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Aspects of p-adic geometry related to entanglement entropy

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in memory of Boris Dubrovin

Abstract. The replica argument for the computation of entanglement entropy is based on Rényi entropies computed over cyclic branched coverings of a Riemann surface. The recent development of a p-adic approach to AdS/CFT holography suggests that this computational method should have an analog in p-adic geometry. The purpose of this survey paper is to outline some aspects of p-adic geometry that are naturally related to the problem of formulating an analog of the usual replica argument for entanglement entropy in the context of p-adic holography.

1. Introduction

The use of p-adic and adelic methods in physics has been broadly developed over several decades, see for instance [**11**], [**18**], [**43**], [**58**], [**60**]. In [**45**] a p-adic model of AdS/CFT holography was proposed, based on Mumford curves as boundary spaces and quotients of Bruhat–Tits trees by p-adic Schottky groups as bulk spaces, by analogy with the complex case of Riemann surfaces with Schottky uniformization as boundaries and hyperbolic handlebodies obtained as quotients of the 3-dimensional real hyperbolic space by a Kleinian Schottky group. This viewpoint proposed in [**45**] was motivated by previous work of Manin on the fiber at infinity in Arakelov geometry, [**42**]. More recently, the p-adic approach to AdS/CFT holography started to be investigated much more systematically in [**25**] and [**29**], and has since been more extensively studied. (The bibliography on the subject has rapidly grown to an extensive size and we will not attempt to summarize it here.) In particular, in this more recent approach, the view of holography in terms of entanglement entropy of boundary state and bulk geometry, and tensor networks and holographic codes have become more prominent, see for instance [**30**].

The purpose of the present paper is to outline some aspects of p-adic geometry related to a possible parallel in the p-adic setting of the "replica argument" for the computation of entanglement entropy. We will review several results in complex and p-adic geometry that have direct relevance to these questions.

From the mathematical perspective, I hope that discussing this problem will emphasize how the current interest in p-adic forms of AdS/CFT holography can suggest some interesting mathematical problems, in particular a rigorous geometric formulation of the replica argument for the computation of entanglement entropy, in both complex and p-adic geometry. From the physics perspective, discussing this problem can present a compelling reason for considering non-archimedean valued physical fields. Indeed, in most of the current literature on p-adic holography, one considers physical theories based on fields defined over p-adic domains (such as Mumford curves or the Drinfeld plane) but with real or complex values. This severely limits the use of p-adic algebraic geometry that can be made in such theories. The development of a good analog of the replica argument provides a strong reason in favor of considering physical fields with non-archimedean values, since the geometry of p-adic curves and their function theory is essential for a possible formulation of a p-adic version of the replica argument.

This is just a survey paper, which collects known results from complex and p-adic geometry aimed at building a dictionary between the geometry underlying the replica argument of [**8**] and a possible p-adic version. We outline certain questions in p-adic geometry that one needs to address in order to be able to formulate an analog of the argument of [**8**]. While we do not provide here an answer to these questions, this survey should be regarded as a way of building the geometric background needed to address the problem. The general question of entanglement entropy in p-adic holography is part of an ongoing research project joint with Matthew Heydeman, Sarthak Parikh, and Ingman Saberi, to which the present paper provides some geometric background.

2. Replica argument and branched coverings

In this section we review some of the fundamental aspects of the replica argument for the computation of entanglement entropy and we identify some basic building blocks that are needed for a similar formulation in the p-adic setting.

Given a conformal field theory (CFT) on a compact Riemann surface $X(\mathbb{C})$, denote by $\mathcal{Z}_1$ its partition function. Consider a cyclic branched covering $\pi_n : X_n(\mathbb{C}) \to X(\mathbb{C})$ and the region A given by the cut branches, that is, a collection of disjoint open intervals with endpoints at the branch points of π_n. Also denote by $\mathcal{Z}_n$ the partition function for the same CFT considered on the n-fold cyclic branched covering $X_n(\mathbb{C})$. If the whole system on $X(\mathbb{C})$ is in a pure state with density matrix ρ, the restriction of the system to a region A determines a reduced density matrix ρ_A obtained by tracing out the contribution of the complementary region. The main quantity of interest is the entanglement entropy $S_A = -\mathrm{Tr}\rho_A \log \rho_A$.

The *Replica Argument* is a method for the evaluation of the entanglement entropy based on the geometry of complex Riemann surfaces, [**8**]. It is based on computing Rényi entropies associated to the CFT on the cyclic branched coverings $X_n(\mathbb{C})$ and then obtaining the desired entanglement entropy as a formal limit of these Rényi entropies when n (considered then as a formal variable) tends to 1.

More precisely, one considers on the cyclic branched covering $X_n(\mathbb{C})$ the partition function $\mathcal{Z}_n(A)$, which is the partition function of the CFT on $X_n(\mathbb{C})$, computed as a path integral, with boundary conditions at the cut branches in A that specify the n-sheeted structure of the branched covering $X_n(\mathbb{C})$. One then shows that the comparison between $\mathcal{Z}_n(A)$ and the partition function $\mathcal{Z}_1^n$, which would correspond to n independent copies of the base $X(\mathbb{C})$, is equal to

$$\mathrm{Tr}(\rho_A^n) = \frac{\mathcal{Z}_n(A)}{\mathcal{Z}_1^n}.$$

This shows that the path integrals computing $\mathcal{Z}_n(A)$ and $\mathcal{Z}_1^n$ can be used to compute the entanglement Rényi entropies

$$S_{n,A} = \frac{1}{1-n}\log \mathrm{Tr}\rho_A^n = \frac{1}{1-n}\log \frac{\mathcal{Z}_n(A)}{\mathcal{Z}_1^n}\,.$$

The resulting $S_{n,A}$ admit analytic continuation to non-integer values $S_{\alpha,A}$, hence it makes sense to take their limit

$$S_A = \lim_{\alpha\to 1} S_{\alpha,A} = -\mathrm{Tr}\rho_A \log\rho_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\frac{\mathcal{Z}_n(A)}{\mathcal{Z}_1^n}.$$

In a CFT the primary fields are those annihilated by all the lowering generators of the conformal algebra. The descendant fields are obtained from the primaries by acting on them with the raising generators. It is shown that the quantity $\mathcal{Z}_n(A)/\mathcal{Z}_1^n$ behaves under conformal transformations like the n-th power of the two-point function of a primary operator, with a specific scaling dimension Δ_n that depends on n. Such two-point functions in turn can be computed in terms of the stress tensor of the theory.

In order to compute the Rényi entropies with the method outlined above, one considers a branched covering specified by the equation

$$z = \prod_i (w-w_i)^{\alpha_i}$$

with $\alpha_i \in \mathbb{Q}$. The holomorphic components $T(w)$ and $T(z)$ of the stress tensor are related through a transformation involving the Schwarzian derivative, so that the expectation value is given by (see [**8**])

$$\langle T(w)\rangle = \frac{c}{12}\{z,w\} = \frac{c}{12}\frac{z'''z' - \frac{3}{2}z''^2}{z'^2},$$

and the two-point function of a primary field of dimension Δ_n can then be computed explicitly in terms of the data of the branched cover. As shown in [**8**], if the branch points w_i are the $2N$ endpoints u_i, v_i of the open intervals $\mathcal{I}_i = (u_i, v_i)$ in $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C}) = X(\mathbb{C})$, with $A = \cup_i \mathcal{I}_i$, and if $\alpha_i = \pm 1/n$, then one obtains

$$\mathrm{Tr}\rho_A^n \sim c_n^N \left(\frac{\prod_{j<k}(u_k-u_j)(v_k-v_j)}{\prod_{j\le k}(v_k-u_j)} \right)^{\frac{c}{6}(n-\frac{1}{n})}. \tag{2.1}$$

Using only the leading term on the right-hand-side of (2.1), one obtains for the entanglement entropy the value

$$S_A = \frac{c}{3}\left(\sum_{j\le k}\log\frac{(v_k-u_j)}{a} - \sum_{j<k}\log\frac{(u_k-u_j)}{a} - \sum_{j<k}\log\frac{(v_k-v_j)}{a} \right) + Nc_1',$$

with a regularization given by the parameter a. A related interesting property shown in [**8**] is the ratio formula

$$\frac{S_{n,A}S_{n,B}}{S_{n,A\cup B}S_{n,A\cap B}} = \left(\frac{(v_1-u_1)(v_2-u_2)}{(u_2-u_1)(v_2-v_1)} \right)^{\frac{c}{6}(n-\frac{1}{n})} \tag{2.2}$$

which is simply given by a power of the cross-ratio.

In fact, (2.1) was regarded in [**8**] as sufficient to determine the entanglement entropy. However, it was then observed in [**9**] that the term in the formula (2.1) needs to be corrected, to account for the geometry of the cyclic branched coverings as higher genus Riemann surfaces. This was done in [**9**], where it was shown that (2.1) acquires an additional factor $\mathcal{F}_n$, which is expressible in terms of Riemann theta functions and the period matrix of the Riemann surface, through the Thomae formula for cyclic branched coverings. We will discuss the geometry in more detail.

2.1. Geometry of branched coverings. Before we consider the question of the p-adic case, we discuss a geometric setting for the complex case that will have a good p-adic analog. In the case of Riemann surfaces, we can work with the projective line $\mathbb{P}^1(\mathbb{C})$ as the base space $X(\mathbb{C})$ and consider n-fold branched coverings $X_n(\mathbb{C})$.

As a general setting to construct branched coverings, consider a divisor $D = \sum_{i=0}^k m_i w_i$ in $\mathbb{P}^1(\mathbb{C})$ with $m_i \in \mathbb{N}$. The datum $(\mathbb{P}^1(\mathbb{C}), D)$ determines a genus zero orbifold with orbifold fundamental group $\pi_1^{orb}(\mathbb{P}^1(\mathbb{C}), D) = \langle \gamma_0, \ldots, \gamma_k \,|\, \gamma_0^{m_0} = \gamma_1^{m_1} = \ldots = \gamma_k^{m_k} = \gamma_0 \cdots \gamma_k = 1\rangle$. We consider branched coverings $\pi : \Sigma \to \mathbb{P}^1(\mathbb{C})$ branched at D, that is, the points w_i are the branch locus and the non-negative integers m_i are the branch indices. We require that Σ is a compact Riemann surface and we denote by G the Galois group of this branch covering. The Riemann-Hurwitz formula relates the Euler characteristics of Σ and $\mathbb{P}^1(\mathbb{C})$ and the branch data

$$\chi(\Sigma) = \#G \cdot (\chi(\mathbb{P}^1(\mathbb{C}) \smallsetminus W) + \sum_{i=0}^k \frac{1}{m_i}) = \#G \cdot (2 + \sum_{i=0}^k (\frac{1}{m_i} - 1)),$$

with $W = \{w_0, \ldots, w_k\}$. The covering $\pi : \Sigma \smallsetminus \pi^{-1}(W) \to \mathbb{P}^1(\mathbb{C}) \smallsetminus W$, with $W = \{w_i\}$, corresponds to a normal subgroup Γ of $\pi_1(\mathbb{P}^1(\mathbb{C}) \smallsetminus W)$ with the Galois group of the covering given by $G = \mathrm{Aut}(\Sigma) = \pi_1(\mathbb{P}^1(\mathbb{C}) \smallsetminus W)/\Gamma$.

For instance, if one considers branched covering $\Sigma \to \mathbb{P}^1(\mathbb{C})$ with either two, three, or four branch points, one has the following cases. The case with two branch points has $\Sigma = \mathbb{P}^1(\mathbb{C})$ and $m_0 = m_1 = m$ with group $\mathbb{Z}/m\mathbb{Z}$. The case of three points is the more interesting case of triangle groups. In this case, depending on whether $m_0^{-1} + m_1^{-1} + m_2^{-1} > 1$ or $= 1$ or < 1, the covering is $\Sigma = \mathbb{P}^1(\mathbb{C})$ and the triangle group is either the dihedral $(2, 2, n)$, tetrahedral $(2, 3, 3)$, octahedral $(2, 3, 4)$, or icosahedral $(2, 3, 5)$ group, or $\Sigma = E(\mathbb{C})$ is an elliptic curve with the square $(2, 4, 4)$, triangular $(3, 3, 3)$, and hexagonal $(2, 3, 6)$ Euclidean triangle groups, or else Σ is a Riemann surface of genus $g > 1$ and a hyperbolic triangle Fuchsian group. The case of four branched points involves uniformizations of genus zero orbifolds with four orbifold points by Fuchsian quadrangle groups.

In general, given an orbifold covering $\pi : \Sigma \to \mathbb{P}^1(\mathbb{C})$ corresponding to a datum $(\mathbb{P}^1(\mathbb{C}), D)$, let $\phi : \mathbb{P}^1(\mathbb{C}) \to \Sigma$ denote the multivalued inverse of the projection map π, also known as the "developing map", determined up to composition with elements of $G = \mathrm{Aut}(\Sigma)$. Let z be a coordinate on Σ and w a coordinate on $\mathbb{P}^1(\mathbb{C})$. Then a developing map $z = \phi(w)$ (period map) is obtained as the ratio $\phi(w) = u_1(w)/u_2(w)$ of two independent non-trivial solutions of the orbifold uniformization differential equation (Proposition 4.2 of [**59**])

$$\frac{d^2u}{dw^2} + \frac{1}{2}\{z, w\}u = 0 \tag{2.3}$$

where $\{z, w\} = \{\phi(w), w\} = \mathcal{S}(\phi)(w)$ denotes the Schwarzian derivative.

Assuming the branch points w_i are on the real locus $\mathbb{P}^1(\mathbb{R})$, the Schwarzian derivative $\mathcal{S}(\phi)(w)$ can be computed by considering the map $z(w)$ as a holomorphic map of $\mathbb{H}$ to a circular arc polygon with angles π/m_i that extends to a continuous map on the boundary. This gives a rational function

$$R(w) = \sum_i \frac{(1 - m_i^{-2})}{2(w - w_i)^2} + \frac{\beta_i}{w - w_i} \tag{2.4}$$

which depends on auxiliary parameters β_i satisfying

$$\sum_i \beta_i = 0, \quad \sum_i 2w_i\beta_i + (1 - m_i^{-2}) = 0, \quad \sum_i w_i^2\beta_i + w_i(1 - m_i^{-2}) = 0 \tag{2.5}$$

and $\mathcal{S}(\phi)(w) = R(w)$.

One then aims at solving $\mathcal{S}(\phi)(w) = R(w)$ for ϕ using the orbifold uniformization equation (2.3). In the case of triangle groups this reduces to a hypergeometric equations, hence the resulting $\phi(w)$ can be described in terms of hypergeometric functions.

In [**8**] the Rényi and Entanglement Entropies are computed explicitly in terms of the Schwarzian derivative (2.4). The explicit solution of the uniformization equation (and the period matrix) enter in the Rényi and Entanglement Entropies also through the Thomae formula, as shown in [**9**]. We discuss this more in detail below.

2.2. Cyclic branched coverings of $\mathbb{P}^1(\mathbb{C})$. A class of branched coverings we can consider in the complex case, which has a good p-adic analog, is the case of cyclic branched coverings of $\mathbb{P}^1(\mathbb{C})$. A cyclic covering $X_n(\mathbb{C})$ of $\mathbb{P}^1(\mathbb{C})$ is explicitly given, as an algebraic curve, by an equation of the form

$$y^n = (x - a_1)^{d_1} \cdots (x - a_\ell)^{d_\ell} \tag{2.6}$$

with exponents $1 \leq d_k \leq n - 1$. The Galois group is $G = \mathbb{Z}/n\mathbb{Z}$ acting by $(x, y) \mapsto (x, \zeta y)$ with ζ an n-th root of unity, and the covering map $\pi : X_n(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ given by the projection $(x, y) \mapsto x$, corresponds to the homomorphism $\pi_1(\mathbb{P}^1(\mathbb{C}) \smallsetminus Z) \to \mathbb{Z}/n\mathbb{Z}$, with $Z = \{a_0, a_1, \ldots, a_\ell\}$ that maps the generator $\gamma_k \mapsto d_k \mod n$ for $k = 1, \ldots, \ell$ and $\gamma_0 \mapsto -\sum_i d_i \mod n$. The covering is unramified at infinity if $n | \sum_i d_i$. The curve (2.6) is singular whenever $d_k > 1$ for some k. We will in particular discuss the nonsingular case (known in the physics literature as the nonsingular Z_n curves)

$$y^n = \prod_{j=1}^{n\ell} (x - a_j). \tag{2.7}$$

The period map for cyclic covers (2.6) of the projective line is given by a multivariable generalization of hypergeometric functions, namely the Lauricella functions, see [**15**], [**40**]. Given a cyclic covering $X_n(\mathbb{C})$ as in (2.6), let $\mu_i = d_i/n$. The multivalued differential

$$\eta_a = (x - a_0)^{-\mu_0} \cdots (x - a_\ell)^{-\mu_\ell} dx$$

lifts to a single-valued differential $y^{-1}dx$ on the curve $X_n(\mathbb{C})$. For every relative arc γ_a in $\mathbb{P}^1 \smallsetminus A$ with $A = \{a_i\}_{i=1}^{\ell}$, and for D_k open disks around a_k, the holomorphic function

$$z = (z_1, \ldots, z_\ell) \in D_1 \times \cdots \times D_\ell \mapsto \int_{\gamma_z} \eta_z$$

are the Lauricella functions. The integrals $\int_{\delta_k} \eta_z$ where δ_k is an oriented arc from z_k to z_{k+1}, with $z_{\ell+1} = \infty$ provide a basis.

Note how, in this setting, the cyclic branched cover is completely determined by the orbifold data of the set of branch points on $\mathbb{P}^1(\mathbb{C})$ and the local monodromy at those points. The cuts, which correspond to the paths γ_k in $\mathbb{P}^1(\mathbb{C})$ connecting successive branch points, only enter in determining a basis in which to represent the period matrix for the differentials corresponding to the Lauricella functions.

In terms of the computation of entanglement entropy, one considers in [**8**] branch points on $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$ and arcs γ_k that are also intervals in $\mathbb{P}^1(\mathbb{R})$, so that one thinks of the entanglement entropy as being associated to a real one-dimensional region consisting of intervals in $\mathbb{P}^1(\mathbb{R})$ with endpoints at branch points. Holographically, the entanglement entropy of such a region should correspond to the geometry (a regularized length/area) of a region in the bulk that shares the same boundary with the boundary region. However, one can just as well think of the entanglement entropy as being determined by the orbifold data of the branch points and their monodromies, or equivalently by the algebraic curves that gives the cyclic branched coverings, since the information about the location of the branch points should also suffice to determine the corresponding bulk region that is relevant for the holographic correspondence. The information about the cuts (the boundary region) is retained as a choice of basis for the period matrix. The reason for thinking of it in these terms is that all of this goes through without difficulty to the p-adic case, unlike the formulation in terms of 1-dimensional real curves in the boundary.

A similar situation arises in the computation of entanglement entropy via a tensor network dual to the Bruhat–Tits tree performed in [**30**], where instead of regions (intervals) on the boundary $\mathbb{P}^1(\mathbb{R})$ one should consider the entanglement entropy as associated to configurations of points on the boundary $\mathbb{P}^1(\mathbb{Q}_p)$.

2.3. The Entanglement Entropy for cyclic covers of $\mathbb{P}^1(\mathbb{C})$. We discuss here the derivation of the Rényi entropies $S_{n,A}$ of [**9**] in terms of the geometry of cyclic branched coverings of $\mathbb{P}^1(\mathbb{C})$ given by the nonsingular Z_n curves (2.7). The explicit expression for the Rényi entropies depends on the Thomae formula for nonsingular Z_n curves. This is a broad generalization of the classical Thomae formula for hyperelliptic curves, relating the branch points to the Jacobi theta functions. The Thomae formula for cyclic branched coverings of $\mathbb{P}^1(\mathbb{C})$, both singular and nonsingular, has been extensively studied, see for instance [**4**], [**24**], [**20**], [**37**], [**49**].

We recall here an essential outline of the argument given in [**9**] [**10**] for the computation of the Rényi entropies. In [**8**] and [**9**], [**10**] for a cyclic branched covering X_n (2.7) with 2ℓ branched points $\{\lambda_k\}_{k=1}^{2\ell} = \{a_i, b_i\}_{i=1}^{\ell}$, a path integral argument is used to compute the partition function $\mathrm{Tr}(\rho_A^n)$ as

$$\mathrm{Tr}(\rho_A^n) = \prod_{\sigma\in\mathbb{Z}/n\mathbb{Z}} \mathcal{Z}_\sigma = \prod_{\sigma\in\mathbb{Z}/n\mathbb{Z}} \langle \mathcal{T}_\sigma(a_1)\mathcal{T}_{\sigma^{-1}}(b_1)\cdots\mathcal{T}_\sigma(a_\ell)\mathcal{T}_{\sigma^{-1}}(b_\ell)\rangle,$$

where the $\mathcal{T}_\sigma$ are twist operators at the branch points, which implement the action of the Galois group $\mathbb{Z}/n\mathbb{Z}$ of the cyclic covering. The argument for the computation of the expectation value

$$\langle \mathcal{T}_\sigma(a_1)\mathcal{T}_{\sigma^{-1}}(b_1)\cdots\mathcal{T}_\sigma(a_\ell)\mathcal{T}_{\sigma^{-1}}(b_\ell)\rangle,$$

which is done in [**9**], [**10**] for the case of 4 branch points a_1, b_1, a_2, b_2, relies on expressing the logarithmic derivative $d\log\mathcal{Z}_\sigma$ in terms of the Szegö kernel associated to the canonical differential $\omega(z_1, z_2)$ on $X_n\times X_n$, which in turn satisfies

$$\lim_{z_1\to z_2}(\omega(z_1,z_2) - \frac{dz_1dz_2}{(z_1-z_2)^2}) =$$

$$\frac{1}{n}\sum_{i,j=1}^{2\ell}\frac{q_{ij}\,dz^2}{(z-\lambda_i)(z-\lambda_j)} - \sum_{i,j=1}^{g}\frac{\partial^2\log\theta[e_A](0)}{\partial z_i\partial z_j}\,\omega_i(z)\omega_j(z),$$

where $\theta[e](z)$ is the Riemann theta function with characteristic, where the characteristic $e\in\mathbb{C}^g$ is of the form $e = 2\pi i\epsilon + \delta\tau(X_n)$, with $\epsilon,\delta\in\mathbb{R}^g$, where $\tau=\tau(X_n)$ is the $g\times g$ period matrix of X_n, with respect to a basis $\{A_i, B_i\}_{i=1}^g$ of $\pi_1(X_n)$ and a basis of normalized holomorphic 1-forms $\omega_i(z)$,

$$\int_{A_i}\omega_j = 2\pi i\delta_{ij}, \quad \int_{B_i}\omega_j = \tau_{ij},$$

and the theta function given by

$$\theta[e](z) = \sum_{m\in\mathbb{Z}^g}\exp(\frac{1}{2}(m+\delta)\tau(m+\delta)^t + (z+2\pi i\epsilon)(m+\delta)^t).$$

When the region A for which we are computing the Rényi entropy $S_{n,A}$ is given by a union of ℓ disjoint paths γ_i connecting the branch a_i to the branch point b_i, the characteristic e_A is given by

$$e_A = \begin{bmatrix}\delta\\ \epsilon\end{bmatrix} = \left(\sum_i\int_{\tilde\gamma_i}\omega_1,\ldots,\sum_i\int_{\tilde\gamma_i}\omega_g\right),$$

where $\tilde\gamma_i$ are lifts to X_n of the paths γ_i in $\mathbb{P}^1(\mathbb{C})$. The coefficients q_{ij} are given by ([**37**], [**49**])

$$q_{ij} = \sum_k q_k(i)q_k(j),$$

where the sum is over $k = -\frac{n-1}{2}, -\frac{n-1}{2}+1, \ldots, \frac{n-1}{2}$ and

$$q_k(j) = \frac{1-n}{2n} + \frac{k+j+\frac{n-1}{2}}{n}.$$

This behavior of the Szegö kernel and its relation to the Riemann theta function is the source of the Thomae formula

$$\theta[e_A](0)^{2n} = C_A(\det\mathcal{A})^n\prod_{i<j}(\lambda_i-\lambda_j)^{2nq_{ij}+n\mu} \tag{2.8}$$

where $\mu = (n-1)(2n-1)/(6n)$ and the matrix $\mathcal{A}$ is the $g\times g$ matrix of periods given by the non-normalized holomorphic differentials evaluated on the a_i loops.

In the case considered in [**9**], these are expressed explicitly in terms of the classical hypergeometric function $F(k/n, 1-k/n, 1|x)$, where we write F for the hypergeometric function ${}_2F_1$. In turn the Thomae formula provides the formula (2.1) for the Rényi entropy (with the correct additional factor $\mathcal{F}_n$), see [**20**], [**37**], [**49**].

2.4. Some general facts on Thomae's formula. For (z,Ω) in $\mathbb{C}^g\times\mathbb{H}_g$, with $\mathbb{H}_g$ the Siegel upper half plane, the Riemann theta function with characteristic is given by

$$\theta\left[\begin{array}{c} 2a \\ 2b \end{array}\right](z,\Omega) = \sum_{n\in\mathbb{Z}^g} e^{\pi i (n+1)^t\Omega(n+a)+2\pi i (z+b)^t(n+a)}$$

with $a,b\in(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$.

In its original form, Thomae's formula relates the branch points of hyperelliptic curves to theta constants, [**56**] and §8 of [**48**]. In this case, for a curve

$$y^2 = \prod_{i=1}^{2m}(x-\lambda_i)$$

the Thomae formula expresses in terms of the parameters λ_i the Riemann theta constant for a partition of the branch points $\{1,\dots,2m\}=\Lambda_1\sqcup\Lambda_2$ into two ordered partitions $\Lambda_1=\{i_1<\cdots<i_m\}$ and $\Lambda_2=\{j_1<\cdots<j_m\}$,

$$\theta[e](0,\Omega)^8 = \left(\frac{\det A}{(2\pi i)^{m-1}}\right)^4 \prod_{k<\ell}(\lambda_{i_k}-\lambda_{i_\ell})^2(\lambda_{j_k}-\lambda_{j_\ell})^2,$$

where $e=2\pi i\epsilon+\delta\Omega$ determined by the partition of the branch points and the two points at infinity that compactify the Riemann surface, and where $\{A_i,B_i\}$ are a basis of the first homology and $A=(\int_{A_i} x^{j-1}dx)_{i,j=1,\dots,m}$ is the period matrix.

A generalization to Z_N-curves of the form

$$y^N = \prod_{i=1}^{Nm}(x-\lambda_i)$$

was proved in [**49**] and similarly relates the Riemann theta constant for a divisor associated to an ordered partition $\Lambda=(\Lambda_0,\dots,\Lambda_{N-1})$ if the branch points to the parameters λ_i as in (2.8). For further results on Thomae formulae for Z_N-curves see [**4**], [**20**], [**37**], [**38**].

It is worth recalling here that there are different approaches to proving such Thomae formulae. In particular, one approach based on the relation between the Riemann theta functions and the asymptotics of the Szegö kernel, for which one can obtain explicit algebraic expressions, as in [**37**], [**49**]. Another approach, used in [**20**] shows that, for a non-singular Z_N-curve $y^n=\prod_{i=1}^{rn-1}(x-\lambda_i)$, there is a polynomial h_D in the parameters of the branched cover such that the ratio

$$\frac{\theta^{2\rho n^2}[D](0,\Omega)}{h_D},$$

with $\rho=1$ for n even and $\rho=2$ if n is odd, is constant in the divisor D. This is shown by checking that it is invariant under some basic transformations of D that are then shown to span the whole space. The case of singular Z_N-curves is treated with a similar approach in §5 of [**20**].

In seeking a general p-adic version of the Thomae formula, beyond the case of Mumford curves of genus two of [**54**] which we will review in §3.5, one can either consider a possible approach via a version of analytic objects like the Szegö kernel or a version along the lines of the argument of [**20**] with the transformations of the divisor space. In the first case, one knows that there is a p-adic Poisson kernel introduced in [**55**], which we will review in §3.14. However, it is possible that the method of [**20**] may turn out to be more suitable to adapt to the case of Mumford curves.

2.5. Fuchsian triangle groups. There is one specific case that is worth discussing more in detail in the complex case, before moving to their p-adic analog, namely the case of triangle groups. These have the advantage that the uniformization map can be explicitly computed in terms of hypergeometric functions, moreover, since these are Fuchsian groups of genus zero, there is a very nice interpretation in terms of Hauptmodul and modular functions. Another advantage of this case is that it does have direct p-adic analogs (although they are rare) and a subclass (the arithmetic Fuchsian triangle groups) that simultaneously exist in the complex and the p-adic setting.

The idea is to look explicitly at these cases that, due to their arithmetic nature give rise to curves defined over number fields. A number field, which is a finite extension of $\mathbb{Q}$, can be embedded in $\mathbb{C}$ (by one of its archimedean embeddings) or in some finite extension of $\mathbb{Q}_p$. This means that these arithmetic algebraic curves give rise simultaneously to a complex Riemann surface and to a p-adic algebraic curve. In such cases it should be possible to compute the entanglement entropy both via the usual archimedean setting with Riemann surfaces and p-adically and they should lead to the same answer, since the branch points and monodromies are given by algebraic data, and the formula for the entropy $S_{n,A}$ in [**8**] in such a case gives an algebraic number (where here we take A to mean the data of branch points and monodromies). Note that when one takes the limit defining S_A in terms of the $S_{n,A}$, then the limit itself will be different in the complex and p-adic case, because in one case it is taken with respect to the topology of $\mathbb{C}$ and in the other with respect to that of $\mathbb{C}_p$, but the data $S_{n,A}$ themselves would be the same, in the arithmetic cases, for both complex and p-adic setting.

Triangle groups $\Delta(m_0, m_1, m_2) = \langle \gamma_0, \gamma_1, \gamma_2 \,|\, \gamma_0^{m_0} = \gamma_1^{m_1} = \gamma_2^{m_2} = \gamma_0\gamma_1\gamma_2 = 1\rangle$ for $m_i \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, are subdivided into spherical (when $m_0^{-1} + m_1^{-1} + m_2^{-1} > 1$), flat (when $m_0^{-1} + m_1^{-1} + m_2^{-1} = 1$) and hyperbolic (when $m_0^{-1} + m_1^{-1} + m_2^{-1} < 1$). There are only the following cases of spherical and flat triangle groups: $\Delta(2, n, n)$, $\Delta(2, 3, 3)$, $\Delta(2, 3, 4)$, $\Delta(2, 3, 5)$ for the spherical case, corresponding as we mentioned above to the dihedral, tetrahedral, octahedral and icosahedral tilings of the sphere, and $\Delta(2, 2, \infty)$, $\Delta(2, 3, 6)$, $\Delta(2, 4, 4)$ and $\Delta(3, 3, 3)$ for the flat case, corresponding to tilings of the plane by rectangles, squares, hexagons, and equilateral triangles. The hyperbolic cases correspond to tilings of the hyperbolic plane by triangles with angles π/a, π/b and π/c. We write $\Delta(m_0, m_1)$ for the triangle groups $\Delta(m_0, m_1, \infty)$. The fact that the third angle is 0 in this case means that one of the vertices of the resulting triangle is a cusp. The hyperbolic triangle groups $\Delta(m_0, m_1)$ embed in $\mathrm{PSL}_2(\mathbb{R})$. In particular, $\mathrm{PSL}_2(\mathbb{Z}) = \Delta(2, 3)$.

Recall that Belyĭ's theorem states that the following are equivalent conditions for a smooth algebraic curve X:

- X is defined over a number field;
- $X(\mathbb{C}) = \mathbb{H}/\Gamma$ for some finite index subgroup Γ of a triangle group $\Delta(m_0, m_1)$;
- there is a branched covering $f : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ branched over $\{0, 1, \infty\}$.

Thus, in particular, treating the case of triangle groups (and finite index subgroups) covers all the arithmetic cases of curves defined over number fields.

Recall that the automorphic functions for a Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ are functions $f : \mathbb{H}_\Gamma \to \mathbb{C}$ (where $\mathbb{H}_\Gamma$ is the union of the upper half plane $\mathbb{H}$ with the cusps of Γ in $\partial\mathbb{H}$) that are Γ-invariant and meromorphic at the cusps. Automorphic functions for Γ form a field $\mathcal{K}_\Gamma$. In the case where the Fuchsian group Γ has genus zero this is the field of rational functions of a single generator j_Γ, called the *Hauptmodul* of Γ, that is $\mathcal{K}_\Gamma = \mathbb{C}(j_\Gamma)$. In the case of the triangle group $\Delta(2,3) = \mathrm{PSL}_2(\mathbb{Z})$ this is the usual j-invariant (the generator of the modular field), hence the notation j_Γ.

The Hauptmodul for a triangle group can be computed via the Schwarzian equation, $\mathcal{S}(\phi)(w) = R(w)$ with

$$R(w) = \sum_{i=0}^{3} \sum_i \frac{(1 - m_i^{-2})}{2(w - w_i)^2} + \frac{\beta_i}{w - w_i}$$

for the Schwarz triangle with vertices w_i and angles π/m_i, as in (2.4), with auxiliary parameters (2.5). In the case of a triangle the auxiliary parameters are determined uniquely by (2.5). The map $z = \phi(w)$ maps the triangle to $\mathbb{H}$ and satisfies $\mathcal{S}(\phi)(w) = R(w)$. The Hauptmodul $w = j_\Gamma(z)$, mapping $\mathbb{H}$ to the triangle, correspondingly satisfies $\mathcal{S}(j_\Gamma)(z) = R(j_\Gamma) \cdot (\frac{dj_\Gamma}{dz})^2 = 0$. By expanding j_Γ into a Fourier series at the cusp, in the variable $q = \exp(2\pi i z/\omega)$ where ω is the cusp width, one has $j_\Gamma(z) = \frac{1}{q} + \sum_{k=0}^\infty c_k q^k$. By plugging this series into the equation $\mathcal{S}(j_\Gamma)(z) = R(j_\Gamma) \cdot (\frac{dj_\Gamma}{dz})^2 = 0$, it is possible to get a system of equations for the c_k coefficients, where the k-th equation can be solved for c_{k-1} as a function of the w_i and m_i. In the case of a triangle group $\Delta(m_0, m_1)$ the Schwarzian equation reduces to a classical hypergeometric equation. See [**17**] for more details.

Explicit examples of Hauptmodul of Fuchsian triangle groups that give the uniformization of the corresponding orbifold are given in [**28**]. In the case of the triangle group $\Delta(2,3) = \mathrm{PSL}_2(\mathbb{Z})$ one has

$$j_\Gamma = \frac{(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)^3}{54\vartheta_2^8\vartheta_3^8\vartheta_4^8},$$

expressed in terms of the null theta functions $\vartheta_2, \vartheta_3, \vartheta_4$, while for example in the case of the triangle group $\Delta(4,2)$ one has

$$\frac{(\vartheta_3^4 + \vartheta_4^4)^4}{16\vartheta_2^8\vartheta_3^4\vartheta_4^4},$$

or in the case of $\Delta(6,2)$ one has

$$\frac{\eta^{12}(\tau) + 27\eta^{12}(3\tau)}{108\eta^{12}(\tau)\eta^{12}(3\tau)},$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^\infty (1 - q^n)$ with $q = e^{2\pi i \tau}$, satisfying $\eta^3(\tau) = \frac{1}{2}\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau)$. see [**28**] for a more extensive list of explicit examples. The explicit expression of the inverse of the Schwarz function in terms of Riemann theta constants can be viewed as a version of Thomae's formula in these cases.

2.6. A holographic approach via Schottky uniformization. In [**3**] a holographic approach to the computation of entanglement entropy via the replica argument was developed using Schottky uniformization of Riemann surfaces.

A brief summary of the argument developed in [**3**] is as follows. One considers the same strategy as discussed above to compute the boundary entanglement entropy using cyclic branched coverings $X_n(\mathbb{C})$ and the Rényi entropies on these branched coverings in the form

$$S_n = -\frac{1}{n-1} \log \frac{\mathcal{Z}_n}{\mathcal{Z}_1^n},$$

with $\mathcal{Z}_1$ the partition function of the CFT on the original boundary Riemann surface $X(\mathbb{C})$ and $\mathcal{Z}_n$ the partition function on the n-sheeted cyclic branched covering $X_n(\mathbb{C})$. In this case, however, one assumes that the boundary CFT is holographic and has a gravity dual in the bulk space, so that by the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence the partition function of the boundary CFT is the same as the partition function on the dual gravitational theory in the bulk. Thus, the goal in this setting becomes to compute the bulk partition function, in the form of a classical term and a one-loop quantum correction term.

The bulk boundary correspondence is seen as in [**39**], [**45**] in terms of Schottky uniformization of the boundary Riemann surface and of the bulk hyperbolic handlebody. The bulk gravity theory is described by an action functional given by an Einstein–Hilbert action with cosmological term and a Gibbons–Hawkings boundary term,

$$S = -\frac{1}{2\kappa^2} \int d^3x \sqrt{g}(R + \frac{2}{L^2}) - \frac{1}{\kappa^2} \int d^2x \sqrt{h}(K - \frac{1}{L}) \tag{2.9}$$

which gives rise to a regularized volume of the hyperbolic handlebody, with boundary terms that depend on a cutoff surface and on the boundary of the fundamental domain of the Schottky group action on hyperbolic 3-space, and a Liouville action term for the field that defines the cutoff surface.

The explicit computational procedure for evaluating this classical contribution and the one-loop corrections require the explicit construction of a Schottky uniformization for the cyclic branched coverings $X_n(\mathbb{C})$. This is done in [**3**] in the case where the n-sheeted coverings are constructed by removing two intervals from $\mathbb{P}^1(\mathbb{R})$ inside the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, with branch points at the intervals endpoints. Among the cyclic coverings singular Z_n-curves of genus $g = (n-1)m$ with equations $y^n = p(x)q(x)^{n-1}$ with $p(x) = \prod_{j=0}^{m}(x - x_{2j+1})$ and $q(z) = \prod_{j=1}^{m}(x - x_{2j})$, with singularities at the $(x_2, 0), \dots, (x_{2m}, 0)$ points and branch points at $(x_1, 0), \dots, (x_{2m+1}, 0), (\infty, \infty)$, this case corresponds to $m = 1$ with $p(x) = (x - x_1)(x - x_3)$ and $q(x) = (x - x_2)(x - x_4)$. For these Z_n curves, an explicit Schottly uniformization is constructed using the uniformization equation (2.3) with (2.4), where in this case in (2.4) the $1 - m_i^{-2} = 1 - n^{-2}$ determines the behavior of solutions near the branch points and $2\beta_i = \gamma_i$ are the accessory parameters of the Schottky group. The independent pair of solutions u_1, u_2 of the uniformization equation (2.3) then transform under a monodromy action, where M_1, M_2 and M_3 with $M_1 M_2 M_3 = 1$ are the $\mathrm{PSL}_2(\mathbb{C})$ matrices that determine the monodromy around the branch points. The Schottky group generators

$\gamma_1, \ldots, \gamma_{n-1}$ are then obtained from these monodromy matrices as $\gamma_1 = M_2 M_1$ and $\gamma_i = M_2^{i-1} \gamma_1 M_2^{-(i-1)} = M_2^i M_1 M_2^{-(i-1)}$ for $i = 2, \ldots, n-1$.

The Rényi entropies for the bulk gravitational theory are then computed in [**3**] through a classical term from the on-shell action obtained as in [**21**], [**61**] by

$$\frac{\partial S}{\partial x_i} = -\frac{cn}{6}\gamma_i, \quad \text{with} \quad c = \frac{12\pi L}{\kappa^2},$$

with S and κ as in (2.9) and with γ_i the accessory parameters of the Schottky uniformization and x_i the four points. The one-loop quantum corrections are obtained in the form of a series over the primitive elements in the Schottky group

$$\log Z|_{\text{one-loop}} = -\sum_{\gamma \in \mathcal{P}} \sum_{m=2}^{\infty} \log |1 - q_\gamma^m|,$$

where $q_\gamma^{\pm 1/2}$ are the two eigenvalues of the loxodromic element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$.

3. Geometric background for a p-adic replica argument

The survey of the complex geometry aspects of the replica argument provided in the previous section shows that the main ingredients that are needed for constructing a p-adic analog are

- A p-adic theory of cyclic orbifold coverings X_n and uniformization.
- A p-adic analog of the period matrix $\tau(X_n)$.
- A p-adic analog of the Thomae formula relating the branch points, the period matrix, and the Riemann theta function.
- A p-adic valued version of the path integral argument that relates the expectation values for the twist operators to the period matrix and the Thomae formula.
- A holographic approach based on p-adic Schottky uniformization.

We survey in this section some known results about p-adic Mumford curves and p-adic Schottky uniformizations, cyclic branched coverings of the projective line, the Schwarzian derivative, known instances of the Thomae formula, and holographic correspondences between boundary and bulk geometries for Mumford curves.

3.1. Graphs and Mumford curves.

Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$, with $\mathcal{O}_{\mathbb{K}}$ its ring of integers and $\mathfrak{m} \subset \mathcal{O}_{\mathbb{K}}$ the maximal ideal, with residue field $\mathbb{F}_q = \mathcal{O}_{\mathbb{K}}/\mathfrak{m}$ of cardinality $q = p^r$ for some $r \in \mathbb{N}$. Let $\mathcal{T}_{\mathbb{K}}$ be the Bruhat–Tits tree of $\mathbb{K}$. Consider a p-adic Schottly group $\Gamma \subset \mathrm{PGL}_2(\mathbb{K})$, namely a finitely generated, discrete, torsion-free subgroup of $\mathrm{PGL}_2(\mathbb{K})$ whose nontrivial elements are all loxodromic (they have two fixed points in $\mathbb{P}^1(\mathbb{K})$). Let $L(\gamma)$ denote the infinite path in $\mathcal{T}_{\mathbb{K}}$ connecting the two fixed points $z^{\pm}(\gamma)$. The limit set $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{K})$ is the closure in $\mathbb{P}^1(\mathbb{K})$ of the set of fixed points $\{z^{\pm}(\gamma) : \Gamma \ni \gamma \neq 1\}$. The quotient of $\Omega_\Gamma = \mathbb{P}^1 \smallsetminus \Lambda_\Gamma$ by the action of Γ is the p-adic Mumford curve X_Γ uniformized by Γ, which is the set of endpoints of the quotient $\mathcal{T}_{\mathbb{K}}/\Gamma$. The quotient of the subtree $\mathcal{T}_\Gamma \subset \mathcal{T}_{\mathbb{K}}$ determined by the paths $L(\gamma)$ of all $\gamma \neq 1$ in Γ has quotient $\mathcal{T}_\Gamma/\Gamma = G_\Gamma$ a finite graph. The graph $\mathcal{T}_{\mathbb{K}}/\Gamma$ consists of the finite graph G_Γ with infinite trees appended to its vertices. The genus of the Mumford curve is the same as the number of generators of Γ and is also the same as the first Betti number of the graph G_Γ. A finite graph G is stable if it is connected and such that each vertex that is not connected to itself by an edge is the source of at least three edges. Any stable graph can occur as the graph G_Γ of a Mumford curve, [**23**, p.124].

The case where $\Gamma = q^{\mathbb{Z}}$, with $0 < |q| < 1$, is of rank one and $\Omega = \mathbb{P}^1 \smallsetminus \{0, \infty\}$ is the case of Tate uniformized p-adic elliptic curves. In this case the graph G_Γ is a polygon. Only the genus zero and genus one cases have so far been investigated in full detail in terms of the p-adic AdS/CFT correspondence and the computation of entanglement entropy and the associated Ryu–Takayanagi formula relating it to bulk geometry, see [**30**]. The higher genus cases can be viewed as p-adic analogs of the higher genus generalizations of the Euclidean BTZ black hole discussed in the complex Riemann surface setting of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence in [**39**], [**45**].

3.2. Currents on graphs and p-adic holomorphic functions. A current on an oriented locally finite graph $\mathcal{G}$ is a map $\mu : E(\mathcal{G}) \to \mathbb{Z}$ from the oriented edges of $\mathcal{G}$ to the integers satisfying the conditions

$$\mu(\bar{e}) = -\mu(e),$$

where $\bar{e}$ denotes the edge with the reverse orientation, and

$$\sum_{e: s(e)=v} \mu(e) = 0.$$

Currents form an abelian group, denoted by $\mathcal{C}(\mathcal{G})$.

Let $\mathcal{H}_{\mathbb{K}}$ denote the Drinfeld p-adic upper half plane, defined as $\mathcal{H}_{\mathbb{K}} = \mathbb{P}^1 \smallsetminus \mathbb{P}^1(\mathbb{K})$ (this means that for an extension $\mathbb{L}$ of $\mathbb{K}$ the set of $\mathbb{L}$-points of $\mathcal{H}$ is given by $\mathcal{H}_{\mathbb{K}}(\mathbb{L}) = \mathbb{P}^1(\mathbb{L}) \smallsetminus \mathbb{P}^1(\mathbb{K})$. In particular, if $\mathbb{C}_{\mathbb{K}}$ denotes the completion of the algebraic closure $\bar{\mathbb{K}}$ (the analog of $\mathbb{C}$ in the p-adic setting) then $\mathcal{H}_{\mathbb{K}}(\mathbb{C}_{\mathbb{K}}) = \mathbb{P}^1(\mathbb{C}_{\mathbb{K}}) \smallsetminus \mathbb{P}^1(\mathbb{K})$ is a p-adic analog of (the complex points of) the upper half plane.

There is an algebra of p-adic holomorphic functions on the Drinfeld upper half plane (see e.g. [**41**]), which we denote by $\mathcal{O}(\mathcal{H}_{\mathbb{K}})$. The group of invertible holomorphic functions is denoted by $\mathcal{O}^*(\mathcal{H}_{\mathbb{K}})$.

Currents on the Bruhat-Tits tree are related to p-adic holomorphic functions on the Drinfeld upper half plane through the short exact sequence (Theorem 2.1 of [**57**])

$$0 \to \mathbb{K}^* \to \mathcal{O}(\mathcal{H})^* \to \mathcal{C}(\mathcal{T}_{\mathbb{K}}) \to 0 \tag{3.1}$$

This means that, up to multiplication by constants, invertible holomorphic functions can be combinatorially constructed using currents on the tree, $\mathcal{C}(\mathcal{T}_{\mathbb{K}}) = \mathcal{O}(\mathcal{H})^*/\mathbb{K}^*$.

One can also consider currents on a locally finite directed graph $\mathcal{G}$ with values in a field $\mathbb{K}$ of characteristic zero, by taking $\mathcal{C}(\mathcal{G}, \mathbb{K}) = \mathcal{C}(\mathcal{G}) \otimes_{\mathbb{Z}} \mathbb{K}$. There is another short exact sequence (Corollary 2.1.2 of [**57**]) relating currents $\mathcal{C}(\mathcal{T}_{\mathbb{K}}, \mathbb{K})$ on the Bruhat-Tits tree and p-adic holomorphic 1-forms on the Drinfeld upper half plane

$$0 \to \mathcal{O}(\mathcal{H}) \xrightarrow{d} \Omega^1(\mathcal{H}) \to \mathcal{C}(\mathcal{T}_{\mathbb{K}}, \mathbb{K}) \to 0. \tag{3.2}$$

Thus, $\mathbb{K}$-valued currents on the Bruhat-Tits tree provide a combinatorial way of describing holomorphic 1-forms modulo exact forms.

The setting described here and the exact sequences (3.1) and (3.2) extend to the case where, instead of the Drinfeld plane $\mathcal{H} = \mathbb{P}^1 \smallsetminus \mathbb{P}^1(\mathbb{K})$ one replaces $\mathbb{P}^1(\mathbb{K})$

by a compact subset Λ of $\mathbb{P}^1(\mathbb{K})$ and the Bruhat-Tits tree $\mathcal{T}_{\mathbb{K}}$ with $\partial\mathcal{T}_{\mathbb{K}} = \mathbb{P}^1(\mathbb{K})$ by a subtree $\mathcal{T}_\Lambda \subset \mathcal{T}_{\mathbb{K}}$ with $\partial\mathcal{T}_\Lambda = \Lambda$. Setting $\Omega = \mathbb{P}^1 \smallsetminus \Lambda$, one has exact sequences

$$0 \to \mathbb{K}^* \to \mathcal{O}(\Omega)^* \to \mathcal{C}(\mathcal{T}_\Lambda) \to 0$$

$$0 \to \mathcal{O}(\Omega) \xrightarrow{d} \Omega^1(\Omega) \to \mathcal{C}(\mathcal{T}_\Lambda, \mathbb{K}) \to 0.$$

In particular, one can consider the case where $\Lambda = \Lambda_\Gamma$ is the limit set of a p-adic Schottky group $\Gamma \subset \mathrm{PGL}(2, \mathbb{K})$ and $\Omega = \Omega_\Gamma$ is the uniformization domain of a Mumford curve $X_\Gamma = \Omega_\Gamma/\Gamma$. In this case the tree $\mathcal{T}_\Lambda = \mathcal{T}_\Gamma$ is the subtree of the Bruhat-Tits tree determined by the paths $L(\gamma)$ connecting the two fixed points of the nontrivial elements $\gamma \in \Gamma$.

There is a homological interpretation of currents. Namely, given a locally finite directed graph $\mathcal{G}$, consider, in addition to the abelian group $\mathcal{C}(\mathcal{G})$ of currents, the abelian group $\mathcal{A}(\mathcal{G})$ of integer valued functions $h : E(\mathcal{G}) \to \mathbb{Z}$ on the set of oriented edges of $\mathcal{G}$ that satisfy $h(\bar{e}) = -h(e)$ under orientation reversal, and the abelian group $\mathcal{F}(\mathcal{G})$ of integer valued function on the set $V(\mathcal{G})$ of vertices of $\mathcal{G}$. Consider the homomorphism $d : \mathcal{A}(\mathcal{G}) \to \mathcal{F}(\mathcal{G})$ given by

$$d(h)(v) = \sum_{e:s(e)=v} h(e).$$

Then the group $\mathcal{C}(\mathcal{G})$ of currents is the kernel of this morphism and fits in the short exact sequence

$$0 \to \mathcal{C}(\mathcal{G}) \to \mathcal{A}(\mathcal{G}) \xrightarrow{d} \mathcal{F}(\mathcal{G}) \to 0. \tag{3.3}$$

3.3. Jacobian of a Mumford curve. The Jacobian of a Mumford curve X_Γ can be described (see [**57**], Lemma 6.3 and Theorem 6.4) by the isomorphism

$$\mathrm{Pic}^0(X_\Gamma) \cong \mathrm{Hom}(\Gamma, \mathbb{K}^*)/c(\Gamma_{ab}), \tag{3.4}$$

where $\Gamma_{ab} = \Gamma/[\Gamma, \Gamma]$ denotes the abelianization, $\Gamma_{ab} \cong \mathbb{Z}^g$, with g the genus, and where the homomorphism

$$c : \Gamma_{ab} \to \mathrm{Hom}(\Gamma_{ab}, \mathbb{K}^*) \tag{3.5}$$

is defined by the first map in the homology exact sequence

$$0 \to \mathcal{C}(\mathcal{T}_\Gamma)^\Gamma \xrightarrow{c} \mathrm{Hom}(\Gamma, \mathbb{K}^*) \to H^1(\Gamma, \mathcal{O}(\Omega_\Gamma)^*) \to H^1(\Gamma, \mathcal{C}(\mathcal{T}_\Gamma)) \to 0, \tag{3.6}$$

associated to the short exact sequence (3.1).

In the sequence (3.6), one uses the fact that $H^i(\Gamma) = 0$ for $i \geq 2$ and the identification

$$\mathcal{C}(\mathcal{T}_\Gamma)^\Gamma = H^0(\Gamma, \mathcal{C}(\mathcal{T}_\Gamma)) = \Gamma_{ab} = \pi_1(\mathcal{T}_\Gamma/\Gamma)_{ab} = H_1(G_\Gamma, \mathbb{Z}), \tag{3.7}$$

see [**57**], Lemma 6.1 and Lemma 6.3.

Consider then the short exact sequence (3.3) applied to the case of the tree $\mathcal{T}_\Gamma$,

$$0 \to \mathcal{C}(\mathcal{T}_\Gamma) \to \mathcal{A}(\mathcal{T}_\Gamma) \xrightarrow{d} \mathcal{F}(\mathcal{T}_\Gamma) \to 0. \tag{3.8}$$

The long exact homology sequence associated to (3.8) is given by

$$0 \to \mathcal{C}(\mathcal{T}_\Gamma/\Gamma) \to \mathcal{A}(\mathcal{T}_\Gamma/\Gamma) \xrightarrow{d} \mathcal{F}(\mathcal{T}_\Gamma/\Gamma) \xrightarrow{\Phi} H^1(\Gamma, \mathcal{C}(\mathcal{T}_\Gamma)) \to 0, \tag{3.9}$$

where one has $H^1(\Gamma, \mathcal{C}(\mathcal{T}_\Gamma)) \cong \mathbb{Z}$ and, under this identification, the last map in the exact sequence is given by

$$\Phi : \mathcal{F}(G_\Gamma) \to H^1(\Gamma, \mathcal{C}(\mathcal{T}_\Gamma)) = \mathbb{Z}, \quad \Phi(f) = \sum_{v \in V(G_\Gamma)} f(v). \tag{3.10}$$

Moreover, one has an identification

$$H^1(\Gamma, \mathcal{O}(\Omega_\Gamma)^*) = H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X),$$

the group of equivalence classes of holomorphic (hence by GAGA algebraic) line bundles on the Mumford curve $X = X_\Gamma$, and the last map in the exact sequence (3.6) is then given by the degree map $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$, whose kernel is the Jacobian $J(X) = \mathrm{Pic}^0(X)$, see [**57**] Lemma 6.3.

3.4. Theta functions on Mumford curves. A theta function for the Mumford curve $X = X_\Gamma$ is an invertible holomorphic function $f \in \mathcal{O}(\Omega_\Gamma)^*$ such that

$$\gamma^* f = c(\gamma) f, \quad \forall \gamma \in \Gamma,$$

with $c \in \mathrm{Hom}(\Gamma, \mathbb{K}^*)$ the automorphic factor. The group $\Theta(\Gamma)$ of theta functions of the Mumford curve $X = X_\Gamma$ is then obtained from the exact sequences (3.1) and (3.6) as in [**57**],

$$0 \to \mathbb{K}^* \to \Theta(\Gamma) \to \mathcal{C}(\mathcal{T}_\Gamma)^\Gamma \to 0. \tag{3.11}$$

More precisely, as before let $\mathcal{H}_\mathbb{K} = \mathbb{P}^1_\mathbb{K} \smallsetminus \mathbb{P}^1(\mathbb{K})$ be Drinfeld's p-adic upper half plane. It is well known (see for instance the detailed discussion given in [**5**] §I.1 and §I.2) that $\mathcal{H}_\mathbb{K}$ is a rigid analytic space endowed with a surjective map

$$\Lambda : \mathcal{H}_\mathbb{K} \to \mathcal{T}_\mathbb{K} \tag{3.12}$$

to the Bruhat–Tits tree $\mathcal{T}_\mathbb{K}$ such that, for vertices $v, w \in \mathcal{T}^0_\mathbb{K}$ with $v = s(e)$ and $w = r(e)$, for an edge $e \in \mathcal{T}^1_\mathbb{K}$, the preimages $\Lambda^{-1}(v)$ and $\Lambda^{-1}(w)$ are open subsets of $\Lambda^{-1}(e)$.

Given a theta function $f \in \Theta(\Gamma)$, the associated current $\mu_f \in \mathcal{C}(\mathcal{T}_\Gamma)^\Gamma$ obtained as in (3.11) is given explicitly by the growth of the spectral norm in the Drinfeld upper half plane when moving along an edge in the Bruhat–Tits tree, that is,

$$\mu(e) = \log_q \|f\|_{\Lambda^{-1}(r(e))} - \log_q \|f\|_{\Lambda^{-1}(s(e))}, \tag{3.13}$$

with $q = \#\mathcal{O}/\mathfrak{m}$ and $\|f\|_{\Lambda^{-1}(v)}$ is the spectral norm

$$\|f\|_{\Lambda^{-1}(v)} = \sup_{z \in \Lambda^{-1}(v)} |f(z)|$$

with $|\cdot|$ the valuation satisfying $|\pi| = q^{-1}$, for π a uniformizer, that is a generator of the maximal ideal $\mathfrak{m} = (\pi)$.

3.5. Riemann theta functions and Mumford curves of genus two. Consider first the p-adic theta functions of [**44**], [**23**] given by

$$\theta(a, b; z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma a}{z - \gamma b}$$

for $a, b \in \Omega_\Gamma$, and with $c \in \mathrm{Hom}(\Gamma, \mathbb{K}^*)$ given by

$$c(a, b; \gamma) = \frac{\theta(z_0, \gamma(z_0); a)}{\theta(z_0, \gamma(z_0); b)}.$$

These are the building blocks of p-adic automorphic forms, [**23**], [**41**]. The bilinear symmetric pairing $\langle\cdot,\cdot\rangle : \Gamma\times\Gamma\to\mathbb{K}^*$ of [**44**]

$$\langle\gamma,\gamma'\rangle = \frac{\theta(z_0,\gamma(z_0);z)}{\theta(z_0,\gamma(z_0),\gamma'(z))}. \tag{3.14}$$

In the case of a Mumford curve of genus two [**54**], the fundamental p-adic periods are computed in terms of the generators γ_1, γ_2 of the Schottky group (and an element γ_3 with $\gamma_1\gamma_2\gamma_3 = 1$) and the pairing above by

$$q_i^{-1} = \langle\gamma_j,\gamma_k\rangle,$$

for $\{i,j,k\}$ a cyclic permutation of $\{1,2,3\}$. These satisfy $\langle\gamma_i,\gamma_i\rangle = q_j q_k$.

As in [**54**], let $W = \{P_1,\dots,P_6\}$ be the Weierstrass points of the genus two Mumford curve $X = X_\Gamma$, defined over $\mathbb{K}$. The Schottky uniformization of X determines a subdivision of W into sets $S_i = \{P_i^+, P_i^-\}$, see §2.1 of [**54**]. For each pair $ij\in\{12,23,31\}$ one has a character of Γ defined by

$$\chi_{ij}(\gamma) = \prod_{\gamma'\in\Gamma} \frac{\gamma' z(P_j^+) - a}{\gamma' z(P_j^+) - \gamma a}\,\frac{\gamma' z(P_i^+) - \gamma a}{\gamma' z(P_i^+) - a}.$$

The values, for $ij\in\{12,23,31\}$,

$$p_i = \chi_{ij}(\gamma_j)$$

are the p-adic half-periods with $p_i^2 = q_i$.

One defines a pairing (γ,γ') by setting $p_i^{-1} = (\gamma_j,\gamma_k)$ for $\{i,j,k\}$ a cyclic permutation of $\{1,2,3\}$ as above, where p_i are the p-adic half-periods. The p-adic Riemann theta functions are functions on $\mathrm{Hom}(\Gamma,\mathbb{K}^*)$ defined (see [**54**]) as the sum over $\Gamma^{ab} = \Gamma/[\Gamma,\Gamma]$,

$$\vartheta(\chi) = \sum_{\gamma\in\Gamma^{ab}} (\gamma,\gamma)\,\chi(\gamma).$$

In particular, there is such a Riemann theta function for each choice of the P_i^+ points in W.

As an algebraic curve, X can be described by the equation

$$y^2 = \alpha x(x-1)(x - x(P_1^-))(x - x(P_2^-))(x - x(P_3^-)),$$

where x is a function on X that has a double pole at P_1^+, a double zero at P_2^+ and with $x(P_3^+) = 1$, see §3 of [**54**].

The Thomae formula, in this setting for genus two Mumford curves, expresses the coordinates $x(P_i^-)$ in terms of the half-periods p_i, or conversely the half-periods in terms of the parameters of the branched covering given by the coordinates of these Weierstrass points. Theorem 28 of [**54**] shows that one has

$$\begin{aligned}
x(P_3^-) &= \frac{\vartheta^2(\chi_{P_2^+,P_2^-})\vartheta^2(\chi_{P_1^-,P_2^+})}{\vartheta^2(\chi_{P_1^+,P_1^-})\vartheta^2(\chi_{P_1^+,P_2^-})}\\
\frac{1}{1-x(P_2^-)} &= \frac{\vartheta^2(\chi_{P_3^+,P_3^-})\vartheta^2(\chi_{P_2^-,P_3^+})}{\vartheta^2(\chi_{P_2^+,P_2^-})\vartheta^2(\chi_{P_2^+,P_3^-})}\\
\frac{x(P_1^-)-1}{x(P_1^-)} &= \frac{\vartheta^2(\chi_{P_1^+,P_1^-})\vartheta^2(\chi_{P_3^-,P_1^+})}{\vartheta^2(\chi_{P_3^+,P_3^-})\vartheta^2(\chi_{P_3^+,P_1^-})}
\end{aligned}$$

In turn, the values $\vartheta(\chi_{P_i^\pm, P_j^\pm})$ have explicit power series expressions in terms of the p-adic half-periods p_i, as shown in §3.1 of [**54**].

There is moreover a direct relation between these p-adic Riemann theta functions and the complex Riemann theta functions. When written as formal power series, as in §3.1 of [**54**], they can be formally identified with the Riemann theta constants with

$$\Omega = \frac{1}{\pi i}\begin{pmatrix} \log(p_2p_3) & -\log(p_3) \\ -\log(p_3) & \log(p_1p_3) \end{pmatrix},$$

see [**27**] and Proposition 4.13 of [**12**].

In view of a p-adic formulation of the replica argument, this Thomae formula for Mumford curves of genus two should be extended to a more general class of Z_N Mumford curves (cyclic branched coverings). We describe the geometry of such curves in the following.

3.6. p-adic orbifolds. In the Archimedean setting, the orbifold fundamental group provides a Galois theory of ramified coverings, in a similar way to how the ordinary topological fundamental group provides a Galois theory of ordinary (unramified) coverings. The topological and the orbifold fundamental group are related by the exact sequence

$$1 \to \pi_1^{top}(\Sigma, s) \to \pi_1^{orb}(X, x) \to G \to 1, \tag{3.15}$$

where $X = (\mathbb{P}^1, D)$ is an (Archimedean) orbifold, Σ is a covering compact Riemann surface and G is the Galois group of the covering $\pi : \Sigma \to X$, as we discussed in the previous section.

Generally, in the theory of p-adic orbifolds and uniformization one works with an algebraically closed extension $\mathbb{K}$ of $\mathbb{Q}_p$, to ensures one can take solutions of polynomial equations without having to pass to field extensions. In this p-adic setting one can still define orbifold data $X = (\mathbb{P}^1(\mathbb{K}), D)$ with $D = \sum_i m_i w_i$, and orbifold coverings $\pi : \Sigma \to X$ such that the restriction $\pi : \Sigma \smallsetminus \pi^{-1}(W) \to \mathbb{P}^1(\mathbb{K}) \smallsetminus W$, where $W = \{w_i\}$ is a "tempered étale covering", while the map π is ramified over w_i with ramification index m_i. The étale condition corresponds to the usual notion of an ordinary finite covering, while the notion of tempered étale covering extends this notion to allow for compositions of étale and (possibly infinite) topological coverings and quotients by equivalence relations (compatible with the covering map). The tempered fundamental group $\pi_1^{temp}(X, x)$ is the Galois group of the category of tempered coverings of X. It surjects onto the topological fundamental group $\pi_1^{top}(X, x)$, by the inclusion of topological coverings in the category of tempered coverings. The reason for working with tempered coverings instead of topological coverings in the p-adic setting lies in the fact that the topological fundamental group is often trivial in cases where the complex counterpart is not simply-connected, while the tempered fundamental group is non-trivial in such cases. The typical example is the fact that $\pi_1^{top}(\mathbb{C}_p \smallsetminus \{0, 1, \infty\}) = 0$ (unlike its complex counterpart) but $\pi_1^{temp}(\mathbb{C}_p \smallsetminus \{0, 1, \infty\}) \neq 0$: its profinite completion $\pi_1^{alg}(\mathbb{C}_p \smallsetminus \{0, 1, \infty\})$ is the profinite free group on two generators. On the other hand, tempered coverings are better behaved than étale coverings for the opposite reason: the étale fundamental group in the p-adic case tends to be too large, as in the case of $\mathbb{P}^1$ which has non-trivial étale coverings, while by analogy to the complex case one expects it to be simply connected (which is the case in the tempered sense). Non-archimedean orbifold coverings are in particular tempered coverings, so one can view orbifold coverings as a subcategory of tempered coverings. The orbifold fundamental group is then a quotient of the tempered fundamental group, with (3.15) replaced by

$$1 \to \pi_1^{temp}(\Sigma, s) \to \pi_1(X, x) \to G \to 1, \tag{3.16}$$

where Σ is an orbifold covering of X with Galois group G.

There is a p-adic version of the Riemann–Hilbert correspondence, which gives an equivalence of Tannakian categories between the category of finite dimensional linear representations of the étale fundamental group of X and the category of vector bundles with integrable connection (E, ∇) on X defining an étale local system E^∇. In particular, the tempered Riemann-Hilbert functor RH^{temp} is the restriction to representations that factor through the tempered fundamental group of X. The essential image of RH^{temp} consists of those (E, ∇) such that, for some finite étale cover $f : Y \to X$, the pullback $f^*(E, \nabla)$ has a full set of multivalued analytic solutions (that is, E^∇ is locally constant on the covering Y). The representation of the tempered fundamental group associated to a given bundles with integrable connection (E, ∇) with a locally constant local system E^∇ is referred to as the monodromy representation. If Y is a Mumford curve, uniformized by a p-adic Schottky group $\Gamma = \pi_1^{top}(Y, s)$, and $f : Y \to X$ an orbifold covering, then the image of the associated monodromy representation $\rho : \pi_1^{orb}(X, x) \to \mathrm{PGL}_2(\mathbb{K})$ is contained in the normalizer of the Schottky group Γ, with $Y = \tilde{Y}/\rho(\pi_1^{orb}(X, x))$, where $\tilde{Y}$ is the topological universal cover (the universal orbifold cover does not exist in the non-archimedean case).

3.7. The p-adic Schwarzian uniformization equation. If $\pi : X \to \mathbb{P}^1$ is a p-adic orbifold covering, with orbifold datum $(\mathbb{P}^1, D)$ with $D = \sum_i m_i w_i$, and with X a Mumford curve, let $w \in \mathbb{K}(\mathbb{P}^1)$ be a rational function with no poles outside of $W = \{w_i\}$. For $\tilde{X}$ the topological universal cover of X and z a coordinate on $\tilde{X}$, we have a meromorphic uniformization map $z \mapsto w(z)$ There is a unique associated unformization differential equation (Proposition III.4.6.3 of [**1**]) of the form

$$y'' + \frac{1}{2} R y = 0, \tag{3.17}$$

with the derivative with respect to w and with $R \in \mathbb{K}(\mathbb{P}^1)$ a rational function, and such that the local inverse $z = \phi(w)$, the developing map, is given by the ratio of two independent solutions of (3.17) and $R(w) = \mathcal{S}(\phi)(w)$, the Schwarzian derivative, defined exactly as in the archimedean case. Note that the notation used here for the Schwarzian, in both the archimedean and the non-archimedean case, differs from [**1**] by a sign and a factor of 2. Also note that, unlike the archimedean case, solutions of p-adic differential equations typically may have smaller domain of convergence.

3.8. Ihara's abstract Schwarzian derivative. Ihara developed in [**32**] a general framework for an algebraic formulation of the Schwarzian derivative that can be defined over arbitrary fields. Given a field $\mathbb{K}$ and a one-dimensional vector space $D(\mathbb{K})$ with a differentiation $d : \mathbb{K} \to D(\mathbb{K})$ satisfying $d(a + b) = da + db$ and $d(ab) = a\, db + b\, da$, consider the tensor products $D^r(\mathbb{K}) = D(\mathbb{K})^{\otimes r}$, with $D^0(\mathbb{K}) = \mathbb{K}$

and $D^1(\mathbb{K}) = D(\mathbb{K})$. For $\alpha, \beta \in D(\mathbb{K})^* = D(\mathbb{K}) \smallsetminus \{0\}$ set

$$\langle \alpha, \beta \rangle = \frac{2x_1 x_3 - 3x_2^2}{x_1^2} \beta^2,$$

where $x_1 = \alpha/\beta$ and $x_{i+1} = dx_i/\beta$. This is the abstract Schwarzian derivative. Equivalently, one can formulate it in a way that is more similar to the classical one by considering a map $\mathcal{S} : D(\mathbb{K})^* \to D^2(\mathbb{K})$ with $\mathcal{S}(\alpha) - \mathcal{S}(\beta) = \langle \alpha, \beta \rangle$. Setting $\mathcal{S}_\gamma(\beta) = \langle \beta, \gamma \rangle$ gives such a function, since $\langle \alpha, \gamma \rangle - \langle \beta, \gamma \rangle = \langle \alpha, \beta \rangle$, and any other differs from it by a constant in $D^2(\mathbb{K})$. Ihara gives several examples of abstract Schwarzian in [**32**], including the standard Schwarzian derivative as well as the Schwarzian derivatives associated to p-adic uniformizations of Shimura curves. Mochizuki in §1 of [**46**] also gives a construction of Schwarzian derivative on p-adic curves of genus $g > 1$.

3.9. Branched coverings and Mumford curves. In this section we revisit the geometry of branched coverings and the Schwarzian equations of uniformization in the p-adic setting. The setting is a lot more restrictive in the non-achimedean case. Indeed, since we want to work in a holographic setting, we need to restrict our attention to those p-adic curves that are Mumford curves, since those have a corresponding bulk space, given by a quotient of the Bruhat–Tits tree. However, not all p-adic curves are Mumford curves, as the condition of having totally split reduction (with dual graph the finite graph $\mathcal{T}_\Gamma/\Gamma \subset \mathcal{T}_\mathbb{K}/\Gamma$) is very restrictive. For instance, it is known that the p-adic elliptic curves $y^2 = x(x-1)(x-\lambda)$ are Mumford curves (Mumford–Tate curves) if and only if $|\lambda - 1| < 1$. Thus, the class of p-adic orbifolds we consider will also be more restrictive than the general case, as we want to consider only those p-adic orbifolds $(\mathbb{P}^1(\mathbb{K}), D)$ that are covered by Mumford curves. It is known that in certain specific cases, which mirror the cases we discussed above for the Archimedean setting, like hyperbolic triangle groups, requiring the existence of a covering by Mumford curves restricts the possibilities to a finite list of cases and to only the values $p = 2$, $p = 3$ and $p = 5$, [**2**], [**34**], [**35**]. On the other hand, from the inverse Galois theory point of view, branched coverings of $\mathbb{P}^1$ by Mumford curve are very general. Indeed a p-adic analog of the classical Riemann Existence Theorem for Riemann surfaces holds: any finite group is a Galois group for a Mumford curve branched covering of $\mathbb{P}^1$ (and the same result holds for Mumford coverings of other Mumford curves), see [**7**], [**51**].

We consider a p-adic orbifold $X = (P^1(\mathbb{K}), D)$, with $D = \sum_i m_i w_i$ and the associated branched covering $\pi : \Sigma \to X$, with Σ a p-adic curve. We assume that Σ is in fact a Mumford curve, that is, there is a p-adic Schottky group $\Gamma \subset \mathrm{PGL}_2(\mathbb{K})$ such that $\Sigma = \Omega/\Gamma$, where $\Omega = \Omega_\Gamma \subset \mathbb{P}^1(\mathbb{K})$ is the domain of discontinuity. The Galois group G of the branched covering then fits into a short exact sequence $1 \to \Gamma \to \mathcal{N} \to G \to 1$, where $\mathcal{N}$ is the p-adic orbifold fundamental group, which is a finitely generated discrete subgroup $\mathcal{N} \subset \mathrm{PGL}_2(\mathbb{K})$. The orbifold uniformization is given by the quotient map $\Omega \to \Omega/\mathcal{N} = X$. There is a tree $\mathcal{T}_\mathcal{N}$ associated to the group $\mathcal{N}$, which is the smallest subtree of $\mathcal{T}_\mathbb{K}$ whose ends contain the fixed points of the elements of $\mathcal{N}$. Since $\mathcal{N}$ is not a Schottky group and the tree has non-trivial stabilizers, one should regard $\mathcal{T}_\mathcal{N}$ as a tree of groups. The quotient $\mathcal{T}_\mathcal{N}/\mathcal{N}$ is a graph of finite groups with finitely many infinite ends corresponding to the branch points of the orbifold cover.

3.10. Cyclic coverings of the p-adic projective line. The case of p-adic cyclic branched coverings of $\mathbb{P}^1$ by Mumford curves was discussed in [**6**], [**52**]. Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$ that is sufficiently large so that all the branch points are $\mathbb{K}$-rational. The p-adic cyclic branched coverings $\pi : X_n \to \mathbb{P}^1$ of order n that admit a Mumford curve structure on X_n are among those of the form

$$y^n = \prod_{i=1}^{\ell} (x - \lambda_{i1})^{a_i} (x - \lambda_{i2})^{n-a_i} ,$$

where up to global reparameterizations of $\mathbb{P}^1$ by $\mathrm{PGL}_2(\mathbb{K})$ one can assume that $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (0, \infty, 1, \lambda)$, with $|\lambda| = 1$. Explicit estimates on $|\lambda - 1|$ in terms of the branch orders is given in [**6**], which ensure that X_n is indeed a Mumford curve.

In general, in considering cyclic branched coverings of $\mathbb{P}^1$, one excludes the singular case. However, in the replica argument in complex geometry the possibility of singular Z_n-curves is sometimes included. It is unclear whether the singular case will have to be included here too or whether it suffices to work under the non-singular assumption.

3.11. p-adic Fuchsian triangle groups. The analog in the p-adic setting of the case of Fuchsian triangle groups is given by the p-adic Schwarz orbifolds, see §5.3 of [**1**] and [**2**], [**34**], [**35**]. A Schwarz p-adic orbifold over an extension $\mathbb{K}$ of $\mathbb{Q}_p$ is a datum $(\mathbb{P}^1, D)$ with D consisting of three points $0, 1, \infty$ with assigned multiplicities m_0, m_1, m_∞. One can work directly over the field $\mathbb{C}_p$, so that one knows the p-adic orbifold $(\mathbb{P}^1, D)$ is certainly uniformizable (see §4.4.5 of [**1**]). One can write the classical hypergeometric equation

$$w(w-1)F'' + ((a+b-1)w - c)F' + abF = 0 \,, \tag{3.18}$$

seen as a p-adic differential equation, for

$$a = \frac{1}{2}(1 - \frac{1}{m_0} - \frac{1}{m_1} + \frac{1}{m_\infty}), \quad b = \frac{1}{2}(1 - \frac{1}{m_0} - \frac{1}{m_1} - \frac{1}{m_\infty}), \quad c = 1 - \frac{1}{m_0}.$$

Note that, when one tries to make sense of periods in the p-adic setting one is faced with the problem of not quite having an analog of integrating along a path or a loop. While periods associated to the generators of the Schootky group uniformization, in the case of Mumford curves, were constructed in [**44**], one typically misses the "other half" of the periods, see the discussion in §4 of [**1**]. An approach that is used, which is especially convenient in our setting, is to regard periods as solutions of a Gauss–Manin connection, [**19**]. We are specifically interested in the case arising from the classical hypergeometric equation. We consider the hypergeometric equation with $a, b, c \in \mathbb{Z}_p$ and $c - a, c - b, b, a \notin \mathbb{Z}$. On the span of $f_{a,b,c,z}$ and $f_{a,b,c,z}/(1-x)$ with

$$f_{a,b,c,z}(x) = x^b (1-x)^{c-b} (1-zx)^{-a}$$

the Gauss–Manin connection associated to the hypergeometric equation is given by

$$\nabla \begin{pmatrix} f_{a,b,c,z} \\ \frac{f_{a,b,c,z}}{1-x} \end{pmatrix} = \begin{pmatrix} -\frac{c}{z} & \frac{c-b}{z} \\ \frac{c-a}{1-z} & \frac{a+b-c}{1-z} \end{pmatrix} \begin{pmatrix} f_{a,b,c,z} \\ \frac{f_{a,b,c,z}}{1-x} \end{pmatrix},$$

and periods are analytic solutions of the Gauss-Manin connection.

The way in which the hypergeometric equation determines a p-adic triangle group $\Delta_p(m_0, m_1, m_\infty)$ in $\mathrm{PGL}_2(\mathbb{K})$ is through the associated connection and local system, which determines a representation

$$\pi_1^{orb}(P^1, D) \to \mathrm{PGL}_2(\mathbb{K})$$

whose image is $\Delta_p(m_0, m_1, m_\infty)$. Unlike in the archimedean case, in the p-adic case the correspondence between finite dimensional representations of the fundamental group and bundles with integrable connections is surjective only on those connections with locally constant sheaves of solutions, see §1.5.2 of [**1**]. There is a finite branched covering $\pi : X \to \mathbb{P}^1$ branched at $\{0, 1, \infty\}$ such that the pullback to X of the p-adic hypergeometric equation has a full set of multivalued analytic solutions (the associated local system is locally constant).

A classification of all the possible p-adic triangle groups $\Delta_p(m_0, m_1, m_\infty)$ for which the branched covering $\pi : X \to \mathbb{P}^1$ branched at $\{0, 1, \infty\}$ is given by a Mumford curve X was announced in [**34**], [**35**]. While there may still be issues with this complete classification, we are interested here, in particular, in those $\Delta_p(m_0, m_1, m_\infty)$ which are *arithmetic* triangle groups, as these can be viewed simultaneously as p-adic and archimedean. The classification of the arithmetic p-adic triangle groups was obtained in [**2**] and we refer only to this case here.

In the case of Fuchsian triangle groups in $\mathrm{PSL}_2(\mathbb{R})$ the arithmetic ones are commensurable to unit groups of quaternion algebras over number fields. A complete list of all arithmetic triangle groups was obtained in [**53**]. In the p-adic case, the arithmetic triangle groups are commensurable to p-unit groups of quaternion algebras over totally real number fields $\mathbb{E}$. Such arithmetic p-adic triangle groups were constructed in [**2**]. These p-adic arithmetic triangle groups embed as subgroups of $\mathrm{PGL}_2(\mathbb{K})$, for $\mathbb{K}$ a sufficiently large finite extension of $\mathbb{Q}_p$ and at the same time they also embed in $\mathrm{PGL}_2(\mathbb{C})$ as an arithmetic Fuchsian triangle group associated, in both cases, to the same totally real number field $\mathbb{E}$. The associated finite orbifold covering $\pi : X \to \mathbb{P}^1$, branched at $\{0, 1, \infty\}$, with X a Mumford curve, is also defined over a number field, hence it can also be viewed as a complex Riemann surface and a complex orbifold covering $\pi : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. The classification given in [**2**] shows that there are exactly 45 2-adic arithmetic triangle groups, 16 3-adic ones, and 9 5-adic, while none exist for larger primes. The complete list of the values (m_0, m_1, m_∞) for all of these cases is given in [**2**], and in §III.5 of [**1**].

Let $\mathbb{F}$ be a number field (a finite degree extension of $\mathbb{Q}$), and let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$ that contains $\mathbb{F}$. If X_n is a branched covering of $\mathbb{P}^1$ that is defined over the number field $\mathbb{F}$, we can associate to the same curve both a complex Riemann surface and a p-adic curve defined over a finite extension $\mathbb{K}$ of $\mathbb{Q}_p$ that contains $\mathbb{F}$. If moreover the covering X_n is associated to a finite index subgroup of an arithmetic triangle group that exists p-adically (which requires restricting to $p = 2, 3, 5$ in the hyperbolic case), then the uniformizing group can be seen as both a Fuchsian triangle group $\Delta(m_0, m_1, m_\infty)$ in $\mathrm{PSL}_2(\mathbb{C})$ and as a p-adic triangle group $\Delta_p(m_0, m_1, m_\infty)$. The Schwarzian uniformization equation is defined over $\bar{\mathbb{Q}}$, hence it can also be viewed both in $\mathbb{C}$ and in $\mathbb{C}_p$. The ratio of two independent solutions (over $\mathbb{C}$ or over $\mathbb{C}_p$) gives rise to the corresponding period map, which we respectively denote by $\phi_{\mathbb{C}}(w)$ and $\phi_{\mathbb{C}_p}(w)$.

3.12. Boundary and bulk geometry. It may seem at first that, in passing from real or complex valued fields to non-archimedean valued fields in the boundary theory, one may lose contact with the corresponding geometry on the bulk Bruhat–Tits tree. However, this is not the case. Indeed, the algebraic geometry of Mumford curves, which relies on working with non-archimedean valued fields for the construction of theta functions, automorphic functions etc. (see [**41**]) is built in such a way that all of these objects have counterparts that live on the Bruhat–Tits tree. We will analyze here the cases that are of direct relevance to the entanglement entropy formulation. The first example where one sees a relation between p-adic valued functions on the boundary and geodesic lengths in the bulk is the cross ratio formula, which is relevant in our setting because of (2.2).

3.13. Cross-ratio and geodesics. The cross-ratio formula (2.2) is suggestive of how one should then pass from the non-Archimedean valued quantities to associated real valued ones, and at the same time from a boundary to a bulk picture. For later use, we recall here the expression of the norm of the p-adic cross-ratio in terms of bulk geodesics compared with its Archimedean counterpart.

In the complex case, one knows (see [**42**], [**45**]) that the cross ratio of points on $\mathbb{P}^1(\mathbb{C})$ can be expressed in terms of oriented geodesic lengths and angles in the hyperbolic space $\mathbb{H}^3$ and that this relation can be made compatible with the action of Schottky groups. The analogous result in the p-adic setting was obtained in [**44**], where an analogous result for p-adic Schottky groups is obtained and used to construct the Jacobian of Mumford curves.

In the p-adic case, let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$ with uniformizer π and normalized valuation $\nu : \mathbb{K}^* \to \mathbb{Z}$ with $\nu(\pi) = 1$. Given three points a, b, c in $\mathbb{P}^1(\mathbb{K})$ let $V(a, b, c)$ denote the vertex in the Bruhat–Tits tree $\mathcal{T}_{\mathbb{K}}$ uniquely determined by a, b, c. Let $x = (x_0 : x_1)$, $y = (y_0 : y_1)$, $u = (u_0 : u_1)$, $v = (v_0 : v_1)$ be four pairwise distinct $\mathbb{K}$-rational points in $\mathbb{P}^1(\mathbb{K})$. The basic bulk/boundary formula for the cross-ratio

$$\mathcal{R}(x, y, u, v) = \frac{(x_1u_0 - x_0u_1)(y_1v_0 - y_0v_1)}{(x_0y_1 - x_1y_0)(u_0v_1 - u_1v_0)} \tag{3.19}$$

is given by the relation

$$|\nu(\mathcal{R}(x, y, u, v))| = \mathrm{dist}_{\mathcal{T}_{\mathbb{K}}}(V(x, y, u), V(y, u, v)), \tag{3.20}$$

namely the valuation of the cross-ratio is expressed as a geodesic distance in the Bruhat–Tits tree.

By comparison, in the archimedean setting, the basic bulk/boundary formula for the cross ratio expresses the log of the absolute value of the cross ratio in terms of a geodesic distance and the argument of the cross ratio in terms of the angled between geodesics (Proposition 2.2 of [**42**]). The fact that the "angle information" is lost in the p-adic cross ratio formula reflects the use of the discretized bulk space given by the Bruhat–Tits tree, as opposed to the continuous bulk space given by the Drinfeld plane.

3.14. Poisson kernel. Another instance of bulk/boundary correspondence in p-adic geometry can be seen in the role of the p-adic Poisson kernel in the construction of p-adic automorphic forms.

The Teitelbaum p-adic Poisson kernel introduced in [**55**] provides an isomorphism between the harmonic cocycles of weight ℓ on the Bruhat–Tits tree and the weight ℓ modular forms for the Schottky group Γ, that is, the holomorphism $\ell/2$-differential forms on the Mumford curve X_Γ.

The cocycles of weight ℓ on the Bruhat–Tits tree are defined as in the case we recalled in §3.2. In the more general setting of [**55**], let $P_k(F)$ is the $(k+1)$-dimensional vector space of polynomials of degree k over a field F. Given a polynomial $P(t) \in P_k(\mathbb{C}_p)$, the group $\mathrm{SL}_2(\mathbb{Q}_p)$ acts on the left by

$$\gamma \cdot P(t) = (bt+d)^k P(\frac{at+c}{bt+d}).$$

A harmonic cocycle of weight ℓ on the Bruhat–Tits tree $\mathcal{T}_{\mathbb{K}}$, for $\mathbb{K}$ a finite extension of $\mathbb{Q}_p$, is a map $c : E(\mathcal{T}_{\mathbb{K}}) \to P_{\ell-2}(\mathbb{C}_p)$, satisfying

$$c(\gamma e)(t) = \gamma \cdot c(e)(t).$$

Let $C_{\mathrm{har}}(\Gamma, \ell)$ denote the space of harmonic cocycles of weight ℓ.

A harmonic cocycle $c \in C_{\mathrm{har}}(\Gamma, \ell)$, which one can write in the form $c(e) = \sum_i c_i(e)\binom{\ell-2}{i} t^i$, determines an associated boundary measure on $\mathbb{P}^1(\mathbb{Q}_p)$ obtained by setting

$$\int_{U(e)} x^i \, d\mu_c(x) = c_i(e), \quad i = 0, \dots, \ell-2,$$

where $U(e)$ is the clopen subset of $\mathbb{P}^1(\mathbb{Q}_p)$ determined by the edge $e \in \mathcal{T}_{\mathbb{K}}$.

The p-adic Poisson kernel is given by $1/(z-x)$, defined for $z \in \mathcal{H}_{\mathbb{K}}(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus \mathbb{P}^1(\mathbb{Q}_p)$, the Drinfeld p-adic upper half plane, and for $x \in \mathbb{P}^1(\mathbb{K})$. Given a harmonic cocycle $c \in C_{\mathrm{har}}(\Gamma, \ell)$ one can construct an associated modular forms of weight ℓ on the Drinfeld p-adic plane using the Poisson kernel integration with respect to the measure μ_c,

$$F_c(z) = \int_{\mathbb{P}^1(\mathbb{K})} \frac{1}{z-x} \, d\mu_c(x)$$

which satisfies $F_c(\gamma z) = (cz+d)^\ell F_c(z)$. Conversely, the residue map from the Drinfeld p-adic plane to the Bruhat–Tits tree provides an inverse map that assigns to a modular form F a harmonic cocycle c_F on the Bruhat–Tits tree, see [**55**].

One obtains a modular forms of weight ℓ in the Mumford curve X_Γ by integrating the Poisson kernel with $z \in \Omega_\Gamma$ and $x \in \Lambda_\Gamma$

$$f_c(z) = \int_{\Lambda_\Gamma} \frac{1}{z-x} \, d\mu_c(x).$$

This correspondence between harmonic cocycles on the Bruhat–Tits tree and modular forms on the Mumford curve generalizes the case with $\ell = 2$ considered in [**44**]. This method was used in [**14**] to prove rigidity results for Mumford curves.

3.15. Manin-Drinfeld periods and bulk geometry. The bulk/boundary formula for the cross ratio described above lies at the heart of the interpretation of the Manin–Drinfeld p-adic periods of Mumford curves in terms of bulk geometry, [**44**]. Indeed this interpretation, alongside its analog in complex geometry [**42**] were the original source of the proposed existence of a p-adic form of AdS/CFT holography in [**45**].

Consider the scalar product on the group of integral 1-chains on the quotient graph $\mathcal{T}_{\mathbb{K}}/\Gamma$, where two edges e_1, e_2 are orthogonal if $e_1 \neq \pm e_2$ and the inner product of an edge with itself is normalized to one. For an element $\gamma \in \Gamma$ with fixed points $z_\gamma^\pm \in \Lambda_\Gamma$ and with axis $L(\gamma) \subset \mathcal{T}_{\mathbb{K}}$ connecting the fixed points, let $e_1, \ldots, e_r$ be the edges of a fundamental domain of the action of $\gamma^{\mathbb{Z}}$ on $L(\gamma)$, oriented from z_γ^+ to z_γ^-. The element $c_\gamma = \sum_{i=1}^r e_i$ defines a 1-cycle in $\mathcal{T}_{\mathbb{K}}/\Gamma$ with homology class the class of γ in $\Gamma^{ab} = \Gamma/[\Gamma,\Gamma]$. It is shown in [**44**] that $\mathrm{ord}_{\mathbb{K}}\langle\gamma,\gamma'\rangle$, for the pairing of (3.14) is expressed in terms of the inner product of the 1-cocycles c_γ and $c_{\gamma'}$, hence in terms of the bulk geometry of $\mathcal{T}_{\mathbb{K}}/\Gamma$. Since in [**44**] the pairing $\langle\gamma,\gamma'\rangle$ is also described in terms of the p-adic theta functions (3.14), this can be seen as another instance of bulk/boundary relation in the geometry of Mumford curves.

Another aspect of a p-adic formulation of the replica argument for the computation of entanglement entropy would be a bulk geometry interpretation of the p-adic Riemann theta function and of a p-adic Thomae formula.

4. Formulating a replica problem for p-adic holography

We have seen in the previous section that several of the geometric tools that enter the replica argument in the case of complex Riemann surfaces have analogs in p-adic geometry, modulo certain open questions, like a general form of Thomae formula for p-adic Z_n-curves.

We outline here some further questions, some of which are currently work in progress and will appear elsewhere, that aim at extending the replica argument in p-adic holography.

4.1. The holographic approach and Schottky uniformization. A first approach that can be followed in the p-adic setting is an adaptation of the holographic argument of [**3**] for the computation of the entanglement entropy via the computation of the Rényi entropies, on the cyclic branched coverings of the p-adic projective line discussed above, of the partition function of the gravitational bulk theory. The general lines of the argument used in [**3**], which we summarized in §2.6, can be adapted to this setting. The bulk gravitational theory on the Bruhat–Tits tree was analyzed, for instance, in [**26**], [**31**]. An explicit procedure for the construction of the Schottky uniformization on the p-adic cyclic branched coverings needs to be developed, analogous to the use used in [**3**] based on the uniformization equation. The classical contribution to the Rényi entropies should then be obtained from a regularized volume of the bulk space $\mathcal{T}_{\mathbb{K}}/\Gamma$, while the one-loop contributions obtained in [**3**] from the eigenvalues of the primitive elements in the Schottky group have a direct analog in the p-adic case, though adapting the argument deriving the one-loop corrections to the p-adic setting is not a straightforward step. This p-adic holographic computation of the Rényi entropies and entanglement entropies is currently work in progress. The explicit relation between algebraic parameters and Schottky group parameters in families of equations is generally a hard problem, but there are currently three examples where these are known explicitly: elliptic curves, four-point covers of $\mathbb{P}^1$, as well as another, more complicated set of curves, see [**13**].

4.2. Archimedean and non-archimedean valued physical fields. Both in the use of the uniformization equation in the holographic approach mentioned above, and more generally when considering the boundary conformal field theory in the p-adic setting, a general problem is how one should conceive of the relevant physical fields.

The current literature on the p-adic AdS/CFT holography has focused on a setting where one considers physical fields on the boundary $\mathbb{P}^1(\mathbb{Q}_p)$ (or higher genus Mumford curves $X(\mathbb{Q}_p)$) with values in either real or complex numbers, see for instance [**29**]. If one works with such choice of physical fields, the properties of a boundary conformal field theory are severely limited by the fact that one only has the global symmetries given by $\mathrm{PGL}_2(\mathbb{Q}_p)$ and does not have a good analog of holomorphic functions and of the Schwarzian derivative. Indeed, in the complex setting, the Schwarzian derivative

$$\mathcal{S}(\phi)(w) = \frac{\phi'''(w)}{\phi'(w)} - \frac{3}{2}\left(\frac{\phi''(w)}{\phi'(w)}\right)^2$$

of a rational map $z = \phi(w)$ measures how well $\phi(w)$ is approximated by a linear fractional transformation in $\mathrm{PGL}_2(\mathbb{C})$. These "local approximations" to fractional linear transformations are missing when one considers real or complex valued fields on $\mathbb{P}^1(\mathbb{Q}_p)$. It is clear that this is a serious limitation in the theory, especially in view of the geometry underlying the replica argument that we discussed in the previous section.

A natural way to bypass this limitation is to construct boundary field theories on $\mathbb{P}^1(\mathbb{Q}_p)$ based on fields with values in (a subfield of) $\mathbb{C}_p$. We refer to these as non-Archimedean physical fields. A general theory of non-archimedean valued physical fields was developed in [**36**]. One can obtain real valued fields from non-archimedean ones by applying the p-adic norm.

As we have discussed in the previous section, all the properties of branched coverings and Riemann surfaces that are involved in the complex case have good analogs in the p-adic case, provided one abandons the idea of working with real and complex valued fields and one uses fields with non-archimedean values, so that the corresponding algebraic geometry makes sense. Thus, in the setting of a p-adic replica argument it makes sense to think that non-archimedean valued physical fields are a natural choice.

With this observation in mind, one should then understand the appropriate formulation of the path integral argument behind the formulation of the Rényi and entanglement entropies in [**8**], [**9**], [**10**] from the point of view of p-adic valued distributions considered in [**36**] and the formulation and properties of objects such as the branch points twist fields of [**8**], [**9**], [**10**]. We sketch some ideas of how such things can be thought of in the next subsections.

4.3. Branch points twist fields. When focusing only on the boundary CFT, without rephrasing the entropies computations in terms of the bulk theory, one is faced with the problem of properly interpreting the branch points twist fields and the path integral formulation of the density matrix of [**8**], [**9**], [**10**].

In the original archimedean setting, if $\{\phi_j\}_{j=1}^n$ denote the real-valued fields on the j-th sheet of the branched cover $X_n(\mathbb{C})$ of $\mathbb{P}^1(\mathbb{C})$, and one writes $\tilde{\phi}_k = \sum_{j=1}^n e^{2\pi i \frac{k}{n} j}\phi_j$, with $\tilde{\phi}_k^* = \tilde{\phi}_{n-k}$, the twist operator acts by $\mathcal{T}_n\tilde{\phi}_k = e^{2\pi k/n}\tilde{\phi}_k$, with

$\tilde{\mathcal{T}}_n = \mathcal{T}_n^{-1}$. If the fields ϕ_j are uncoupled, then the twist fields are a product $\mathcal{T}_n = \prod_{k=0}^{n-1} \mathcal{T}_{n,k}$ with $\mathcal{T}_{n,k}$ acting as the identity on $\tilde{\phi}_{k'}$ for $k' \neq k$ and as multiplication by $e^{2\pi k/n}$ on $\tilde{\phi}_k$. This gives a partition function

$$\mathcal{Z}_n = \prod_{k=0}^{n-1} \langle \mathcal{T}_{n,k}(u_1,0)\tilde{\mathcal{T}}_{n,k}(v_1,0)\cdots \mathcal{T}_{n,k}(u_N,0)\tilde{\mathcal{T}}_{n,k}(v_N,0)\rangle$$

for the case of N intervals $[u_r, v_r]$. The twist fields $\mathcal{T}_{n,k}$ and $\tilde{\mathcal{T}}_{n,k}$ thus perform the gluing of the fields ϕ_j at the points u_r, v_r where the different sheets of the covering are joined, see Section 2 of [**9**].

In the p-adic setting, one can consider $\mathbb{Q}_p$-valued fields ϕ_j on the j-th sheet of an n-fold cyclic branched covering of $\mathbb{P}^1(\mathbb{Q}_p)$. In order to introduce twist fields that have the same effect as in the archimedean case, one can pass to an extension $\mathbb{K}$ of $\mathbb{Q}_p$ that contains the n-th roots of unity and define the $\tilde{\phi}_k$ and $\mathcal{T}_n$ and $\mathcal{T}_{n,k}$ in a similar way. The formalism needed to deal with $\mathbb{Q}_p$-valued fields ϕ_j in the partition functions is summarized below.

4.4. The path integral problem. The formulation of path integrals in a setting where one works with p-adic valued physical fields is rendered more complicated than in the archimedean case by the properties of p-adic valued measures, as discussed in [**36**]. The Mahler procedure for the construction of additive p-adic valued measures (Chapter I, Section 7 of [**36**]) consists of sampling a continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ at the dense subset given by the natural numbers and defining the interpolation coefficients a_n by

$$a_n = \sum_{k=0}^{n} (-1)^k C_n^k\, f(n-k)$$

with the combinatorial coefficients $C_n^k := n!/(k!(n-k)!) = n(n-1)\cdots(n-k+1)/k!$. The series

$$f^*(x) = \sum_{n=0}^{\infty} a_n C_x^n,$$

with $C_x^n := x(x-1)\cdots(x-k+1)/k!$ converges uniformly on $\mathbb{Z}_p$ to the function $f(x)$, hence one can define a measure μ, as a continuous linear functional $\mu : C(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p$ by setting

$$\mu(f) := \sum_{n=0}^{\infty} a_n \mu_n, \quad \text{with} \quad \mu_n = \mu(C_x^n)$$

The measures constructed in this way can be equivalently described as additive functions on the clopen subsets of $\mathbb{Z}_p$.

A problem with this construction is that there is no translation invariant Haar measure on $\mathbb{Z}_p$ that can be obtained in this way. The lack of a good analog of the Lebesgue measure creates difficulties in defining the finite dimensional Gaussian and oscillatory integrals on which the infinite dimensional Feynman integrals are modelled. However, this problem can be circumvented (Chapter II of [**36**]) using the fact that in the archimedean case a Gaussian distribution is determined, by Fourier or Laplace transform and the Parseval identity, by the exponential of a quadratic form. Thus, one defines a space $\mathcal{A}'$ of distribution dual to analytic functions on an extension $\mathbb{K}$ of $\mathbb{Q}_p$, by defining the distributions $\delta^{(\alpha)}$ as usual by $\langle \delta^{(\alpha)}, \phi\rangle =$

$(-1)^\alpha \phi^{(\alpha)}(0)$ and considering more general operators of the form $P = \sum_\alpha P_\alpha \delta^{(\alpha)}$. The Laplace transform on $\mathcal{A}'$ is defined as $\langle \mathcal{L}(P)(t), \phi\rangle := \langle P, \exp(t\cdot\phi)\rangle$. A Fourier transform is defined in a similar way, up to passing to a quadratic extension of $\mathbb{K}$. The Laplace transform $\mathcal{L}'(\mu)$ of a p-adic valued measure is then defined by imposing the Parseval identity

$$\int_{\mathbb{K}} \mathcal{L}(P)(t)\, d\mu(t) = \int_{\mathbb{K}} P(x)\, \mathcal{L}'(d\mu)(x)$$

where the identity is understood as an identity of distributions applied to an arbitrary test function ϕ. The Gaussian distribution on $\mathbb{K}$ is then defined as the distribution $G_{a,B} \in \mathcal{A}'$ such that $\mathcal{L}'(G_{a,B}) = \exp(\frac{1}{2}Bx^2 + ax)$. (The case in n-dimensions is analogous with a covariance matrix B and a mean value $a \in \mathbb{K}^n$.) One uses the notation

$$\int_{\mathbb{K}} \phi(x) \exp(\frac{1}{2}Bx^2 + ax)\, dx := \langle G_{a,B}, \phi\rangle$$

to denote the pairing of the Gaussian distribution $G_{a,B}$ with a test function ϕ. Unlike its archimedean counterpart, the Gaussian distribution obtained in this way is an unbounded operator on the space of continuous function, see Chapter II, Section 7 of [**36**]. There is an infinite dimensional generalization of this construction (Chapter III of [**36**]) that can be used to define Feynman integrals for p-adic valued physical fields. With this formalism, the action functional for a non-archimedean scalar boson field $\phi : \mathbb{K}^n \to \mathbb{K}$ is written in the form

$$S(\phi) = \frac{\tau}{2} \int_{\mathbb{K}^n} (\partial_\mu \phi(x) \partial_\mu \phi(x) + m^2 \phi^2(x)) dx + \tau g \int_{\mathbb{K}^n} V(\phi(x)) dx$$

with $g, m \in \mathbb{K}$ and $\mathbb{K}(\sqrt{\tau})$ a quadratic extension, with the associated functional integral

$$\int_{\mathcal{A}'(\mathbb{K}^n,\mathbb{K})} G(\phi) \exp(\sqrt{\tau} S(\phi)) d\phi(x).$$

As shown in Chapter III, Section 2 of [**36**], these functional integrals admit a momentum space representation, and free-field propagators which can be used to make sense of perturbation theory.

In the setting we are considering here, with the purpose of computing entanglement entropies via the replica argument, one needs to describe in the distributional sense recalled above the partition functions

$$\mathcal{Z}_n = \int \exp(\sqrt{\tau}(S(\phi_1) + \cdots + S(\phi_n))) G(\phi_1) \cdots G(\phi_n) d\phi_1 \cdots d\phi_n$$

where ϕ_j are the p-adic valued fields on the j-th sheet of the branched covering and the integration is over the field configurations satisfying a gluing condition at the branch points (see Section 2 of [**9**]) and with the action functional $S(\phi)$ of the CFT on the boundary $\mathbb{P}^1(\mathbb{Q}_p)$. A key part of the argument here is obtaining a p-adic analog of the relation of [**8**] between the holomorphic component of the stress tensor and the Schwarzian derivative of the uniformization equation recalled in §2 above.

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