(A, H, D) finitely summable spectral triple
Simple poles in Dirac spec.

Regular: a and (D, a) satisfy
\[ \sigma_\pm (a) = e^{\pm \frac{i}{\hbar} D}, \quad a \in \sigma_\pm (D, a) \]

even: \( \gamma \)

Then consider:

\[ \psi_n (a_0, \ldots, a_n) = \sum_{\mathbf{k} = (k_0, \ldots, k_n)} c_{n, \mathbf{k}} \int \gamma a_0 \cdots a_n [D, a_0]^{(k_0)} \cdots [D, a_n]^{(k_n)} |D|^{-n-2/|k|} \]

for \( |k_0| = k_1 + \cdots + k_n \) and \( \mathbf{T}^{(k_i)} = \nabla^{k_i}(T) \). \( \nabla(T) = D^2 T - T D^2 = [D^2, T] \)

- Only finitely many nonzero terms in sum above
- \( (\psi_n)_{n=0,2,4,\ldots} \) defines a cocycle in \((b, B)\)-complex for \( A \)
- Pairing of this cocycle with \( K_0(C^\infty(A)) \) gives

\[ \text{HC}(A) \]

Index of \( D \)

(twisted by connection from \( e_{A^N} \in \mathcal{E} \) \( \in \) \( K_0(A) \))

i.e. realizes index pairing using

\[ \text{Residues } f \text{ instead of } \text{Tr} \]
Monte equivalences and inner fluctuations of the metric for spectral triples

$A$, $B$ Monte equivalent via bimodule $E_A^B$

$(A, H, D)$ spectral triple

$\pi : A \to B(H)$

Take $H'= E \otimes \mathcal{H}$ have $\pi' : B \to B(H')$

Dirac operator: note

$D' = 1 \otimes D$ does not suffice

because $[D, a] 
eq 0$ so not compatible with $\otimes A$

defining $H'$

$H' = E \otimes \mathcal{H}$ $\exists a \otimes \psi = \frac{1}{2} \otimes \pi(a) \frac{1}{2}$

have $(1 \otimes D)(\frac{1}{2} \otimes \psi) = \frac{1}{2} \otimes D\psi = \frac{1}{2} \otimes \pi(a) D\psi$

while $(1 \otimes D)(\frac{1}{2} \otimes \pi(a) \psi) = \frac{1}{2} \otimes D \pi(a) \psi$

but $\pi(a) D \psi \neq D \pi(a) \psi$ so $1 \otimes D$ not well defined as operator on $H' = E \otimes \mathcal{H}$

To make it well defined need to correct by difference

a connection on the bimodule $E$

$\text{-linear map } \nabla : E \to E \otimes \Omega^1_D$
\[ D : E \rightarrow E \otimes_A S_D^1 \text{ satisfying Leibnitz rule} \]

\[ \nabla (\xi \cdot a) = (\nabla \xi) a + \xi \otimes da \quad da = [D, a] \]

\[ S_D^1(A) = \{ \sum a_i [D, b_i]; i \} \]

Then define new Dirac operator on \( H' = E \otimes_A H \)
by setting
\[ D'(\xi \otimes \eta) = \xi \otimes \text{Dy} + (\nabla \xi) \eta \]

\[ \text{D}' = 1 \otimes \text{D} + \nabla \otimes 1 \]

Self - Morita equivalences

trivial one with \( E = A \)

then \( H' = E \otimes_A H \cong H \) here

and only change is in Dirac operator

with connection - gauge potential \( A \)

\[ D \rightarrow D + A \]

\[ A = \sum a_i [D, b_i]; i \] a self adjoint element of \( S_D^1(A) \)

\[ A^* = A \]

(some algebras also have non-trivial self Morita equivalences)

with connection - gauge potential \( A \)

\[ D \rightarrow D + A \]

\[ A = \sum a_i [D, b_i]; i \] a self adjoint element of \( S_D^1(A) \)

\[ A^* = A \]

(so new Dirac op. still self - adj)
If have also a real structure $J$ on $(A,H,D)$
\[
D \to D + A + \varepsilon JAJ^{-1}
\]
\[
\text{sign } \pm \text{ of } JD = \varepsilon DJ
\]

So that new Dirac still same commutation relation

**Action of unitaries on gauge potentials**

\[
D + A + \varepsilon JAJ^{-1}
\]

\[
u \in U(A) \text{ unitary}
\]

\[
\text{Ad}(u)(D + A + \varepsilon JAJ^{-1}) \text{ Ad}(u^*)
\]

\[
\begin{align*}
&= D + \gamma_u(A) + \varepsilon J\gamma_u(A)J^{-1} \\
&= (\text{right } & \text{action})
\end{align*}
\]

\[
\gamma_u(A) = u[D,u^*] + uA u^*
\]

\[
V = uV = uJuJ^{-1}
\]

\[
UDU^* = u(VDV^*)u^* = u(D + u[D,V^*])u^*
\]

\[
= uDu^* + u[D,v^*] = D + u[D,u^*] + u[D,V^*]
\]

\[
\varepsilon J v[D,v^*] = JuvJ^{-1} [D,JuvJ^{-1}] = \varepsilon J uv[D,u^*]J^{-1}
\]

**Note:** Inner fluctuations \(\text{\underline{NOT}}\) an equivalence relation.

E.g. find dim example, can fluctuate $D$ so that $D + A = 0$ but then no $a[D,b] \neq 0$ if $D' = 0$. So cannot fluctuate back to $D$; not symmetric.
Spectral action functional for finitely summable spectral triples 

\( (A, \nabla, \mathcal{D} ) \) possibly with additional \( \gamma, J \)

\[ \text{Tr} \left( f \left( \frac{D}{\Lambda} \right) \right) \quad D = \text{Dirac operator} \]

\( \Lambda \in \mathbb{R}^* \) energy scale so \( \frac{D}{\Lambda} \) dimensionless

\( f \) positive real function (even)

Assume that the Dirac operator has heat kernel expansion

\[ \text{Tr} \left( e^{-tD^2} \right) \sim \sum a_\alpha t^\alpha \quad t \to 0 \quad (\forall) \]

if points of dim spectrum are simple

(only simple poles of \( \xi \) function)

then indeed no \( \log(t) \) terms in expansions

\[ \xi_D (s) = \text{Tr} \left( |D|^{-s} \right) \]

(1) A non-zero term in \((\ast)\) \( a_\alpha \neq 0 \) with \( \alpha < 0 \)

\Rightarrow \text{pole of } \xi_D \text{ at } s = -2\alpha

\[ \text{Res} \quad s = -2\alpha \quad \xi_D (s) = \frac{2\alpha}{\Gamma (-\alpha)} \]

(2) Since no \( \log(t) \) terms \( \Rightarrow \xi_D \) regular at \( s = 0 \)

\[ \xi_D (0) + (\text{dim } \text{Ker } D) = a_0 \]

\text{assume } \text{Ker } D = \infty
\[1D^{\delta} = D^{\delta / 2} = \frac{1}{\Gamma(\delta / 2)} \int_0^\infty e^{-tD} t^{\delta/2 - 1} dt \]  

Mollin transform

(1) \[\int_0^1 t^{\alpha + \delta / 2 - 1} dt = (\alpha + \delta / 2)^{-1}\]

(2) \[\frac{1}{\Gamma(\delta / 2)} \sim \frac{\delta}{2} \quad s \to 0\]

\[\Rightarrow \text{pole part at } s=0 \text{ of } \int_0^\infty \text{Tr}(e^{-t\Delta}) t^{\delta/2 - 1} dt \text{ given by} \]

\[a_0 \int_0^1 t^{\delta/2 - 1} dt = a_0 \frac{2}{\delta}\]

\[\text{given } \xi_D(0)\]

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**Asymptotic expansion of the spectral action**

\[\text{Tr} (f(\frac{\Delta}{\lambda})) \sim \sum_{\beta \in \text{DimSp}(A/H, D)} \int_{\text{pols of } \xi_D} \frac{\lambda^\beta}{\beta!} \int 1D^{\beta} + f(0) \xi_D(0) \]

\[+ \cdots \text{ neg. terms } \lambda^{-\beta}\]

\[\text{DimSp} < 0 \]

\[\int_0^\infty f(v) v^{\beta - 1} dv\]

\[\beta^I\]

Following terms are

\[f_{-2k} a_{2k} (\lambda^{-2k})\]

\[f_{-2k} = (-1)^k \frac{(2k)!}{(4+2k)!} f^{(4+2k)} (0) \quad (\text{usual)}\]

\[2k \quad a_{2k} \quad \text{again residues}\]

Suppressed for \( \lambda \to \infty \)