Introduction:
A Mathematical Promenade
& Gromov's Ergobrain Program

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Ma191b: Geometry of Neuroscience
Lecture 1

About this "Geometry of Neuroscience" class

This class is based on a class designed and jointly taught by Matilde Marcolli (math) and Doris Tsao (neuroscience) in 2017, later taught by M.M. at the University of Toronto in 2018, and is the basis for an ongoing book project *Geometry of Neuroscience* with Doris Tsao (UC Berkeley) this class will cover:

1. Structures in the Brain: Neurons, Networks, Codes
   - Single neuron as a dynamical system
   - Hopfield equation and neural computation
   - Neural codes and homotopy types
   - Brain networks and random graphs
   - Directed algebraic topology and microcircuitry
   - Models of learning
   - Artificial and brain networks
   - Category theory of networks and resources
   - Integrated Information

2. Vision: Visual Cortex, Segmentation/Tracking
   - The Visual System and deep networks
   - Gabor frames and signal analysis
   - Conformal geometry of the visual cortex
   - Contact geometry of the visual cortex
   - Gabor frames and contact geometry, Gabor bundles
   - Segmentation and tracking: differential topology
   - Segmentation: variational analysis
   - Tracking: algebraic geometry
   - Pattern Theorey
2. Language: Syntax as Brain Structure
   - Universal Grammar hypothesis
   - Merge and syntax
   - Syntax in the brain

In this lecture: Motivational Introduction, Gromov's idea of "Ergosystems"

- Misha Gromov, *Structures, Learning, and Ergosystems*, 2011
- Misha Gromov, *Ergostructures, Ergologic and the Universal Learning Problem*, 2013

Two provocative programmatic long papers about the goal of mathematical understanding of learning and *structures* in the biological brain

Gromov's Ergobrain

• Gromov conjectures the existence of a *mathematical* structure implementing the transformation of incoming signal to representation in the brain: ergobrain

• a dynamic entity continuously built by the brain: (goal free) structure learning

• with constraints from network architecture

• Gromov's proposed terminology:

- neuro-brain: a (mathematical) model of the physiology of the brain (chemical, electrical, connectivity, etc.)
- ergo-brain: a "dynamical reduction" of neuro-brain (quotient)
- ergo-system: like ergobrain but not necessarily derived from a neurobrain
- ego-mind: mental processes, interactions of organisms with outside world
- ergo-learning: spontaneous structure building process

Why "ergosystem"?

Cogito ergo sum

I see therefore I am

I envy therefore I am

I love therefore I am

I dream therefore I am

"Therefore" = hope to understand consciousness through reasoning/mathematics

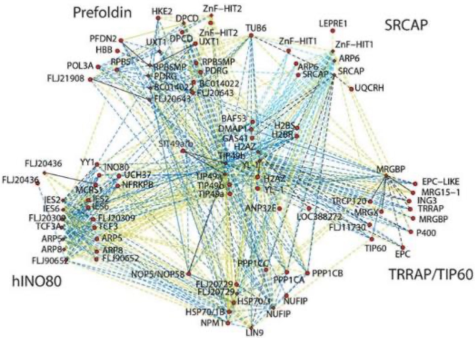
Ergosystem is a machine for extracting *structure* from incoming *signals*, through a process of *goal-free learning* (in contrast to reinforcement learning)

an analogy with the biology of cells (Gromov)

different levels: case of cells

1. interaction with the environment (cytology)
2. selection of beneficial mutations (molecular genetics)
3. biochemistry of proteins and cell structures

third level example: protein interaction networks (structure and information)



case of neuroscience: analogs of first two levels

1. psychology: observation of output signals following interaction with input signals
2. neurophysiology: nervous system function from molecules and cells to systems

missing the analog of the "biochemistry level" for cells

understanding of *structures* and the mechanisms that govern their formation and functionality ("structures" in a mathematical sense: mathematics is the study of structures)

language is an example of a structure in the human brain, but many questions about the realization of its computational and mathematical structure in the brain physiology

Language as a model ergosystem

- Linguistic information entering the ergobrain does not much depend on the physical carrier of this information. This suggest a universal class of structures encoding this information; our main objective is describing these structures, which we call syntactic ergo-structures.
- Such a structure is a combinatorial object X, a kind of a network made of finitely many "atomic units = ergo-ideas". This network structure generalizes/refines that of a graph, where some patterns are similar to those found in the mathematical theory of n-categories.
- The combinatorial structure is intertwined with a geometric one; both an individual network X and the totality of these, say X, carry natural distance-like geometries which are derived from combinatorics and, at the same time, are used for constructing combinatorial relations.
- The learning process is modeled by some transformation(s) L in the space X; the resulting "educated learner" appears as an attractive fixed point x (or a class of points) under this transformation(s).
- The (ergo)learning process (transformation) L starts from a space of signals and eventually compresses (folds) the information they carry by some coclustering algorithms into our x.

Vision as a model ergosystem

- The input of the visual system amounts, roughly, to arrays of pixels changing over time.
- How does brain segment groups of pixels into objects, and learn that these objects are invariant under O(3) transformations?

"The mathematics of building/identifying the O(3)-symmetry of the visual perception field is similar to but more complicated than how Alfred Sturtevant reconstructed in 1913 linearity of the gene arrangements on the basis of distributions of phenotype linkages long before the advent of the molecular biology and discovery of DNA."

The role of Mathematics

- long history of very successful interactions of mathematics and physics
  - "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" (Eugene Wigner, 1960): referring to physical science and how mathematics can drive new advances in physics
  - in more recent years we have also seen the "unreasonable effectiveness of physics in the mathematical sciences" with new progress in mathematics driven by physics
  - there is also an ongoing "unreasonable effectiveness of mathematics in linguistics" which suggests a broader role in brain structures/ergosystems

- Biology has traditionally not put too much emphasis on theoretical ideas as a guiding principle for progress in the field; interactions between mathematics and biology are recent and still largely underdeveloped

- Gromov's speculation on mathematics and neuroscience: "basic mental processes can be meaningfully described, if at all, only in a broader mathematical context and this mathematics does not exist at the present day" ... or does it?

Example of mathematical viewpoint: symmetry

• several molecules occur with symmetries (icosahedral symmetry of viruses; helix and double helix symmetries in proteins and DNA, etc.)...

• *energy and symmetry*: configuration space $\mathcal{M}$ of molecules with group action G, invariant energy functional E, typically local minima over G-symmetric configurations are local minima for all configuration

• *information and symmetry*: a symmetric form is specified by fewer parameters, preferable from the Shannon information viewpoint

Symmetry constraints and simplifies the space of possibilities

What is the architecture of the ergobrain? An analogy to viruses

"In the end of the day, the symmetry of viruses depends on the structural constrains imposed by the geometry of the physical space which allows the existence of such improbable objects as icosahedra.

Similarly, we expect that the constrains of the mathematical space (which we have not defined yet) where ergosystems reside will have a strong structural imprint on a possible architecture of the ergobrain. For example, we want to understand what in the (ergo)brain is responsible for the unreasonable power of our visual system which allowed a discovery of icosahedra by humans (at least) as early as –400."

Example of how algebra constraints systems behavior and reveals universal laws

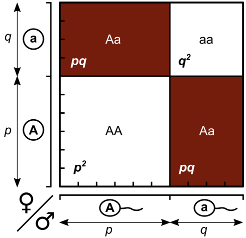
a biology example: Hardy-Weinberg principle

• allele and genotype frequencies in a population would remain constant through generations in the absence of other evolutionary influences (mutation, selection, etc.)

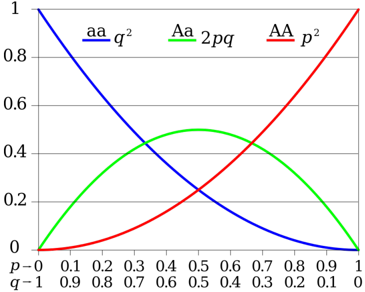
• two alleles with frequencies $f_0(A) = p$ and $f_0(a) = q$ (with $p + q = 1$) (similar for n alleles a_i)

• expected genotype proportions next generation

$$f_1(AA) = p^2, \ \ f_1(aa) = q^2 \ \text{homozygotes}, \quad f_1(Aa) = 2pq \ \text{heterozygotes}$$



- still satisfying $f_1(AA) + f_1(aa) + f_1(Aa) = p^2 + q^2 + 2pq = 1$
- ok for $q = 1 - p$ giving



- new distribution of alleles depending on distribution of genotypes (at k-th generation)

$$f_k(A) = f_k(AA) + \frac{1}{2} f_k(Aa) \quad \text{and} \quad f_k(a) = f_k(aa) + \frac{1}{2} f_k(Aa)$$

- so at first new generation

$$f_1(A) = p^2 + pq = p = f_0(A) \quad \text{and} \quad f_1(a) = q^2 + pq = q = f_0(a)$$

$$f_1(AA) = p^2 = f_0(A)^2, \; f_1(aa) = q^2 = f_0(a)^2, \; f_1(Aa) = 2pq = 2 f_0(A) f_0(a)$$

then to compute $(f_{n+1}(AA), f_{n+1}(Aa), f_{n+1}(aa))$

- probability of, say, AA–aa combination is $2f_n(AA)f_n(aa)$ and results in Aa: write result as $(0,1,0)$
- summing analogous count for all six possible combinations

$$(f_{n+1}(AA), f_{n+1}(Aa), f_{n+1}(aa)) =$$

$$((f_n(AA)+\frac{1}{2}f_n(Aa))^2, (f_n(AA)+\frac{1}{2}f_n(Aa))(f_n(aa)+\frac{1}{2}f_n(Aa)), (f_n(aa)+\frac{1}{2}f_n(Aa))^2)$$

$$= (f_n(A)^2, 2f_n(A)f_n(a), f_n(a)^2)$$

- while $(f_1(AA), f_1(Aa), f_1(aa))$ need not be the same as $(f_0(AA), f_0(Aa), f_0(aa))$, since $(f_1(A), f_1(a)) = (f_0(A), f_0(a))$ will then have all

$$(f_n(AA), f_n(Aa), f_n(aa)) = (f_1(AA), f_1(Aa), f_1(aa))$$

everyrthing stabilizes by the second generation

- gene recombination not sufficient to explain evolution, in the absence of additional phenomena like gene mutation

What kind of mathematics?

General framework:

1. Combinatorial objects: from *graphs* to *n-categories* (networks and relations)
2. Transformations and symmetries: *groups* and generalizations (groupoids, categories, Hopf algebras, operads, etc.)
3. Probabilities and information/entropy and complexity: algebraic structures with superimposed probabilistic and thermodynamical (statistical physics) structures
4. Topological and Metric: proximity, similarity, deformation, homotopy, distance, measurement, manifolds, shapes
5. Grammars: formal languages, logic, types, compositional structures

we'll meet several of these along the way, for now just a quick glimpse of some notions

Categories

• traditional setting of mathematics based on Set Theory, first example of categories: Sets (or Finite Sets)

• Category $\mathcal{C}$: Objects $X, Y, \cdots \in \mathcal{O}(\mathcal{C})$, Morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

associative composiiton $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ and identity $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ unit for composition

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g\circ f}\downarrow & & \downarrow{\scriptstyle h\circ g} \\ Z & \xrightarrow[h]{} & W \end{array}$$

• Functors $F : \mathcal{C} \to \mathcal{C}'$, on objects $\mathcal{O}(\mathcal{C}) \ni X \mapsto F(X) \in \mathcal{O}(\mathcal{C}')$, on morphisms $F(f) : F(X) \to F(Y)$ (covariant; also contravariant) with $F(g \circ f) = F(g) \circ F(f)$ and $F(1_X) = 1_Y$

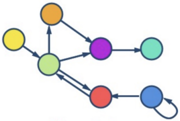
• Natural Transformations: $\eta : F \to G$, to every object a morphism $\eta_X : F(X) \to G(X)$ with $\eta_Y \circ F(f) = G(f) \circ \eta_X$

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

• Categories of Vector Spaces, Topological Spaces, Smooth Manifolds, Groups, Rings, etc.

... concept of mathematical structure

Example: directed graphs are functors



- category **2** with two objects V, E and two non-identity morphisms $s, t : E \to V$
- functor $G : \mathbf{2} \to \mathrm{Sets}$, sets V_G and E_G (vertices, edges) and maps of sets $s_G, t_G : E_G \to V_G$ source and target maps of directed edges
- natural transformation $\eta : G \to G'$ is usual notion of map of directed graphs

$$\eta_V : V_G \to V_{G'} \quad \text{and} \quad \eta_E : E_G \to E_{G'}$$

preserving incidence relations

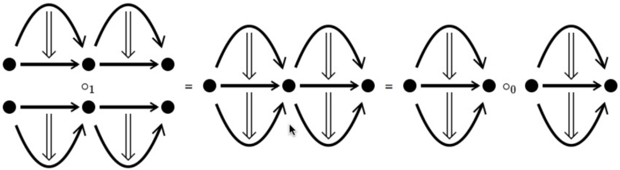
$$\eta_V \circ s_G = s_{G'} \circ \eta_E \quad \text{and} \quad \eta_V \circ t_G = t_{G'} \circ \eta_E$$

Higher Categorical Structures

• 2-Categories: a category $\mathcal{C}$ where the sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are themselves the objects of a category (ie there are morphisms between morphisms: 2-morphisms)

• vertical and horizontal composition of 2-morphisms with exchange relation

$$(\alpha \circ_0 \beta) \circ_1 (\gamma \circ_0 \delta) = (\alpha \circ_1 \gamma) \circ_0 (\beta \circ_1 \delta)$$



• objects = vertices; morphisms = 1-cells; 2-morphisms = 2-cells

Homotopy and Higher Category Theory

• Note: 2-category composition of 1-morphisms associative; bicategory associative up to 2-isomorphism

• Example: Objects = points in an open set in the plane; Morphisms = oriented paths connecting points; 2-morphisms = homotopies of paths; composition up to homotopy... can also compose homotopies up to homotopy etc. ... higher *n*-categories (with *n* different levels of "isomorphism": 0-isomorphism equality, 1-isomorphism realized by invertible 1-morphisms; 2-isomorphism by 2-morphisms etc.)... ∞-categories

• Contemporary mathematical viewpoint is shifting more and more towards these higher (or ∞) structures and homotopy

Groups, Semigroups, Groupoids, Algebras, and Categories

• Group: small category with one object and all morphisms invertible (product, associativity, unit)
... symmetries (action by automorphisms)

• Group algebra $\mathbb{C}[G]$ (discrete group G) finitely supported functions $f : G \to \mathbb{C}$ with convolution

$$(f_1 \star f_2)(g) = \sum_{g=g_1 g_2} f_1(g_1) f_2(g_2)$$

involution $f^*(g) \equiv \overline{f(g^{-1})}$

• Semigroup: small category with one object (not always inverses)
... actions by endomorphisms

• Semigroup algebra: $f : S \to \mathbb{C}$ with convolution

$$(f_1 \star f_2)(s) = \sum_{s=s_1 s_2} f_1(s_1) f_2(s_2)$$

no longer necessarily involutive

• Groupoid: small category where all morphisms are invertible (product is defined only when target of first arrow is source of second)

... another type of symmetry

• Groupoid algebra $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, s, t)$ (objects and morphisms, source and target); algebra of functions $f : \mathcal{G}^{(1)} \to \mathbb{C}$ with convolution

$$(f_1 \star f_2)(\gamma) = \sum_{\gamma = \gamma_1 \circ \gamma_2} f_1(\gamma_1) f_2(\gamma_2)$$

and involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$

• Semigroupoid: a small category (associative composition of morphisms)

• Semigroupoid algebra: functions of morphisms with convolution

$$(f_1 \star f_2)(\phi) = \sum_{\phi = \phi_1 \circ \phi_2} f_1(\phi_1) f_2(\phi_2)$$

Topology and invariants

• Topological spaces and continuous functions (the study of shapes up to continuous deformations)

• Invariants: ways of distinguishing between topological spaces

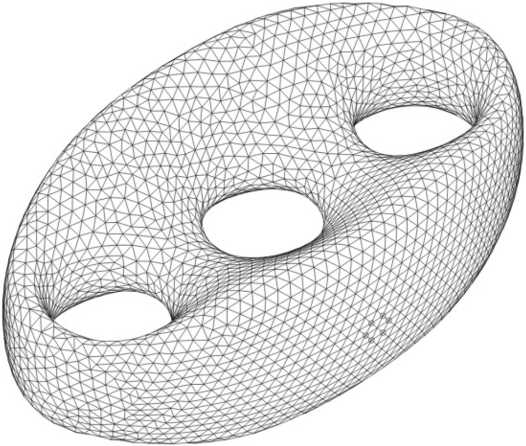
• Invariants are functors from the category $\mathcal{T}$ of Topological Spaces to another category (vector spaces, groups, rings, etc.)

• Homology: functor $H_* : \mathcal{T} \to \mathcal{V}_{\mathbb{Z}}$ to $\mathbb{Z}$-graded vector spaces

$$H_n(X, \mathbb{Z}) = \frac{\mathrm{Ker}(\partial_n : C_n(X, \mathbb{Z}) \to C_{n-1}(X, \mathbb{Z}))}{\mathrm{Image}(\partial_{n+1} : C_{n+1}(X, \mathbb{Z}) \to C_n(X, \mathbb{Z}))}$$

$C_n(X, \mathbb{Z})$ abelian group spanned by n-simplexes of a triangulation of X; ∂_n (oriented) boundary map

Homology is independent of the choice of triangulation; it measures “holes” and “connectivity” in various dimensions



a triangulation of a surface of genus 3

Metric Spaces, Riemannian Manifolds

• Topological spaces with a quantitative measure of proximity: topology is induced by a metric (open sets generated by open balls in the metric)

• distance function (metric) $d : X \times X \to \mathbb{R}_+$ with

- $d(x,y) = d(y,x)$ for all $x,y \in X$
- $d(x,y) = 0$ iff $x = y$
- triangle inequality $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$

• smooth manifold M (covered by local charts homeomorphic to $\mathbb{R}^n$ with $\mathcal{C}^\infty$ changes of coordinates in charts overlap); tangent spaces $T_x M$ at all points (tangent bundle TM)

• Riemannian manifold: smooth manifold, metric structure determined by a metric tensor $g = (g_{\mu\nu})$ symmetric positive, section of $T^*M \otimes T^*M$

• length of curves $L(\gamma) = \int g(\gamma'(t), \gamma'(t))^{1/2}\, dt$, geodesic distance:

$$d(x,y) = \inf_\gamma L(\gamma)$$

Probabilities and Entropy

• mathematical structures (especially algebraic structures) endowed with probabilities

• a successful approach in formal languages and generative grammars: probabilistic grammars (generative rules applied with certain probabilities)

• other algebraic structures can be made probabilistic: groups, semigroups, groupoids, semigroupoids... all like directed graphs: assign probabilities at each successive choice of next oriented edge in a path... like Markov processes

• also attach information measures (entropy) to algebraic structures (operations weighted by entropy functionals): information algebras

The Key Idea: in applications to Biology all mathematical structures should be endowed with probabilistic weights and entropy/information weights

Complexity and Patterns

- Kolmogorov Complexity

shortest length of a program required to compute the given output (theory of computation, Turing machines)

- Gell-Mann Effective Complexity

description length of "regularities" (structured patterns) contained in the object

Ambiguity, Galois Symmetry, Category Theory (math examples of interplay between properties of categories and geometric symmetries/ambiguities)

• Symmetries describe ambiguities (up to isomorphism, up to invertible transformations, up to homotopy, etc.)

• Categorical viewpoint on Symmetry: need categories with good properties... very much like the category of vector spaces: abelian (kernels and cokernels), tensor, rigid (duality, internal Hom)

• fiber functor $\omega : \mathcal{C} \to \mathcal{V}$ to vector spaces preserving all properties (tensor, etc.) and symmetries $G = \mathrm{Aut}(\omega)$ the invertible natural transformations of the fiber functor

• $(\mathcal{C}, \omega)$ as above: Tannakian category with Galois group G

• this includes: usual Galois theory; Motives; Regular-singular differential systems; symmetries of Quantum Field Theory, etc.

• what interplay between categorical compositional structures and symmetries in neuroscience?

Structures in the brain

Mathematical tools have expanded and evolved as more levels of structure in the brain are studied from small scales (single neurons, microcircuitry) to large scales (connectome) and in modeling neural codes

- differential equations, dynamical systems: functioning of single neuron, ion channels, synaptic connections
- topology: random graphs, connectivity, directed algebraic topology concurrent and distributed computing, simplicial complexes (activation complex, microcircuitry)
- homotopy: neural codes, patterns of brain activity and reconstruction of stimulus space
- category theory: allocation of resources to networks, interactions between different types of resources
- information structures: integrated information, quantitative approaches to consciousness

Visual system

Modeling the structure of the V1 visual cortex involves a novel interplay of different mathematical tools: signal analysis (Gabor frames) and contact geometry (modeling the connectivity of the V1 cortex) and conformal geometry of the retinal map. The main problem of segmentation and tracking in vision also involves an interplay of very different areas of mathematics: differential topology, variational analysis, algebraic geometry

- signal analysis: receptor fields of V1 neurons
- conformal geometry: retinotopic mapping to the V1 cortex
- contact geometry: connectivity of the V1 cortex
- differential topology: segmentation and tracking
- variational analysis: image segmentation
- algebraic geometry: image tracking

Language system

A good mathematical modeling of language as a computational system can be obtained by algebraic methods. The problem of identifying neurocomputational mechanisms realizing this computational model is developing and involves an interplay of algebra, analysis, topology

- algebra: Hopf algebras, magmas, semirings, operads play a role in the modeling of syntax of human language
- topology: semantic spaces rely on notions of proximity and similarity
- representations: realizations of the algebra of syntax in neurocomputational settings