$\mathbb{F}_1$-geometry of Moduli Spaces

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Main reference:


Moduli spaces $\overline{M}_{0,n}$ together with their operad structure descend to $\mathbb{F}_1$, with descent data given in terms of “constructible sets over the field with one element”
What is the “field with one element”? 

Finite geometries \((q = p^k, p \text{ prime})\)

\[
\#\mathbb{P}^{n-1}(\mathbb{F}_q) = \frac{\#(\mathbb{A}^n(\mathbb{F}_q) \setminus \{0\})}{\#\mathbb{G}_m(\mathbb{F}_q)} = \frac{q^n - 1}{q - 1} = [n]_q
\]

\[
\#\text{Gr}(n,j)(\mathbb{F}_q) = \#\{\mathbb{P}^j(\mathbb{F}_q) \subset \mathbb{P}^n(\mathbb{F}_q)\}
\]

\[
= \frac{[n]_q!}{[j]_q![n-j]_q!} = \binom{n}{j}_q
\]

\[
[n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad [0]_q! = 1
\]

The origin of \(\mathbb{F}_1\)-geometry: Jacques Tits observed if take \(q = 1\)

\(\mathbb{P}^{n-1}(\mathbb{F}_1) := \) finite set of cardinality \(n\)

\(\text{Gr}(n,j)(\mathbb{F}_1) := \) set of subsets of cardinality \(j\)

Is there an algebraic geometry over \(\mathbb{F}_1\)?
Extensions $\mathbb{F}_1^n$ (Kapranov-Smirnov)

Monoid $\{0\} \cup \mu_n$ (n-th roots of unity)

- Vector space over $\mathbb{F}_1^n$: pointed set $(V, v)$ with free action of $\mu_n$ on $V \setminus \{v\}$
- Linear maps: permutations compatible with the action

$$\mathbb{F}_1^n \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[t, t^{-1}]/(t^n - 1)$$

Counting of points: for geometries $X$ over $\mathbb{Z}$, reductions mod $p$

$$N_q(X) = \#X(\mathbb{F}_q), \quad q = p^r$$

Polynomially countable if $N_q(X) = P_X(q)$ polynomial in $q$. Counting of “points over the field with one element and its extensions”

$$P_X(m + 1) = \#X(\mathbb{F}_1^m)$$

Different approaches to $\mathbb{F}_1$-geometry: Soulé, Haran, Dourov, Toen–Vaquie, Connes–Consani, López-Peña–Lorscheid
We follow the approach to $\mathbb{F}_1$-geometry via torifications (López-Peña–Lorscheid)

Idea: *geometerize* the expected behavior of the “counting of points” function

Levels of torified structures:

- Torification of the class in the Grothendieck ring
- Geometric torification
- Affine torification
- Regular torification

Then we’ll also need a *weaker* form of geometric torification: *constructible torifications*
The Grothendieck ring of varieties $K_0(\mathcal{V}_{\mathbb{Z}})$

- generators $[X]$ isomorphism classes

- $[X] = [X \setminus Y] + [Y]$ for $Y \subset X$ closed

- $[X] \cdot [Y] = [X \times Y]$ 

Tate (virtual) motives: $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V}_{\mathbb{Z}})$, with $\mathbb{L} = [\mathbb{A}^1]$ 

Universal Euler characteristics:
- Any additive invariant of varieties: $\chi(X) = \chi(Y)$ if $X \cong Y$

\[
\chi(X) = \chi(Y) + \chi(X \setminus Y), \quad Y \subset X
\]

\[
\chi(X \times Y) = \chi(X)\chi(Y)
\]

values in a commutative ring $\mathcal{R}$ is same thing as a ring homomorphism

\[
\chi : K_0(\mathcal{V}_{\mathbb{Z}}) \to \mathcal{R}
\]
Examples of additive invariants:

- Topological Euler characteristic
- *Counting points over finite fields*
- Gillet–Soulé motivic $\chi_{mot}(X)$:

\[
\chi_{mot} : K_0(V)[L^{-1}] \to K_0(M), \quad \chi_{mot}(X) = [(X, id, 0)]
\]

for $X$ smooth projective; complex $\chi_{mot}(X) = W(X)$

**Note:** counting points over finite fields factors through the Grothendieck ring, geometrize expected behavior of the counting function as a condition on $[X] \in K_0(V_\mathbb{Z})$. 
• Grothendieck class torification

The class \([X] \in K_0(\mathcal{V}_\mathbb{Z})\) in the Grothendieck ring satisfies

\[ [X] = \sum_k a_k T^k \]

with \(T = [\mathbb{G}_m] = \mathbb{L} - 1\), and with coefficients \(a_k \geq 0\).

• If \([X] \in \mathbb{Z}[\mathbb{L}]\) Tate motive, then polynomially countable \(N_q(X) = P_X(q)\), then

\[ \#X(\mathbb{F}_1) = P_X(1) = \lim_{q \to 1} N_q(X) = a_0 \]

points over \(\mathbb{F}_1\) and points over \(\mathbb{F}_1^m\):

\[ \#X(\mathbb{F}_1^m) = P_X(m + 1) = \sum_k a_k m^k \]

where \(N_q(T) = N_q(\mathbb{A}^1 \setminus \{pt\}) = q - 1\)
Example: $\mathbb{F}_1^m$-points of $\mathbb{P}^1$

- Class $[\mathbb{P}^1] = 1 + L = 2 + T$ and counting $N_q(\mathbb{P}^1) = 1 + q$
- $\mathbb{F}_1$-points: $\#\mathbb{P}^1(\mathbb{F}_1) = 2$, say $\mathbb{P}^1 = \mathbb{G}_m \cup \{0, \infty\}$
- $\mathbb{F}_1^m$-points: $\#\mathbb{P}^1(\mathbb{F}_1^m) = m + 2$, given by $\{0, \infty\}$ and $m$-th roots of unity in $\mathbb{G}_m$

This suggest a more geometric notion of torification, as a decomposition of the variety, not only of the Grothendieck class.
• Geometric torification (López-Peña–Lorscheid)

Morphism of schemes $e_X : T \to X$ from (finite) disjoint union of tori $T = \bigsqcup_{j \in I} T_j$, $T_j = \mathbb{G}^d_{m}$, with restriction of $e_X$ to each torus an immersion inducing bijection of $k$–points, $e_X(k) : T(k) \to X(k)$, for every field $k$

• Affine torification

$\exists$ affine covering $\{U_\alpha\}$ of $X$ compatible with $e_X$: $\forall U_\alpha$, $\exists$ subfamily $\{T_j | j \in I_\alpha\}$ of torification such that restriction $e_X|_{\bigcup_{j \in I_\alpha} T_j}$ torification of $U_\alpha$

• Regular torification

Closure of tori $T_j$ is union of other tori of the torification

Examples: $\mathbb{P}^n = \mathbb{A}^n \cup \cdots \cup \mathbb{A}^1 \cup \mathbb{A}^0$ affinely torified; Grassmanians $\text{Gr}(n,j)$ torified (cell decomposition), but not affine; smooth toric varieties regular torification (torus orbits)
Morphisms in $\mathbb{F}_1$-geometry

- **torified morphism:**
  \[ \Phi : (X, e_X : T_X \to X) \to (Y, e_Y : T_Y \to Y) \]
a triple \( \Phi = (\phi, \psi, \{\phi_i\}) \) with \( \phi : X \to Y \) morphism of $\mathbb{Z}$-varieties, \( \psi : I_X \to I_Y \) map of indexing sets and \( \phi_j : T_{X,j} \to T_{Y,\psi(j)} \) morphism of algebraic groups, \( \phi \circ e_X|_{T_{X,j}} = e_Y|_{T_{Y,\psi(j)} \circ \phi_j} \)

- **affinely torified morphism:**
  for every \( j \) the image of \( U_j \) under \( \Phi \) is an affine subscheme of \( Y \)

- **Remark:** a torified morphism of affinely torified varieties is an affinely torified morphism (Lorscheid)

**Problem:** When two torifications determine the same $\mathbb{F}_1$-structure? Need a notion of equivalence (isomorphism) of torifications
Torifications giving same $\mathbb{F}_1$-structure on a $\mathbb{Z}$-variety $X$

- **Strong equivalence**: identity morphism is torified.
- **Ordinary Equivalence**: $\exists$ isomorphism of $X$ that is torified.
- **Weak equivalence**: $X$ has a decompositions into disjoint unions $X = \bigcup_j X_j$ and $X = \bigcup_j X'_j$, compatible with torifications, and $\exists$ isomorphisms $\phi_i : X_i \to X'_i$ that are torified.

Which equivalence relation determines what “changes of coordinates” are allowed within $\mathbb{F}_1$-geometry

This also determines what morphisms of $\mathbb{F}_1$-varieties
Morphisms of torified \( \mathbb{Z} \)-varieties

- **Strong \( \mathbb{F}_1 \)-morphisms** (strongly torified): torified morphisms
- **Ordinary \( \mathbb{F}_1 \)-morphisms** (ordinarily torified): arbitrary compositions of torified morphisms and isomorphisms
- **Weak \( \mathbb{F}_1 \)-morphisms** (weakly torified): arbitrary compositions of torified morphisms and weak equivalences

Obtain in this way three different categories \( \mathcal{G} \mathcal{T}^s \subset \mathcal{G} \mathcal{T}^o \subset \mathcal{G} \mathcal{T}^w \)

- Objects are pairs \((X_\mathbb{Z}, \mathcal{T})\) variety over \( \mathbb{Z} \) and torification
- Morphisms are strong, ordinary, or weak morphisms
Moduli Space $\overline{M}_{0,n}$

- moduli space of stable genus zero curves with $n$ marked points
- Open strata: $M_{0,n}$ ($n \geq 4$) complement of diagonals in product $n - 3$ copies of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$
- Compactification by boundary strata $\prod_k M_{0,n_k+1}$ with $\sum_k n_k = n$
Generalization $T_{d,n}$ (Chen–Gibney–Krashen)

- moduli space of $n$–pointed stable rooted trees of $d$–dimensional projective spaces ($T_{1,n} = \overline{M}_{0,n+1}$)
- oriented rooted trees $\tau$, to each vertex $v \in V_\tau$ a $X_v \simeq \mathbb{P}^d$, to unique outgoing tail at $v$ a choice of hyperplane $H_v \subset X_v$, to each incoming tail $f$ at $v$ a point $p_{v,f}$ in $X_v$ with $p_{v,f} \neq p_{v,f'}$ for $f \neq f'$ and with $p_{v,f} \notin H_v$

- Open stratum: $\text{TH}_{d,n}$ of $T_{d,n}$ is configuration space of $n$ distinct points in $\mathbb{A}^d$ up to translation and homothety, complement of diagonals

$$\text{TH}_{d,n} \simeq (\mathbb{A}^d \setminus \{0, 1\})^{n-2} \setminus \Delta$$

- Compactification: boundary strata $\prod_i \text{TH}_{d,n_i}$ with $\sum_i n_i = n$
- Description as iterated blowup (related to Fulton-MacPherson compactification of configuration spaces)
Moduli Space $\overline{M}_{0,n}$: torification of Grothendieck class

$$[M_{0,n}] = \sum_{k=0}^{n-2} s(n - 2, k) \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) \mathbb{T}^j$$

with $s(m, k) =$ Stirling numbers of the first kind

$$(-1)^{m-k} s(m, k) = \# \{ \sigma \in S_m : \sigma = k\text{-cycles} \}$$

This follows from class of open strata

$$[M_{0,n}] = (\mathbb{T} - 1)(\mathbb{T} - 2) \cdots (\mathbb{T} - n + 2) = \binom{\mathbb{T} - 1}{n - 3} (n - 3)!$$

$$= (-1)^n (1 - \mathbb{T})_n^{-2}, \text{ with } (x)_m = \Gamma(x + m)/\Gamma(x) = \text{Pochhammer symbol with } (x)_m = \sum_{k=0}^{m} (-1)^{m-k} s(m, k) x^k$$

Note: some coeffs of $[M_{0,n}]$ negative, so not torified, but when adding strata together torified
Generating series in $K_0(\mathcal{V}_Z)_\mathbb{Q}[[t]]$

$$\varphi(t) = t + \sum_{n=2}^{\infty} [M_{0,n}] \frac{t^n}{n!}$$

unique solution in $t + t^2 K_0(\mathcal{V}_Z)_\mathbb{Q}[[t]]$ of

$$(1 + \mathbb{L} t - \mathbb{L} \varphi(t)) \varphi'(t) = 1 + \varphi(t)$$

Note: analogous to Manin’s result on Poincaré polynomial, since Hodge-Tate ($h^p,q(X_{\mathbb{C}}) = 0$ for $p \neq q$) hence if $[X] = \sum_k b_k \mathbb{L}^k$ Poincaré polynomial is $P_X(q) = \sum_k b_k q^{2k}$

$\mathbb{F}_1$-points: $p_{n,m} = M_{0,n}(\mathbb{F}_1^m)$

$$\varphi_m(t) = \sum_{n \geq 1} p_{n,m} \frac{t^n}{n!}$$

solution of

$$(1 + (m+1) t - (m+1) \varphi_m(t)) \varphi'_m(t) = 1 + \varphi_m(t)$$
Moduli Spaces $T_{d,n}$: torification of Grothendieck class

- Generating function

$$
\psi(t) = \sum_{n \geq 1} [T_{d,n}] \frac{t^n}{n!}
$$

unique solution in $t + t^2 K_0(\mathcal{V}_\mathbb{Z})_\mathbb{Q}[[t]]$ of

$$(1 + \mathbb{L}^d t - \mathbb{L} [\mathbb{P}^{d-1}] \psi(t)) \psi'(t) = 1 + \psi(t)$$

with $[\mathbb{P}^{d-1}] = \frac{\mathbb{L}^{d-1}}{\mathbb{L} - 1}$

- Classes $[T_{d,n}] \in K_0(\mathcal{V}_\mathbb{Z})$ have decomposition into $\mathbb{T}^k$ with $a_k \geq 0$: use repeated blowup description and blowup formula

$$
[B_{\mathcal{Y}}(X)] = [X] + [Y]([\mathbb{P}^{\text{codim}_{X}(Y)-1}] - 1)
$$

Get recursive relation

$$
[T_{d,n+1}] = ([\mathbb{P}^d] + n\mathbb{L} [\mathbb{P}^{d-2}])[T_{d,n}] + \mathbb{L} [\mathbb{P}^d] \sum_{i+j=n+1, 2 \leq i \leq n-1} \binom{n}{i} [T_{d,i}] [T_{d,j}]
$$
$\mathbb{F}_1^m$-points of $T_{d,n}$

$p_{n,m} = N_{T_{d,n}}(m+1) = T_{d,n}(\mathbb{F}_1^m)$ formally replacing $\mathbb{L}$ with $m+1$ in Grothendieck class: generating function

$$\eta_m(t) = \sum_{n \geq 1} \frac{p_{n,m}}{n!} t^n.$$  

solution of differential equation

$$(1 + (m+1)^d t - (m+1)\kappa_d(m+1)\eta_m)\eta'_m = 1 + \eta_m,$$

with $\kappa_d(q^2) = \frac{q^{2d} - 1}{q^2 - 1}$

**Question then:** from Grothendieck classes to more geometric notion of torification?
Complemented subvarieties in $F_1$-geometry

- Example of $\mathbb{P}^1 = \mathbb{G}_m \cup \{0, \infty\}$: points $\{0, \infty\}$ have complement that is still torified, but not if removing additional points.

- $X$ over $\mathbb{Z}$ with geometric torification $\mathcal{T}$. A subvariety $Y \subset X$ is (strongly, ordinarily, weakly) complemented if both $Y$ and $X \setminus Y$ have geometric torifications and inclusions $Y \hookrightarrow X$ and $X \setminus Y \hookrightarrow X$ are (strongly, ordinarily, weakly) morphisms.

- Usual in algebraic geometry: complement of a subvariety in a variety not always a variety, but a constructible set.

- A notion of constructible torifications: more general complements of torifications inside other torifications with positivity of Grothendieck classes.

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$F_1$-geometry of Moduli Spaces
Constructible sets over $\mathbb{F}_1$

- $C_{\mathbb{F}_1} = \text{class of constructible sets over } \mathbb{Z} \text{ that can be obtained, starting from } \mathbb{G}_m$, through of products, disjoint unions, and complements.

- $X = \text{constructible set over } \mathbb{Z}$. Constructible torification of $X$ is morphism of constructible sets $e_X : C \to X$, for some $C \in C_{\mathbb{F}_1}$, with restriction of $e_X$ to each component of $C$ an immersion inducing a bijections of $k$–points, for every field $k$.

- $\mathbb{F}_1$-constructible set = constructible set over $\mathbb{Z}$ with a constructible torification, with Grothendieck class $[X] = \sum_k a_k \mathbb{T}^k$ with $a_k \geq 0$

Note: Grothendieck class condition is needed in definition (does not follow, unlike for geometric torifications)
Categories of $\mathbb{F}_1$-constructible sets $\mathcal{CT}^s \subset \mathcal{CT}^o \subset \mathcal{CT}^w$

- Objects: pairs $(X_Z, \mathcal{C})$, with $X_Z$ a constructible set over $\mathbb{Z}$ and $\mathcal{C} = \{C_i\}$ is a constructible torification of $X_Z$

- morphisms strong, ordinary, or weak morphisms of constructibly torified spaces

Blowups: $X_Z$ variety with a constructible torification, $Y \subset X$ be a closed subvariety with geometric torification and $X \setminus Y$ with constructible torification (inclusions are strong, ordinary, weak morphisms). Then blowup $\text{Bl}_Y(X)$ has constructible torification so that $\pi : \text{Bl}_Y(X) \to X$ (strong, ordinary, weak) morphism of constructibly torified spaces.

If $Y \subset X$ has complemented geometric torification then $\text{Bl}_Y(X)$ has geometric torification.
Constructible torification of $\overline{M}_{0,n}$

- Choice of a constructible torification of $\mathbb{P}^1$ minus three points
- $\mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{G}_m$ has geometric torification, $\mathbb{P}^1 \setminus \{0, 1, \infty\} = \mathbb{G}_m \setminus \{1\}$ has constructible torification
- $M_{0,n}$ ($n \geq 4$) complement of diagonals in product $n - 3$ copies of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$: removing diagonals taking complements of sets in $C_{\mathbb{F}_1}$ inside others: still in $C_{\mathbb{F}_1}$, constructible torification
- *not* an $\mathbb{F}_1$-constructible set structure on $M_{0,n}$ (no positivity of Grothendieck class)
- *but* when combining strata $[\overline{M}_{0,n}]$ has positivity

$\Rightarrow$ moduli spaces $\overline{M}_{0,n}$ are $\mathbb{F}_1$-constructible sets
Constructible torification of $T_{d,n}$

- Start with choice of constructible torification of $\mathbb{A}^d$ minus two points
- Open stratum $TH_{d,n}$ of $T_{d,n}$ is complement of diagonals

$$TH_{d,n} \cong (\mathbb{A}^d \setminus \{0, 1\})^{n-2} \setminus \Delta$$

with $0 = (0, \ldots, 0)$ and $1 = (1, \ldots, 1)$

- constructible torification on $\mathbb{A}^d \setminus \{0, 1\}$ determines constructible torifications on products, diagonals, and complements
- positivity of Grothendieck class holds after assembling together strata to get full $T_{d,n}$

$\Rightarrow$ moduli spaces $T_{d,n}$ are $\mathbb{F}_1$-constructible sets
Operad structure of $\overline{M}_{0,n}$

- Operad $\mathcal{M}(n) = \overline{M}_{0,n+1}$ with compositions

$$\mathcal{M}(n) \times \mathcal{M}(m_1) \times \cdots \times \mathcal{M}(m_n) \to \mathcal{M}(m_1 + \cdots + m_n)$$

- Composition maps are *strong* morphisms of constructibly torified spaces

- Symmetric group $S_n$ acts on $\overline{M}_{0,n}$ permuting marked points

- Action of $S_n$ *ordinary* morphisms of $\mathbb{F}_1$-constructible sets
Operad structure of $T_{d,n}$

- composition (inclusion of boundary strata)

$$T_{d,k} \times T_{d,n_1} \times \cdots \times T_{d,n_k} \rightarrow T_{d,n_1+\cdots+n_k}$$

- compositions are \emph{strong} morphisms of constructible torifications

- Isomorphisms $T_{d,S} \xrightarrow{\sim} T_{d,S'}$ for $S' \xrightarrow{\sim} S$ with $S$ set of marked points, $\#S = n$. These are \emph{ordinary} morphisms of constructible torifications

- Forgetful morphisms: $T_{d,S} \rightarrow T_{d,S'}$ for $S' \subset S$ with $\#S' \geq 2$. These are \emph{strong} morphisms of $\mathbb{F}_1$-constructible sets
Blueprint approach to $\mathbb{F}_1$-geometry (Lorscheid)

Blueprint: $\mathcal{A} \rightharpoonup \mathcal{R}$

- a commutative multiplicative monoid $\mathcal{A}$
- associated semiring $\mathbb{N}[\mathcal{A}]$
- set of relations

$$\mathcal{R} \subset \mathbb{N}[\mathcal{A}] \times \mathbb{N}[\mathcal{A}]$$

written as $\sum a_i \equiv \sum b_j$, for $(\sum a_i, \sum b_j) \in \mathcal{R}$.

**Example:** Plücker embedding of Grassmannian $G(2, n) \hookrightarrow \mathbb{P}^{\binom{n}{2} - 1}$ gives blueprint structure to $G(2, n)$ with $\mathcal{R}$ generated by Plücker relations

$$x_{ij}x_{kl} + x_{il}x_{jk} = x_{ik}x_{jl}, \quad \text{for} \quad 1 \leq i < j < k < l \leq n$$

**Idea:** Use explicit equations for $\overline{M}_{0,n}$ obtained by an embedding into a toric variety to give blueprint structure to $\overline{M}_{0,n}$
Blueprint structure of $\overline{M}_{0,n}$

- Can embed $\overline{M}_{0,n} \hookrightarrow X_\Delta$ a toric variety, so that intersection with torus is $M_{0,n}$ (Gibney–Maclagan and Tevelev)
- Can obtain explicit equations for $\overline{M}_{0,n}$ in the Cox ring of the toric variety $X_\Delta$ (Gibney–Maclagan and Keel–Tevelev)
- monoid
  \[ A = F_1[x_I : I \in \mathcal{I}] := \left\{ \prod x_I^{n_I} \right\}_{n_I \geq 0}, \]
  with $\mathcal{I} = \{ I \subset \{1, \ldots, n\}, 1 \in I, \#I \geq 2, \#I^c \geq 2 \}$ where $\mathbb{Q}[x_I, : I \in \mathcal{I}]$ Cox ring of $X_\Delta$
- relations
  \[ \mathcal{R}' = \left\{ \prod_{ij \in I, kl \notin I} x_I + \prod_{il \in I, jk \notin I} x_I \equiv \prod_{ik \in I, jl \notin I} x_I : 1 \leq i < j < k < l \leq n \right\}, \]
• $S_f^{-1}R'$ localization with respect to submonoid generated by element $f = \prod I \chi_I$

• Obtain blueprint structure on $\bar{M}_{0,n}$

$$\mathcal{O}_{\mathbb{F}_1}(\bar{M}_{0,n}) = \mathcal{A}/\mathcal{R}$$

with blueprint relations $\mathcal{R} = S_f^{-1}R' \cap \mathcal{A}$

Note: the blueprint notion of $\mathbb{F}_1$-structure is weaker than notions based on torification (no constraints on motive and zeta functions)
What about higher genus? Moduli spaces $\overline{M}_{g,n}$

- $\overline{M}_{g,n}$ stacks rather than schemes
- Morphisms: inclusion of boundary strata
  $$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}$$

forgetting markings
  $$\overline{M}_{g,n} \to \overline{M}_{g,n-1},$$

gluing two marked points together
  $$\overline{M}_{g,n+2} \to \overline{M}_{g+1,n}$$

- Stable curves of genus zero with marked points have constructible torifications, but not higher genus
- Can’t have $\mathbb{F}_1$-constructible set structure on $\overline{M}_{g,n}$ (motive not Tate in general, so can’t have decomposition of Grothendieck class into tori)
Genus zero locus $\overline{M}_{0,g,n}$

- Inside higher genus moduli space there is a “genus zero part”
- $\overline{M}_{0,g,n}$ is closure of locus of irreducible $g$-nodal curves in $\overline{M}_{g,n}$
- These curves have normalization given by a smooth rational curve with $2g + n$ marked points
- Subgroup $G \subset S_{2g}$ of permutations of these $2g$ additional marked points that commute with the product $(12)(34) \cdots (2g − 1 \ 2g)$ of $g$ transpositions
- Normalization of $\overline{M}_{0,g,n}$ is identified with quotient $\overline{M}_{0,2g+n}/G$. 
Blueprint structure for $\overline{M}_{0, 2g+n}/G$

- group $G$ acts on a blueprint $A//\mathcal{R}$ by automorphisms if automorphisms of monoid $A$ preserving blueprint relations $\mathcal{R}$
- action of $G$ on $\overline{M}_{0, 2g+n}$ automorphisms of blueprint $\mathcal{O}_{\mathbb{F}_1}(\overline{M}_{0, 2g+n})$
- Noncommutative geometry point of view: replace quotient with crossed product
- Monoid crossed product $A \rtimes G$ with $(a, g)(a', g') = (ag(a'), gg')$
- Semiring crossed product $\mathbb{N}[A] \rtimes G$ finite sums $\sum (a_i, g_i)$ with $a_i \in A$ and $g_i \in G$ with $(a_i, g_i)(a_j, g_j) = (a_ig_i(a_j), g_ig_j)$
- $\mathcal{R}_G$ elements $((\sum a_i, g), (\sum b_j, g))$
- crossed product blueprint $(A//\mathcal{R}) \rtimes G$ pair $(A \rtimes G, \mathcal{R}_G)$
- Blueprint structure of rational locus: $\mathcal{O}_{\mathbb{F}_1}(\overline{M}_{0, 2g+n}) \rtimes G$