$\mathbb{F}_1$-geometry of Moduli Spaces

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Main reference:

- Yuri Manin, M.M., *Moduli Operad over* $\mathbb{F}_1$, arXiv:1302.6526

Moduli spaces $\overline{M}_{0,n}$ together with their operad structure descend to $\mathbb{F}_1$, with descent data given in terms of "constructible sets over the field with one element"

What is the "field with one element"?

Finite geometries $\quad (q = p^k,\ p \text{ prime})$

$$\#\mathbb{P}^{n-1}(\mathbb{F}_q) = \frac{\#(\mathbb{A}^n(\mathbb{F}_q) \smallsetminus \{0\})}{\#\mathbb{G}_m(\mathbb{F}_q)} = \frac{q^n - 1}{q - 1} = [n]_q$$

$$\#\mathrm{Gr}(n,j)(\mathbb{F}_q) = \#\{\mathbb{P}^j(\mathbb{F}_q) \subset \mathbb{P}^n(\mathbb{F}_q)\}$$

$$= \frac{[n]_q!}{[j]_q![n-j]_q!} = \binom{n}{j}_q$$

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad [0]_q! = 1$$

The origin of $\mathbb{F}_1$-geometry: Jacques Tits observed if take $q = 1$

$$\mathbb{P}^{n-1}(\mathbb{F}_1) := \text{ finite set of cardinality } n$$

$$\mathrm{Gr}(n,j)(\mathbb{F}_1) := \text{ set of subsets of cardinality } j$$

Is there an algebraic geometry over $\mathbb{F}_1$?

Extensions $\mathbb{F}_{1^n}$ (Kapranov-Smirnov)
Monoid $\{0\} \cup \mu_n$ (n-th roots of unity)
- Vector space over $\mathbb{F}_{1^n}$: pointed set (V, v) with free action of μ_n on $V \smallsetminus \{v\}$
- Linear maps: permutations compatible with the action

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[t, t^{-1}]/(t^n - 1)$$

Counting of points: for geometries X over $\mathbb{Z}$, reductions mod p

$$N_q(X) = \#X(\mathbb{F}_q), \qquad q = p^r$$

Polynomially countable if $N_q(X) = P_X(q)$ polynomial in q.
Counting of "points over the field with one element and its extensions"

$$P_X(m+1) = \#X(\mathbb{F}_{1^m})$$

Different approaches to $\mathbb{F}_1$-geometry: Soulé, Haran, Dourov, Toen–Vaquie, Connes–Consani, López-Peña–Lorscheid

We follow the approach to $\mathbb{F}_1$-geometry via torifications (López-Peña–Lorscheid)

Idea: *geometerize* the expected behavior of the "counting of points" function

Levels of torified structures:

- Torification of the class in the Grothendieck ring
- Geometric torification
- Affine torification
- Regular torification

Then we'll also need a *weaker* form of geometric torification: *constructible torifications*

The Grothendieck ring of varieties $K_0(\mathcal{V}_{\mathbb{Z}})$

- generators $[X]$ isomorphism classes
- $[X] = [X \smallsetminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

Tate (virtual) motives: $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V}_{\mathbb{Z}})$, with $\mathbb{L} = [\mathbb{A}^1]$

Universal Euler characteristics:
Any additive invariant of varieties: $\chi(X) = \chi(Y)$ if $X \cong Y$

$$\chi(X) = \chi(Y) + \chi(X \smallsetminus Y), \quad Y \subset X$$

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

values in a commutative ring $\mathcal{R}$ is same thing as a ring homomorphism

$$\chi : K_0(\mathcal{V}_{\mathbb{Z}}) \to \mathcal{R}$$

Examples of additive invariants:

- Topological Euler characteristic
- *Couting points over finite fields*
- Gillet–Soulé motivic $\chi_{mot}(X)$:

$$\chi_{mot} : K_0(\mathcal{V})[\mathbb{L}^{-1}] \to K_0(\mathcal{M}), \quad \chi_{mot}(X) = [(X, id, 0)]$$

for X smooth projective; complex $\chi_{mot}(X) = W^{\cdot}(X)$

Note: counting points over finite fields factors through the Grothendieck ring, geometerize expected behavior of the counting function as a condition on $[X] \in K_0(\mathcal{V}_{\mathbb{Z}})$.

• Grothendieck class torification

The class $[X] \in K_0(\mathcal{V}_{\mathbb{Z}})$ in the Grothendieck ring satisfies

$$[X] = \sum_k a_k \mathbb{T}^k$$

with $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$, and with coefficients $a_k \geq 0$.

• If $[X] \in \mathbb{Z}[\mathbb{L}]$ Tate motive, then polynomially countable $N_q(X) = P_X(q)$, then

$$\#X(\mathbb{F}_1) = P_X(1) = \lim_{q \to 1} N_q(X) = a_0$$

points over $\mathbb{F}_1$ and points over $\mathbb{F}_{1^m}$:

$$\#X(\mathbb{F}_{1^m}) = P_X(m+1) = \sum_k a_k \, m^k$$

where $N_q(\mathbb{T}) = N_q(\mathbb{A}^1 \smallsetminus \{pt\}) = q - 1$

Example: $\mathbb{F}_{1^m}$-points of $\mathbb{P}^1$

- Class $[\mathbb{P}^1] = 1 + \mathbb{L} = 2 + \mathbb{T}$ and counting $N_q(\mathbb{P}^1) = 1 + q$
- $\mathbb{F}_1$-points: $\#\mathbb{P}^1(\mathbb{F}_1) = 2$, say $\mathbb{P}^1 = \mathbb{G}_m \cup \{0, \infty\}$
- $\mathbb{F}_{1^m}$-points: $\#\mathbb{P}^1(\mathbb{F}_{1^m}) = m + 2$, given by $\{0, \infty\}$ *and* m-th roots of unity in $\mathbb{G}_m$

This suggest a more *geometric* notion of torification, as a decomposition of the variety, not only of the Grothendieck class.

• Geometric torification (López-Peña–Lorscheid)

Morphism of schemes $e_X : T \to X$ from (finite) disjoint union of tori $T = \coprod_{j\in I} T_i$, $T_j = \mathbb{G}_m^{d_j}$, with restriction of e_X to each torus an immersion inducing bijection of k–points, $e_X(k) : T(k) \to X(k)$, for every field k

• Affine torification

$\exists$ affine covering $\{U_\alpha\}$ of X compatible with e_X: $\forall\ U_\alpha$, $\exists$ subfamily $\{T_j \,|\, j \in I_\alpha\}$ of torification such that restriction $e_X|_{\cup_{j\in I_\alpha} T_j}$ torification of U_α

• Regular torification

Closure of tori T_j is union of other tori of the torification

Examples: $\mathbb{P}^n = \mathbb{A}^n \cup \cdots \cup \mathbb{A}^1 \cup \mathbb{A}^0$ affinely torified; Grassmanians $\mathrm{Gr}(n, j)$ torified (cell decomposition), but not affine; smooth toric varieties regular torification (torus orbits)

Morphisms in $\mathbb{F}_1$-geometry

- torified morphism:

$\Phi : (X, e_X : T_X \to X) \to (Y, e_Y : T_Y \to Y)$ a triple
$\Phi = (\phi, \psi, \{\phi_i\})$ with $\phi : X \to Y$ morphism of $\mathbb{Z}$-varieties,
$\psi : I_X \to I_Y$ map of indexing sets and $\phi_j : T_{X,j} \to T_{Y,\psi(j)}$ morphism of algebraic groups, $\phi \circ e_X|_{T_{X,j}} = e_Y|_{T_{Y,\psi(j)}} \circ \phi_j$

- affinely torified morphism:

for every j the image of U_j under Φ is an affine subscheme of Y

- Remark: a torified morphism of affinely torified varieties is an affinely torified morphism (Lorscheid)

Problem: When two torifications determine *the same* $\mathbb{F}_1$-structure? Need a notion of equivalence (isomorphism) of torifications

Torifications giving *same* $\mathbb{F}_1$-structure on a $\mathbb{Z}$-variety X

- *Strong equivalence*: identity morphism is torified.
- *Ordinary Equivalence*: $\exists$ isomorphism of X that is torified.
- *Weak equivalence*: X has a decompositions into disjoint unions $X = \cup_j X_j$ and $X = \cup_j X'_j$, compatible with torifications, and $\exists$ isomorphisms $\phi_i : X_i \to X'_i$ that are torified.

Which equivalence relation determines what "changes of coordinates" are allowed within $\mathbb{F}_1$-geometry

This also determines what morphisms of $\mathbb{F}_1$-varieties

Morphisms of torified $\mathbb{Z}$-varieties

- *Strong* $\mathbb{F}_1$*-morphisms* (strongly torified): torified morphisms
- *Ordinary* $\mathbb{F}_1$*-morphisms* (ordinarily torified): arbitrary compositions of torified morphisms and isomorphisms
- *Weak* $\mathbb{F}_1$*-morphisms* (weakly torified): arbitrary compositions of torified morphisms and weak equivalences

Obtain in this way three different categories $\mathcal{GT}^s \subset \mathcal{GT}^o \subset \mathcal{GT}^w$

• Objects are pairs $(X_{\mathbb{Z}}, \mathcal{T})$ variety over $\mathbb{Z}$ and torification

• Morphisms are strong, ordinary, or weak morphisms

Moduli Space $\overline{M}_{0,n}$

- moduli space of stable genus zero curves with n marked points

- Open strata: $M_{0,n}$ $(n \geq 4)$ complement of diagonals in product $n-3$ copies of $\mathbb{P}^1 \smallsetminus \{0,1,\infty\}$

- Compactification by boundary strata $\prod_k M_{0,n_k+1}$ with $\sum_k n_k = n$

Generalization $T_{d,n}$ (Chen–Gibney–Krashen)

• moduli space of n–pointed stable rooted trees of d–dimensional projective spaces ($T_{1,n} = \overline{M}_{0,n+1}$)

• oriented rooted trees τ, to each vertex $v \in V_\tau$ a $X_v \simeq \mathbb{P}^d$, to unique outgoing tail at v a choice of hyperplane $H_v \subset X_v$, to each incoming tail f at v a point $p_{v,f}$ in X_v with $p_{v,f} \neq p_{v,f'}$ for $f \neq f'$ and with $p_{v,f} \notin H_v$

• Open stratum: $TH_{d,n}$ of $T_{d,n}$ is configuration space of n distinct points in $\mathbb{A}^d$ up to translation and homothety, complement of diagonals

$$TH_{d,n} \simeq (\mathbb{A}^d \smallsetminus \{\mathbf{0}, \mathbf{1}\})^{n-2} \smallsetminus \Delta$$

• Compactification: boundary strata $\prod_i TH_{d,n_i}$ with $\sum_i n_i = n$

• Description as iterated blowup (related to Fulton-MacPherson compactification of configuration spaces)

Moduli Space $\overline{M}_{0,n}$: torification of Grothendieck class

$$[\overline{M}_{0,n}] = \sum_{k=0}^{n-2} s(n-2,k) \sum_{j=0}^{k} \binom{k}{j} \mathbb{T}^j$$

with $s(m,k)$ = Stirling numbers of the first kind
$(-1)^{m-k} s(m,k) = \#\{\sigma \in S_m \, : \, \sigma = \; k\text{-cycles} \, \}$

This follows from class of open strata

$$[M_{0,n}] = (\mathbb{T}-1)(\mathbb{T}-2)\cdots(\mathbb{T}-n+2) = \binom{\mathbb{T}-1}{n-3}(n-3)!$$

$= (-1)^n (1-\mathbb{T})_{n-2}$, with $(x)_m = \Gamma(x+m)/\Gamma(x)$ = Pochhammer symbol with $(x)_m = \sum_{k=0}^m (-1)^{m-k} s(m,k)\, x^k$

Note: some coeffs of $[M_{0,n}]$ negative, so not torified, but when adding strata together torified

Generating series in $K_0(\mathcal{V}_{\mathbb{Z}})_{\mathbb{Q}}[[t]]$

$$\varphi(t) = t + \sum_{n=2}^{\infty} [\overline{M}_{0,n}] \frac{t^n}{n!}$$

unique solution in $t + t^2 K_0(\mathcal{V}_{\mathbb{Z}})_{\mathbb{Q}}[[t]]$ of

$$(1 + \mathbb{L}\, t - \mathbb{L}\, \varphi(t))\, \varphi'(t) = 1 + \varphi(t)$$

Note: analogous to Manin's result on Poincaré polynomial, since Hodge-Tate ($h^{p,q}(X_{\mathbb{C}}) = 0$ for $p \neq q$) hence if $[X] = \sum_k b_k \mathbb{L}^k$ Poincaré polynomial is $\mathcal{P}_X(q) = \sum_k b_k q^{2k}$

$\mathbb{F}_{1^m}$-points: $p_{n,m} = \overline{M}_{0,n}(\mathbb{F}_{1^m})$

$$\varphi_m(t) = \sum_{n \geq 1} p_{n,m} \frac{t^n}{n!}$$

solution of

$$(1 + (m+1)\, t - (m+1)\, \varphi_m(t))\, \varphi_m'(t) = 1 + \varphi_m(t)$$

Moduli Spaces $T_{d,n}$: torification of Grothendieck class

- Generating function

$$\psi(t) = \sum_{n\geq 1} [T_{d,n}] \frac{t^n}{n!}$$

unique solution in $t + t^2 K_0(\mathcal{V}_{\mathbb{Z}})_{\mathbb{Q}}[[t]]$ of

$$(1 + \mathbb{L}^d\, t - \mathbb{L}\, [\mathbb{P}^{d-1}]\psi(t))\, \psi'(t) = 1 + \psi(t)$$

with $[\mathbb{P}^{d-1}] = \frac{\mathbb{L}^d - 1}{\mathbb{L} - 1}$

- Classes $[T_{d,n}] \in K_0(\mathcal{V}_{\mathbb{Z}})$ have decomposition into $\mathbb{T}^k$ with $a_k \geq 0$: use repeated blowup description and blowup formula

$$[\mathrm{Bl}_Y(X)] = [X] + [Y]([\mathbb{P}^{\mathrm{codim}_X(Y)-1}] - 1)$$

Get recursive relation

$$[T_{d,n+1}] = ([\mathbb{P}^d] + n\mathbb{L}\, [\mathbb{P}^{d-2}])[T_{d,n}] + \mathbb{L}\, [\mathbb{P}^d] \sum_{i+j=n+1, 2\leq i\leq n-1} \binom{n}{i} [T_{d,i}]\, [T_{d,j}]$$

$\mathbb{F}_{1^m}$-points of $T_{d,n}$

$p_{n,m} = N_{T_{d,n}}(m+1) = T_{d,n}(\mathbb{F}_{1^m})$ formally replacing $\mathbb{L}$ with $m+1$ in Grothendieck class: generating function

$$\eta_m(t) = \sum_{n\geq 1} \frac{p_{n,m}}{n!} t^n.$$

solution of differential equation

$$(1 + (m+1)^d t - (m+1)\kappa_d(m+1)\eta_m)\eta_m' = 1 + \eta_m,$$

with $\kappa_d(q^2) = \frac{q^{2d}-1}{q^2-1}$

Question then: from Grothendieck classes to more geometric notion of torification?

Complemented subvarieties in $\mathbb{F}_1$-geometry

- Example of $\mathbb{P}^1 = \mathbb{G}_m \cup \{0, \infty\}$: points $\{0, \infty\}$ have complement that is still torified, but not if removing additional points
- X over $\mathbb{Z}$ with geometric torification $\mathcal{T}$. A subvariety $Y \subset X$ is (strongly, ordinarily, weakly) *complemented* if both Y and $X \smallsetminus Y$ have geometric torifications and inclusions $Y \hookrightarrow X$ and $X \smallsetminus Y \hookrightarrow X$ are (strongly, ordinarily, weakly) morphisms
- Usual in algebraic geometry: complement of a subvariety in a variety not always a variety, but a *constructible set*
- A notion of *constructible torifications*: more general complements of torifications inside other torifications *with positivity of Grothendieck classes*

Constructible sets over $\mathbb{F}_1$

• $\mathcal{C}_{\mathbb{F}_1}$ = class of constructible sets over $\mathbb{Z}$ that can be obtained, starting from $\mathbb{G}_m$, through of products, disjoint unions, and complements.

• X = constructible set over $\mathbb{Z}$. *Constructible torification* of X is morphism of constructible sets $e_X : C \to X$, for some $C \in \mathcal{C}_{\mathbb{F}_1}$, with restriction of e_X to each component of C an immersion inducing a bijections of k–points, for every field k.

• $\mathbb{F}_1$-constructible set = constructible set over $\mathbb{Z}$ with a constructible torification, *with Grothendieck class* $[X] = \sum_k a_k \mathbb{T}^k$ *with* $a_k \geq 0$

Note: Grothendieck class condition is needed in definition (does not follow, unlike for geometric torifications)

Categories of $\mathbb{F}_1$-constructible sets $\mathcal{CT}^s \subset \mathcal{CT}^o \subset \mathcal{CT}^w$

• Objects: pairs $(X_{\mathbb{Z}}, \mathcal{C})$, with $X_{\mathbb{Z}}$ a constructible set over $\mathbb{Z}$ and $\mathcal{C} = \{C_i\}$ is a constructible torification of $X_{\mathbb{Z}}$

• morphisms strong, ordinary, or weak morphisms of constructibly torified spaces

Blowups: $X_{\mathbb{Z}}$ variety with a constructible torification, $Y \subset X$ be a closed subvariety with geometric torification and $X \smallsetminus Y$ with constructible torification (inclusions are strong, ordinary, weak morphisms). Then blowup $\mathrm{Bl}_Y(X)$ has constructible torification so that $\pi : \mathrm{Bl}_Y(X) \to X$ (strong, ordinary, weak) morphism of constructibly torified spaces.

If $Y \subset X$ has complemented geometric torification then $\mathrm{Bl}_Y(X)$ has geometric torification.

Constructible torification of $\overline{M}_{0,n}$

- Choice of a constructible torification of $\mathbb{P}^1$ minus three points

- $\mathbb{P}^1 \smallsetminus \{0, \infty\} = \mathbb{G}_m$ has geometric torification, $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\} = \mathbb{G}_m \smallsetminus \{1\}$ has constructible torification

- $M_{0,n}$ ($n \geq 4$) complement of diagonals in product $n-3$ copies of $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$: removing diagonals taking complements of sets in $\mathcal{C}_{\mathbb{F}_1}$ inside others: still in $\mathcal{C}_{\mathbb{F}_1}$, constructible torification

- *not* an $\mathbb{F}_1$-constructible set structure on $M_{0,n}$ (no positivity of Grothendieck class)

- *but* when combining strata $[\overline{M}_{0,n}]$ has positivity

$\Rightarrow$ moduli spaces $\overline{M}_{0,n}$ are $\mathbb{F}_1$-constructible sets

Constructible torification of $T_{d,n}$

• Start with choice of constructible torification of $\mathbb{A}^d$ minus two points

• Open stratum $TH_{d,n}$ of $T_{d,n}$ is complement of diagonals

$$TH_{d,n} \simeq (\mathbb{A}^d \smallsetminus \{\mathbf{0}, \mathbf{1}\})^{n-2} \smallsetminus \Delta$$

with $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$

• constructible torification on $\mathbb{A}^d \smallsetminus \{\mathbf{0}, \mathbf{1}\}$ determines constructible torifications on products, diagonals, and complements

• positivity of Grothendieck class holds after assembling together strata to get full $T_{d,n}$

$\Rightarrow$ moduli spaces $T_{d,n}$ are $\mathbb{F}_1$-constructible sets

Operad structure of $\overline{M}_{0,n}$

• Operad $\mathcal{M}(n) = \overline{M}_{0,n+1}$ with compositions

$$\mathcal{M}(n) \times \mathcal{M}(m_1) \times \cdots \times \mathcal{M}(m_n) \to \mathcal{M}(m_1 + \cdots + m_n)$$

• Composition maps are *strong* morphisms of constructibly torified spaces

• Symmetric group S_n acts on $\overline{M}_{0,n}$ permuting marked points

• Action of S_n *ordinary* morphisms of $\mathbb{F}_1$-constructible sets

Operad structure of $T_{d,n}$

• composition (inclusion of boundary strata)

$$T_{d,k} \times T_{d,n_1} \times \cdots \times T_{d,n_k} \to T_{d,n_1+\cdots+n_k}$$

• compositions are *strong* morphisms of constructible torifications

• Isomorphisms $T_{d,S} \stackrel{\simeq}{\to} T_{d,S'}$ for $S' \stackrel{\simeq}{\to} S$ with S set of marked points, $\#S = n$. These are *ordinary* morphisms of constructible torifications

• Forgetful morphisms: $T_{d,S} \to T_{d,S'}$ for $S' \subset S$ with $\#S' \geq 2$. These are *strong* morphisms of $\mathbb{F}_1$-constructible sets

Blueprint approach to $\mathbb{F}_1$-geometry (Lorscheid)

Blueprint: $\mathcal{A}//\mathcal{R}$

- a commutative multiplicative monoid $\mathcal{A}$
- associated semiring $\mathbb{N}[\mathcal{A}]$
- set of relations

$$\mathcal{R} \subset \mathbb{N}[\mathcal{A}] \times \mathbb{N}[\mathcal{A}]$$

written as $\sum a_i \equiv \sum b_j$, for $(\sum a_i, \sum b_j) \in \mathcal{R}$.

Example: Plücker embedding of Grassmannian $G(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ gives blueprint structure to $G(2,n)$ with $\mathcal{R}$ generated by Plücker relations

$$x_{ij}x_{kl} + x_{il}x_{jk} = x_{ik}x_{jl}, \quad \text{for} \quad 1 \leq i < j < k < l \leq n$$

idea: Use explicit equations for $\overline{M}_{0,n}$ obtained by an embedding into a toric variety to give blueprint structure to $\overline{M}_{0,n}$

Blueprint structure of $\overline{M}_{0,n}$

• Can embed $\overline{M}_{0,n} \hookrightarrow X_\Delta$ a toric variety, so that intersection with torus is $M_{0,n}$ (Gibney–Maclagan and Tevelev)

• Can obtain explicit equations for $\overline{M}_{0,n}$ in the Cox ring of the toric variety X_Δ (Gibney–Maclagan and Keel–Tevelev)

• monoid

$$\mathcal{A} = F_1[x_I \,:\, I \in \mathcal{I}] := \{\prod_I x_I^{n_I}\}_{n_I \geq 0},$$

with $\mathcal{I} = \{I \subset \{1, \dots n\}, 1 \in I, \; \#I \geq 2, \#I^c \geq 2\}$ where $\mathbb{Q}[x_I, \,:\, I \in \mathcal{I}]$ Cox ring of X_Δ

• relations

$$\mathcal{R}' = \left\{ \prod_{ij \in I,\, kl \notin I} x_I + \prod_{il \in I,\, jk \notin I} x_I \equiv \prod_{ik \in I,\, jl \notin I} x_I \,:\, 1 \leq i < j < k < l \leq n \right\},$$

• $\mathcal{S}_f^{-1}\mathcal{R}'$ *localization* with respect to submonoid generated by element $f = \prod_I x_I$

• Obtain blueprint structure on $\overline{M}_{0,n}$

$$\mathcal{O}_{\mathbb{F}_1}(\overline{M}_{0,n}) = \mathcal{A}//\mathcal{R}$$

with blueprint relations $\mathcal{R} = \mathcal{S}_f^{-1}\mathcal{R}' \cap \mathcal{A}$

Note: the *blueprint* notion of $\mathbb{F}_1$-structure is weaker than notions based on torification (no constraints on motive and zeta functions)

What about higher genus? Moduli spaces $\overline{M}_{g,n}$

- $\overline{M}_{g,n}$ *stacks* rather than schemes
- Morphisms: inclusion of boundary strata

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}$$

forgetting markings

$$\overline{M}_{g,n} \to \overline{M}_{g,n-1},$$

gluing two marked points together

$$\overline{M}_{g,n+2} \to \overline{M}_{g+1,n}$$

- Stable curves of genus zero with marked points have constructible torifications, but not higher genus
- Can't have $\mathbb{F}_1$-constructible set structure on $\overline{M}_{g,n}$ (motive not Tate in general, so can't have decomposition of Grothendieck class into tori)

Genus zero locus $\overline{M}^0_{g,n}$

- Inside higher genus moduli space there is a "genus zero part"
- $\overline{M}^0_{g,n}$ is closure of locus of irreducible g-nodal curves in $\overline{M}_{g,n}$
- These curves have normalization given by a smooth rational curve with $2g + n$ marked points
- subgroup $G \subset S_{2g}$ of permutations of these $2g$ additional marked points that commute with the product $(12)(34)\cdots(2g-1\ 2g)$ of g transpositions
- normalization of $\overline{M}^0_{g,n}$ is identified with quotient $\overline{M}_{0,2g+n}/G$.

Blueprint structure for $\overline{M}_{0,2g+n}/G$

- group G acts on a blueprint $\mathcal{A}//\mathcal{R}$ by automorphisms if automorphisms of monoid $\mathcal{A}$ preserving blueprint relations $\mathcal{R}$
- action of G on $\overline{M}_{0,2g+n}$ automorphisms of blueprint $\mathcal{O}_{\mathbb{F}_1}(\overline{M}_{0,2g+n})$
- Noncommutative geometry point of view: replace quotient with crossed product
- Monoid crossed product $\mathcal{A} \rtimes G$ with $(a,g)(a',g') = (ag(a'), gg')$
- Semiring crossed product $\mathbb{N}[\mathcal{A}] \rtimes G$ finite sums $\sum(a_i, g_i)$ with $a_i \in \mathcal{A}$ and $g_i \in G$ with $(a_i, g_i)(a_j, g_j) = (a_i g_i(a_j), g_i g_j)$
- $\mathcal{R}_G$ elements $((\sum a_i, g), (\sum b_j, g))$
- crossed product blueprint $(\mathcal{A}//\mathcal{R}) \rtimes G$ pair $(\mathcal{A} \rtimes G, \mathcal{R}_G)$
- Blueprint structure of rational locus: $\mathcal{O}_{\mathbb{F}_1}(\overline{M}_{0,2g+n}) \rtimes G$