of Feynman amplitudes:

Step 1: explicit chains in X^{V_Γ}

- Acyclic orientations: Γ no looping edges, $\Omega(\Gamma)$ set of acyclic orientations; Stanley: $(-1)^{V_\Gamma} P_\Gamma(-1)$ acyclic orientations where $P_\Gamma(t)$ chromatic polynomial
- orientation $\bullet \in \Omega(\Gamma) \Rightarrow$ partial ordering of vertices $w \geq_\bullet v$
- chain with boundary $\partial \Sigma_\bullet \subset \cup_{e \in E_\Gamma} \Delta_e$

$$\Sigma_\bullet := \{(x_v) \in X^{V_\Gamma}(\mathbb{R}) : r_w \geq r_v \text{ whenever } w \geq_\bullet v\}$$

middle dimensional relative homology class

$$[\Sigma_\bullet] \in H_{|V_\Gamma|}(X^{V_\Gamma}, \cup_{e \in E_\Gamma} \Delta_e)$$

- $\Sigma_\bullet \setminus \cup_v \{r_v = 0\}$ bundle fiber $(S^{D-1})^{V_\Gamma}$ base

$$\bar{\Sigma}_\bullet = \{(r_v) \in (\mathbb{R}_+^*)^{V_\Gamma} : r_w \geq r_v \text{ whenever } w \geq_\bullet v\}$$

Step 2: Gegenbauer polynomials

- Generating function and orthogonality ($|t| < 1$ and $\lambda > -1/2$)

$$\frac{1}{(1 - 2tx + t^2)^\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) t^n$$

$$\int_{-1}^1 C_n^{(\lambda)}(x) C_m^{(\lambda)}(x) (1 - x^2)^{\lambda-1/2} dx = \delta_{n,m} \frac{\pi 2^{1-2\lambda} \Gamma(n + 2\lambda)}{n!(n + \lambda) \Gamma(\lambda)^2}$$

- $D = 2\lambda + 2$ Newton potential expansion in Gegenbauer polynomials:

$$\begin{aligned} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2\lambda}} &= \frac{1}{\rho_e^{2\lambda} (1 + (\frac{r_e}{\rho_e})^2 - 2 \frac{r_e}{\rho_e} \omega_{s(e)} \cdot \omega_{t(e)})^\lambda} \\ &= \rho_e^{-2\lambda} \sum_{n=0}^{\infty} \left(\frac{r_e}{\rho_e}\right)^n C_n^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}), \end{aligned}$$

with $\rho_e = \max\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and $r_e = \min\{\|x_{s(e)}\|, \|x_{t(e)}\|\}$ and with $\omega_v \in S^{D-1}$

Step 3: angular and radial integrals

- on chain of integration $\sigma_\Gamma = X(\mathbb{R})^{V_\Gamma}$ Feynman integral becomes (Version N.1)

$$\sum_{\mathbf{o} \in \Omega(\Gamma)} m_{\mathbf{o}} \int_{\Sigma_{\mathbf{o}}} \prod_{e \in E_\Gamma} r_{t_{\mathbf{o}}(e)}^{-2\lambda} \left(\sum_n \left(\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}} \right)^n C_n^{(\lambda)}(\omega_{s_{\mathbf{o}}(e)} \cdot \omega_{t_{\mathbf{o}}(e)}) \right) dV$$

with positive integers $m_{\mathbf{o}}$ (multiplicities) and volume form

$$dV = \prod_v d^D x_v = \prod_v r_v^{D-1} dr_v d\omega_v$$

- **angular integrals:**

$$\mathcal{A}_{(n_e)_{e \in E_\Gamma}} = \int_{(S^{D-1})^{V_\Gamma}} \prod_e C_{n_e}^{(\lambda)}(\omega_{s(e)} \cdot \omega_{t(e)}) \prod_v d\omega_v$$

- **radial integrals:**

$$\sum_{\mathbf{o} \in \Omega(\Gamma)} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \prod_{e \in E_\Gamma} \mathcal{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) \prod_v r_v^{D-1} dr_v$$

$$\mathcal{F}(r_{s_{\mathbf{o}}(e)}, r_{t_{\mathbf{o}}(e)}) = r_{t_{\mathbf{o}}(e)}^{-2\lambda} \sum_{n_e} \mathcal{A}_{n_e} \left(\frac{r_{s_{\mathbf{o}}(e)}}{r_{t_{\mathbf{o}}(e)}} \right)^{n_e}$$

Example: polygons and polylogarithms

- Γ polygon with k edges, $D = 2\lambda + 2$:

$$\mathcal{A}_n = \left(\frac{\lambda 2\pi^{\lambda+1}}{\Gamma(\lambda+1)(n+\lambda)} \right)^k \cdot \dim \mathcal{H}_n(S^{2\lambda+1})$$

$\mathcal{H}_n(S^{2\lambda+1})$ space of harmonic functions deg n on $S^{2\lambda+1}$
(Gegenbauer polynomial and zonal spherical harmonics)

- when $D = 4$, Feynman amplitude:

$$(2\pi^2)^k \sum_{\mathbf{o}} m_{\mathbf{o}} \int_{\bar{\Sigma}_{\mathbf{o}}} \text{Li}_{k-2} \left(\prod_i \frac{r_{w_i}^2}{r_{v_i}^2} \right) \prod_v r_v dr_v$$

polylogarithm functions

$$\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

vertices v_j , w_i sources and tails of oriented paths of \mathbf{o}

Step 4: stars of vertices and isoscalars

- star (corolla) of a vertex, with unpaired half-edges: angular integral

$$\mathcal{A}_{\underline{n}}(\underline{\omega}) = \int_{S^{D-1}} \prod_j C_{n_j}^{(\lambda)}(\omega_j \cdot \omega) d\omega$$

with $\underline{n} = (n_j)_{e_j \in E_\Gamma}$ and $\underline{\omega} = (\omega_j)_{e_j \in E_\Gamma}$

- integrals of products of spherical harmonics:

$$\mathcal{A}_{(n_j)}(\omega_{v_j}) = c_{D,n_1} \cdots c_{D,n_k} \tilde{\mathcal{A}}_{(n_j)}(\omega_{v_j})$$

$$\tilde{\mathcal{A}}_{(n_j)}(\omega_{v_j}) = \sum_{\ell_1, \dots, \ell_k} \overline{Y_{\ell_1}^{(n_1)}(\omega_1) \cdots Y_{\ell_k}^{(n_k)}(\omega_k)} \int_{S^{D-1}} Y_{\ell_1}^{(n_1)}(\omega) \cdots Y_{\ell_k}^{(n_k)}(\omega) d\omega$$

$\{Y_\ell^{(n)}\}_{\ell=1, \dots, d_n}$ orthonormal basis of $\mathcal{H}_n(S^{D-1})$; $d_n = \dim \mathcal{H}_n(S^{D-1})$

and

$$c_{D,n} = \frac{\text{Vol}(S^{D-1})(D-2)}{2n + D - 2}$$

isoscalar factors

- reduce to trivalent vertices: Gaunt coefficients $\langle Y_{\ell_1}^{(n_1)}, Y_{\ell_2}^{(n_2)}, Y_{\ell_3}^{(n_3)} \rangle_D$
Racah's factorization in terms of *isoscalar factors*

$$\langle Y_{\ell_1}^{(n_1)}, Y_{\ell_2}^{(n_2)}, Y_{\ell_3}^{(n_3)} \rangle_D = \begin{pmatrix} n_1 & n_2 & n_3 \\ n'_1 & n'_2 & n'_3 \end{pmatrix}_{D:D-1} \langle Y_{\ell'_1}^{(n'_1)}, Y_{\ell'_2}^{(n'_2)}, Y_{\ell'_3}^{(n'_3)} \rangle_{D-1}$$

$$\ell_i = (n'_i, \ell'_i) \text{ with } n'_i = m_{D-2,i} \text{ and } \ell'_i = (m_{D-3,i}, \dots, m_{1,i})$$

there are general explicit expressions for the isoscalar factors

Step 5: gluing trivalent stars by matching half edges

- integrate on variables of matched half-edges:

$$\mathcal{A}_{(n_i)_{i=1,\dots,4}}((\omega_i)_{i=1,\dots,4}) = \sum_{\ell_j} \prod_{i=1}^4 c_{D,n_i} \overline{Y_{\ell_j}^{(n_i)}(\omega_i)} \mathcal{K}_{n_i,\ell_j}(n)$$

$$\mathcal{K}_{n_i,\ell_j}(n) = c_{D,n}^2 \sum_{\ell=1}^{d_n} \langle Y_{\ell}^{(n)}, Y_{\ell_1}^{(n_1)}, Y_{\ell_2}^{(n_2)} \rangle_D \cdot \langle Y_{\ell}^{(n)}, Y_{\ell_3}^{(n_3)}, Y_{\ell_4}^{(n_4)} \rangle_D$$

- when $D = 4$ and $\ell_j = 0$

$$\mathcal{K}_{\underline{n},0}^{(D=4)}(n) = \left(\prod_{i=1}^4 \frac{1}{(n_i + 1)^{1/2}} \right) \frac{4\pi^4}{(n + 1)^3},$$

in range $n + n_1 + n_2$ and $n + n_3 + n_4$ even and $|n_j - n_k| \leq n_i \leq n_j + n_k$ for (n_i, n_j, n_k) equal to (n, n_1, n_2) or (n, n_3, n_4) and transpositions; zero otherwise

- radial integral for matched half-edges:

$$r^9 \prod_{i=1}^3 t_i^{\alpha_i} \sum_{n_1, n_2, n_3} \mathcal{A}_{(n_1, n_2, n_3)}(\omega_1, \omega_2, \omega_3) t_1^{\epsilon_1 n_1} t_2^{\epsilon_2 n_2} t_3^{\epsilon_3 n_3} dr \prod_{i=1}^3 dt_i$$

$\alpha_i = 1$ and $\epsilon_i = 1$ outgoing; $\alpha_i = 3$ and $\epsilon_i = -1$ incoming

- leading term of integral for matched half-edges ($D = 4$):

$$\sum_{\underline{n}} \left(\prod_{i=1}^4 c_{D, n_i} \overline{Y_0^{(n_i)}(\omega_i)} \frac{t_i^{\alpha_i + \epsilon_i n_i} dt_i}{(n_i + 1)^{1/2}} \right) \int_{\underline{\Sigma}} t^4 dt \sum_n \frac{4\pi^2}{(n+1)^3} t^{\epsilon n}$$

sum with constraints $n + n_1 + n_2$ and $n + n_3 + n_4$ even and $|n_j - n_k| \leq n_i \leq n_j + n_k$ for (n_i, n_j, n_k) equal to (n, n_1, n_2) or (n, n_3, n_4) and transpositions

Step 6: gluing all half edges and nested sums

- \mathcal{R} a domain of summation for integers (n_1, \dots, n_k)

$$\mathcal{R} = \mathcal{R}_P^{(k)} := \{(n_1, \dots, n_k) \mid n_i > 0, i = 1, \dots, k\}$$

$$\mathcal{R} = \mathcal{R}_{MP}^{(k)} := \{(n_1, \dots, n_k) \mid n_k > \dots > n_2 > n_1 > 0\}$$

$$\mathcal{R} = \mathcal{R}_T^{(3)} := \{(n_1, n_2, n_3) \mid n_2 > n_1, n_2 - n_1 < n_3 < n_2 + n_1\}.$$

associated series

$$\text{Li}_{s_1, \dots, s_k}^{\mathcal{R}}(z_1, \dots, z_k) = \sum_{(n_1, \dots, n_k) \in \mathcal{R}} \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1^{s_1} \dots n_k^{s_k}}$$

includes products of polylogs, multiple polylogs, etc.

- even/odd: $\text{Li}_{s_1, \dots, s_k}^{\mathcal{R}, \text{even}}(z_1, \dots, z_k)$ and $\text{Li}_{s_1, \dots, s_k}^{\mathcal{R}, \text{odd}}(z_1, \dots, z_k)$ respectively:

$$\frac{1}{2} \left(\text{Li}_{s_1, \dots, s_k}^{\mathcal{R}}(z_1, \dots, z_k) + \text{Li}_{s_1, \dots, s_k}^{\mathcal{R}}(-z_1, \dots, -z_k) \right)$$
$$\frac{1}{2} \left(\text{Li}_{s_1, \dots, s_k}^{\mathcal{R}}(z_1, \dots, z_k) - \text{Li}_{s_1, \dots, s_k}^{\mathcal{R}}(-z_1, \dots, -z_k) \right).$$

- more general odd/even summations ($\mathcal{E}_i = 2\mathbb{N}$ or $\mathcal{E}_i = \mathbb{N} \setminus 2\mathbb{N}$)

$$\mathrm{Li}_{s_1, \dots, s_k}^{\mathcal{R}, \mathcal{E}_1, \dots, \mathcal{E}_k}(z_1, \dots, z_k) = \sum_{(n_1, \dots, n_k) \in \mathcal{R}, n_i \in \mathcal{E}_i} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

- Example: matching all half-edges

$$\int_0^1 t^9 (2^6 \mathrm{Li}_{6,3}^{\mathcal{R}_{MP}, \text{odd}, \text{even}}(t, t) + 2 \mathrm{Li}_{3,3,3}^{\mathcal{R}_T, \text{even}}(t, t, t)) dt$$

- then relate $\mathrm{Li}_{s_1, s_2, s_3}^{\mathcal{R}_T}(z_1, z_2, z_3)$ to well known generalizations of multiple zeta values and multiple polylogarithms

- **Mordell–Tornheim** multiple series

$$\zeta_{MT,k}(s_1, \dots, s_k; s_{k+1}) = \sum_{(n_1, \dots, n_k) \in \mathcal{R}_P^{(k)}} n_1^{-s_1} \cdots n_k^{-s_k} (n_1 + \cdots + n_k)^{-s_{k+1}}$$

and function $\text{Li}_{s_1, \dots, s_k; s_{k+1}}^{MT}(z_1, \dots, z_k; z_{k+1})$

$$\sum_{(n_1, \dots, n_k) \in \mathcal{R}_P^{(k)}} \frac{z_1^{n_1} \cdots z_k^{n_k} z_{k+1}^{(n_1 + \cdots + n_k)}}{n_1^{s_1} \cdots n_k^{s_k} (n_1 + \cdots + n_k)^{s_{k+1}}}$$

- **Apostol–Vu** multiple series

$$\zeta_{AV,k}(s_1, \dots, s_k; s_{k+1}) = \sum_{(n_1, \dots, n_k) \in \mathcal{R}_{MP}^{(k)}} n_1^{-s_1} \cdots n_k^{-s_k} (n_1 + \cdots + n_k)^{-s_{k+1}}$$

and function $\text{Li}_{s_1, \dots, s_k; s_{k+1}}^{AV}(z_1, \dots, z_k; z_{k+1})$

$$\sum_{(n_1, \dots, n_k) \in \mathcal{R}_{MP}^{(k)}} \frac{z_1^{n_1} \cdots z_k^{n_k} z_{k+1}^{(n_1 + \cdots + n_k)}}{n_1^{s_1} \cdots n_k^{s_k} (n_1 + \cdots + n_k)^{s_{k+1}}}$$

Euler–Maclaurin summation formula for $f(t) = x^t t^{-s}$

$$f^{(k)}(t) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \binom{s+k-j-1}{k-j} (k-j)! t^{-(s+k-j)} x^t \log(x)^j$$

gives

$$\begin{aligned} \sum_{n=a}^b f(n) &= \int_a^b f(t) dt + \frac{1}{2}(f(b) + f(a)) \\ &+ \sum_{k=2}^N \frac{b_k}{k!} (f^{(k-1)}(b) - f^{(k-1)}(a)) \\ &- \int_a^b \frac{B_N(t - [t])}{N!} f^{(N)}(t) dt, \end{aligned}$$

b_k Bernoulli numbers and B_k Bernoulli polynomials

- applied to $\text{Li}_{s_1, s_2, s_3}^{\mathcal{R}}(z_1, z_2, z_3)$ with $\mathcal{R} = \mathcal{R}_T^{(3)}$ summation terms

$$\pm F_{j,k}(s_3, z_3) \text{Li}_{s_1, s_2; s_3+k-j}^{AV}(z_1, z_2; z_3)$$

$$\pm F_{j,k}(s_3, z_3) \text{Li}_{s_1, s_3+k-j; s_2}^{MT}(z_1, z_2; z_3)$$

with

$$F_{j,k}(s, z) = \frac{b_k}{k!} \binom{k}{j} \binom{s+k-j-1}{k-j} (k-j)! \log(z)^j$$

Conclusion: by this method can see explicit integrals leading to multiple zeta values, but computations become easily extremely complicated even for simple graphs!