Noncommutative geometry and the field with one element

Matilde Marcolli

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The field with one element

Finite geometries   ($q = p^k$, p prime)

$$\#\mathbb{P}^{n-1}(\mathbb{F}_q) = \frac{\#(\mathbb{A}^n(\mathbb{F}_q) \smallsetminus \{0\})}{\#\mathbb{G}_m(\mathbb{F}_q)} = \frac{q^n - 1}{q - 1} = [n]_q$$

$$\#\mathrm{Gr}(n, j)(\mathbb{F}_q) = \#\{\mathbb{P}^j(\mathbb{F}_q) \subset \mathbb{P}^n(\mathbb{F}_q)\}$$

$$= \frac{[n]_q!}{[j]_q![n-j]_q!} = \binom{n}{j}_q$$

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad [0]_q! = 1$$

J. Tits: take $q = 1$

$$\mathbb{P}^{n-1}(\mathbb{F}_1) := \text{ finite set of cardinality } n$$

$\mathrm{Gr}(n, j)(\mathbb{F}_1) :=$ set of subsets of cardinality j

Algebraic geometry over $\mathbb{F}_1$?

Extensions $\mathbb{F}_{1^n}$ (Kapranov-Smirnov)

Monoid $\{0\} \cup \mu_n$ (n-th roots of unity)

- Vector space over $\mathbb{F}_{1^n}$: pointed set (V, v) with free action of μ_n on $V \smallsetminus \{v\}$
- Linear maps: permutations compatible with the action

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[t, t^{-1}]/(t^n - 1)$$

Zeta functions (Manin) as variety over $\mathbb{F}_1$:

$$Z(\mathbb{T}^n, s) = \frac{s-n}{2\pi} \quad \text{with } \mathbb{T}^n = \mathbb{G}_m^n$$

$\Rightarrow$ $\mathbb{F}_1$-geometry and infinite primes (Arakelov)

$$\Gamma_{\mathbb{C}}(s)^{-1} = ((2\pi)^{-s}\Gamma(s))^{-1} = \prod_{n \geq 0} \frac{s+n}{2\pi}$$

like a "dual" infinite projective space $\oplus_{n=0}^{\infty} \mathbb{T}^{-n}$ over $\mathbb{F}_1$

Note: Spec($\mathbb{Z}$) not a (fin. type) scheme over $\mathbb{F}_1$: what is $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$??

<u>More recently:</u> Soulé, Haran, Dourov, Toen–Vaquie, Connes–Consani, Borger

Focus on two approaches:
Descent data for rings from $\mathbb{Z}$ to $\mathbb{F}_1$:
- by cyclotomic points (Soulé)
- by Λ-ring structure (Borger)

Gadgets over $\mathbb{F}_1$

C. Soulé, *Les variétés sur le corps à un élément*, Mosc. Math. J. Vol.4 (2004) N.1, 217–244.

$$(X, \mathcal{A}_X, e_{x,\sigma})$$

- $X : \mathcal{R} \to Sets$ covariant functor,
$\mathcal{R}$ finitely generated flat rings

- $\mathcal{A}_X$ complex algebra

- evaluation maps: for all $x \in X(R)$, $\sigma : R \to \mathbb{C}$
$\Rightarrow e_{x,\sigma} : \mathcal{A}_X \to \mathbb{C}$ algebra homomorphism

$$e_{f(y),\sigma} = e_{y,\sigma\circ f}$$

for $f : R' \to R$ ring homomorphism

Affine varieties $V_{\mathbb{Z}} \Rightarrow$ gadget $X = G(V_{\mathbb{Z}})$ with $X(R) = \mathrm{Hom}(O(V), R)$ and $\mathcal{A}_X = O(V) \otimes \mathbb{C}$

Affine variety over $\mathbb{F}_1$ (Soulé)
Gadget with $X(R)$ finite; variety $X_{\mathbb{Z}}$ and morphism of gadgets

$$X \to G(X_{\mathbb{Z}})$$

such that all $X \to G(V_{\mathbb{Z}})$ come from $X_{\mathbb{Z}} \to V_{\mathbb{Z}}$

Should think: $R = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] \Rightarrow X(R)$ cyclotomic points

Λ-rings, endomotives, and $\mathbb{F}_1$
J. Borger, *Λ-rings and the field with one element*, preprint 2009

- Grothendieck: characteristic classes, Riemann–Roch

- Λ-ring structure: "descent data" for a ring from $\mathbb{Z}$ to $\mathbb{F}_1$

Torsion free R with action of semigroup $\mathbb{N}$ by endomorphisms lifting Frobenius

$$s_p(x) - x^p \in pR, \quad \forall x \in R$$

Morphisms: $f \circ s_k = s_k \circ f$

$\mathbb{Q}$-algebra $A \Rightarrow$ Λ-ring
iff action of Gal$(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{N}$ on $\mathcal{X}$ = Hom$(A, \bar{\mathbb{Q}})$ factors through an action of $\hat{\mathbb{Z}}$

What is noncommutative geometry?

A tool to handle "bad quotients"

Equivalence relation $\mathcal{R}$ on X:
quotient $Y = X/\mathcal{R}$.

Even for very good $X \Rightarrow X/\mathcal{R}$ pathological!

Classical: functions on the quotient
$\mathcal{A}(Y) := \{f \in \mathcal{A}(X) \,|\, f$ is $\mathcal{R}-$invariant$\}$

$\Rightarrow$ often too few functions
$\mathcal{A}(Y) = \mathbb{C}$ only constants

NCG: $\mathcal{A}(Y)$ noncommutative algebra

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation
(compact support or rapid decay)

Convolution product

$$(f_1 * f_2)(x,y) = \sum_{x \sim u \sim y} f_1(x,u) f_2(u,y)$$

involution $f^*(x,y) = \overline{f(y,x)}$.

$\mathcal{A}(\Gamma_{\mathcal{R}})$ noncommutative algebra $\Rightarrow Y = X/\mathcal{R}$ *noncommutative space*

Recall: $C_0(X) \Leftrightarrow X$ Gelfand–Naimark equiv of categories abelian C^*-algebras, loc comp Hausdorff spaces

Result of NCG:

$Y = X/\mathcal{R}$ *noncommutative space* with NC algebra of functions $\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$ is

- as good as X to do geometry
  (deRham forms, cohomology, vector bundles, connections, curvatures, integration, points and subvarieties)

- but with *new* phenomena
  (time evolution, thermodynamics, quantum phenomena)

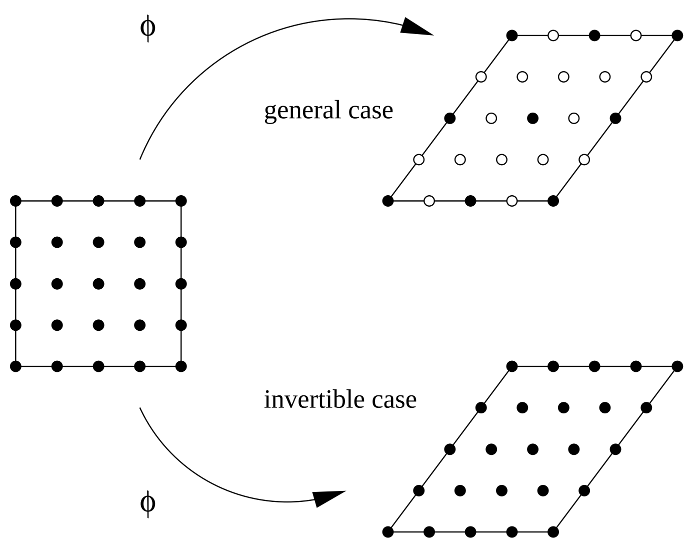
An example: $\mathbb{Q}$-lattices

Alain Connes, M.M., *Quantum statistical mechanics of $\mathbb{Q}$-lattices* in "Frontiers in number theory, physics, and geometry. I", 269–347, Springer, Berlin, 2006.

Definition: (Λ, ϕ) $\mathbb{Q}$-lattice in $\mathbb{R}^n$
lattice $\Lambda \subset \mathbb{R}^n$ + labels of torsion points

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

group *homomorphism* (invertible $\mathbb{Q}$-lat if isom)



Commensurability $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$
iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$

$\mathbb{Q}$-lattices / Commensurability $\Rightarrow$ NC space

More concretely: <u>1-dimension</u>

$$(\Lambda, \phi) = (\lambda \mathbb{Z}, \lambda \rho) \quad \lambda > 0$$

$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$

Up to scaling λ: algebra $C(\hat{\mathbb{Z}})$

<u>Commensurability</u> Action of $\mathbb{N} = \mathbb{Z}_{>0}$

$$\alpha_n(f)(\rho) = f(n^{-1}\rho) \qquad \text{zero otherwise}$$

(partially defined action of $\mathbb{Q}_+^*$)

<u>1-dimensional $\mathbb{Q}$-lattices up to scale</u> / <u>Commens.</u>

$$\Rightarrow \ \text{NC space} \quad C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$$

Crossed product algebra

$$f_1 * f_2(r, \rho) = \sum_{s \in \mathbb{Q}_+^*, s\rho \in \hat{\mathbb{Z}}} f_1(rs^{-1}, s\rho) f_2(s, \rho)$$

$$f^*(r, \rho) = \overline{f(r^{-1}, r\rho)}$$

Bost–Connes algebra

J.B. Bost, A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*. Selecta Math. (N.S.) Vol.1 (1995) N.3, 411–457.

Algebra $\mathcal{A}_{\mathbb{Q},BC} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ generators and relations

$$\begin{array}{ll} \mu_n \mu_m = \mu_{nm} & \\ \mu_n \mu_m^* = \mu_m^* \mu_n & \text{when } (n,m)=1 \\ \mu_n^* \mu_n = 1 & \end{array}$$

$$e(r+s) = e(r)e(s), \quad e(0) = 1$$

$$\rho_n(e(r)) = \mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$

Time evolution

$$\sigma_t(e(r)) = e(r), \quad \sigma_t(\mu_n) = n^{it} \mu_n$$

Hamiltonian $\mathrm{Tr}(e^{-\beta H}) = \zeta(\beta)$

Quantum statistical mechanics

$(\mathcal{A}, \sigma_t)$ C^*-algebra and time evolution

State: $\varphi : \mathcal{A} \to \mathbb{C}$ linear $\varphi(1) = 1$, $\varphi(a^* a) \geq 0$

Time evolution $\sigma_t \in \mathrm{Aut}(\mathcal{A})$
rep. on Hilbert space $\mathcal{H} \Rightarrow$ Hamiltonian $H = \frac{d}{dt}\sigma_t|_{t=0}$

Equilibrium states (inverse temperature $\beta = 1/kT$)

$$\frac{1}{Z(\beta)} \mathrm{Tr}\left(a\, e^{-\beta H}\right) \qquad Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right)$$

Classical points of NC space

KMS states $\quad \varphi \in \mathrm{KMS}_\beta$ $(0 < \beta < \infty)$

$\forall a, b \in \mathcal{A}$ $\exists$ holom function $F_{a,b}(z)$ on strip: $\forall t \in \mathbb{R}$

$F_{a,b}(t) = \varphi(a\sigma_t(b)) \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$

More on Bost–Connes

- Representations π_ρ on $\ell^2(N)$:

$$\mu_n \epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(r))\epsilon_m = \zeta_r^m \epsilon_m$$

$\zeta_r = \rho(e(r))$ root of 1, for $\rho \in \hat{\mathbb{Z}}^*$

- Low temperature extremal KMS $(\beta > 1)$

$$\varphi_{\beta,\rho}(a) = \frac{\mathrm{Tr}(\pi_\rho(a) e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}, \quad \rho \in \hat{\mathbb{Z}}^*$$

High temperature: unique KMS state

- Zero temperature: evaluations $\varphi_{\infty,\rho}(e(r)) = \zeta_r$

$$\varphi_{\infty,\rho}(a) = \langle \epsilon_1, \pi_\rho(a)\epsilon_1 \rangle$$

Intertwining: $a \in \mathcal{A}_{\mathbb{Q},BC}$, $\gamma \in \hat{\mathbb{Z}}^*$

$$\varphi_{\infty,\rho}(\gamma a) = \theta_\gamma(\varphi_{\infty,\rho}(a))$$

$$\theta : \hat{\mathbb{Z}}^* \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

Class field theory isomorphism

Bost–Connes endomotive

A.Connes, C.Consani, M.M., *Noncommutative geometry and motives: the thermodynamics of endomotives*, Adv. in Math. 214 (2) (2007), 761–831

$A = \varinjlim_n A_n$ with $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$
abelian semigroup action $S = \mathbb{N}$ on $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$

Endomotives (A, S) from self maps of algebraic varieties $s : Y \to Y$, $s(y_0) = y_0$ unbranched, $X_s = s^{-1}(y_0)$, $X = \varprojlim X_s = \mathrm{Spec}(A)$

$$\xi_{s,s'} : X_{s'} \to X_s, \quad \xi_{s,s'}(y) = r(y), \quad s' = rs \in S$$

Bost–Connes endomotive:
$\mathbb{G}_m$ with self maps $u \mapsto u^k$

$$s_k : P(t, t^{-1}) \mapsto P(t^k, t^{-k}), \quad k \in \mathbb{N}, \quad P \in \mathbb{Q}[t, t^{-1}]$$

$$\xi_{k,\ell}(u(\ell)) = u(\ell)^{k/\ell}, \quad u(\ell) = t \mod t^\ell - 1$$

$X_k = \mathrm{Spec}(\mathbb{Q}[t, t^{-1}]/(t^k - 1)) = s_k^{-1}(1)$ and $X = \varprojlim_k X_k$

$$u(\ell) \mapsto e(1/\ell) \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}], \quad C(X(\bar{\mathbb{Q}})) = C(\hat{\mathbb{Z}})$$

Integer model of the Bost–Connes algebra

A.Connes, C.Consani, M.M., *Fun with* $\mathbb{F}_1$, to appear in J.Number Theory, arXiv:0806.2401

$\mathcal{A}_{\mathbb{Z},BC}$ generated by $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and μ_n^*, $\tilde{\mu}_n$

$$\begin{array}{l}\tilde{\mu}_n\tilde{\mu}_m = \tilde{\mu}_{nm}\\ \mu_n^*\mu_m^* = \mu_{nm}^*\\ \mu_n^*\tilde{\mu}_n = n\\ \tilde{\mu}_n\mu_m^* = \mu_m^*\tilde{\mu}_n \quad (n,m)=1.\end{array}$$

$$\mu_n^* x = \sigma_n(x)\mu_n^* \qquad \text{and} \qquad x\tilde{\mu}_n = \tilde{\mu}_n\sigma_n(x)$$

where $\sigma_n(e(r)) = e(nr)$ for $r \in \mathbb{Q}/\mathbb{Z}$

Note: $\rho_n(x) = \mu_n x \mu_n^*$ ring homomorphism but not $\tilde{\rho}_n(x) = \tilde{\mu}_n x \mu_n^*$ (correspondences "crossed product" $\mathcal{A}_{\mathbb{Z},BC} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\tilde{\rho}} \mathbb{N}$)

NCG and Soulé's $\mathbb{F}_1$-geometry:

Roots of unity as varieties over $\mathbb{F}_1$:

$$\underline{\mu}^{(k)}(R) = \{x \in R \mid x^k = 1\} = \mathrm{Hom}_{\mathbb{Z}}(A_k, R)$$

$A_k = \mathbb{Z}[t, t^{-1}]/(t^k - 1)$

- Inductive system $\mathbb{G}_m$ over $\mathbb{F}_1$:

$$\underline{\mu}^{(n)}(R) \subset \underline{\mu}^{(m)}(R), \quad n|m, \qquad A_m \twoheadrightarrow A_n$$

$$\mathcal{A}_X = C(S^1)$$

- Projective system (BC): $\xi_{m,n} : X_n \twoheadrightarrow X_m$

$$\xi_{m,n} : \underline{\mu}^{(n)}(R) \twoheadrightarrow \underline{\mu}^{(m)}(R), \qquad n|m$$

$$\underline{\mu}^{\infty}(R) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Q}/\mathbb{Z}], R)$$

$\Rightarrow$ projective system of affine varieties over $\mathbb{F}_1$

$$\xi_{m,n} : \mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} \to \mathbb{F}_{1^m} \otimes_{\mathbb{F}_1} \mathbb{Z}$$

$$\mathcal{A}_X = \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$$

Bost–Connes and $\mathbb{F}_1$

A.Connes, C.Consani, M.M., *Fun with* $\mathbb{F}_1$, arXiv:0806.2401

Affine varieties $\mu^{(n)}$ over $\mathbb{F}_1$ defined by gadgets $G(\mathrm{Spec}(\mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]))$; projective system

Endomorphisms σ_n (of varieties over $\mathbb{Z}$, of gadgets, of $\mathbb{F}_1$-varieties)

Extensions $\mathbb{F}_{1^n}$: free actions of roots of 1
(Kapranov–Smirnov)

$\zeta \mapsto \zeta^n,\ n \in \mathbb{N}$ and $\zeta \mapsto \zeta^\alpha \leftrightarrow e(\alpha(r)),\ \alpha \in \hat{\mathbb{Z}}$

Frobenius action on $\mathbb{F}_{1^\infty}$

In reductions mod p of integral Bost–Connes endomotive $\Rightarrow$ Frobenius

Bost–Connes = extensions $\mathbb{F}_{1^n}$ plus Frobenius

Characteristic p versions of the BC endomotive

$$\mathbb{Q}/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}/\mathbb{Z})^{(p)}$$

denom = power of p; denom = prime to p

$$\mathbb{K}[\mathbb{Q}_p/\mathbb{Z}_p] \rtimes p^{\mathbb{Z}^+}$$

endomorphisms σ_n for $n = p^\ell$, $\ell \in \mathbb{Z}^+$

$\varphi_{\mathbb{F}_p}(x) = x^p$ Frobenius of $\mathbb{K}$ char p

$$(\sigma_{p^\ell} \otimes \varphi^\ell_{\mathbb{F}_p})(f) = f^{p^\ell}$$

$f \in \mathbb{K}[\mathbb{Q}/\mathbb{Z}]$

$$(\sigma_{p^\ell} \otimes \varphi^\ell_{\mathbb{F}_p})(e(r) \otimes x) = e(p^\ell r) \otimes x^{p^\ell} = (e(r) \otimes x)^{p^\ell}$$

$\Rightarrow$ BC endomorphisms restrict to Frobenius on mod p reductions: σ_{p^ℓ} Frobenius correspondence on pro-variety $\mu^\infty \otimes_{\mathbb{Z}} \mathbb{K}$

NCG and Borger's $\mathbb{F}_1$-geometry

M.M., *Cyclotomy and endomotives*, preprint arXiv:0901.3167

Multivariable Bost–Connes endomotives

Variety $\mathbb{T}^n = (\mathbb{G}_m)^n$ endomorphisms $\alpha \in M_n(\mathbb{Z})^+$

$$X_\alpha = \{t = (t_1, \ldots, t_n) \in \mathbb{T}^n \,|\, s_\alpha(t) = t_0\}$$

$$\xi_{\alpha,\beta} : X_\beta \to X_\alpha, \quad t \mapsto t^\gamma, \quad \alpha = \beta\gamma \in M_n(\mathbb{Z})^+$$

$$t \mapsto t^\gamma = \sigma_\gamma(t) = (t_1^{\gamma_{11}} t_2^{\gamma_{12}} \cdots t_n^{\gamma_{1n}}, \ldots, t_1^{\gamma_{n1}} t_2^{\gamma_{n2}} \cdots t_n^{\gamma_{nn}})$$

$X = \varprojlim_\alpha X_\alpha$ with semigroup action

$C(X(\bar{\mathbb{Q}})) \cong \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ generators $e(r_1) \otimes \cdots \otimes e(r_n)$

$$\mathcal{A}_n = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes n} \rtimes_\rho M_n(\mathbb{Z})^+$$

generated by $e(\underline{r})$ and μ_α, μ_α^*

$$\rho_\alpha(e(\underline{r})) = \mu_\alpha e(\underline{r}) \mu_\alpha^* = \frac{1}{\det \alpha} \sum_{\alpha(\underline{s}) = \underline{r}} e(\underline{s})$$

$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r}) \mu_\alpha = e(\alpha(\underline{r}))$$

Endomorphisms:

$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r}) \mu_\alpha$$

The Bost–Connes endomotive is a direct limit of Λ-rings

$R_n = \mathbb{Z}[t, t^{-1}]/(t^n - 1) \quad s_k(P)(t, t^{-1}) = P(t^k, t^{-k})$

Action of $\hat{\mathbb{Z}}$:

$$\alpha \in \hat{\mathbb{Z}} : \quad (\zeta : x \mapsto \zeta x) \mapsto (\zeta : x \mapsto \zeta^\alpha x)$$

$\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$

$\Rightarrow$ Frobenius over $\mathbb{F}_{1^\infty}$ (Haran)

Embeddings of Λ-rings (Borger–de Smit)
Every torsion free finite rank Λ-ring embeds in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes n}$ with action of $\mathbb{N}$ compatible with $S_{n,diag} \subset M_n(\mathbb{Z})^+$

BC systems as universal Λ-rings

$\Rightarrow$ multivariable Bost–Connes endomotives as universal Λ-rings

Quantum statistical mechanics of multivariable Bost–Connes endomotives

Representations on $\ell^2(M_n(\mathbb{Z})^+)$: $\zeta_i^\alpha = \prod_j \zeta_i^{\alpha_{ij}}$

$$\zeta = (\zeta_1, \ldots, \zeta_n) = \rho(\underline{r}), \quad \rho \in \mathrm{GL}_n(\hat{\mathbb{Z}})$$

$$\mu_\beta \, \epsilon_\alpha = \epsilon_{\beta\alpha}$$

$$\pi_\rho(e(\underline{r}))\epsilon_\alpha = \prod_i \zeta_i^{\alpha^\tau} \epsilon_\alpha$$

Time evolution:

$$\sigma_t(e(\underline{r})) = e(\underline{r}), \quad \sigma_t(\mu_\alpha) = \det(\alpha)^{it} \, \mu_\alpha$$

Hamiltonian:

$$H\epsilon_\alpha = \log\det(\alpha) \; \epsilon_\alpha$$

<u>Problem:</u> Infinite multiplicities in the spectrum: $\mathrm{SL}_n(\mathbb{Z})$-symmetry

Groupoid and convolution algebra

Similar to: A.Connes, M.M. *Quantum statistical mechanics of $\mathbb{Q}$-lattices*, in "Frontiers in Number Theory, Physics, and Geometry, I" pp.269–350, Springer, 2006.

$$\mathcal{U}_\Gamma = \{(\alpha,\rho) \in \Gamma\backslash \mathrm{GL}_n(\mathbb{Q})^+ \times_\Gamma \widehat{\mathbb{Z}}^n \,|\, \alpha(\rho) \in \widehat{\mathbb{Z}}^n\}$$

Quotient $\mathcal{U}_\Gamma$ by $\mathrm{SL}_n(\mathbb{Z}) \times \mathrm{SL}_n(\mathbb{Z})$

$$(\gamma_1,\gamma_2) : (\alpha,\rho) \mapsto (\gamma_1\alpha\gamma_2^{-1}, \gamma_2(\rho))$$

Convolution algebra

$$(f_1 \star f_2)(\alpha,\rho) = \sum_{(\alpha,\rho)=(\alpha_1,\rho_1)\circ(\alpha_2,\rho_2)\in\mathcal{U}_\Gamma} f_1(\alpha_1,\rho_1) f_2(\alpha_2,\rho_2)$$

$f^*(\alpha,\rho) = \overline{f(\alpha^{-1},\alpha(\rho))}$ and $\sigma_t(f)(\alpha,\rho) = \det(\alpha)^{it} f(\alpha,\rho)$

$$(\pi_\rho(f)\xi)(\alpha) = \sum_{\beta\in\Gamma\backslash \mathrm{GL}_n(\mathbb{Q})^+ \,:\, \beta\rho\in\widehat{\mathbb{Z}}^*} f(\alpha\beta^{-1},\beta(\rho))\xi(\beta)$$

On $\ell^2(\Gamma\backslash G_\rho)$. If $\rho \in (\widehat{\mathbb{Z}}^*)^n$:

$$G_\rho = \{\alpha \in \mathrm{GL}_n(\mathbb{Q})^+ \,|\, \alpha(\rho) \in \widehat{\mathbb{Z}}^n\} = M_n(\mathbb{Z})^+$$

$$Z(\beta) = \sum_{m\in\Gamma\backslash M_n(\mathbb{Z})^+} \det(m)^{-\beta}$$

Why QSM?

A.Connes, C.Consani, M.M., *The Weil proof and the geometry of the adeles class space*, arXiv:math/0703392

For original BC system $\Rightarrow$ dual system $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$. Restriction map:

$$\delta : \hat{\mathcal{A}}^\natural_\beta \to \mathcal{S}^\natural(\Omega_\beta)$$

Cyclic module: cokernel $HC_0(D(\mathcal{A}, \varphi))$
$\Rightarrow$ Spectral realization of zeros of Riemann zeta function and trace formula

<u>Multivariable case:</u>

Manin (1994): problem of $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} =$???
cf. Weil's proof for function fields $C \times_{\mathbb{F}_q} C$

Zeta function viewpoint: Kurokawa
tensor product zeta function: summing over zeros

$$\prod_{\lambda \in \Xi} (s-\lambda)^{m_\lambda} \otimes \prod_{\mu \in \Theta} (s-\mu)^{n_\mu} := \prod_{(\lambda,\mu)} (s-\lambda-\mu)^{m_\lambda + n_\mu}$$

as zeta-regularized infinite products

Question: Kurokawa zeta functions from multivariable dual systems?

Thermodynamics of endomotives and RH

A.Connes, C.Consani, M.M., *Noncommutative geometry and motives: the thermodynamics of endomotives*, Adv. in Math. 214 (2) (2007), 761–831

- Dual system: $\hat{\mathcal{A}} = \mathcal{A}_{BC} \rtimes_\sigma \mathbb{R}$
- Scaling action: $\theta_\lambda(\int x(t)\, U_t\, dt) = \int \lambda^{it}\, x(t)\, U_t\, dt$
- Classical points: $\tilde{\Omega}_\beta \simeq \Omega_\beta \times \mathbb{R}_+^*$ low T KMS states
- Restriction map: (morphism of cyclic modules)

$$\delta : \hat{\mathcal{A}}_\beta \xrightarrow{\pi} C(\tilde{\Omega}_\beta, \mathcal{L}^1) \xrightarrow{\mathrm{Tr}} C(\tilde{\Omega}_\beta)$$

- Cokernel (abelian category) $D(\mathcal{A}, \varphi) = \mathrm{Coker}(\delta)$
- Cyclic homology (with scaling action) $HC_0(D(\mathcal{A}, \varphi))$
- Galois action+ scaling $C_{\mathbb{Q}} = \hat{\mathbb{Z}}^* \times \mathbb{R}_+^*$

Weil explicit formula as trace formula

on $\mathcal{H}^1 = HC_0(D(\mathcal{A}, \varphi))$:

$$\mathrm{Tr}(\vartheta(f)|_{\mathcal{H}^1}) = \hat{f}(0) + \hat{f}(1) - \Delta \bullet \Delta f(1) - \sum_v \int'_{(\mathbb{K}_v^*, e_{K_v})} \frac{f(u^{-1})}{|1-u|} d^*u$$

$$\vartheta(f) = \int_{C_{\mathbb{Q}}} f(g) \vartheta_g\, d^*g \quad f \in S(C_{\mathbb{Q}})$$

Self inters of diagonal $\Delta \bullet \Delta = \log|a| = -\log|D|$

($D =$ discriminant for #-field, Euler char $\chi(C)$ for $\mathbb{F}_q(C)$)

Note:

- Connes' 1997 RH paper: Tr$(R_\Lambda U(f))$:

- only critical line zeros
- Trace formula (global) $\Leftrightarrow$ RH

- Tr$(\vartheta(f)|_{\mathcal{H}^1})$:

- all zeros involved
- RH $\Leftrightarrow$ positivity

$$\mathrm{Tr}\left(\vartheta(f \star f^\sharp)|_{\mathcal{H}^1}\right) \geq 0 \quad \forall f \in S(C_{\mathbb{Q}})$$

where

$$(f_1 \star f_2)(g) = \int f_1(k) f_2(k^{-1}g) d^*g$$

multiplicative Haar measure d^*g and adjoint

$$f^\sharp(g) = |g|^{-1}\overline{f(g^{-1})}$$

$\Rightarrow$ Better for comparing with Weil's proof for function fields

Weil's proof in a nutshell

$\mathbb{K} = \mathbb{F}_q(C)$ function field, $\Sigma_{\mathbb{K}}$ = places deg n_v = # orbit of Fr on fiber $C(\bar{\mathbb{F}}_q) \to \Sigma_{\mathbb{K}}$

$$\zeta_{\mathbb{K}}(s) = \prod_{\Sigma_{\mathbb{K}}} (1 - q^{-n_v s})^{-1} = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

$P(T) = \prod(1 - \lambda_n T)$ char polynomial of Fr^* on $H^1_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell)$

$$C(\bar{\mathbb{F}}_q) \supset \mathrm{Fix}(\mathrm{Fr}^j) = \sum_k (-1)^k \mathrm{Tr}(\mathrm{Fr}^{*j} | H^k_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell))$$

RH $\Leftrightarrow$ eigenvalues λ_n with $|\lambda_j| = q^{1/2}$

Correspondences: divisors $Z \subset C \times C$; degree, codegree, trace:

$$d(Z) = Z \bullet (P \times C) \qquad d'(Z) = Z \bullet (C \times P)$$

$$\mathrm{Tr}(Z) = d(Z) + d'(Z) - Z \bullet \Delta$$

RH $\Leftrightarrow$ Weil positivity $\mathrm{Tr}(Z \star Z') > 0$

$\Rightarrow$ Dictionary between NCG and Alg Geom:
$C \times_{\mathbb{F}_q} C \Leftrightarrow$ NCG version of $\mathrm{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \mathrm{Spec}(\mathbb{Z})$

A possible intermediate step: Witt vectors

Affine ring scheme of big Witt vectors

$$\mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots]$$

symmetric functions λ_i, ghost coordinates

$$\psi_n = \sum_{d|n} d\, u_d^{n/d}, \quad \text{with} \quad \psi_n = x_1^n + x_2^n + \cdots$$

$W(R) = \prod_{n\geq 1} R$ addition/multipl of ghost coords

Truncated Witt schemes with $\mathbb{Z}[u_1, \ldots, u_N] \Rightarrow$ gadgets $\mathcal{W}_{\mathbb{Z}}^{(N)}$ over $\mathbb{F}_1$ (Manin)

Also Λ-ring structure given by the ψ_n

Borger's suggestion: Witt schemes related to

$$\text{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \cdots \times_{\mathbb{F}_1} \text{Spec}(\mathbb{Z})$$

$\Rightarrow$ Realize inside multivariable BC endomotives

Other aspects of $\mathbb{F}_1$-geometry:

Analytic geometry

Yu.I. Manin, *Cyclotomy and analytic geometry over $\mathbb{F}_1$*, arXiv:0809.1564.

The Habiro ring

K. Habiro, *Cyclotomic completions of polynomial rings*, Publ. RIMS Kyoto Univ. (2004) Vol.40, 1127–1146.

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q)_n)$$

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

$\mathbb{Z}[q]/((q)_n) \twoheadrightarrow \mathbb{Z}[q]/((q)_k)$ for $k \leq n$ since $(q)_k | (q)_n$

<u>Evaluation maps</u> at roots of 1: surj ring homom

$$ev_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta]$$

give an *injective* homomorphism:

$$ev : \widehat{\mathbb{Z}[q]} \to \prod_{\zeta \in \mathcal{Z}} \mathbb{Z}[\zeta]$$

<u>Taylor series expansions</u>

$$\tau_\zeta : \widehat{\mathbb{Z}[q]} \to \mathbb{Z}[\zeta][[q-\zeta]]$$

injective ring homomorphism

Ring of "analytic functions on roots of unity" $\Rightarrow$ Another model for the NCG of the cyclotomic tower, replacing $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ with $\widehat{\mathbb{Z}[q]}$

Multivariable Habiro rings (Manin)

$$\widehat{\mathbb{Z}[q_1,\ldots,q_n]} = \varprojlim_N \mathbb{Z}[q_1,\ldots,q_n]/I_{n,N}$$

where $I_{n,N}$ is the ideal

$\big((q_1-1)(q_1^2-1)\cdots(q_1^N-1),\ldots,(q_n-1)(q_n^2-1)\cdots(q_n^N-1)\big)$

<u>Evaluations at roots of 1</u>

$$ev_{(\zeta_1,\ldots,\zeta_n)} : \widehat{\mathbb{Z}[q_1,\ldots,q_n]} \to \mathbb{Z}[\zeta_1,\ldots,\zeta_n]$$

<u>Taylor expansions</u>

$T_Z : \widehat{\mathbb{Z}[q_1,\ldots,q_n]} \to \mathbb{Z}[\zeta_1,\ldots,\zeta_n][[q_1-\zeta_1,\ldots,q_n-\zeta_n]]$

$Z = (\zeta_1,\ldots,\zeta_n)$ in $\mathcal{Z}^n$

Endomotives: Habiro ring version M.M., *Cyclotomy and endomotives*, preprint arXiv:0901.3167

One variable:

$$\sigma_n(f)(q) = f(q^n)$$

lifts $P(\zeta) \mapsto P(\zeta^n)$ in $\mathbb{Z}[\zeta]$ through ev_ζ

Multivariable:

$$\mathbb{Z}[\widehat{q_1, \ldots, q_n}] = \varprojlim_N \mathbb{Z}[q_1, \ldots, q_n, q_1^{-1}, \ldots, q_n^{-1}]/\mathcal{J}_{n,N}$$

$\mathcal{J}_{n,N}$ ideal generated by the $(q_i - 1)\cdots(q_i^N - 1)$, for $i = 1, \ldots, n$ and the $(q_i^{-1} - 1)\cdots(q_i^{-N} - 1)$

Torus $\mathbb{T}^n = (\mathbb{G}_m)^n$, algebra $\mathbb{Q}[t_i, t_i^{-1}]$

$$t^\alpha = (t_i^\alpha)_{i=1,\ldots,n} \quad \text{with} \quad t_i^\alpha = \prod_j t_j^{\alpha_{ij}}$$

Semigroup action $\alpha \in M_n(\mathbb{Z})^+$:

$$q \mapsto \sigma_\alpha(q) = \sigma_\alpha(q_1, \ldots, q_n) =$$

$$(q_1^{\alpha_{11}} q_2^{\alpha_{12}} \cdots q_n^{\alpha_{1n}}, \ldots, q_1^{\alpha_{n1}} q_2^{\alpha_{n2}} \cdots q_n^{\alpha_{nn}}) = q^\alpha$$

Question: 3-manifolds and $\mathbb{F}_1$-geometry?

Universal Witten–Reshetikhin–Turaev invariant

K.Habiro, *A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres*, Invent. Math. 171 (2008) 1–81

- Chern–Simons path integral (Witten)
- quantum groups at roots of 1 (Reshetikhin–Turaev)

$$\tau(M) : \mathcal{Z} \to \mathbb{C}, \quad \tau_\zeta(M)$$

- Ohtsuki series

$$\tau^O(M) = 1 + \sum_{n=1}^{\infty} \lambda_n(M)(q-1)^n$$

Unified view (Habiro): $J_M(q) = J_L(q)$

$$J_M(q) \in \widehat{\mathbb{Z}[q]}$$

function in the (one-variable) Habiro ring:

$$ev_\zeta(J_M(q)) = \tau_\zeta(M)$$

$$\tau_1(J_M(q)) = \tau^O(M)$$

Using: 3-dimensional integral homology sphere M surgery presentation $M = S^3_L$, algebraically split link $L = L_1 \cup \cdots \cup L_\ell$ in S^3 framing ± 1

$$S^3_L \cong S^3_{L'} \Leftrightarrow L \sim L' \text{ Fenn–Rourke moves}$$

Integral homology 3-spheres

$\mathbb{Z}hs =$ free ab group generated by orientation-preserving homeomorphism classes of integral homology 3-spheres
Ring with product $M_1 \# M_2$ connected sum

$$J_{M_1 \# M_2}(q) = J_{M_1}(q) J_{M_2}(q), \quad J_{S^3}(q) = 1$$

$$J_{-M}(q) = J_M(q^{-1})$$

$\Rightarrow$ WRT ring homomorphism

$$J : \mathbb{Z}hs \to \widehat{\mathbb{Z}[q]}$$

Ohtsuki filtration

$$\mathbb{Z}hs = F_0 \supset F_1 \supset \cdots F_k \supset \cdots$$

F_k $\mathbb{Z}$-submodule spanned by

$$[M, L_1, \ldots, L_k] = \sum_{L' \subset \{L_1, \ldots, L_k\}} (-1)^{|L'|} M_{L'}$$

$L_i =$ alg split links ± 1-framed

Habiro conjecture

$$J : \widehat{\mathbb{Z}hs} \to \widehat{\mathbb{Z}[q]}$$

$\widehat{\mathbb{Z}hs} = \varprojlim_d \mathbb{Z}hs/F_d$ with $d : \mathbb{N} \to \mathbb{Z}_{\geq 0}$
$d(n)$-components of link have framing $\pm n$

Integral homology spheres and $\mathbb{F}_1$?

$X_{\mathbb{Z}hs}(R) := \{\phi : \mathbb{Z}hs \to R \,|\, \exists \tilde{\phi} : \widehat{\mathbb{Z}[q]} \to R,\ \phi = \tilde{\phi} \circ J\}$

$J : \mathbb{Z}hs \to \widehat{\mathbb{Z}[q]}$ WRT invariant;

$X_{\mathbb{Z}hs}(R) =$ set of "coarser" R-valued invariants

$\mathcal{A}_X := \mathbb{Z}hs \otimes \mathbb{C}$ WRT evaluations as cyclotomic points: $\sigma \circ \phi : \mathbb{Z}hs \to \mathbb{C}$ factors through some evaluation ev_ζ

$\Rightarrow X_{\mathbb{Z}hs}$ gadget over $\mathbb{F}_1$

Question: Using Habiro conjecture $X_{\widehat{\mathbb{Z}hs}}$ inductive limit of affine varieties over $\mathbb{F}_1$? (finiteness)

Question: 3-manifolds and endomotives?

Semigroup action? Is it possible to construct

$$M = S^3_{(L,m)} \mapsto \sigma_n(M) = S^3_{(L_n,m_n)}$$

(inv. under Fenn–Rourke moves)

$$\sigma_n(M_1 \# M_2) = \sigma_n(M_1)\sigma_n(M_2)$$

Question: $\exists$? $\Upsilon_M \in \widehat{\mathbb{Z}[q]}$

$$\Upsilon_{\sigma_n(M)} = \sigma_n(\Upsilon_M)$$

Is it possible to use endomotives (eg multivariable BC) to construct 3-manifold invariants ?

$$\mathbb{Z}hs \rtimes \mathbb{N} \to \mathcal{A} = A \rtimes \mathbb{N}$$

Quantum statistical mechanics?