Feynman integrals and algebraic geometry

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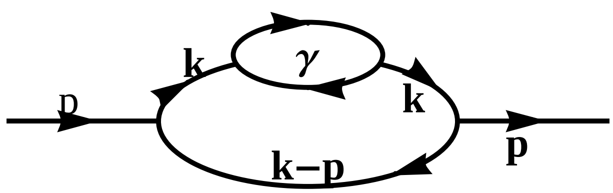
Based on: joint work with P. Aluffi, arXiv:0807.1690, arXiv:0811.2514, arXiv:0901.2107.

Quantum Fields and Motives (an unlikely match)

- *Feynman diagrams: graphs and integrals*

$$(2\pi)^{-2D}\int \frac{1}{k^4}\frac{1}{(k-p)^2}\frac{1}{(k+\ell)^2}\frac{1}{\ell^2}\,d^Dk\,d^D\ell$$



Divergences ⇒ Renormalization

- *Algebraic varieties and motives*

$\mathcal{V}_{\mathbb{K}}$ smooth proj alg varieties over $\mathbb{K}$ ⇒ category of pure motives $\mathcal{M}_{\mathbb{K}}$: $\mathrm{Hom}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{m-n}_{/\sim}(X,Y)\,p$ with $p^2=p$, $q^2=q$, $\mathbb{Q}(m)$ = Tate motives
Universal cohomology theory for algebraic varieties

What do they have in common?
Main question: are residues of Feynman integrals periods of mixed Tate motives?

Supporting evidence

- Multiple zeta values from Feynman integral calculations (Broadhurst–Kreimer)

- Parametric Feynman integrals as periods (Bloch–Esnault–Kreimer)

- Graph hypersurfaces and their motives (Belkale–Brosnan)

- Hopf algebras of renormalization (Connes–Kreimer)

- Flat equisingular connections and Galois symmetries (Connes-M.)

- Feynman integrals and Hodge structures (Bloch–Kreimer; M.)

Perturbative QFT in a nutshell

$\mathcal{T}$ = scalar field theory in spacetime dimension D

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

with Lagrangian density

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi)$$

Effective action and perturbative expansion (1PI graphs)

$$S_{eff}(\phi) = S_0(\phi) + \sum_\Gamma \frac{\Gamma(\phi)}{\#\mathrm{Aut}(\Gamma)}$$

$$\Gamma(\phi) = \frac{1}{N!} \int_{\sum_i p_i = 0} \widehat{\phi}(p_1) \cdots \widehat{\phi}(p_N) U_\mu^z(\Gamma(p_1, \ldots, p_N)) dp_1 \cdots dp_N$$

$$U(\Gamma(p_1, \ldots, p_N)) = \int I_\Gamma(k_1, \ldots, k_\ell, p_1, \ldots, p_N) d^D k_1 \cdots d^D k_\ell$$

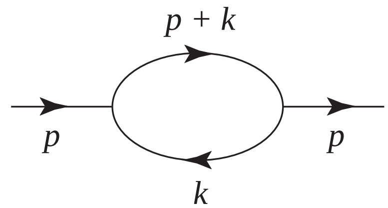
$\ell = b_1(\Gamma)$ loops

Dimensional Regularization: $U_\mu^z(\Gamma(p_1, \ldots, p_N))$

$$= \int \mu^{z\ell} d^{D-z} k_1 \cdots d^{D-z} k_\ell I_\Gamma(k_1, \ldots, k_\ell, p_1, \ldots, p_N)$$

(Laurent series in $z \in \Delta^* \subset \mathbb{C}^*$)

Regularization (Dim Reg)



$$\int \frac{1}{k^2+m^2} \frac{1}{((p+k)^2+m^2)} \, d^D k$$

ϕ^3-theory $D = 4$ divergent

Schwinger parameters

$$\frac{1}{k^2+m^2} \frac{1}{(p+k)^2+m^2} = \int_{s>0,\, t>0} e^{-s(k^2+m^2)-t((p+k)^2+m^2)} \, ds \, dt$$

diagonalize quadratic form in exp

$$-Q(k) = -\lambda \left((k+xp)^2 + ((x-x^2)p^2+m^2)\right)$$

with $s = (1-x)\lambda$ and $t = x\,\lambda \Rightarrow$ Gaussian $q = k + xp$

$$\int e^{-\lambda\, q^2} \, d^D q \;=\; \pi^{D/2} \, \lambda^{-D/2}$$

$$\int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2+\lambda\, m^2)} \int e^{-\lambda\, q^2} \, d^D q \;\lambda \, d\lambda \; dx$$

$$= \pi^{D/2} \int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2+\lambda\, m^2)} \, \lambda^{-D/2} \, \lambda \, d\lambda \, dx$$

$$= \pi^{D/2} \, \Gamma(2-D/2) \int_0^1 ((x-x^2)p^2 + m^2)^{D/2-2} \, dx$$

Feynman rules

Construction of $I_\Gamma(k_1,\ldots,k_\ell,p_1,\ldots,p_N)$:

- Internal lines $\Rightarrow$ propagator = quadratic form q_i

$$\frac{1}{q_1\cdots q_n}, \quad q_i(k_i)=k_i^2+m^2$$

- Vertices: conservation (valences = monomials in $\mathcal{L}$)

$$\sum_{e_i\in E(\Gamma):s(e_i)=v} k_i = 0$$

- Integration over k_i, internal edges

$$U(\Gamma)=\int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i}k_i+\sum_{j=1}^N \epsilon_{v,j}p_j)}{q_1\cdots q_n}\, d^Dk_1\cdots d^Dk_n$$

$n=\#E_{int}(\Gamma)$, $N=\#E_{ext}(\Gamma)$

$$\epsilon_{e,v}=\begin{cases} +1 & t(e)=v \\ -1 & s(e)=v \\ 0 & \text{otherwise,}\end{cases}$$

- Connected graphs: $\Gamma=\cup_{v\in T}\Gamma_v$

$$U(\Gamma_1\amalg\Gamma_2,p)=U(\Gamma_1,p_1)U(\Gamma_2,p_2)$$

- 1PI graphs:

$$U(\Gamma,p)=\prod_{v\in T}U(\Gamma_v,p_v)\frac{\delta((p_v)_e-(p_{v'})_e)}{q_e((p_v)_e)}$$

Parametric Feynman integrals

- Schwinger parameters $q_1^{-k_1} \cdots q_n^{-k_n} =$

$$\frac{1}{\Gamma(k_1)\cdots\Gamma(k_n)} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \cdots + s_n q_n)} \, s_1^{k_1-1} \cdots s_n^{k_n-1} \, ds_1 \cdots ds_n.$$

- Feynman trick

$$\frac{1}{q_1 \cdots q_n} = (n-1)! \int \frac{\delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n} \, dt_1 \cdots dt_n$$

then change of variables $k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k$

$$\eta_{ik} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{ loop } \ell_k \\ 0 & \text{otherwise} \end{cases}$$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n - D\ell/2}}$$

$\sigma_n = \{t \in \mathbb{R}_+^n | \sum_i t_i = 1\}$, vol form ω_n

• Graph polynomials

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e \notin T} t_e$$

$$(M_\Gamma)_{kr}(t) = \sum_{i=0}^{n} t_i \eta_{ik} \eta_{ir}$$

Massless case $m = 0$:

$$V_\Gamma(t,p) = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)} \quad \text{and} \quad P_\Gamma(p,t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

cut-sets C (compl of spanning tree plus one edge)
$s_C = (\sum_{v \in V(\Gamma_1)} P_v)^2$ with $P_v = \sum_{e \in E_{ext}(\Gamma), t(e)=v} p_e$
for $\sum_{e \in E_{ext}(\Gamma)} p_e = 0$ $\quad$ $\deg \Psi_\Gamma = b_1(\Gamma) = \deg P_\Gamma - 1$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2} \omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

stable range $-n + D\ell/2 \geq 0$

$$\int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}$$

log divergent $n = D\ell/2$, or massive $m \neq 0$ with $p = 0$
$\Rightarrow \Psi_\Gamma(t)^{n-D(\ell+1)/2} \omega_n$

Feynman integrals and periods

Residue of $U(\Gamma)$ (up to divergent Gamma factor)

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

Graph hypersurfaces

$$\hat{X}_\Gamma = \{t \in \mathbb{A}^n \,|\, \Psi_\Gamma(t) = 0\}$$

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} \,|\, \Psi_\Gamma(t) = 0\} \quad \deg = b_1(\Gamma)$$

$$\hat{Y}_\Gamma(p) = \{t \in \mathbb{A}^n \,|\, P_\Gamma(t,p) = 0\}$$

$$Y_\Gamma(p) = \{t \in \mathbb{P}^{n-1} \,|\, P_\Gamma(t,p) = 0\} \quad \deg = b_1(\Gamma)+1$$

Relative cohomology (range $-n + D\ell/2 \geq 0$)

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma))$$

$\Sigma_n = \{\prod_i t_i = 0\} \supset \partial\sigma_n$

Realization of mixed Tate motive? $\mathfrak{m}(X,Y)$
(Bloch–Esnault–Kreimer)

Mixed Tate motives $\mathcal{MT}(\mathbb{K})$

• Pure motives $\mathcal{M}$: smooth projective varieties

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = qCorr^{m-n}_{/\sim}(X,Y)p$$

p, q projectors, morphisms alg cycles codim $= \dim X - m + n$, numerical equivalence
$\mathbb{Q}(1) = \mathbb{L}^{-1}$ Tate motive
$\mathcal{M} =$ abelian category, rigid tensor (Tannakian)

• Mixed motives $\mathcal{DM}$ triangulated category (Voevodsky, Levine, Hanamura)

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1]$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2]$$

$\mathcal{DMT} \subset \mathcal{DM}$ generated by the $\mathbb{Q}(m)$

• Examples of mixed Tate: constructed with stratifications and locally trivial fibrations from affine spaces

• Over $\mathbb{K}$ numer field: t-structure $\mathcal{MT}(\mathbb{K})$ abelian (Tannakian: $G = U \rtimes \mathbb{G}_m$, prounipotent U)

The Grothendieck ring of varieties $K_0(\mathcal{V})$

- generators $[X]$ isomorphism classes
- $[X] = [X \smallsetminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

<u>Additive invariant</u> $\chi(X) = \chi(Y)$ if $X \cong Y$

$$\chi(X) = \chi(Y) + \chi(X \smallsetminus Y), \quad Y \subset X$$

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

same as assigning

$$\chi : K_0(\mathcal{V}) \to \mathcal{R}$$

ring homomorphism

$\Rightarrow$ *Universal Euler characteristics*

- Example 1: topological Euler characteristic
- Example 2: Gillet–Soulé:

$K_0(\mathcal{M})$ (abelian category of pure motives: virtual motives)

$$\chi : K_0(\mathcal{V})[\mathbb{L}^{-1}] \to K_0(\mathcal{M}), \quad \chi(X) = [(X, id, 0)]$$

for X smooth projective; complex $\chi(X) = W^{\cdot}(X)$

Tate motives: $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathcal{M})$

Computing in the Grothendieck group

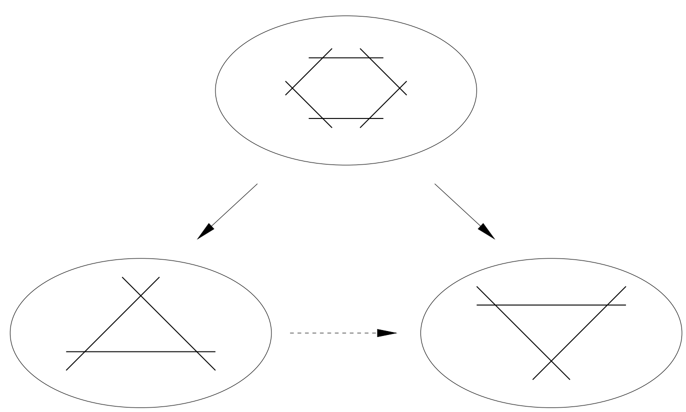
Dual graph and Cremona map

$$\mathcal{C} : (t_1 : \cdots : t_n) \mapsto (\frac{1}{t_1} : \cdots : \frac{1}{t_n})$$

outside $\mathcal{S}_n$ singularities locus of

$$\Sigma_n = \{\prod_i t_i = 0\}$$

ideal $I_{\mathcal{S}_n} = (t_1 \cdots t_{n-1}, t_1 \cdots t_{n-2} t_n, \cdots, t_1 t_3 \cdots t_n)$



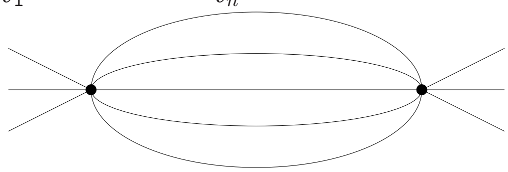
$$\Psi_\Gamma(t_1, \ldots, t_n) = (\prod_e t_e) \Psi_{\Gamma^\vee}(t_1^{-1}, \ldots, t_n^{-1})$$

$$\mathcal{C}(X_\Gamma \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)$$

isomorphism of X_Γ and $X_{\Gamma^\vee}$ outside of Σ_n

Example: Banana graphs

$\Psi_\Gamma(t) = t_1 \cdots t_n (\frac{1}{t_1} + \cdots + \frac{1}{t_n})$



Class in the Grothendieck group

$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L} - 1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L} - 1)^{n-2}$$

where $\mathbb{L} = [\mathbb{A}^1]$ Lefschetz motive

$X_{\Gamma^\vee} = \mathcal{L}$ hyperplane in $\mathbb{P}^{n-1}$ ($\Gamma^\vee$= polygon)

$$[\mathcal{L} \smallsetminus \Sigma_n] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T} + 1}$$

$\mathbb{T} = [\mathbb{G}_m] = [\mathbb{A}^1] - [\mathbb{A}^0]$

$$X_{\Gamma_n} \cap \Sigma_n = \mathcal{S}_n$$

$$[\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^{n-2}$$

$$[X_{\Gamma_n}] = [X_{\Gamma_n} \cap \Sigma_n] + [X_{\Gamma_n} \smallsetminus \Sigma_n]$$

Using Cremona: $[X_{\Gamma_n}] = [\mathcal{S}_n] + [\mathcal{L} \smallsetminus \Sigma_n]$

$$\Rightarrow \chi(X_{\Gamma_n}) = n + (-1)^n$$

Belkale–Brosnan's universality

• Classes $[X_\Gamma]$ generate $K_0(\mathcal{V})$ Grothendieck ring of varieties

• but is the part of the motive involved in the period simpler? a mixed Tate motive?

(Note: Tate part of $K_0(\mathcal{M})$ is just $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$)

Bloch's sum over graphs (using Cremona)

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\mathrm{Aut}(\Gamma)} \in \mathbb{Z}[\mathbb{L}],$$

Tate motive (though $[X_\Gamma]$ individually need not be)

Feynman rules in algebraic geometry

$U(\Gamma) \in \mathcal{R}$ (comm. ring $\mathcal{R}$, finite graph Γ)

$U(\Gamma) = U(\Gamma_1) \cdots U(\Gamma_k)$ for $\Gamma = \Gamma_1 \amalg \cdots \amalg \Gamma_k$

$$U(\Gamma) = U(L)^{\# E(T)} \prod_{v \in V(T)} U(\Gamma_v)$$

non-1PI: $\Gamma = \cup_{v \in V(T)} \Gamma_v$

Inverse propagator: $U(L)$ for $L =$ single edge

(Ring homomorphism $U : \mathcal{H}_{CK} \to \mathcal{R}$ plus choice of $U(L)$:
$\mathcal{H}_{CK} =$ Connes–Kreimer Hopf algebra)

<u>Algebro-geometric Feynman rules</u> $\Gamma = \Gamma_1 \amalg \Gamma_2$

$$\mathbb{A}^{n_1+n_2} \smallsetminus \hat{X}_\Gamma = (\mathbb{A}^{n_1} \smallsetminus \hat{X}_{\Gamma_1}) \times (\mathbb{A}^{n_2} \smallsetminus \hat{X}_{\Gamma_2})$$

$$\Psi_\Gamma(t_1, \ldots, t_n) = \Psi_{\Gamma_1}(t_1, \ldots, t_{n_1}) \Psi_{\Gamma_2}(t_{n_1+1}, \ldots, t_{n_1+n_2})$$

In projective space not product but *join*:

$$\mathbb{P}^{n_1+n_2-1} \smallsetminus X_\Gamma \to (\mathbb{P}^{n_1-1} \smallsetminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \smallsetminus X_{\Gamma_2})$$

$\mathbb{G}_m$-bundle (assume Γ_i not a forest)

Ring of immersed conical varieties $\mathcal{F}$

$V \subset \mathbb{A}^N$ N not fixed, homogeneous ideals (conical), $[V]$ up to linear changes of coordinates (less than up to isomorphism)

$$[V \cup W] = [V] + [W] - [V \cap W]$$

$$[V] \cdot [W] = [V \times W]$$

embedded version of Grothendieck ring

- Mod by isomorphisms $\Rightarrow$ maps to $K_0(\mathcal{V})$

- Maps to polynomial invariant

$$I_{CSM} : \mathcal{F} \to \mathbb{Z}[T]$$

not factoring through Grothendieck group (characteristic classes of singular varieties)

<u>Algebro-geometric Feynman rules:</u> homomorphisms

$$I : \mathcal{F} \to \mathcal{R}, \qquad \mathbb{U}(\Gamma) := I([\mathbb{A}^n]) - I([\hat{X}_\Gamma])$$

$\Rightarrow I([\mathbb{A}^n \smallsetminus \hat{X}_\Gamma])$ Feynman rule with

$$\mathbb{U}(L) = I([\mathbb{A}^1])$$

Inverse propagator = affine line $[\mathbb{A}^1]$
$\Rightarrow$ Lefschetz motive $\mathbb{L}$

Universal algebro-geometric Feynman rule

$$\mathbb{U}(\Gamma) = [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] \in \mathcal{F}$$

<u>Motivic</u> = factors through $K_0(\mathcal{V})$

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = (\mathbb{L} - 1)[\mathbb{P}^{n-1} \smallsetminus X_\Gamma] \in K_0(\mathcal{V})$$

(if Γ not a forest)
since $[\hat{X}_\Gamma] = (\mathbb{L} - 1)[X_\Gamma] + 1$ affine cone

• Euler characteristic as Feynman rule? Notice
$\chi(\mathbb{A}^n \smallsetminus \hat{X}_\Gamma) = 0$ but nontrivial χ from I_{CSM}

Characteristic classes of singular varieties

- Nonsingular: $c(V) = c(TV) \cap [V]$

$$\int c(TV) \cap [V] = \chi(V)$$

deg of zero dim component = Poincaré–Hopf

- Singular: M.H. Schwartz: radial vector fields;

MacPherson: functoriality $\Rightarrow c_{CSM}(X)$

- Constructible functions $\mathbb{F}(X)$ functor

$f_*(1_W) = \chi(W \cap f^{-1}(p))$

- Natural transformation to homology (Chow)

$c_*(1_X) = c(TX) \cap [X]$ for smooth

(Mather classes and local Euler obstructions)

Hypersurfaces with isolated singularities

$\Rightarrow$ Milnor numbers (X_Γ non-isolated singularities)

- Inclusion-exclusion (not isomorphism-invariant)

$$c_{CSM}(X) = c_{CSM}(Y) + c_{CSM}(X \smallsetminus Y)$$

- classes $c_{CSM}(X_\Gamma)$ in ambient $\mathbb{P}^{n-1}$

(equiv to Eul.char. of iterated hyperplane sections)

Example: banana graphs: $\chi(X_{\Gamma_n})$ = top deg term

$c_{CSM}(X_{\Gamma_n}) = ((1+H)^n - (1-H)^{n-1} - nH - H^n) \cdot [\mathbb{P}^{n-1}]$

Feynman rules from CSM classes

$$c_*(1_{\hat{X}}) = a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + \cdots + a_N[\mathbb{P}^N] \in A(\mathbb{P}^N)$$

natural transformation from constructible function $1_{\hat{X}}$ for $\hat{X} \subset \mathbb{A}^N$ loc closed in $\mathbb{P}^N$ to Chow group $A(\mathbb{P}^N)$

$$G_{\hat{X}}(T) := a_0 + a_1 T + \cdots + a_N T^N$$

indep of N, stops at $\dim \hat{X}$; invariant coord. changes;

$$G_{\hat{X}\cup\hat{Y}}(T) = G_{\hat{X}}(T) + G_{\hat{Y}}(T) - G_{\hat{X}\cap\hat{Y}}(T)$$

(from inclusion-exlusion of CSM)

$$I_{CSM}([\hat{X}]) = G_{\hat{X}}(T), \quad I_{CSM} : \mathcal{F} \to \mathbb{Z}[T]$$

Not easy to see: *ring homomorphism*

$$G_{\hat{X}\times\hat{Y}}(T) = G_{\hat{X}}(T) \cdot G_{\hat{Y}}(T)$$

need CSM classes of joins $J(X,Y) \subset \mathbb{P}^{m+n-1}$

$$(sx_1 : \cdots : sx_m : ty_1 : \cdots : ty_n), \quad (s:t) \in \mathbb{P}^1$$

$\hat{X} \times \hat{Y}$ affine cone over $J(X,Y)$:

$c_*(1_{J(X,Y)}) = ((f(H)+H^m)(g(H)+H^n) - H^{m+n}) \cap [\mathbb{P}^{m+n-1}]$

$c_*(1_X) = H^n f(H) \cap [\mathbb{P}^{n+m-1}]$, $c_*(1_Y) = H^m g(H) \cap [\mathbb{P}^{n+m-1}]$

CSM Feynman rule:

$$\mathbb{U}_{CSM}(\Gamma) = C_\Gamma(T) = I_{CSM}([\mathbb{A}^n]) - I_{CSM}([\hat{X}_\Gamma])$$

algebro geometric but not motivic:

$$C_{\Gamma_1}(T) = T(T+1)^2 \quad C_{\Gamma_2}(T) = T(T^2+T+1)$$

$$[\mathbb{A}^n \smallsetminus \hat{X}_{\Gamma_i}] = [\mathbb{A}^3] - [\mathbb{A}^2] \in K_0(\mathcal{V})$$

Properties of $C_\Gamma(T)$:

- $C_\Gamma(T)$ monic of deg n
- Γ = forest $\Rightarrow C_\Gamma(T) = (T+1)^n$
- Inverse propagator $\mathbb{U}_{CSM}(L) = T+1$
- Coeff of T^{n-1} is $n - b_1(\Gamma)$
- $C'_\Gamma(0) = \chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma)$

$\Rightarrow$ modification of $\chi(\mathbb{P}^{n-1} \smallsetminus \hat{X}_\Gamma)$ giving Feynman rule

Graphs and determinant hypersurfaces

$$\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik} \eta_{ir}$$

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

determinant hypersurface $\hat{\mathcal{D}}_\ell = \{\det(x_{ij}) = 0\}$

$$[\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1) \Rightarrow \text{ mixed Tate}$$

When Υ embedding

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2} \omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}$$

If $\hat{\Sigma}_\Gamma$ normal crossings divisor in $\mathbb{A}^{\ell^2}$
with $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma \Rightarrow$ Question on periods:

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)) \quad \text{mixed Tate?}$$

(motive whose realization is relative cohomology)

Combinatorial conditions for embedding

$$\Upsilon : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \hookrightarrow \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$$

- Closed 2-cell embedded graph:

$$\iota : \Gamma \hookrightarrow S_g$$

$S_g \smallsetminus \Gamma$ union of open disks (faces); closure of each is a disk.

- Two faces have at most one edge in common
- Every edge in the boundary of two faces

Sufficient: Γ 3-edge-connected with closed 2-cell embedding of face width ≥ 3.

Face width: largest $k \in \mathbb{N}$, every non-contractible simple closed curve in S_g intersects Γ at least k times (∞ for planar).

Note: 2-edge-connected =1PI; 2-vertex-connected conjecturally implies face width ≥ 2

Identifying the motive $\mathfrak{m}(X,Y)$

$\hat{\Sigma}_\Gamma \subset \hat{\Sigma}_{\ell,g}$ $(f = \ell - 2g + 1)$

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}$$

$$\begin{cases} x_{ij} & = 0 \quad 1 \leq i < j \leq f-1 \\ x_{i1} + \cdots + x_{i,f-1} & = 0 \quad 1 \leq i \leq f-1 \end{cases}$$

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\Sigma}_{\ell,g} \cap \hat{\mathcal{D}}_\ell))$$

$\hat{\Sigma}_{\ell,g}$ = normal crossings divisor $\Upsilon_\Gamma(\partial\sigma_n) \subset \hat{\Sigma}_{\ell,g}$
depends only on $\ell = b_1(\Gamma)$ and g = min genus of S_g

Sufficient conditions for mixed Tate:

- Varieties of frames: mixed Tate?

$$\mathbb{F}(V_1, \ldots, V_\ell) := \{(v_1, \ldots, v_\ell) \in \mathbb{A}^{\ell^2} \mid v_k \in V_k\}$$

- Two subspaces: ($d_{12} = \dim(V_1 \cap V_2)$)

$$[\mathbb{F}(V_1, V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}$$

- Three subspaces ($D = \dim(V_1 + V_2 + V_3)$)

$$[\mathbb{F}(V_1, V_2, V_3)] = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1)$$

$$-(\mathbb{L}-1)((\mathbb{L}^{d_1}-\mathbb{L})(\mathbb{L}^{d_{23}}-1)+(\mathbb{L}^{d_2}-\mathbb{L})(\mathbb{L}^{d_{13}}-1)+(\mathbb{L}^{d_3}-\mathbb{L})(\mathbb{L}^{d_{12}}-1)$$

$$+(\mathbb{L} - 1)^2(\mathbb{L}^{d_1+d_2+d_3-D} - \mathbb{L}^{d_{123}+1}) + (\mathbb{L} - 1)^3$$

Higher: difficult to find suitable induction

Other formulation: $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ locus of complete flags $0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell = E$

$$\dim E_i \cap V_i = d_i, \quad \dim E_i \cap V_{i+1} = e_i$$

Are these mixed Tate? (for all choices of d_i, e_i)

$\mathbb{F}(V_1, \ldots, V_\ell)$ fibration over $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$: class $[\mathbb{F}(V_1, \ldots, V_\ell)]$

$= [Flag_{\ell,\{d_i,e_i\}}(\{V_i\})](\mathbb{L}^{d_1}-1)(\mathbb{L}^{d_2}-\mathbb{L}^{e_1})(\mathbb{L}^{d_3}-\mathbb{L}^{e_2}) \cdots (\mathbb{L}^{d_r}-\mathbb{L}^{e_{r-1}})$

$Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ intersection of unions of Schubert cells in flag varieties $\Rightarrow$ Kazhdan–Lusztig?

Removing singularities

• DimReg: local Igusa L-functions (Belkale–Brosnan)

$$I(s) = \int_\sigma f(t)^s \omega \Rightarrow \text{Laurent series}$$

coefficients are periods

(log divergent case: more general Bogner–Weinzierl)

• Blowups (Bloch–Esnault–Kreimer) $X_\Gamma \cap \Sigma_n$
where singularities can occur: separate via blowups

• Leray coboundaries (M.M.)

$$D_\epsilon(X) = \cup_{s \in \Delta^*_\epsilon} X_s$$

$X_s = f^{-1}(s)$, circle bundle $\pi_\epsilon : \partial D_\epsilon(X) \to X_\epsilon$

integrate around singularities in $\pi_\epsilon^{-1}(\sigma \cap X_\epsilon)$

$\Rightarrow$ Laurent series in ϵ

Regularization and renormalization

Removing divergences from Feynman integrals by adjusting bare parameters in the Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2(1-\delta Z) + \left(\frac{m^2-\delta m^2}{2}\right)\phi^2 - \frac{g+\delta g}{6}\phi^3$$

Regularization: replace divergent integral by function with pole

($z \in \mathbb{C}^*$ in DimReg, ϵ deformation of X_Γ, etc.)

Renormalization: consistency over subgraphs

$\Rightarrow$ BPHZ method:

- Preparation:

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma\in\mathcal{V}(\Gamma)} C(\gamma)U(\Gamma/\gamma)$$

- Counterterm: projection onto polar part

$$C(\Gamma) = -T(\bar{R}(\Gamma))$$

- Renormalized value:

$$R(\Gamma) = \bar{R}(\Gamma) + C(\Gamma)$$

$$= U(\Gamma) + C(\Gamma) + \sum_{\gamma\in\mathcal{V}(\Gamma)} C(\gamma)U(\Gamma/\gamma)$$

Hopf algebra structures (Connes–Kreimer)

$\mathcal{H} = \mathcal{H}(\mathcal{T})$ (depend on theory $\mathcal{L}(\phi)$)

Free commutative algebra in generators
Γ 1PI Feynman graphs

Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Extended to gauge theories (van Suijlekom):
Ward identities as Hopf ideals

Connes–Kreimer theory

• $\mathcal{H}$ dual to affine group scheme G (diffeographisms)

• $G(\mathbb{C})$ pro-unipotent Lie group $\Rightarrow$

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z)$$

Birkhoff factorization of loops exists

• Recursive formula for Birkhoff = BPHZ

• loop = $\phi \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}(\{z\}))$
(germs of meromorphic functions)

• Feynman integral $U(\Gamma) = \phi(\Gamma)$
counterterms $C(\Gamma) = \phi_-(\Gamma)$
renormalized value $R(\Gamma) = \phi_+(\Gamma)|_{z=0}$

Algebro-geometric Feynman rules: $\mathbb{U}(\Gamma)$ gives

$$\phi : \mathcal{H} \to \mathcal{R}, \quad \phi(\Gamma) = \mathbb{U}(\Gamma)$$

algebra homomorphism

Birkhoff factorization works whenever $\mathcal{R}$ has a Rota–Baxter structure of weight $\lambda = -1$ (Ebrahimi-Fard, Guo, Kreimer)

Rota–Baxter weight λ: $\exists$ linear map $\mathfrak{P}$ on $\mathcal{R}$ such that

$$\mathfrak{P}(X)\mathfrak{P}(Y) = \mathfrak{P}(X\mathfrak{P}(Y)) + \mathfrak{P}(\mathfrak{P}(X)Y) + \lambda\mathfrak{P}(XY)$$

For Laurent series: $\mathfrak{P} =$ polar part projection: in CK recursive formula

$$\phi_-(X) = -\mathfrak{P}(\phi(X) + \sum \phi_-(X')\phi(X''))$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$ gives $\phi_-(XY) = \phi_-(X)\phi_-(Y)$

Question: Is there an interesting Rota–Baxter structure on $K_0(\mathcal{V})$ or on $\mathcal{F}$?

Renormalization and motivic Galois theory

(A.Connes–M.M. 2004)

Compare renormalization and motives
by comparing Tannakian categories

- Counterterms as iterated integrals
('t Hooft–Gross relations)
- Solutions of irregular singular differential equations (flat equisingular connections)
- Flat equisingular vector bundles
form a neutral Tannakian category $\mathcal{E}$
- Free graded Lie algebra $\mathcal{L} = \mathcal{F}(e_{-n}; n \in \mathbb{N})$

$$\mathcal{E} \simeq Rep_{\mathbb{U}^*}, \quad \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$$

$\mathbb{U} = \mathrm{Hom}(\mathcal{H}_{\mathbb{U}}, -)$, with $\mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^\vee$

- Motivic Galois group (Deligne–Goncharov)

$$\mathbb{U}^* \simeq \mathrm{Gal}(\mathcal{M}_S)$$

$\mathcal{M}_S$ mixed Tate motives on $S = \mathrm{Spec}(\mathbb{Z}[i][1/2])$

Question: Why $\mathrm{Spec}(\mathbb{Z}[i][1/2])$ from X_Γ?
Equisingular (irregular singular) connections and Hodge structures (regular singular)?

Additional considerations: (M.M. 2008)
What does DimReg mean geometrically?

A tentative motivic approach: Kummer motive

$$M = [u : \mathbb{Z} \to \mathbb{G}_m] \in \mathrm{Ext}^1_{\mathcal{DM}(\mathbb{K})}(\mathbb{Q}(0), \mathbb{Q}(1))$$

with $u(1) = q \in \mathbb{K}^*$ and period matrix

$$\begin{pmatrix} 1 & 0 \\ \log q & 2\pi i \end{pmatrix}$$

Kummer extension of Tate sheaves

$$\mathcal{K} \in \mathrm{Ext}^1_{\mathcal{DM}(\mathbb{G}_m)(\mathbb{Q}_{\mathbb{G}_m}(0), \mathbb{Q}_{\mathbb{G}_m}(1))}$$

$$\mathbb{Q}_{\mathbb{G}_m}(1) \to \mathcal{K} \to \mathcal{Q}_{\mathbb{G}_m}(0) \to \mathbb{Q}_{\mathbb{G}_m}(1)[1]$$

Logarithmic motives $\mathrm{Log}^n = \mathrm{Sym}^n(\mathcal{K})$

$$\mathrm{Log}^\infty = \varprojlim_n \mathrm{Log}^n$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ \log(s) & (2\pi i) & 0 & \cdots & 0 & \cdots \\ \frac{\log^2(s)}{2!} & (2\pi i)\log(s) & (2\pi i)^2 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ \frac{\log^n(s)}{n!} & (2\pi i)\frac{\log^{n-1}(s)}{(n-1)!} & (2\pi i)^2\frac{\log^{n-2}(s)}{(n-2)!} & \cdots & (2\pi i)^{n-1} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix}$$

Graph polynomials and motivic sheaves M_Γ

$(\Psi_\Gamma : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \to \mathbb{G}_m, \hat{\Sigma}_n \smallsetminus \hat{X}_\Gamma \cap \hat{\Sigma}_n, n-1, n-1)$

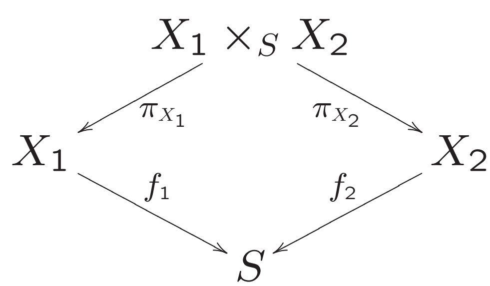
Arapura's category of motivic sheaves $(f : X \to S, Y, i, w)$

DimReg = product $M_\Gamma \times \mathrm{Log}^\infty$ fibered product

$$(X_1 \times_S X_2 \to S, Y_1 \times_S X_2 \cup X_1 \times_S Y_2, i_1 + i_2, w_1 + w_2)$$

$$\int \pi^*_{X_1}(\omega) \wedge \pi^*_{X_2}(\eta) = \int \omega \wedge f_1^*(f_2)_*(\eta)$$

on $\sigma_1 \times_S \sigma_2$ for $\sigma_i \subset X_i$, $\partial\sigma_i \subset Y_i$



DimReg integral $\int_\sigma \Psi_\Gamma^z \alpha$ period on $M_\Gamma \times \mathrm{Log}^\infty$

NCG explanation of DimReg in A.Connes, M.M. *Anomalies, Dimensional Regularization and noncommutative geometry*, unpublished manuscript, 2005, available at `www.its.caltech.edu/ matilde/work.html`