Quantum Hall effect models in NCG

Electrons in a crystal:

\[ \Gamma \subset \mathbb{R}^d \] lattice, \( \mathbb{Z}^d \) co-compact; \( \mathbb{R}^d \) compact \( (d = 2, 3) \)

periodic potential (electron-ion interaction)

\[ U(x) = \sum_{\gamma \in \Gamma} u(x - \gamma) \]

invariant under translations by \( \Gamma \)

\[ T_\gamma U = U \quad \forall \gamma \in \Gamma \]

N-electrons: N-particle Hamiltonian

\[ N \sum_{i=1}^{N} \left( -\Delta_{x_i} + V(x_i) \right) + \frac{1}{2} \sum_{i \neq j} W(x_i - x_j) \]

Simplify to a single particle problem using

"independent electron approximation"

\[ \sum_{i=1}^{N} \left( -\Delta_{x_i} + V(x_i) \right) + \frac{1}{2} \sum_{i \neq j} W(x_i - x_j) \]

Correct \( U \) by an average effect of all other electrons on a given one

\[ \rightarrow \text{Usually } V(x) \text{ unbounded (Coulomb potential well)} \]

but effective potential of independent-electron approx \( \underline{V(x)} \) bounded function

(Condensed matter physics)
then wave function
\[ \psi(x_1, \ldots, x_N) = \det(\psi_{ij}(x_j)) \]
\[ (-\Delta_{\mathbb{R}^d} + V(x)) \phi_i = E_i \phi_i \]
\[ E = \sum_i E_i \]

reduces completely to a single electron problem

(Usually inverse problem of determining \( V \): not known explicitly)

\[ H = -\Delta + V \quad T_Y = \text{translations, } y \in \mathcal{X} \]
\[ (\text{unitary operators}) \]
\[ \mathcal{X} = \mathcal{L}^2(\mathbb{R}^d) \]

\[ T_y H T_y^{-1} = H \quad \forall y \in \mathcal{X} \]

\[ \Rightarrow T_Y \text{ commutes w/ } H \quad \text{simultaneously diagonalize in basis of eigenstates of } H \]

\[ T_y \hat{\psi} = c(y) \hat{\psi} \quad T_y, T_z = T_y, T_z \]

\[ \Rightarrow c : \mathcal{X} \rightarrow \mathbb{U}(1) \quad \text{group homomorphism} \]

\[ c(y) = \exp(i\langle k, y \rangle) \quad \forall k \in \mathcal{F} = \text{Poincaré dual of } \mathcal{X} \]

\[ \mathcal{X} = \mathbb{Z}^d \quad \Rightarrow \mathcal{F} \cong \mathbb{T}^d \quad \text{thus} \]

\[ T^d \cong \mathcal{F} = \mathbb{R}^d / \mathbb{Z}^d \]

\[ \mathcal{F} = \{ k \in \mathbb{R}^d : \langle k, y \rangle \in 2\pi \mathbb{Z}, \forall y \in \mathcal{X} \} \]

dual lattice (reciprocal lattice)
Brillouin zones of the crystal: fundamental domains of reciprocal lattice \( \Gamma^* \)

(identify \( w \) with \( T^d \))

Classical Bloch theory of electrons in solids:

\[
\begin{align*}
\{ \begin{array}{l}
(\Delta + V) \psi &= E \psi \\
\psi(x+y) &= e^{ik \cdot y} \psi(x)
\end{array} \}
\]


spectral problems

for given \( k \): eigenvalues \( E_1(k), E_2(k), \ldots, E_n(k), \ldots \)

\[ E(k) = E(k + u); \ u \in \Gamma^* \]

\( k \mapsto E(k) \quad k \in \mathbb{R}^d / \Gamma^* \)

energy-crystal momentum dispersion relation

Discretization of the problem \( \mathcal{E} \):

Replace \( \mathbb{R}^d \) by \( \mathbb{Z}^d \) lattice

\( \Delta \) Laplacian replaced by finite difference \( \Delta \) (random walk in a lattice)

\( R \psi(n_1, \ldots, n_d) = \frac{d}{i=1} \psi(n_1, \ldots, n_i+1, \ldots, n_d) \)

\[ + \sum_{i=1}^{d} \psi(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d) \]

\( \Delta_{\text{disc}} \psi(n_1, \ldots, n_d) = (2d - R) \psi(n_1, \ldots, n_d) \)

becomes

\[ \begin{align*}
\Psi \in L^2(\Gamma) \text{ satisfying } \\
(R + V) \Psi &= (\lambda + 2d) \Psi \\
R \Psi &= \sum_{i=1}^{d} R_{\Psi_i} \Psi
\end{align*} \]

\( (R_{\Psi_i} \Psi)(n_1, \ldots, n_d) = \Psi(n_1, \ldots, n_i+a_i, \ldots, n_d) \)

\( \Psi_{\Psi_i} = \Psi \) a.e. \( \Gamma \)
This classical theory of electron motion in solids does not work anymore when transverse magnetic field \( B \) is applied.

Classical Hall effect

Current density \( \vec{j} \) and electric field \( \vec{E} \) (Hall current)

\[
0 = \vec{E} + \vec{j} \times \vec{B}
\]

Equation of equilibrium of forces

- \( \vec{j} \times \vec{B} \) : intensity of Hall current
- \( \vec{E} \) : intensity electric field

Hall conductance

\[
\sigma_H = \frac{N_e s}{B}
\]

\( N_e \): density of charges

\[
\sigma_H = \frac{\gamma}{R_H}
\]

\( \gamma = \frac{e^2}{\hbar} \) filling factor (dimensionless)

\( R_H = \frac{h}{e^2} \) Hall resistance

* Integer Quantum Hall effect:

\( \sigma_H \) has quantized values at integer multiples of \( \frac{e^2}{h} \)

Klitzing, 1980

Laughlin, 1981

* Fractional QHE

Strom-

- Tsui, 1982

- Fractional values also occur

(\( \text{lower } T, \text{stronger } B \))
Magnetic field 2-form \( \omega = d\eta \)

\((B = \text{curl } A)\)

Schrödinger operator \(\Delta^\eta + V\)

\(\Delta^\eta = (d-i\eta)^*(d-i\eta) \quad \forall \text{ same independent approx. electric potential}\)

\(\gamma^* \omega = \omega \quad \text{translation invariance for 2-form of magnetic field}\)

but \(\omega - \gamma^* \omega = d(\eta - \gamma^* \eta)\)

does not mean invariant magnetic potential

\(d(\eta - \gamma^* \eta) = 0 \quad \text{only implies}\)

\(\eta - \gamma^* \eta = d\phi^\gamma \quad \text{(because } \mathbb{R}^\gamma \text{ non-cohom. closed form } \Rightarrow \text{exact)}\)

\(\phi^\gamma(x) = \int_{x_0}^x (\eta - \gamma^* \eta)\)

\(= \) \(T_\gamma \) translations no longer commute with \(\Delta^\eta\)

but twisted by phase \(\phi\) again commute

\(T_\delta \phi \gamma := \exp(i\phi_\delta) \cdot T_\delta \gamma\)
\[
(d-i\gamma) T^\phi_\gamma = T^\phi_y (d-i\gamma) \rightarrow \text{commute with } \Delta^1
\]

\( \gamma \in \Gamma: \)

\[
T^\phi_\gamma T^\phi_{\gamma'} = \sigma(\gamma, \gamma') T^\phi_{\gamma \gamma'}
\]

Don't form a commutative algebra anymore

with \( \sigma(\gamma, \gamma') = \exp(-i\phi(\gamma \times_0)) \) cocycle

and \( \phi_\gamma(x) + \phi_{\gamma'}(\gamma x) = \phi_{\gamma \gamma'}(x) \) independent of \( x \)

Notice usual \( T^\gamma \) generate \( C^*(\Gamma) \) group C*-alg.

Since \( \Gamma \cong \mathbb{Z}^d \) lattice (abelian grp.)

\[
C^*(\Gamma) = C(\hat{\Gamma}) \quad \text{Pontryagin duality}
\]

\[
\hat{\Gamma} = \Gamma = \mathbb{R}^d / \Gamma^d \rightarrow C^*(\Gamma) = C(\text{Brillouin zone})
\]

Now with magnetic field \( T^\phi_\gamma \) generate a C*-algebra non-commutative

replaces Brillouin zone

* In the presence of a magnetic field

Brillouin zone becomes noncommutative
Discretized model on lattice $\Gamma = \mathbb{Z}^2$

Harper operator $\leftrightarrow$ Magnetic Laplacian

(like Random walk operator $\leftrightarrow$ Laplacian)

$$H_{\alpha_1, \alpha_2} \psi(m,n) = e^{-i\alpha_1 n} \psi(m+1,n)$$
$$+ e^{i\alpha_1 n} \psi(m-1,n)$$
$$+ e^{-i\alpha_2 m} \psi(m,n+1)$$
$$+ e^{i\alpha_2 m} \psi(m,n-1)$$

Magnetic translations $\sigma((m',n'),(m,n)) = \exp(-i(\alpha_1 m' + \alpha_2 n'))$

$$U = T^\sigma_{\gamma_1}, \quad V = T^\sigma_{\gamma_2}$$

$$\gamma_1 = (0,1), \quad (U \psi)(m,n) = \psi(m,n+1) e^{-i\alpha_2 m}$$
$$\gamma_2 = (1,0), \quad (V \psi)(m,n) = \psi(m+1,n) e^{-i\alpha_1 n}$$

$$H_{\alpha_1, \alpha_2} = U + U^* + V + V^*$$

$$UV = e^{i\theta} VU \quad \theta = \alpha_2 - \alpha_1$$

$\Rightarrow$ Brillouin zone replaced by a noncommutative torus $T^2 \quad \Lambda_\theta$
In general $\Gamma$ (discrete group)

$\sigma: \Gamma \times \Gamma \rightarrow U(1)$ multiplier:

$\sigma(\gamma_1, \gamma_2) \sigma(\gamma_1 \gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2) \sigma(\gamma_2, \gamma_3) \quad \blacktriangleright$

$\sigma(\gamma_i, 1) = \sigma(1, \gamma_i) = 1$

$H = \chi^2(\Gamma)$

$$(L_\gamma \psi)(\gamma') = \psi(\gamma' \gamma) \sigma(\gamma, \gamma' \gamma')$$

$$(R^\sigma_\gamma \psi)(\gamma') = \psi(\gamma' \gamma) \sigma(\gamma', \gamma)$$

$$(L^{\sigma}_{\gamma} L^{\sigma}_{\gamma'}) = \sigma(\gamma, \gamma') L^{\sigma}_{\gamma \gamma'}$$

$$R^{\sigma}_{\gamma} R^{\sigma}_{\gamma'} = \sigma(\gamma, \gamma') R^{\sigma}_{\gamma \gamma'}$$

$\gamma_i \in \{\text{set of symmetric generators of } \Gamma \}$

$\text{generators & their inverses}$

$R_{\sigma} = \sum_{i=1}^{r} R^{\sigma}_{\gamma_i}$ Harper operator

$\gamma \cdot R_{\sigma}$ discretization of magnetic Laplacian on $\Gamma$
Algebra of observables

$C^*(\Gamma, \sigma)$ twisted group ring
generated by magnetic translations $T^\gamma$

equivalently $f : \Gamma \to \mathbb{C}$ fin. support

$$(f_1 \ast f_2)(\gamma) = \sum_{\gamma = \gamma_1 \gamma_2} f_1(\gamma_1) f_2(\gamma_2) \sigma(\gamma_1 \gamma_2)$$

(coycle id. = associativity)

$C^*_{\gamma}(\Gamma, \sigma)$ $C^*$-completuin in rep. on $l^2(\Gamma)$

(For $\Gamma = \mathbb{Z}^2$, $C^*_{\gamma}(\Gamma, \sigma) = A_\theta$ NC torus)

discrete analog of spectral problem for magnetic Laplacian

$R_\sigma \psi + V \psi = E \psi$

$(\frac{\partial^2}{\partial x^2} \psi = R_\sigma \psi + V \psi \ \psi \in l^2(\Gamma)$

Schrödinger eq.

Spec ($R_\sigma$) complement: open sets (band structure)

for many $\psi \to$ bands

so many $\sim$ Cantor set as spectrum

Hofstadter butterfly: $\theta \in \mathbb{Q}$ or $\mathbb{R}\setminus\mathbb{Q}$

Counting gaps in the spectrum

$\uparrow$

Counting projections in $C^*_{\gamma}(\Gamma, \sigma)$

$P_E = \chi_{[0, E]} (H_\gamma \psi)$ spectral projections $\uparrow$ here if $E$ in a gap
\[ P_E = \int_C \frac{d\lambda}{\lambda - H_{\gamma, V}} = \int_C R_\lambda \, d\lambda \quad \text{if } C \supset \text{Spec} \quad \text{i.e. } E \text{ not in Spec} \]

\[ R_\lambda = (\lambda - H_{\gamma, V})^{-1} \quad \text{resolvent} \]

\[ G_\lambda^*(\Gamma, \sigma) \text{ closed under holomorphic functional calculus} \]

\[ \Rightarrow P_E \subset G_\lambda^*(\Gamma, \sigma) \]

Canonical faithful trace

\[ \tau : \mathcal{M}(\Gamma, \sigma) \to \mathbb{C} \]

von Neumann alg.

\[ \text{closure of } C(\mathcal{F}, \sigma) \quad \text{in } \mathcal{B}(L^2(\Gamma)) \text{ weak top.} \]

\[ \tau(a) = \langle a s_1, s_1 \rangle_{L^2(\Gamma)} \quad \{ s_1 \text{ canonical basis of } L^2(\Gamma) \} \]

\[ \text{extended to } \quad \tau \otimes \text{Tr} : K_0(G_\lambda^*(\Gamma, \sigma)) \to \mathbb{R} \]

Range of the trace

\[ \text{e.g. for NC torus } \quad \boxed{\mathbb{Z}_0 + \mathbb{Z} \subseteq \mathbb{C} \subseteq \mathbb{R}} \]

So when \( \theta \in \mathbb{Q} \) know there are only fin. many gaps

When \( \theta \notin \mathbb{R} \cup \mathbb{Q} \) indication that so-many but not sure as values could be on other projections

Conjectural
\( S_0 = \mathbb{L}(s) \delta \phi \) \\
\( S = \{ 0, 1 \} \) \\
\( R = \bigcap_{k \in \mathbb{N}} \text{Dom}(s^k) \)

**Conductance cocycle.** Kubo formula.

\[
\sigma_H = \tau \left( P F \left[ s_1 P, s_2 P \right] \right)
\]

(from transport theory, current density in \( y \) direction)

= functional derivative \( s_1 \) of \( H_0 \) by \( A_1 \)-component of magnetic potential

\[
\text{value of current} \quad tr(P s H) \quad \text{proj state of system}
\]

\[
i \, tr(P[s_3 P, s_i P]) = -i \, E_z \, tr(P[s_2 P, s_i P])
\]

\[
E = -\frac{\partial A}{\partial t} \quad \text{will be a cyclic cocycle}
\]

**Conductance cocycle**

\[
tr_K(f_0, f_1, f_2) = tr(f_0 (s_1 f_1) s_2 (f_2) - s_2 (f_1) s_1 (f_2))
\]

for elements \( f_0, f_1, f_2 \in \mathcal{C}(\Gamma, \sigma) \)

\[
\sigma_E = tr_K(P_E, P_E, P_E)
\]

Values of conductance: range of this "trace" (index pairing) of cyclic cohomology & \( K \)-theory.

b) **Theorem on ordinary traces**

\[
\therefore \Rightarrow \mathbb{Z} \text{-valued}
\]