Differential Topology of Image Segmentation and Invariance

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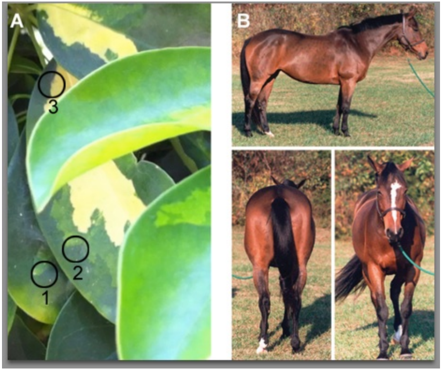
Ma191b: Geometry of Neuroscience

This part is based on:

References:

- Thomas J. Tsao, Doris Y. Tsao, *A topological solution to object segmentation and tracking*, Proc. Natl. Acad. Sci. U.S.A. 119 (41) e2204248119

- segmentation problem: how visual pixels grouped into distinct objects within a single image
- invariance problem: how objects can be identified across images despite changing appearance



Main problem of invariance: appearance of the same object can change drastically with change in perspective

• can hand pick multiple views of same object, and train classifiers to perform view-invariant detection, but this delegates to the hand picker responsible for training (which is the same biological visual system one wants to understand)

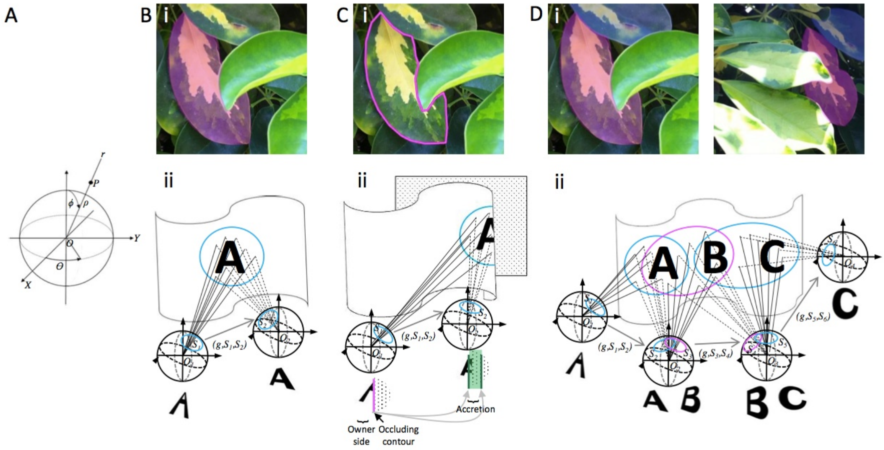
• instead, main idea for mathematical approach:

- segmentation of an image into separate surfaces through detection of occluding contours (spatial separation of visible surfaces)
- tracking of invariant surfaces by detection of stereo diffeomorphisms (overlap relations between surfaces visible from different views)

- Two aspects of the problem:
  - how surfaces in space are projected into ray spaces and how diffeomorphisms in ray spaces encode surfaces
  - how to compute these diffeomorphisms from images

The first problem is an "encoding" problem, mathematically well-posed, the second a "detection" problem subject to noise and ambiguity.

- perspective projection as a mapping from a $2D$ surface to a $2D$ ray space using differential topology to view perspective projections as 2D to 2D diffeomorphisms on regular domains separated by critical points



(A) points project to ray space, (B) surface contiguity faithfully encoded, (C) occluding contour and breaking of local diffeomorphisms, (D) surface persistence

Main Properties:

- 3D space homeomorphically mapped through biprojection to a 3D submanifold within the product of a pair of 2D ray spaces
- a scene in 3D space is a collection of 2D surfaces: these also continuously mapped into product of two ray spaces
- topology of surfaces coded by perspective transformations between pairs of ray spaces
- mapping from a domain in one ray space to another ray space fully specifies a surface in the 3D environment
- conditions under which partial samples can be associated to a single surface
- transition vectors model movements of the eye and perspective transitions: perspective mappings and occluding contours derived from changes of images

• setting: a scene in Euclidean 3D space, compact objects (and planes) with boundary a collection of smooth, compact, orientable 2D surfaces (approximate rough surfaces with smooth)

• ray: a half line in Euclidean 3D space

• ray space $\mathcal{S}(P)$ set of all rays starting at a point $P \in \mathbb{R}^3$

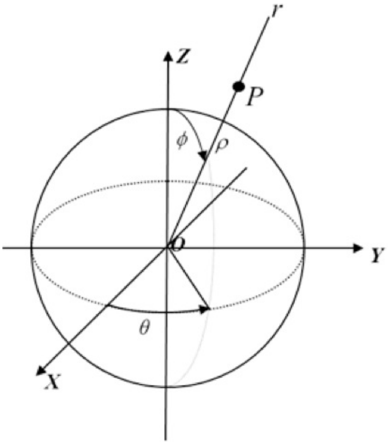
• identification $\mathcal{S}(P) \cong S^2$ sphere (Note: half-lines not lines, so parameterizing space is S^2 not $\mathbb{P}^2(\mathbb{R})$)

• spherical coordinates (θ, ϕ) with $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$ for ray space

• observation domain: open connected region $\Omega \subset \mathbb{R}^3$

• visual space: sphere bundle $\mathcal{S}(\Omega)$ with fiber $\mathcal{S}(P) \cong S^2$ over each $P \in \Omega$ and projection map $\pi : \mathcal{S}(\Omega) \to \Omega$

ray projection: given an observation point $O \in \mathbb{R}^3$ a point $P \neq O \in \mathbb{R}^3$ projects to the ray $r = (\theta, \phi)$ in the ray space $\mathcal{S}(O)$



• visible point P: interior of ray segment from P to O does not intersect any surface Σ in the given *scene*

• visible scene for a given observation point O: set of points P on the surfaces Σ in the scene that are visible for O

• perspective projection from a 2D surface Σ in $\mathbb{R}^3$ to a ray space $\mathcal{S}(O)$ with $O \notin \Sigma$ is ray projection from each *visible* point $P \in \Sigma$

• ambient perspective projection: perspective projection to $\mathcal{S}(O)$ of all points of the visible 3D environment

• ambient perspective projection to a visual space $\mathcal{S}(\Omega)$: ambient perspective projections to each $\mathcal{S}(O)$ with $O \in \Omega$

Groupoid of Movement in Observation Space

• not all arbitrary rigid motions of ambient $\mathbb{R}^3$ are available as shift of perspective of the observer: the observer motion is restricted by the scene

• set of arrows corresponding to allowed motion, with composition rule (source of next equal target of previous)

• groupoid: $\mathcal{G}$ small category where all morphisms (arrows) are invertible

- set of objects $\mathcal{G}^{(0)}$
- set of morphisms (arrows) $\mathcal{G}^{(1)}$
- source and target maps $s, t : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$
- associative composition

$$(x_2 = s(\gamma_2), \gamma_2, x_3 = t(\gamma_2)) \circ (x_1 = s(\gamma_1), \gamma_1, x_2 = t(\gamma_1))$$
$$= (x_1 = s(\gamma_2 \circ \gamma_1), \gamma_2 \circ \gamma_1, x_3 = t(\gamma_2 \circ \gamma_1))$$

• perspective transition groupoid (P, γ, Q) with $P, Q \in \Omega$ and γ a rigid motion of $\mathbb{R}^3$ that maps $s(\gamma) = P$ to $t(\gamma) = Q$

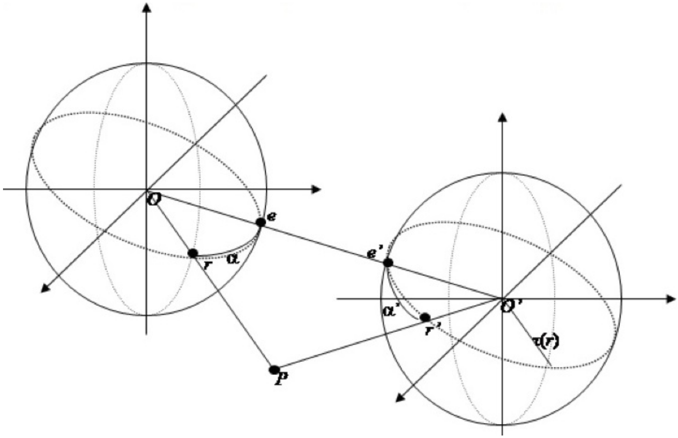
• stereo projection: given a perspective transition (O, γ, O') and the ray spaces $\mathcal{S}(O)$ and $\mathcal{S}(O')$: a source point P in the visual scene (visible from both O and O') determines rays $r_O(P) \in \mathcal{S}(O)$ and $r_{O'}(P) \in \mathcal{S}(O')$, transformed $r_{O'}(P) = \gamma_* r_O(P)$ by induced $\gamma_* : \mathcal{S}(O) \to \mathcal{S}(O')$ transformation (diffeomorphism) of S^2

• stereo set for a perspective transition (O, γ, O')

$$\mathcal{S}_\Sigma(O, O') = \{(r_O(P), r_{O'}(P)) \in \mathcal{S}(O) \times \mathcal{S}(O') \,|\, P \in \Sigma_O \cap \Sigma_{O'}\}$$

with $\Sigma_O, \Sigma_{O'} \subset \Sigma$ visible part of scene Σ (union of 2D surfaces) for O and O'

• $\mathcal{S}(O) \times \mathcal{S}(O') \cong S^2 \times S^2$ is a 4D space and a stereo set $\mathcal{S}_\Sigma(O, O')$ is a 2D surface in this 4D space



the geometry of a perspective transition and the stereo projection (biprojection) from a point P in the visual scene

• 3D embedding: $\ell_{OO'}$ line segment from O to O' and $\mathcal{M} = \mathbb{R}^3 \smallsetminus \ell_{OO'}$, biprojection from $\mathcal{M}$ to $\mathcal{S}(O) \times \mathcal{S}(O')$ is a diffeomorphic embedding

• this can be checked directly in local coordinates for the map $(x, y, z) \mapsto (\theta, \phi, \theta', \phi')$

$$\theta = \arccos\left(\frac{x}{(x^2 + y^2)^{1/2}}\right), \quad \phi = \arccos\left(\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right)$$

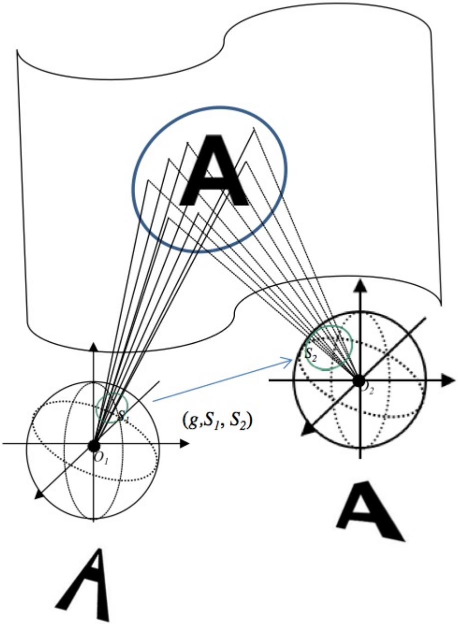
$$\theta' = \arccos\left(\frac{x}{(x^2 + y^2)^{1/2}}\right), \quad \phi' = \arccos\left(\frac{z - \ell}{(x^2 + y^2 + (z - \ell)^2)^{1/2}}\right)$$

where assume O at $(0, 0, 0)$ and O' at $(0, 0, \ell)$

• a domain $S_1 \subset \mathcal{S}(O)$ and a diffeomorphism $g : S^2 \to S^2$ applied to $g : \mathcal{S}(O) \to \mathcal{S}(O')$ that maps S_1 to a diffeomorphic image $S_2 = g(S_1)$ in $\mathcal{S}(O')$

• 2D submanifold $\Gamma(g, S_1, S_2) = \{(P, Q) \in S_1 \times S_2 \,|\, Q = g(P)\}$ embedded inside 4D space $\mathcal{S}(O) \times \mathcal{S}(O')$

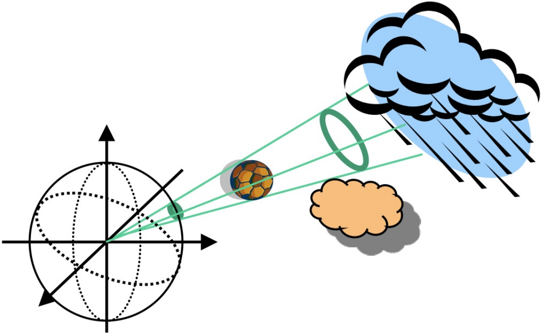
• thus in a perspective transition the stereo projection encodes (without loss of information) a surface as a 2D submanifold of $\mathcal{S}(O) \times \mathcal{S}(O')$

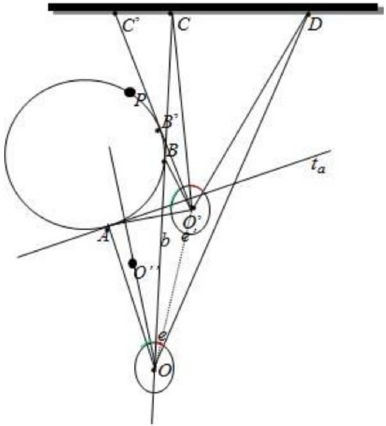

A
S_1
O_1
(g, S_1, S_2)
S_2
O_2
A
A

Occluding contours

• almost all points in a ray space are regular points with respect to a perspective projection (regular points of a differentiable map are general: Sard's theorem)

• fold cone: surface of fold cone is transverse to surfaces Σ of scene except at occluding contours (where transversality fails)





point B is on an occluding contour for O but a regular point for O'

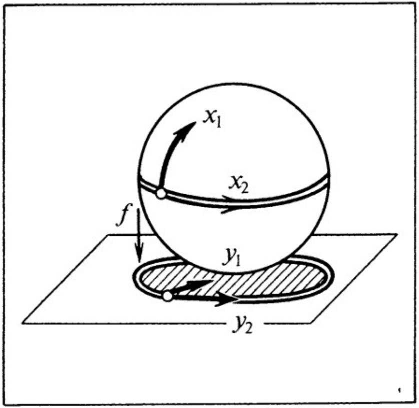
Singularities

- H. Whitney, *On Singularities of Mappings of Euclidean Spaces. I. Mappings of the Plane*, Annals of Mathematics 62 (1955) N.3, 374–410.
- V.I. Arnold, *Catastrophe Theory*, Springer, 1984

Whitney showed: *generically* only two kinds of singularities; all other singularities reduce to these or cancel under small changes in the direction of projection, while these two types are stable and persist after small deformations of the map

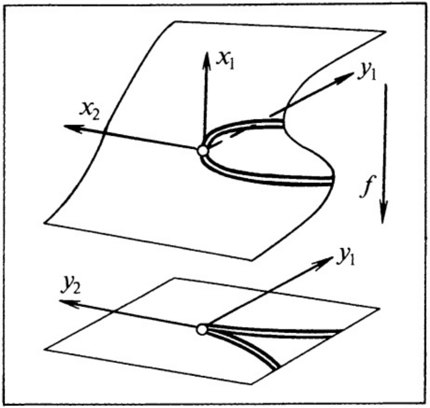
- stable singularities of smooth mappings between 2D surfaces
  - folds
  - cusps

Folds



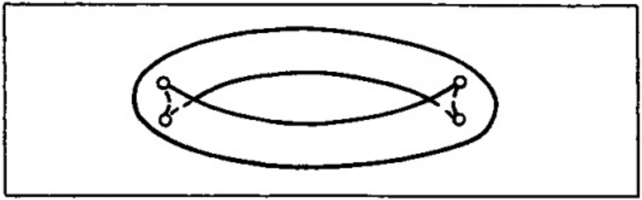
The fold of the projection of a sphere onto the plane

Folds and Cusps



. The cusp of a projection of a surface onto the plane

Occlusion Contours and Fold and Cusp singularities

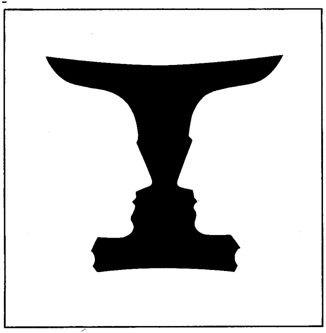


• every point in a border of accretion is a critical value of the perspective projection

• singularities of a projection from a surface to the ray space are either fold points or cusp points (at the end of fold contour)

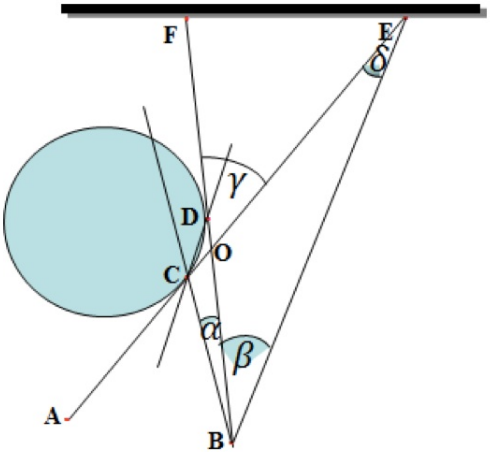
Ownership of occluding contours

- figure-background perception problems



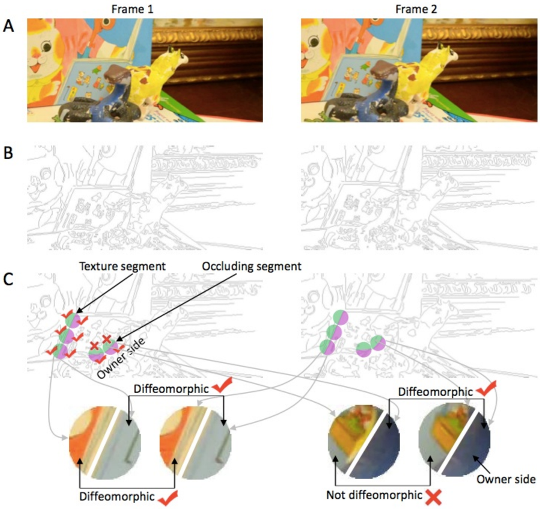
- ownership problem with 2D images, but usually resolved when allowed to change the perspective (mapping 3D space with biprojections to $S^2 \times S^2$)

Accretion



observation moves from A to B, occlusion contour point moves from C to D, accretion area spanned by angle α in $\mathcal{S}(B)$

Illusory countours versus occluding contours



in a segmentation of a 2D image some contours are "illusory" (they do not separate different objects in 3D space)

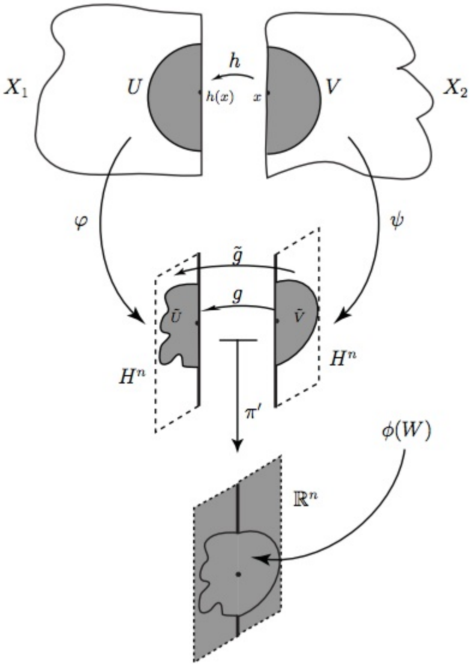
Manifolds with boundary

- surface with boundary: local charts $\phi_i : U_i \to \mathbb{R} \times \mathbb{R}_+$ homeomorphisms of U_i with open sets of $\mathbb{R} \times \mathbb{R}_+$ in the induced topology from $\mathbb{R}^2$, with smooth coordinate changes $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ (smooth near point in $\mathbb{R} \times \mathbb{R}_+$ if extends to smooth on neighborhood in $\mathbb{R}^2$)

- regions in a 2D image on two sides of a contour seen as surfaces with boundary, mapped by perspective change to two other regions, each image under a diffeomorphism of *surface with boundary*

- these may or may not be restriction of a single diffeomorphism of a 2D surface *without boundary*

- if the glue: illusory contour; if they do not glue to a single diffeomorphism: obstruction contour with intervening accretion region


X_1
U
h
$h(x)$
x
V
X_2
φ
ψ
$\tilde{g}$
g
$\tilde{U}$
$\tilde{V}$
H^n
H^n
π'
$\phi(W)$
$\mathbb{R}^n$

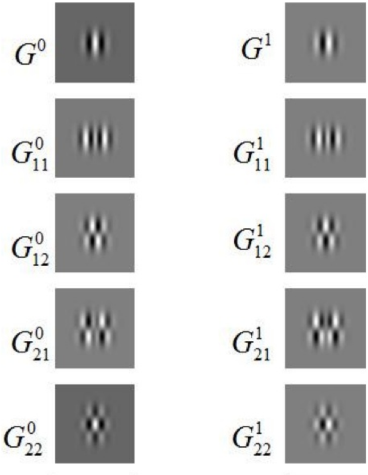
• in a perspective transition accretion/deletion happens only on the *background* side: disambiguation of the image-background ownership problem

• given a set of frame images of same scene from varying observation points, set of associated perspective transitions (O, γ_k, O'_k) and surfaces Σ_i in the 3D scene, with projection images $S_i \subset \mathcal{S}(O)$ and $S'_{i,k} \subset \mathcal{S}(O'_k)$ with mapping $g_k : S_i \to S'_i$ with $g_k = \gamma_{k,*}$ induced on $\mathcal{S}(O) \to \mathcal{S}(O'_k)$ by γ_k

• contours on S_i and accretion zones on $\mathcal{S}(O'_k)$ to determine occlusion contours and border ownership

Computational aspect

• how receptor fields solve for correspondence between images taken from different perspectives



vertically oriented Gabor filters even/odd

Receptor fields (Gabor frames)

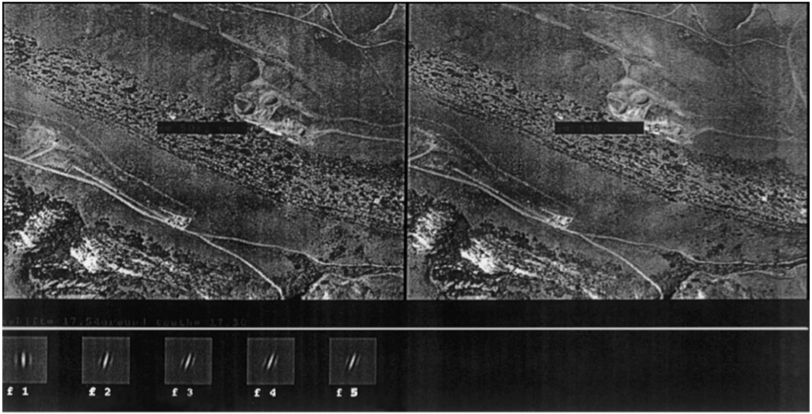
• components of image function on Gabor filters $\langle G_i, F\rangle$

• transformation $\langle G_i, F\circ g\rangle = \langle G_i, L(g)F\rangle = \langle L(g)^* G_i, F\rangle$

• diffeomorphism mapping $g : S_1 \to S_2$, locally linearization by an affine transformation J_g

• $\langle J_g^* G_i, F\rangle$ affine transformed Gabor filters $J_g^* G_i$

• perspective transition and measured response (by first order approximation of deformation) in shift of Gabor frame of a simple cell's receptive field, apparently following an energy minimization process



measured changes of one of the Gabor simple cell's receptive field further mathematical considerations on the Tsao-Tsao model

- two main geometric aspects of the problem:
  1. how surfaces in space are projected into ray spaces and how diffeomorphisms in ray spaces encode surfaces;
  2. how to compute these diffeomorphisms from images.

  The first problem is an "encoding" problem, mathematically well-posed, the second is a "detection" problem subject to noise and ambiguity
- *perspective projection*: mapping from a $2D$ surface to a $2D$ ray space, differential topology $2D$-to-$2D$ diffeomorphisms on regular domains separated by critical points
- stereovision: two independent $2D$ ray spaces and mapping $3D$ space to a submanifold of the 4-dimensional product of two ray spaces, topology of surfaces encoded through "perspective transformations" between ray spaces

Geometries

- *ray* = half-line in Euclidean $3D$ space, ray space is $2D$ sphere $\mathcal{S}(P) \cong S^2$ (half-lines so not $\mathbb{P}^2(\mathbb{R})$)
- *visual space* sphere bundle with projection $\pi : \mathcal{S}(\Omega) \to \Omega$ fiber $\mathcal{S}(P) \simeq S^2$ over $P \in \Omega$: points of $\Omega \subset \mathbb{R}^3$ (open and connected) together with set of all possible observation directions (rays)
- rigid motions: special Euclidean group $\mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$

$$1 \to \mathbb{R}^3 \to \mathrm{SE}(3) \stackrel{\pi}{\to} \mathrm{SO}(3) \to 1$$

- *change of perspective*: observation points $O, O' \in \Omega$ related by rigid motion γ of the ambient $\mathbb{R}^3$ that moves O to $O' = \gamma(O)$; also acting on visual space sphere bundle $\mathcal{S}(\Omega)$ mapping rays $r \in \mathcal{S}(O)$ to rays $\gamma(r) \in \mathcal{S}(\gamma(O))$

Perspective transition groupoid for a visual scene $\Omega \subset \mathbb{R}^3$

- objects: $\mathcal{G}^{(0)}(\Omega)$ = total space of visual space sphere bundle $\mathcal{S}(\mathbb{R}^3 \smallsetminus \Omega)$
- morphisms (arrows) $\mathcal{G}^{(1)}(\Omega)$ = triples $((O,r),\gamma,(O',r'))$, with $\gamma \in \mathrm{SE}(3)$ and $(O,r),(O',r') \in \mathcal{S}(\mathbb{R}^3 \smallsetminus \Omega)$ with $O, O' \in \mathbb{R}^3 \smallsetminus \Omega$, $r \in \mathcal{S}(O)$ and $r' \in \mathcal{S}(O')$ with $\gamma(O) = O'$ and $\gamma(r) = r'$
- source and target maps $s,t : \mathcal{G}^{(1)}(\Omega) \to \mathcal{G}^{(0)}(\Omega)$

$$s((O,r),\gamma,(O',r')) = (O,r) \quad \text{and} \quad t((O,r),\gamma,(O',r')) = (O',r')$$

- morphisms are invertible

$$((O,r),\gamma,(O',r'))^{-1} = ((O',r'),\gamma^{-1},(O,r))$$

- composition of morphisms

$$((O',r'),\gamma',(O'',r''))\circ((O,r),\gamma,(O',r')) = ((O,r),\gamma'\gamma,(O'',r''))$$

Ray projections

- observation point $O \in \mathbb{R}^3$, for any point $P \neq O$ in $\mathbb{R}^3$ *ray projection*: r_{OP} in $\mathcal{S}(O)$ pointing in direction of P
- region $\Omega \subset \mathbb{R}^3$ and different observation points $O, O' \in \mathbb{R}^3 \smallsetminus \Omega$: stereo-projection (biprojection) map

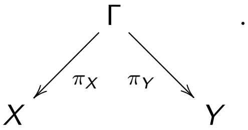
$$\rho : \Omega \to \mathcal{S}(O) \times \mathcal{S}(O'), \quad \rho : P \mapsto (r_{OP}, r_{O'P})$$

- $\ell_{OO'}$ straight line in $\mathbb{R}^3$ that contains O and O'
- map $\rho : \Omega \to \mathcal{S}(O) \times \mathcal{S}(O')$ fails to be injective along $\Omega \cap \ell_{OO'}$: embedding $\rho : \mathbb{R}^3 \smallsetminus \ell_{OO'} \hookrightarrow \mathcal{S}(O) \times \mathcal{S}(O')$
- $2D$ surface $\Sigma \subset \mathbb{R}^3 \smallsetminus \ell_{OO'}$ has image $\rho(\Sigma) \subset \mathcal{S}(O) \times \mathcal{S}(O')$ embedding in $4D$ space $S^2 \times S^2$
- restriction to $\rho(\Sigma)$ of projections π_1, π_2 of $\mathcal{S}(O) \times \mathcal{S}(O')$ to the two factors determine smooth projections $\pi_i : \rho(\Sigma) \to S^2$ between $2D$ spaces
- generally assume visual scene limited in $3D$: does not simultaneously encounter all ray directions from the observer

$$r_O(\Omega) \subset (S^2 \smallsetminus \{x_\Omega\}) \simeq \mathcal{S}(O)$$

Correspondences

- product of two spaces $X \times Y$: *correspondence* between X and Y = subspace $\Gamma \subset X \times Y$ with $\mathrm{codim}(\Gamma) = \dim(Y)$
- projection maps restricting to Γ projections π_X, π_Y of $X \times Y$,



- function $f : X \to Y$ is a correspondence given by its graph $\Gamma(f) = \{(x, y) \in X \times Y \,|\, y = f(x)\}$
- view correspondences as more general "multivalued" functions

Composition of correspondences

- functions $f : X \to Y$ and $g : Y \to W$ have composition $g \circ f : X \to W$
- the graphs $\Gamma(f) \subset X \times Y$ and $\Gamma(g) \subset Y \times W$ compose as

$$\pi_{X\times W}\left(\pi^{-1}_{X\times Y}(\Gamma(f)) \cap \pi^{-1}_{Y\times W}(\Gamma(g))\right) =$$

$$\{(x,w) \in X \times W \,|\, w = g(f(x))\} = \Gamma(g \circ f)$$

- so general definition of composition of correspondences $Z \in \mathcal{Z}(X,Y)$ and $Z' \in \mathcal{Z}(Y,W)$ (intersection product)

$$Z \bullet Z' := \pi_{X\times W}\left(\pi^{-1}_{X\times Y}(Z) \cap \pi^{-1}_{Y\times W}(Z')\right)$$

- when working with smooth manifolds this requires the intersection $\pi^{-1}_{X\times Y}(Z) \cap \pi^{-1}_{Y\times W}(Z')$ to give a smooth manifold
- not always true but transversality in differential topology ensures "generally true"

Transversality in differential topology

- smooth n-dim submanifolds Z, Z' of same smooth $2n$-dim manifold X are transversal if at each $x \in Z \cap Z'$

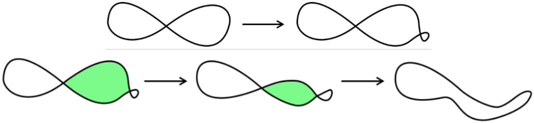
$$T_x Z + T_x Z' = T_x X$$

- if Z and Z' intersect transversely in X, then their intersection $Z \cap Z'$ is a smooth manifold
- smooth functions $f : Z \to X$ and $g : Z' \to X$ are transversal if $df(T_z Z) + dg(T_{z'} Z') = T_x X$ when $f(z) = g(z') = x$
- Sard's theorem: transversality is a general property: given smooth maps $f : Z \to X$ and $g : Z' \to X$, given a function $F : Z \times S \to X$ with $F(z, s_0) = f(z)$ with F and g transversal, then $f_s : Z \to X$ and $g : Z' \to X$ transversal for all s in complement of measure zero set in S
- transversality of F and g achievable taking $\dim S > \dim X$ with $F(z, \cdot) : S \to X$ submersions
- general transversality implies that up to small deformations of the embeddings the pullbacks $\pi^{-1}_{X\times Y}(Z)$ and $\pi^{-1}_{Y\times W}(Z')$ intersect transversely in the triple product $X \times Y \times W$

Whitney's theorems

- transversality $\Rightarrow \pi^{-1}_{X\times Y}(Z) \cap \pi^{-1}_{Y\times W}(Z')$ smooth manifold
- then map $\pi_{X\times W} : \pi^{-1}_{X\times Y}(Z) \cap \pi^{-1}_{Y\times W}(Z') \to X \times W$
- assume for simplicity $\dim X = \dim Y = \dim W = n$ and $\dim Z = \dim Z' = n$ (more general case similar), then restriction of projection $\pi_{X\times W}$ map of n-dimensional to $2n$-dimensional manifold
- *Whitney's immersion theorem*: smooth $f : X \to Y$ between smooth manifolds with $\dim X = n > 1$ and $\dim Y = m$ can be approximated by an immersion with only transversal self-intersections when $m > 2n - 1$
- *Whitney's embedding theorem*: a manifold X of dimension n can be embedded inside a Euclidean space $\mathbb{R}^{2n}$

- first small deformation of $\pi_{X\times W}$ to immersion with double points (with sign for orientation); then Whitney trick: pairs of double points with opposite signs can be cancelled by deforming the immersion (small deformation creating opposite double points)



- so possibly after a smooth deformation $\pi_{X\times W}(\pi^{-1}_{X\times Y}(Z)\cap\pi^{-1}_{Y\times W}(Z'))$ is an embedding of an n-dimensional submanifold in $X\times W$
- equivalence classes of correspondences $\mathcal{Z}(X,Y)/\sim$: up to equivalence that allows for such deformations to achieve transversality and embeddings
- semigroupoid: in general correspondences are not invertible, invertible ones are (equivalent to) graphs of diffeomorphisms

Correspondences and visibility

- correspondences given by surfaces $\Gamma = \rho(\Sigma)$ inside product $\mathcal{S}(O) \times \mathcal{S}(O')$
- sub-correspondence $\mathcal{S}_\Sigma(O, O') \subset \rho(\Sigma)$ accounting for *visibility* condition for Σ with respect to observation points O, O'
- non-transparent $\Sigma \subset \mathbb{R}^3$ part of a visual scene
- *visible point* $P \in \Sigma$ if ray from O to P contains no other point of $\Sigma \Rightarrow$ *visible scene*
- $\Sigma \subset \mathbb{R}^3 \smallsetminus \ell_{OO'} \Rightarrow$ two visible surfaces Σ_O and $\Sigma_{O'}$
- *stereo set correspondence*

$$\mathcal{S}_\Sigma(O, O') := \rho(\Sigma_O \cap \Sigma_{O'}) =$$

$$\{(r_{OP}, r_{O'P}) \in \mathcal{S}(O) \times \mathcal{S}(O') \,|\, P \in \Sigma_O \cap \Sigma_{O'}\}$$

- stereo set correspondence

$$\begin{array}{ccccc} & & \mathcal{S}_\Sigma(O, O') & & \\ & \swarrow^{r_O} & & \searrow^{r_{O'}} & \\ \mathcal{S}(O) & & & & \mathcal{S}(O') \end{array}$$

- assuming surfaces Σ_O and $\Sigma_{O'}$ and $\Sigma_O \cap \Sigma_{O'}$ are compact smooth (possibly with boundary)
- correspondences $\mathcal{S}_\Sigma(O, O')$ also smooth surfaces in $\mathcal{S}(O) \times \mathcal{S}(O')$ ray projections smooth maps between smooth $2D$ manifolds (possibly with boundary)

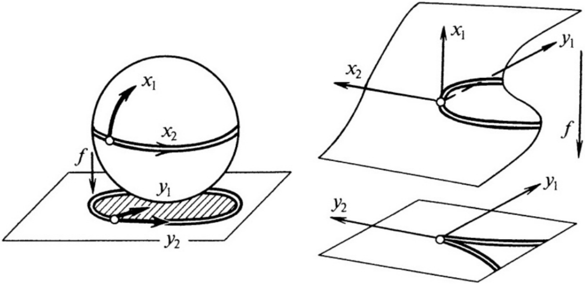
Critical sets and stable types

- smooth manifolds X, Y (without boundary), both of dimension n and a smooth map $f : X \to Y$
- set C_f of *critical points* of f: points $x \in X$ where differential $df|_x : T_xX \to T_{f(x)}Y$' set of *critical values* of f image $f(C_f) \subset Y$ has rank *smaller* than $n = \dim Y$ as linear map between the tangent spaces
- *Sard's theorem*: critical values $f(C_f)$ set of measure zero in Y
- Whitney: singularities of smooth maps $2D$ surface to plane (up to small deformations) reduced to *stable types*: *folds* and *cusps*

$$\begin{cases} y_1 = x_1^2 \\ y_2 = x_2 \end{cases} \quad \text{or} \quad \begin{cases} y_1 = x_1^3 + x_1 x_2 \\ y_2 = x_2 \end{cases}$$

(up to diffeo change of coordinates)

folds and cusps



local quadratic or cubic equation near folds and cusps $\Rightarrow$ fiber of the projection in general two points near a fold curve or three points near a cusp: only one of these points will be in the visible surface Σ_O

fold curves (cusp endpoints) are the occluding contours of visual scene

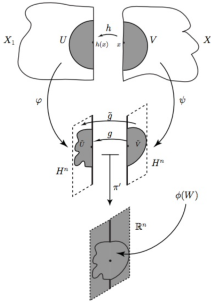
Occluding contours

presence of folds and cusps for the two projections $r_O : \rho(\Sigma) \to \mathcal{S}(O)$ and $r_{O'} : \rho(\Sigma) \to \mathcal{S}(O')$ detect how far the correspondence $\rho(\Sigma)$ is from being the graph of an invertible function.

When occluding contours are present, the correspondence describing the change of perspective fails to be the graph of a diffeomorphism

Surfaces with boundary, accretion

- visible surfaces correspondence $\mathcal{S}_\Sigma(O, O')$ instead of whole $\rho(\Sigma)$
- surfaces Σ_O and $\Sigma_{O'}$ with boundary: boundary partly overlaps with occluding contours of $\rho(\Sigma)$ projections r_O and $r_{O'}$
- *surface with boundary* local charts $\phi_i : U_i \to H^2 := \mathbb{R} \times \mathbb{R}_+$ with homeomorphisms of U_i with open sets of $H^2 = \mathbb{R} \times \mathbb{R}_+$ (induced topology from $\mathbb{R}^2$); smooth coordinate changes $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$; smooth near a point in $\mathbb{R} \times \mathbb{R}_+$ means extends to smooth on a neighborhood in $\mathbb{R}^2$



regions of $2D$ image on two sides of a contour = surfaces with boundary, mapped by perspective change to two other regions (each image under a diffeo of surfaces with boundary)

- restrictions of a single diffeomorphism of a 2D surface *without boundary*: *illusory contour*
- not restrictions of single diffeo (not glue to surface without boundary): actual *obstruction contour*

$\Rightarrow$ accretion region disambiguates image/background ownership and illusory/real contours

Thin semigroupoid of ray projections

- *thin category* is a category with at most one morphism between any pair of objects (Example: partially ordered sets)
- correspondences from ray projections of a surface Σ in visual space

$$\rho_{OO'}(\Sigma) \in \mathcal{Z}(\mathcal{S}(O), \mathcal{S}(O')) \quad \text{and} \quad \rho_{O'O''}(\Sigma) \in \mathcal{Z}(\mathcal{S}(O'), \mathcal{S}(O''))$$

- composition satisfies

$$\pi_{\mathcal{S}(O)\times\mathcal{S}(O'')}(\pi^{-1}_{\mathcal{S}(O)\times\mathcal{S}(O')}(\rho_{OO'}(\Sigma)) \cap \pi^{-1}_{\mathcal{S}(O')\times\mathcal{S}(O'')}(\rho_{O'O''}(\Sigma)))$$
$$= \rho_{OO''}(\Sigma) = \rho_{OO'}(\Sigma) \bullet \rho_{O'O''}(\Sigma)$$

- *thin semigroupoid* $\mathcal{S}_\Sigma$ with objects the observation points

$$\mathcal{S}_\Sigma^{(0)} = \mathrm{Obj}(\mathcal{S}_\Sigma) = \mathbb{R}^3 \smallsetminus \Omega$$

and with a single morphism

$$\rho_{OO'}(\Sigma) \in \mathrm{Mor}_{\mathcal{S}_\Sigma}(O, O')$$

- just keeps track of changes of stereo-perspective on a fixed surface Σ in visual scene

Semigroupoid of visibility under ray projections

- correspondences with visibility condition: $\mathcal{S}_\Sigma(O, O') = \rho_{OO'}(\Sigma_O \cap \Sigma_{O'})$
- visibility condition creates a problem with composition: only have (instead of $=$)

$$\mathcal{S}_\Sigma(O, O') \bullet \mathcal{S}_\Sigma(O', O'') \subset \mathcal{S}_\Sigma(O, O'')$$

- so to have composability (semigroupoid structure) need to add correspondences of the form

$$\mathcal{S}_\Sigma(O, O_1, \dots, O_n, O') = \rho_{OO'}(\Sigma_O \cap \Sigma_{O_1} \cap \cdots \cap \Sigma_{O_n} \cap \Sigma_{O'})$$

  describing the part of the surface Σ that remains visible from all the given intermediate viewpoints
- richer structure than the thin semigroupoid of the $\rho_{OO'}(\Sigma)$