FROM EXCEPTIONAL COLLECTIONS TO MOTIVIC DECOMPOSITIONS VIA NONCOMMUTATIVE MOTIVES

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ABSTRACT. Making use of noncommutative motives we relate exceptional collections (and more generally semi-orthogonal decompositions) to motivic decompositions. On one hand we prove that the Chow motive $M(\mathcal{X})_\mathbb{Q}$ of every smooth and proper Deligne-Mumford stack $\mathcal{X}$, whose bounded derived category $\mathcal{D}^b(\mathcal{X})$ of coherent schemes admits a full exceptional collection, decomposes into a direct sum of tensor powers of the Lefschetz motive. Examples include projective spaces, quadrics, toric varieties, homogeneous spaces, Fano threefolds, and moduli spaces. On the other hand we prove that if $M(\mathcal{X})_\mathbb{Q}$ decomposes into a direct direct sum of tensor powers of the Lefschetz motive and moreover $\mathcal{D}^b(\mathcal{X})$ admits a semi-orthogonal decomposition, then the noncommutative motive of each one of the pieces of the semi-orthogonal decomposition is a direct sum of $\otimes$-units. As an application we obtain a simplification of Dubrovin's conjecture.

Dedicated to Yuri Manin, on the occasion of his 75^{th} *birthday.*

INTRODUCTION

Let $\mathcal{X}$ be a smooth and proper Deligne-Mumford (=DM) stack [11]. In order to study it we can proceed in two distinct directions. On one direction we can associate to $\mathcal{X}$ its different Weil cohomologies $H^*(\mathcal{X})$ (Betti, de Rham, l-adic, and others; see [4, §8]) or more intrinsically its Chow motive $M(\mathcal{X})_\mathbb{Q}$ (with rational coefficients); see §1. On another direction we can associate to $\mathcal{X}$ its bounded derived category $\mathcal{D}^b(\mathcal{X}) := \mathcal{D}^b(\mathrm{Coh}(\mathcal{X}))$ of coherent sheaves; see [36].

In several cases of interest (projective spaces, quadrics, toric varieties, homogeneous spaces, Fano threefolds, moduli spaces, and others; see §2) the derived category $\mathcal{D}^b(\mathcal{X})$ admits a "weak decomposition" into simple pieces. The precise formulation of this notion goes under the name of a *full exceptional collection*; consult [19, §1.4] for details. This motivates the following general questions:

Question A: *What can it be said about the Chow motive $M(\mathcal{X})_\mathbb{Q}$ of a smooth proper DM stack $\mathcal{X}$ whose bounded derived category $\mathcal{D}^b(\mathcal{X})$ admits a full exceptional collection ? Does $M(\mathcal{X})_\mathbb{Q}$ also decomposes into simple pieces ?*

Question B: *Conversely, what can it be said about the bounded derived category $\mathcal{D}^b(\mathcal{X})$ of a smooth proper DM stack $\mathcal{X}$ whose Chow motive $M(\mathcal{X})_\mathbb{Q}$ decomposes into simple pieces ?*

In this article, making use of the theory of noncommutative motives, we provide a precise and complete answer to Question A and a partial answer to Question B.

Date: April 2, 2012.
2000 *Mathematics Subject Classification.* 13D09, 14A22, 14C15.
Key words and phrases. Noncommutative algebraic geometry, Deligne-Mumford stacks, exceptional collections, semi-orthogonal decompositions, Chow motives, noncommutative motives.

1. Statement of results

Throughout the article we will work over a perfect base field k. Let us denote by $\mathcal{DM}(k)$ the category of smooth proper DM stacks (over $\mathrm{Spec}(k)$) and by $\mathcal{P}(k)$ its full subcategory of smooth projective varieties. Recall from [4, §8] [1, §4.1.3] the construction of the (contravariant) functors

$$h(-)_{\mathbb{Q}} : \mathcal{DM}(k)^{\mathsf{op}} \longrightarrow \mathrm{DMChow}(k)_{\mathbb{Q}} \qquad M(-)_{\mathbb{Q}} : \mathcal{P}(k)^{\mathsf{op}} \longrightarrow \mathrm{Chow}(k)_{\mathbb{Q}}$$

with values in the categories of Deligne-Mumford-Chow motives and Chow motives, respectively. There is a natural commutative diagram

$$\begin{tikzcd} \mathcal{DM}(k)^{\mathsf{op}} \arrow[d, "h(-)_{\mathbb{Q}}"'] & \mathcal{P}(k)^{\mathsf{op}} \arrow[l, hook'] \arrow[d, "M(-)_{\mathbb{Q}}"] \\ \mathrm{DMChow}(k)_{\mathbb{Q}} & \mathrm{Chow}(k)_{\mathbb{Q}} \arrow[l, "\sim"] \end{tikzcd} \tag{1.1}$$

and as shown in [40, Thm. 2.1] the lower horizontal functor is a $\mathbb{Q}$-linear equivalence. By inverting it we obtain then a well-defined functor

$$\mathcal{DM}(k)^{\mathsf{op}} \longrightarrow \mathrm{Chow}(k)_{\mathbb{Q}} \qquad \mathcal{X} \mapsto M(\mathcal{X})_{\mathbb{Q}} . \tag{1.2}$$

Our first main result, which provides an answer to Question A, is the following:

Theorem 1.3. *Let $\mathcal{X} \in \mathcal{DM}(k)$. Assume that $\mathcal{D}^b(\mathcal{X})$ admits a full exceptional collection $(E_1, \ldots, E_m)$ of length m. Then, there is a choice of integers (up to permutation) $l_1, \ldots, l_m \in \{0, \ldots, \dim(\mathcal{X})\}$ giving rise to a canonical isomorphism*

$$M(\mathcal{X})_{\mathbb{Q}} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m} , \tag{1.4}$$

where $\mathbf{L} \in \mathrm{Chow}(k)_{\mathbb{Q}}$ denotes the Lefschetz motive and $\mathbf{L}^{\otimes l}, l \geq 0$, its tensor powers (with the convention $\mathbf{L}^{\otimes 0} = M(\mathrm{Spec}(k))_{\mathbb{Q}}$); see [1, §4.1.5].

Intuitively speaking, Theorem 1.3 shows that the existence of a full exceptional collection on $\mathcal{D}^b(\mathcal{X})$ "quasi-determines" the Chow motive $M(\mathcal{X})_{\mathbb{Q}}$. The indeterminacy is only on the number of tensor powers of the Lefschetz motive. Note that this indeterminacy *cannot* be refined. For instance, the bounded derived categories of $\mathrm{Spec}(k) \amalg \mathrm{Spec}(k)$ and $\mathbb{P}^1$ admit full exceptional collections of length 2 but the corresponding Chow motives are distinct

$$M(\mathrm{Spec}(k))_{\mathbb{Q}}^{\oplus 2} = M(\mathrm{Spec}(k) \amalg \mathrm{Spec}(k))_{\mathbb{Q}} \neq M(\mathbb{P}^1)_{\mathbb{Q}} = M(\mathrm{Spec}(k))_{\mathbb{Q}} \oplus \mathbf{L} .$$

Hence, Theorem 1.3 furnish us the maximum amount of data, concerning the Chow motive, that can be extracted from the existence of a full exceptional collection.

Corollary 1.5. *Let $\mathcal{X}$ be a smooth and proper DM stack satisfying the conditions of Theorem 1.3. Then, for every Weil cohomology $H^*(-) : \mathcal{DM}(k)^{\mathsf{op}} \to \mathrm{VecGr}_K$ (with K a field of characteristic zero) we have $H^n(\mathcal{X}) = 0$ for n odd and $\dim H^n(\mathcal{X}) \leq m$ for n even.*

Corollary 1.5 can be used in order to prove negative results concerning the existence of a full exceptional collection. For instance, if there exists an odd number n such that $H^n(\mathcal{X}) \neq 0$, then the category $\mathcal{D}^b(\mathcal{X})$ cannot admit a full exception collection. This is illustrated in Corollary 2.3. Moreover, Corollary 1.5 implies that a possible full exceptional collection on $\mathcal{D}^b(\mathcal{X})$ has length always greater or equal to the maximum of the dimensions of the K-vector spaces $H^n(\mathcal{X})$, with n even.

Remark 1.6. After the circulation of this manuscript, Bernardara kindly informed us that an alternative proof of Theorem 1.3 in the particular case where $\mathcal{X}$ is a smooth projective *complex* variety can be found in [5]; see also [6]. Moreover, Kuznetsov kindly explained us an alternative proof of Theorem 1.3 following some of Orlov's ideas. Our approach is rather different and can also be viewed as a further step towards the complete understanding of the relationship between commutative and noncommutative motives.

Recall from [38][39, §4] the construction of the *universal additive invariant*

$$\mathcal{U}_a(-) : \mathsf{dgcat}(k) \longrightarrow \mathsf{Hmo}_0(k)\,.$$

Roughly speaking, $\mathcal{U}_a(-)$ is the *universal* functor from dg categories (see §3.1) towards an additive category which inverts derived Morita equivalences and satisfies additivity. Examples of additive invariants include algebraic K-theory, Hochschild homology, cyclic homology (and all its variants), and even topological cyclic homology. Because of this universal property, which is reminiscent from the theory of motives, $\mathsf{Hmo}_0(k)$ is called the category of *noncommutative motives*. In order to simplify the exposition let us denote by $\mathbf{1}$ the noncommutative motive $\mathcal{U}_a(k)$. Our second main result, which provides a partial answer to Question B, is the following: (recall from [19, §1.4] the notion of semi-orthogonal decomposition[1])

Theorem 1.7. *Let $\mathcal{X}$ be a DM stack such that $M(\mathcal{X})_{\mathbb{Q}} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}$ (for some choice of integers $l_1, \ldots, l_m \in \{0, \ldots, \dim(\mathcal{X})\}$). Assume that $\mathcal{D}^b(\mathcal{X})$ admits a semi-orthogonal decomposition $\langle \mathcal{C}^1, \ldots, \mathcal{C}^j, \ldots, \mathcal{C}^p \rangle$ of length p and let $\mathcal{C}^j_{\mathsf{dg}}$ be the natural dg enhancement of $\mathcal{C}_j$. Then, we have canonical isomorphisms*

$$\mathcal{U}_a(\mathcal{C}^j_{\mathsf{dg}})_{\mathbb{Q}} \simeq \underbrace{\mathbf{1}_{\mathbb{Q}} \oplus \cdots \oplus \mathbf{1}_{\mathbb{Q}}}_{n_j} \qquad 1 \leq j \leq p\,. \tag{1.8}$$

Moreover, $\sum_{j=1}^{p} n_j = m$ and for every additive invariant E (with values in a $\mathbb{Q}$-linear category) the equality $E(\mathcal{C}^j_{\mathsf{dg}}) = E(k)^{\oplus n_j}$ holds.

Note that Theorem 1.7 imposes strong conditions on the shape of a possible semi-orthogonal decomposition of $\mathcal{D}^b(\mathcal{X})$, whenever $M(\mathcal{X})_{\mathbb{Q}}$ decomposes into a direct sum of tensor powers of the Lefschetz motive. Intuitively speaking, if a semi-orthogonal exists then each one of its pieces is rather simple from the noncommutative viewpoint.

The proofs of Theorems 1.3 and 1.7 rely on the "bridge" between Chow motives and noncommutative motives established in [37]. In what concerns Theorem 1.3, we prove first that the dg enhancement $\mathcal{D}^b_{\mathsf{dg}}(\mathcal{X})$ of $\mathcal{D}^b(\mathcal{X})$ becomes isomorphic in the category of noncommutative motives to the direct sum of m copies of the $\otimes$-unit. Then, making use of the mentioned "bridge" we show that all the possible lifts of this direct sum are the Chow motives of shape $\simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}$ with $l_1, \ldots, l_m \in \{0, \ldots, \dim(\mathcal{X})\}$. In what concerns Theorem 1.7, the canonical isomorphisms (1.8) follow from the mentioned "bridge" and from the decomposition of the noncommutative motive associated to $\mathcal{D}^b_{\mathsf{dg}}(\mathcal{X})$; consult §4 for details.

[1] A natural generalization of the notion of full exceptional collection.

Dubrovin conjecture. At his ICM address [12], Dubrovin conjectured a striking connection between Gromov-Witten invariants and derived categories of coherent sheaves. The most recent formulation of this conjecture, due to Hertling-Manin-Teleman [18], is the following:

Conjecture: *Let X be a smooth projective complex variety. (i) The quantum cohomology of X is (generically) semi-simple if and only if X is Hodge-Tate (i.e. its Hodge numbers $h^{p,q}(X)$ are zero for $p \neq q$) and the bounded derived category $\mathcal{D}^b(X)$ admits a full exceptional collection. (ii) The Stokes matrix of the structure connection of the quantum cohomology identifies with the Gram matrix of the exceptional collection.*

Thanks to the work of Bayer, Golyshev, Guzzeti, Ueda, and others (see [2, 15, 17, 41]), items (i)-(ii) are nowadays known to be true in the case of projective spaces (and its blow-ups) and Grassmannians, while item (i) is also known to be true for minimal Fano threefolds. Moreover, Hertling-Manin-Teleman proved that the Hodge-Tate property follows from the semi-simplicity of quantum cohomology. Making use of Theorem 1.3 we prove that the Hodge-Tate property follows also from the existence of a full exceptional collection.

Proposition 1.9. *Let X be a smooth projective complex variety. If the bounded derived category $\mathcal{D}^b(X)$ admits a full exceptional collection then X is Hodge-Tate. Moreover, if X is defined over a number field k, then all the complex varieties X_α associated to the different field embeddings $\alpha : k \hookrightarrow \mathbb{C}$ are also Hodge-Tate.*

By Proposition 1.9 we conclude then that the Hodge-Tate property is unnecessary in the above conjecture, and hence can be removed.

2. Examples

In this section we illustrate Theorems 1.3 and 1.7 by discussing several examples.

Projective spaces. In a pioneering work [3], Beilinson constructed a full exceptional collection $(\mathcal{O}(-n), \ldots, \mathcal{O}(0))$ of length $n+1$ on the bounded derived category $\mathcal{D}^b(\mathbb{P}^n)$ of the n^{th} projective space. Hence, by Theorem 1.3 the Chow motive $M(\mathbb{P}^n)_{\mathbb{Q}}$ decomposes into a direct sum of tensor powers of the Lefschetz motive. In this particular case it is well known that

$$M(\mathbb{P}^n)_{\mathbb{Q}} \simeq M(\mathrm{Spec}(k))_{\mathbb{Q}} \oplus \mathbf{L} \oplus \cdots \oplus \mathbf{L}^{\otimes n} .$$

Quadrics. In this family of examples we assume that k is of characteristic $\neq 2$. Let (V, q) be a non-degenerate quadratic form of dimension $n \geq 3$ and $Q_q \subset \mathbb{P}(V)$ the associated (smooth) projective quadric of dimension $d = n-2$. In the case where k is algebraically closed and of characteristic zero, Kapranov [22] constructed the following full exceptional collection on the derived category $\mathcal{D}^b(Q_q)$:

$$\begin{aligned} &(\Sigma(-d), \mathcal{O}(-d+1), \ldots, \mathcal{O}(-1), \mathcal{O}) && \text{if } d \text{ is odd} \\ (\Sigma_+(-d), &\Sigma_-(-d), \mathcal{O}(-d+1), \ldots, \mathcal{O}(-1), \mathcal{O}) && \text{if } d \text{ is even}, \end{aligned}$$

where $\Sigma_\pm$ (and Σ) denote the spinor bundles. By Theorem 1.3 we conclude then that the Chow motive $M(Q_q)_{\mathbb{Q}}$ decomposes into a direct sum of tensor powers of the Lefschetz motive. We would like to mention that this motivic decomposition was also obtained by Rost [34, 35] using groundbreaking computations of the Chow groups of quadrics. As explained in *loc. cit.*, we have

$$M(Q_q)_{\mathbb{Q}} \simeq \begin{cases} M(\mathrm{Spec}(k))_{\mathbb{Q}} \oplus \mathbf{L} \oplus \cdots \oplus \mathbf{L}^{\otimes n} & \text{if } d \text{ is odd} \\ M(\mathrm{Spec}(k))_{\mathbb{Q}} \oplus \mathbf{L} \oplus \cdots \oplus \mathbf{L}^{\otimes n} \oplus \mathbf{L}^{\otimes (d/2)} & \text{if } d \text{ is even}. \end{cases}$$

In the case where k is of characteristic zero but no longer algebraically closed, Kapranov's full exceptional collection was generalized by Kuznetsov [27] into a semi-orthogonal decomposition

$$\langle \mathcal{D}^b(Cl_0(Q_q)), \mathcal{O}(-d+1), \ldots, \mathcal{O}\rangle ,$$

where $Cl_0(Q_q)$ denotes the even part of the Clifford algebra associated to Q_q. As explained in [7, Remark 2.1], the above motivic decomposition of $M(Q_q)_{\mathbb{Q}}$ still holds over an arbitrary field k of characteristic zero and so by Theorem 1.7 one concludes that

$$\mathcal{U}_a(\mathcal{D}^b_{\mathsf{dg}}(Cl_0(Q_q))_{\mathbb{Q}} \simeq \mathcal{U}_a(Cl_0(Q_q))_{\mathbb{Q}} \simeq \begin{cases} \mathbf{1}_{\mathbb{Q}} & \text{if } d \text{ is odd} \\ \mathbf{1}_{\mathbb{Q}} \oplus \mathbf{1}_{\mathbb{Q}} & \text{if } d \text{ is even}. \end{cases}$$

In particular the rationalized Grothendieck group $K_0(Cl_0(Q_q))_{\mathbb{Q}}$ identifies with $\mathbb{Q}$ if d is odd and with $\mathbb{Q} \oplus \mathbb{Q}$ if d is even.

Remark 2.1. ($\mathbb{Z}$-coefficients) It follows from the proof of Lemma 4.1 that

$$\mathcal{U}_a(\mathcal{D}^b_{\mathsf{dg}}(Q_q)) \simeq \mathcal{U}_a(Cl_0(Q_q)) \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}$$

in $\mathsf{Hmo}_0(k)$. On the other hand, Rost [34, 35] proved that the motive $M(Q_q)_{\mathbb{Z}}$ decomposes into a direct sum of tensor powers of the Lefschetz motive plus a well-defined *Rost motive* $R(Q_q)_{\mathbb{Z}}$. Moreover, $R(Q_q)_{\mathbb{Z}}$ decomposes into a direct sum of tensor powers of the Lefschetz motive if and only if Q_q is similar to a *split* quadratic

$$\tau := \begin{cases} \sum_{i=1}^m x_i y_i & \text{if} \quad n = 2m \\ \sum_{i=1}^m x_i y_i + z^2 & \text{if} \quad n = 2m+1. \end{cases}$$

Since the "bridge" between Chow motives and noncommutative motives does not admit a $\mathbb{Z}$-coefficients version, the Rost motive $R(Q_q)_{\mathbb{Z}}$ and the noncommutative motive $\mathcal{U}_a(Cl_0(Q_q))$ cannot be explicitly compared. Nevertheless, it is natural to consider $\mathcal{U}_a(Cl_0(Q_q))$ as the noncommutative analogue of the Rost motive and to ask the following: *is Q_q similar to a split quadratic τ if and only if $\mathcal{U}_a(Cl_0(Q_q))$ decomposes into a direct sum of copies of $\mathbf{1}$?*

Toric varieties. Let X be a projective toric variety with at most quotient singularities and B an invariant $\mathbb{Q}$-divisor whose coefficients belong to the set $\{\frac{r-1}{r}; r \in \mathbb{Z}_{>0}\}$. In these general cases, Kawamata [24] constructed a full exceptional collection on the bounded derived category $\mathcal{D}^b(\mathcal{X})$ of the stack $\mathcal{X}$ associated to the pair (X, B). By Theorem 1.3 we conclude then that the Chow motive $M(\mathcal{X})_{\mathbb{Q}}$ decomposes into a direct sum of tensor powers of the Lefschetz motive.

Homogeneous spaces. In a recent work [28], Kuznetsov and Polishchuk conjectured the following important result:

Conjecture: *Assume that the base field k is algebraically closed and of characteristic zero. Then, for every semisimple algebraic group G and parabolic subgroup $P \subset G$ the bounded derived category $\mathcal{D}^b(G/P)$ admits a full exceptional collection.*

As explained by the authors in [28, page 3], this conjecture is known to be true in several cases. For instance, when G is a simple algebraic group of type A, B, C, D, E_6, F_4 or G_2 and P is a certain maximal parabolic subgroup, a full exceptional collection on $\mathcal{D}^b(G/P)$ has been constructed. The case of an arbitrary maximal parabolic subgroup P of a simply connected simple group G of type B, C or D was also treated by the authors in [28, Thm. 1.2]. By Theorem 1.3 we conclude then that in all these (conjectural) cases the Chow motive $M(G/P)_{\mathbb{Q}}$ decomposes into a direct sum of tensor powers of the Lefschetz motive.

We would like to mention that this motivic decomposition was also obtained by Brosnan, Chernousov-Gille-Merkurjev, Karpenko, Köck, and others; see [7, 9, 23, 26]. Moreover, since we are working with rational coefficients this decomposition holds also over an arbitrary field of characteristic zero. Therefore, by Theorem 1.7, if $\mathcal{D}^b(G/P)$ admits a semi-orthogonal decomposition then the noncommutative motive of each one of its pieces is a direct sum of copies of $\mathbf{1}_{\mathbb{Q}}$.

Remark 2.2. Similarly to the case of quadrics one can ask the following: is the noncommutative motive (with $\mathbb{Z}$-coefficients) of each one of the pieces of a possible semi-orthogonal decomposition a direct sum of copies of $\mathbf{1}$ if and only if $M(G/P)_{\mathbb{Z}}$ decomposes into a direct sum of powers of the Lefschetz motive; consult [13, §3.3] for a detailed discussion of splitting fields and motivic decompositions.

Fano threefolds. In this family of examples we assume that k is algebraically closed and of characteristic zero. Fano threefolds X have been classified by Iskovskih and Mori-Mukai into 105 different deformation classes; see [20, 21, 33]. Making use of Orlov's results, Ciolli [10] constructed for each one of the 59 Fano threefolds X which have vanishing odd cohomology a full exceptional collection on $\mathcal{D}^b(X)$ of length equal to the rank of the even cohomology. By combining these results with Corollary 1.5 we obtain then the following characterization:

Corollary 2.3. *Let X be a Fano threefold. Then, the derived category $\mathcal{D}^b(X)$ admits a full exceptional collection if and only if the odd cohomology of X vanishes.*

By Theorem 1.3 we conclude also that for each one of the 59 Fano threefolds which have vanishing odd cohomology the associated Chow motive decomposes into a direct sum of tensor powers of the Lefschetz motive.

We would like to mention that the motivic decomposition of a Fano threefold X was computed by Gorchinskiy-Guletskii [16]. As explained in *loc. cit.*, we have

$$M(X)_{\mathbb{Q}} = M(\mathrm{Spec}(k))_{\mathbb{Q}} \oplus M^1(X) \oplus \mathbf{L}^{\oplus b} \oplus (M^1(J)\otimes \mathbf{L}) \oplus (\mathbf{L}^{\otimes 2})^{\oplus b} \oplus M^5(X) \oplus \mathbf{L}^{\otimes 3}$$

where $M^1(X)$ and $M^5(X)$ are the Picard and Albanese motives respectively, $b = b_2(X) = b_4(X)$ is the Betti number, and J is a certain abelian variety defined over k, which is isogenous to the intermediate Jacobian $J^2(X)$ if $k = \mathbb{C}$. Since the motives $M^1(X)$, $M^5(X)$ and $M^1(J)\otimes \mathbf{L}$ underlie the odd cohomology of X we obtain then the following result:

Corollary 2.4. *Let X be a Fano threefold. Then, the bounded derived category $\mathcal{D}^b(X)$ admits a full exceptional collection if and only if the motives $M^1(X)$, $M^5(X)$ and $M^1(J)$ are trivial.*

Moduli spaces. In a recent work, Manin and Smirnov [32] constructed a full exceptional collection on the bounded derived category $\mathcal{D}^b(\overline{\mathcal{M}}_{0,n})$ of the moduli space of n-pointed stable curves of genus zero. This was done by an inductive blow-up procedure which combines Keel's presentation of $\overline{\mathcal{M}}_{0,n}$ with Orlov's decomposition theorem. By Theorem 1.3 we conclude then that the Chow motive $M(\overline{\mathcal{M}}_{0,n})_{\mathbb{Q}}$ decomposes into a direct sum of tensor powers of the Lefschetz motive.

We would like to mention that this motivic decomposition was also obtained by Chen-Gibney-Krashen [8], where the authors used evolved combinatorial arguments in order to give an inductive description of the Chow motive $M(\overline{\mathcal{M}}_{0,n})_{\mathbb{Q}}$. As pointed out to us by Manin, this motivic decomposition can also be obtained from Manin's identity principle [31, §9] using the above description of $\overline{\mathcal{M}}_{0,n}$ in terms of inductive blow-ups. The same arguments apply to some of the toric and Fano threefolds.

3. Preliminaries

3.1. **Dg categories.** A *differential graded (=dg) category*, over our base field k, is a category enriched over cochain complexes of k-vector spaces (morphisms sets are complexes) in such a way that composition fulfills the Leibniz rule $d(f\circ g) = d(f)\circ g + (-1)^{\deg(f)} f\circ d(g)$; consult Keller's ICM address [25]. The category of dg categories will be denoted by $\mathsf{dgcat}(k)$. Given a dg category $\mathcal{A}$ we will write $\mathsf{H}^0(\mathcal{A})$ for the associated k-linear category with the same objects as $\mathcal{A}$ and morphisms given by $\mathsf{H}^0(\mathcal{A})(x,y) := \mathsf{H}^0\mathcal{A}(x,y)$, where H^0 denotes the 0^{th}-cohomology. A dg category $\mathcal{A}$ is called *pre-triangulated* if the associated category $\mathsf{H}^0(\mathcal{A})$ is triangulated. Finally, a *Morita equivalence* is a dg functor $\mathcal{A}\to\mathcal{B}$ which induces an equivalence $\mathcal{D}(\mathcal{A}) \stackrel{\sim}{\to} \mathcal{D}(\mathcal{B})$ on the associated derived categories; see [25, §4.6].

3.2. **Orbit categories.** Let $\mathcal{C}$ be an additive symmetric monoidal category and $\mathcal{O}\in\mathcal{C}$ a $\otimes$-invertible object. As explained in [37, §7], one can then consider the *orbit category* $\mathcal{C}/_{-\otimes\mathcal{O}}$. It has the same objects as $\mathcal{C}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(X,Y) := \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y\otimes\mathcal{O}^{\otimes r})\,.$$

The composition law is induced from $\mathcal{C}$. Concretely, given objects X, Y and Z and morphisms

$$\underline{f} = \{f_r\}_{r\in\mathbb{Z}} \in \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y\otimes\mathcal{O}^{\otimes r}) \qquad \underline{g} = \{g_s\}_{s\in\mathbb{Z}} \in \bigoplus_{s\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(Y, Z\otimes\mathcal{O}^{\otimes s})\,,$$

the l^{th}-component of the composition $\underline{g}\circ\underline{f}$ is the finite sum

$$\sum_r (g_{l-r}\otimes\mathcal{O}^{\otimes r})\circ f_r\,.$$

Under these definitions, we obtain an additive category $\mathcal{C}/_{-\otimes\mathcal{O}}$ and a canonical additive projection functor $\pi(-):\mathcal{C}\to\mathcal{C}/_{-\otimes\mathcal{O}}$. Moreover, $\pi(-)$ is endowed with a natural 2-isomorphism $\pi(-)\circ(-\otimes\mathcal{O}) \stackrel{\sim}{\Rightarrow} \pi(-)$ and is 2-universal among all such functors.

4. Proof of Theorem 1.3

Let us denote by $\langle E_j\rangle, 1\leq j\leq m$, the smallest triangulated subcategory of $\mathcal{D}^b(\mathcal{X})$ generated by the object E_j. As explained in [19, Example 1.60] the full exceptional collection $(E_1,\ldots,E_j,\ldots,E_m)$ of length m gives rise to the semi-orthogonal decomposition

$$\mathcal{D}^b(\mathcal{X}) = \big\langle\langle E_1\rangle,\ldots,\langle E_j\rangle,\ldots,\langle E_m\rangle\big\rangle\,,$$

with $\langle E_j \rangle \simeq \mathcal{D}^b(k)$ for $1 \leq j \leq m$. Recall from [30] that the triangulated category $\mathcal{D}^b(\mathcal{X})$ admits a (unique) differential graded (=dg) enhancement $\mathcal{D}^b_{\mathsf{dg}}(\mathcal{X})$. In particular we have an equivalence $\mathsf{H}^0(\mathcal{D}^b_{\mathsf{dg}}(\mathcal{X})) \simeq \mathcal{D}^b(\mathcal{X})$ of triangulated categories. Let us denote by $\langle E_j \rangle_{\mathsf{dg}}$ the dg enhancement of $\langle E_j \rangle$. Note that $\mathsf{H}^0(\langle E_j \rangle_{\mathsf{dg}}) \simeq \langle E_j \rangle$ and that $\langle E_j \rangle_{\mathsf{dg}} \simeq \mathcal{D}^b_{\mathsf{dg}}(k)$. Recall from the introduction the *universal additive invariant*

$$\mathcal{U}_a(-) : \mathsf{dgcat}(k) \longrightarrow \mathsf{Hmo}_0(k)\,.$$

Lemma 4.1. *The inclusions of dg categories $\mathcal{D}^b_{\mathsf{dg}}(k) \simeq \langle E_j \rangle_{\mathsf{dg}} \hookrightarrow \mathcal{D}^b_{\mathsf{dg}}(\mathcal{X}), 1 \leq j \leq m$, give rise to an isomorphism*

$$\bigoplus_{j=1}^{m} \mathcal{U}_a(\mathcal{D}^b_{\mathsf{dg}}(k)) \stackrel{\sim}{\longrightarrow} \mathcal{U}_a(\mathcal{D}^b_{\mathsf{dg}}(\mathcal{X}))\,. \tag{4.2}$$

Proof. For every $1 \leq i \leq m$, let $\langle E_i, \ldots, E_m \rangle$ be the full triangulated subcategory of $\mathcal{D}^b(\mathcal{X})$ generated by the objects $E_i, \ldots, E_m$. Since by hypothesis $(E_1, \ldots, E_m)$ is a full exceptional collection of $\mathcal{D}^b(\mathcal{X})$, we obtain the following semi-orthogonal decomposition

$$\langle E_i, \ldots, E_m \rangle = \big\langle \langle E_i \rangle, \langle E_{i+1}, \ldots, E_m \rangle \big\rangle\,.$$

Now, let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be pre-triangulated dg categories (with $\mathcal{B}$ and $\mathcal{C}$ full dg subcategories of $\mathcal{A}$) inducing a semi-orthogonal decomposition $\mathsf{H}^0(\mathcal{A}) = \big\langle \mathsf{H}^0(\mathcal{B}), \mathsf{H}^0(\mathcal{C}) \big\rangle$. Then, as proved in [38, Thm. 6.3], the inclusions $\mathcal{B} \hookrightarrow \mathcal{A}$ and $\mathcal{C} \hookrightarrow \mathcal{A}$ give rise to an isomorphism $\mathcal{U}_a(\mathcal{B}) \oplus \mathcal{U}_a(\mathcal{C}) \stackrel{\sim}{\to} \mathcal{U}_a(\mathcal{A})$ in $\mathsf{Hmo}_0(k)$. By applying this result to the dg enhancements

$$\mathcal{A} := \langle E_i, \cdots, E_m \rangle_{\mathsf{dg}} \quad \mathcal{B} := \langle E_i \rangle_{\mathsf{dg}} \quad \mathcal{C} := \langle E_{i+1}, \ldots, E_m \rangle_{\mathsf{dg}}$$

we then obtain an isomorphism

$$\mathcal{U}_a(\mathcal{D}^b_{\mathsf{dg}}(k)) \oplus \mathcal{U}_a(\langle E_{i+1}, \ldots, E_m \rangle_{\mathsf{dg}}) \stackrel{\sim}{\longrightarrow} \mathcal{U}_a(\langle E_i, \ldots, E_m \rangle_{\mathsf{dg}}) \tag{4.3}$$

for every $1 \leq i \leq m$. A recursive argument using (4.3) and the fact that $\mathcal{D}^b_{\mathsf{dg}}(\mathcal{X}) = \langle E_1, \ldots, E_m \rangle_{\mathsf{dg}}$ gives then rise to the above isomorphism (4.2). $\square$

Consider the following commutative diagram :

$$\begin{tikzcd}
\mathcal{DM}(k)^{\mathsf{op}} \arrow[drr, "M(-)_{\mathbb{Q}}"'] \arrow[rrrr, bend left, "\mathcal{D}^{\mathsf{perf}}_{\mathsf{dg}}(-)"] & & \mathcal{P}(k)^{\mathsf{op}} \arrow[ll, hook'] \arrow[rr, "\mathcal{D}^{\mathsf{perf}}_{\mathsf{dg}}(-)"] \arrow[d, "M(-)_{\mathbb{Q}}"] & & \mathsf{dgcat}(k) \arrow[d, "\mathcal{U}_a(-)"] \\
& & \mathrm{Chow}(k)_{\mathbb{Q}} \arrow[d, "\pi(-)"'] & & \mathsf{Hmo}_0(k) \arrow[d, "(-)_{\mathbb{Q}}"] \\
& & \mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \arrow[rr, "R(-)"'] & & \mathsf{Hmo}_0(k)_{\mathbb{Q}}\,.
\end{tikzcd} \tag{4.4}$$

Some explanations are in order. The lower-right rectangle was constructed in [37, Thm. 1.1]. The category $\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$ is the orbit category associated to the Tate motive $\mathbb{Q}(1)$ (which is the $\otimes$-inverse of the Lefschetz motive $\mathbf{L}$) and $\mathsf{Hmo}_0(k)_{\mathbb{Q}}$ is obtained from $\mathsf{Hmo}_0(k)$ by tensoring each abelian group of morphisms with $\mathbb{Q}$. Moreover, the functor $R(-)$ is additive and fully-faithful and $\mathcal{D}^{\mathsf{perf}}_{\mathsf{dg}}(-)$ stands for the derived dg category of perfect complexes. Since $\mathcal{X}$ is regular every bounded complex of coherent sheaves is perfect (up to quasi-isomorphism) and so we have a natural Morita equivalence $\mathcal{D}^{\mathsf{perf}}_{\mathsf{dg}}(\mathcal{X}) \simeq \mathcal{D}^b_{\mathsf{dg}}(\mathcal{X})$. Finally, the upper-left triangle in the one diagram is the one associated to (1.1)-(1.2).

The commutativity of the above diagram (4.4), the natural Morita equivalence $\mathcal{D}^b_{\mathsf{dg}}(k) \simeq \mathcal{D}^b_{\mathsf{dg}}(\mathrm{Spec}(k))$ of dg categories, and the fact that the functor $R(-)$ is additive and fully faithful, imply that the image of (4.2) under $(-)_{\mathbb{Q}}$ can be identified with the isomorphism

$$\bigoplus_{j=1}^{m} \pi(M(\mathrm{Spec}(k))_{\mathbb{Q}}) \xrightarrow{\sim} \pi(M(\mathcal{X})_{\mathbb{Q}}) \tag{4.5}$$

in the orbit category $\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$. Hence, since $M(\mathrm{Spec}(k))_{\mathbb{Q}}$ is the $\otimes$-unit of $\mathrm{Chow}(k)_{\mathbb{Q}}$ and the automorphism $-\otimes\mathbb{Q}(1)$ of $\mathrm{Chow}(k)_{\mathbb{Q}}$ is additive, there are morphisms

$$\underline{f} = \{f_r\}_{r\in\mathbb{Z}} \in \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Chow}(k)_{\mathbb{Q}}}(M(\mathcal{X})_{\mathbb{Q}}, \bigoplus_{j=1}^{m} \mathbb{Q}(1)^{\otimes r})$$

and

$$\underline{g} = \{g_s\}_{s\in\mathbb{Z}} \in \bigoplus_{s\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Chow}(k)_{\mathbb{Q}}}(\bigoplus_{j=1}^{m} M(\mathrm{Spec}(k))_{\mathbb{Q}}, M(\mathcal{X})_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes s})$$

verifying the equalities $\underline{g}\circ\underline{f} = \mathrm{id}$ and $\underline{f}\circ\underline{g} = \mathrm{id}$. The equivalence of categories $\mathrm{Chow}(k)_{\mathbb{Q}} \simeq \mathrm{DMChow}(k)_{\mathbb{Q}}$, combined with the construction of the category of Deligne-Mumford-Chow motives (see [4, §8]), implies that

$$\mathrm{Hom}_{\mathrm{Chow}(k)_{\mathbb{Q}}}(M(\mathcal{X})_{\mathbb{Q}}, \bigoplus_{j=1}^{m} \mathbb{Q}(1)^{\otimes r}) \simeq \bigoplus_{j=1}^{m} A^{\dim(\mathcal{X})+r}(\mathcal{X}\times \mathrm{Spec}(k))$$

and that

$$\mathrm{Hom}_{\mathrm{Chow}(k)_{\mathbb{Q}}}(\bigoplus_{j=1}^{m} M(\mathrm{Spec}(k))_{\mathbb{Q}}, M(\mathcal{X})_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes s}) \simeq \bigoplus_{j=1}^{m} A^{s}(\mathrm{Spec}(k)\times\mathcal{X})\,,$$

where $A^*(-)$ denotes the rational Chow ring of DM stacks defined by Vistoli in [42]. Hence, we conclude that $f_r = 0$ for $r \neq \{-\dim(\mathcal{X}), \ldots, 0\}$ and that $g_s = 0$ for $s \neq \{0, \ldots, \dim(\mathcal{X})\}$. The sets of morphisms

$$\{f_{-l}\,|\,0\leq l\leq \dim(\mathcal{X})\} \quad \text{and} \quad \{g_l \otimes \mathbb{Q}(1)^{\otimes(-l)}\,|\,0\leq l\leq\dim(\mathcal{X})\}$$

give then rise to well-defined morphisms

$$\Phi: M(\mathcal{X})_{\mathbb{Q}} \to \bigoplus_{l=0}^{\dim(\mathcal{X})}\bigoplus_{j=1}^{m}\mathbb{Q}(1)^{\otimes(-l)} \qquad \Psi: \bigoplus_{l=0}^{\dim(\mathcal{X})}\bigoplus_{j=1}^{m}\mathbb{Q}(1)^{\otimes(-l)} \to M(\mathcal{X})_{\mathbb{Q}}$$

in $\mathrm{Chow}(k)_{\mathbb{Q}}$. The composition $\Psi\circ\Phi$ agrees with the 0^{th}-component of the composition $\underline{g}\circ\underline{f} = \mathrm{id}_{\pi(M(\mathcal{X})_{\mathbb{Q}})}$, *i.e.* it agrees with $\mathrm{id}_{M(\mathcal{X})_{\mathbb{Q}}}$. Since $\mathbb{Q}(1)^{\otimes(-l)} = \mathbf{L}^{\otimes l}$ we conclude then that $M(\mathcal{X})_{\mathbb{Q}}$ is a direct summand of the Chow motive $\bigoplus_{l=0}^{\dim(\mathcal{X})}\bigoplus_{j=1}^{m}\mathbf{L}^{\otimes l}$. By definition of the Lefschetz motive we have the following equalities

$$\mathrm{Hom}_{\mathrm{Chow}(k)_{\mathbb{Q}}}(\mathbf{L}^{\otimes p}, \mathbf{L}^{\otimes q}) = \delta_{pq}\cdot\mathbb{Q} \qquad p, q\geq 0\,,$$

where δ_{pq} stands for the Kronecker symbol. As a consequence, $M(\mathcal{X})_{\mathbb{Q}}$ is in fact isomorphic to a subsum of $\bigoplus_{l=0}^{\dim(X)} \bigoplus_{j=1}^{m} \mathbf{L}^{\otimes l}$ indexed by a certain subset S of $\{0, \ldots, \dim(X)\} \times \{1, \ldots, m\}$. By construction of the orbit category we have natural isomorphisms

$$\pi(\mathbf{L}^{\otimes l}) \xrightarrow{\sim} \pi(M(\mathrm{Spec}(k))_{\mathbb{Q}}) \qquad l \geq 0 .$$

Hence, since the direct sum in the left-hand-side of (4.5) contains m terms we conclude that the cardinality of S is also m. This means that there is a choice of integers (up to permutation) $l_1, \ldots, l_m \in \{0, \ldots, \dim(\mathcal{X})\}$ giving rise to a canonical isomorphism

$$M(\mathcal{X})_{\mathbb{Q}} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}$$

in $\mathrm{Chow}(k)_{\mathbb{Q}}$. The proof is then achieved.

5. Proof of Theorem 1.7

As the proof of Lemma 4.1 shows, one can replace $\langle E_j \rangle_{\mathsf{dg}}$ by the dg category $\mathcal{C}^j_{\mathsf{dg}} \subset \mathcal{D}^b_{\mathsf{dg}}(\mathcal{X})$ and hence obtain the isomorphism $\bigoplus_{j=1}^{p} \mathcal{U}_a(\mathcal{C}^j_{\mathsf{dg}}) \simeq \mathcal{U}_a(\mathcal{D}^b_{\mathsf{dg}}(\mathcal{X}))$ in Hmo_0. Since by hypothesis $M(\mathcal{X})_{\mathbb{Q}} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}$ one concludes from the above commutative diagram (4.4) and the fact that $R(-)$ is fully-faithful that $\bigoplus_{j=1}^{p} \mathcal{U}_a(\mathcal{C}^j_{\mathsf{dg}})_{\mathbb{Q}}$ is isomorphic to the direct sum $\mathbf{1}_{\mathbb{Q}} \oplus \cdots \oplus \mathbf{1}_{\mathbb{Q}}$ (with m-terms). As a consequence, for every $1 \leq j \leq p$, there exists a well-defined non-negative integer n_j such that

$$\mathcal{U}_a(\mathcal{C}^j_{\mathsf{dg}})_{\mathbb{Q}} \simeq \underbrace{\mathbf{1}_{\mathbb{Q}} \oplus \cdots \oplus \mathbf{1}_{\mathbb{Q}}}_{n_j} . \tag{5.1}$$

Moreover $\sum_{j=1}^{p} n_j = m$. Now, let $E : \mathsf{dgcat}(k) \to \mathsf{A}$ be an additive invariant; see [39, Def. 4.1]. By [39, Thm. 4.2] it factors through $\mathcal{U}_a(-)$. Since by hypothesis A is moreover $\mathbb{Q}$-linear, we obtain then an additive $\mathbb{Q}$-linear functor

$$\overline{E} : \mathsf{Hmo}_0(k)_{\mathbb{Q}} \longrightarrow \mathsf{A}$$

such that $\overline{E}(\mathcal{U}_a(\mathcal{A})_{\mathbb{Q}}) \simeq E(\mathcal{A})$ for every dg category $\mathcal{A}$. By the above isomorphism (5.1) we conclude then that $E(\mathcal{C}^j_{\mathsf{dg}}) \simeq E(k)^{\oplus n_j}$ as claimed. This concludes the proof.

6. Proof of Corollary 1.5

Since by hypothesis K is a field of characteristic zero, the universal property of the category $\mathrm{Chow}(k)_{\mathbb{Q}}$ of Chow motives with rational coefficients (see [1, Prop. 4.2.5.1] with $F = \mathbb{Q}$) furnish us a (unique) additive symmetric monoidal functor $\overline{H^*}(-)$ making the right-hand-side triangle of the following diagram

$$\begin{tikzcd}
\mathcal{DM}(k)^{\mathsf{op}} \arrow[dr, "M(-)_{\mathbb{Q}}"'] & \mathcal{P}(k)^{\mathsf{op}} \arrow[l, hook'] \arrow[r, "H^*(-)"] \arrow[d] & \mathrm{VecGr}_K \\
& \mathrm{Chow}(k)_{\mathbb{Q}} \arrow[ur, "\overline{H^*}(-)"'] &
\end{tikzcd}$$

commutative. Note that the commutativity of the left-hand-side triangle holds by construction. By hypothesis $\mathcal{D}^b(\mathcal{X})$ admits a full exceptional collection of length m

and so by Theorem 1.3 there is a choice of integers (up to permutation) $l_1, \ldots, l_m \in \{0, \ldots, \dim(X)\}$ giving rise to a canonical isomorphism

$$M(\mathcal{X})_{\mathbb{Q}} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}\,.$$

Since the functor $\overline{H^*}(-)$ is additive and symmetric monoidal and $H^*(\mathcal{X}) = \overline{H^*}(M(\mathcal{X})_{\mathbb{Q}})$ we obtain then the following identification

$$H^*(\mathcal{X}) \simeq \overline{H^*}(\mathbf{L})^{\otimes l_1} \oplus \cdots \oplus \overline{H^*}(\mathbf{L})^{\otimes l_m}\,.$$

As proved in [1, Prop. 4.2.5.1] we have

$$\overline{H^n}(\mathbf{L}) \simeq \begin{cases} K & n = 2 \\ 0 & n \neq 2 \end{cases}$$

and so we conclude that $H^n(\mathcal{X}) = 0$ for n odd and that $\dim H^n(\mathcal{X}) \leq m$ for n even.

7. Proof of Proposition 1.9

By Theorem 1.3 one knows that there is a choice of integers (up to permutation) $l_1, \ldots, l_m \in \{0, \ldots, \dim(X)\}$ giving rise to a canonical isomorphism

$$M(X)_{\mathbb{Q}} \simeq \mathbf{L}^{\otimes l_1} \oplus \cdots \oplus \mathbf{L}^{\otimes l_m}\,. \tag{7.1}$$

Recall from [29] the construction of the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of varieties. As proved in [14], the functor

$$\mathcal{P}(\mathbb{C})^{\mathsf{op}} \longrightarrow \mathrm{Chow}(\mathbb{C})_{\mathbb{Q}} \qquad Y \mapsto M(Y)_{\mathbb{Q}}$$

gives rise to a motivic measure, *i.e.* to a ring homomorphism $\chi_{GS} : K_0(\mathcal{V}_{\mathbb{C}}) \to K_0(\mathrm{Chow}(\mathbb{C})_{\mathbb{Q}})$. Let us denote by $\mathbb{L}$ the class $[\mathbb{A}^1]$ of the affine line $\mathbb{A}^1$ in the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ and by $\mathbf{L}$ the Lefschetz motive as well as its class in $K_0(\mathrm{Chow}(\mathbb{C})_{\mathbb{Q}})$. As explained in [14, §3.2.3] the image of $\mathbb{L}$ under the motivic measure χ_{GS} is $\mathbf{L}$ and so the homomorphism χ_{GS} restricts to an isomorphism $\mathbb{Z}[\mathbb{L}] \xrightarrow{\sim} \mathbb{Z}[\mathbf{L}]$. Hence, by the above motivic decomposition (7.1) one concludes that $[X] = \mathbb{L}^{l_1} + \cdots + \mathbb{L}^{l_m}$ in $K_0(\mathcal{V}_{\mathbb{C}})$. Now, recall from [29] the construction of the Hodge-Deligne motivic measure $\chi_{HD} : K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[u,v]$. Its value on a smooth projective variety Y is the polynomial in two-variables $\sum_{p,q}(-1)^{p+q}h^{p,q}(Y)u^p v^q$ (where $h^{p,q}(Y)$ are the Hodge numbers) and its value on the class $\mathbb{L}$ is the polynomial uv. Hence, the following equality holds

$$\chi_{HD}([X]) = (uv)^{l_1} + \cdots + (uv)^{l_m}\,.$$

In particular all the Hodge numbers $h^{p,q}(X)$, with $p \neq q$, are zero and so we conclude that X is Tate-Hodge.

Now, let us assume that X is defined over a number field k. The same argument as above, with $\mathbb{C}$ replaced by k, shows us that $[X] = \mathbb{L}^{l_1} + \cdots + \mathbb{L}^{l_m}$ in $K_0(\mathcal{V}_k)$. As explained in [29, §4], the field embedding $\alpha : k \hookrightarrow \mathbb{C}$ gives rise to a ring homomorphism $K_0(\mathcal{V}_k) \to K_0(\mathcal{V}_{\mathbb{C}})$. Hence, using the Hodge-Deligne motivic measure one concludes, as above, that all the Hodge numbers $h^{p,q}(X_\alpha)$, with $p \neq q$, are zero and so that X_α is also Tate-Hodge. This concludes the proof.

Acknowledgments: The authors are very grateful to Roman Bezrukavnikov and Yuri Manin for stimulating discussions and precise references. They would like also to thank Marcello Bernardara, Alexander Kuznetsov, John Alexander Cruz Morales and Kirill Zainoulline for detailed comments on a previous draft.

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