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Coverings, correspondences, and noncommutative geometry

Matilde Marcolli [a,*], Ahmad Zainy al-Yasry [b,a]

[a] Max-Planck Institute for Mathematics, Bonn, Germany

[b] University of Baghdad, Iraq

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ABSTRACT

We construct an additive category where objects are embedded graphs in the 3-sphere and morphisms are *geometric correspondences* given by 3-manifolds realized in different ways as branched covers of the 3-sphere, up to branched cover cobordisms. We consider dynamical systems obtained from associated convolution algebras endowed with time evolutions defined in terms of the underlying geometries. We describe the relevance of our construction to the problem of spectral correspondences in noncommutative geometry.

1. Introduction

In this paper we construct an additive category whose objects are embedded graphs in the 3-sphere and where morphisms are formal linear combinations of 3-manifolds. Our definition of correspondences relies on the Alexander branched covering theorem [1], which shows that all closed oriented 3-manifolds can be realized as branched coverings of the 3-sphere, with branched locus an embedded (not necessarily connected) graph. The way in which a given 3-manifold is realized as a branched cover is highly not unique. It is precisely this lack of uniqueness that makes it possible to regard 3-manifolds as correspondences. In fact, we show that, by considering a 3-manifold M realized in two different ways as a covering of the 3-sphere as defining a correspondence between the branch loci of the two covering maps, we obtain a well defined associative composition of correspondences given by the fibered product.

An equivalence relation between correspondences given by 4-dimensional cobordisms is introduced to conveniently reduce the size of the spaces of morphisms. We construct a 2-category where morphisms are coverings as above and 2-morphisms are cobordisms of branched coverings. We discuss how to pass from embedded graphs to embedded links using the relation of b-homotopy on branched coverings, which is a special case of the cobordism relation.

We associate to the set of correspondences with composition a convolution algebra and we describe natural time evolutions induced by the multiplicity of the covering maps. We prove that, when considering correspondences modulo the equivalence relation of cobordism, these time evolutions are generated by a Hamiltonian with discrete spectrum and finite multiplicity of the eigenvalues.

* Corresponding author.
E-mail addresses: marcolli@mpim-bonn.mpg.de (M. Marcolli), zainy@mpim-bonn.mpg.de (A. Zainy al-Yasry).

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Similarly, in the case of the 2-category, we construct an algebra of functions of cobordisms, with two product structures corresponding to the vertical and horizontal composition of 2-morphisms. We consider a time evolution on this algebra, which is compatible with the vertical composition of 2-morphism given by gluing of cobordisms, that corresponds to the Euclidean version of Hartle–Hawking gravity. This has the effect of weighting each cobordism according to the corresponding Einstein–Hilbert action. We also show that evolutions compatible with the vertical composition of 2-morphisms can be obtained from numerical invariants satisfying an inclusion–exclusion principle. In particular, we construct an example based on the splitting formula for the index of an elliptic operator of Dirac type with APS boundary conditions, of the type that arises, for instance, in the linearized version of the gluing formulae for gauge theoretic moduli spaces on 4-manifolds.

The fact that we have a vertical time evolution coming from an index theorem and suggests that time evolutions compatible with the horizontal compositions may also be found by considering an index pairing, this time obtained by applying the bivariant Chern character to the KK-classes associated to the geometric correspondences. We outline the argument for such a construction.

Our category constructed using 3-manifolds as morphisms is motivated by the problem of developing a suitable notion of *spectral correspondences* in noncommutative geometry, outlined in the last chapter of the book [13]. The spectral correspondences described in [13] will be the product of a finite noncommutative geometry by a "manifold part". The latter is a smooth closed oriented 3-manifold that can be seen as a correspondence in the sense described in the present paper. We discuss the problem of extending the construction presented here to the case of products of manifolds by finite noncommutative spaces in the last section of the paper.

2. Three-manifolds as correspondences

For the moment, we only work in the PL (piecewise linear) category, with proper PL maps. This is no serious restriction as, in the case of 3-dimensional and 4-dimensional manifolds, there is no obstruction in passing from the PL to the smooth category. When we refer to embedded graphs in S^3, we mean PL embeddings of 1-complexes in S^3 with no order zero or order one vertices. We use the notation

$$G \subset E \subset S^3 \xleftarrow{\pi_G} M \xrightarrow{\pi_{G'}} S^3 \supset E' \supset G' \tag{2.1}$$

to denote a closed 3-manifold M that is realized in two ways as a covering of S^3, respectively branched along (not necessarily connected) embedded graphs E and E' containing fixed subgraphs G and G'.

More precisely, we consider PL manifolds endowed with a combinatorial structure. Possibly up to passing to a subdivision of the triangulation, we assume that the 1-complexes E and E' are simplicial subcomplexes of the triangulation. The assumption on the proper PL covering maps is that they map simplicial complexes to simplicial complexes and that the preimage of a simplicial complex is also a simplicial subcomplex of the triangulation, possibly after subdivisions.

In particular, this setting includes the case where the branch loci are knots or links. As an example, in the simpler case where the branch loci are knots, we consider the case of the Poincaré homology sphere.

Example 2.1. The Poincaré homology sphere M can be viewed as a 5-fold covering of S^3 branched along the trefoil $K_{2,3}$, or as a 3-fold cover branched along the $(2, 5)$ torus knot $K_{2,5}$ of also as a 2-fold cover branched along the $(3, 5)$ torus knot $K_{3,5}$. Thus we can see M as a correspondence $K_{2,3} \subset S^3 \leftarrow M \rightarrow S^3 \supset K_{2,5}$, or as $K_{2,5} \subset S^3 \leftarrow M \rightarrow S^3 \supset K_{3,5}$, etc.

2.1. The set of geometric correspondences

We define the set of geometric correspondences $\mathcal{C}(G, G')$ between two embedded graphs G and G' in the following way.

Definition 2.2. Given two embedded graphs G and G' in S^3, let $\mathcal{C}(G, G')$ denote the set of 3-manifolds M that can be represented as branched covers as in (2.1), for some graphs E and E', respectively containing G and G' as subgraphs. In the case where $G = G'$, the set $\mathcal{C}(G, G)$ also contains the trivial unbranched covering $id : S^3 \rightarrow S^3$.

The following observations are meant to show that, in general, the $\mathcal{C}(G, G')$ defined as above tend to be very large.

In fact, restricting for simplicity to the case where G and G' are knots, we first have, as an immediate consequence of [31], the following simple observation.

Lemma 2.3. *Let M be a closed 3-manifold that is realized as a branched cover of S^3, branched along a knot K. Then the manifold M belongs to $\mathcal{C}(K, K')$, for all knots K' that are obtained from K by the covering moves of* [31].

It follows immediately from this that the $\mathcal{C}(G, G')$ can be very large. In fact, because of the existence of universal knots (*cf.* [23]) we have the following result.

Lemma 2.4. *If the branch loci are universal knots K and K', then $\mathcal{C}(K, K')$ contains all closed oriented connected 3-manifolds.*

To avoid logical complications in dealing with the "set" of all 3-manifolds, we describe the $\mathcal{C}(G, G')$ in terms of the following set of representation theoretic data.

It is well known [16] that a branched covering $p : M \to S^3$ is uniquely determined by the restriction to the complement of the branch locus $E \subset S^3$. This gives an equivalent description of branched coverings in terms of representations of the fundamental group of the complement of the branch locus [17]. Namely, assigning a branched cover $p : M \to S^3$ of order m branched along an embedded graph E is the same as assigning a representation

$$\sigma_E : \pi_1(S^3 \smallsetminus E) \to S_m, \tag{2.2}$$

where S_m denotes the group of permutations of m elements. The representation is determined up to inner automorphisms, hence there is no dependence on the choice of a base point for the fundamental group in (2.2).

Thus, in terms of these representations, the spaces of morphisms $\mathcal{C}(G, G')$ are identified with the set of data

$$\mathcal{R}_{G,G'} \subset \bigcup_{n,m,G\subset E,G'\subset E'} \mathrm{Hom}(\pi_1(S^3 \smallsetminus E), S_n) \times \mathrm{Hom}(\pi_1(S^3 \smallsetminus E'), S_m), \tag{2.3}$$

where the E, E' are embedded graphs, $n, m \in \mathbb{N}$, and where the subset $\mathcal{R}_{G,G'}$ is determined by the condition that the pair of representations (σ_1, σ_2) define the same 3-manifold. This latter condition is equivalent, in the case where $n = m = 3$ and where the branch loci are knots, to the knots being related by covering moves (colored Reidemeister moves), as in [31].

Notice how the use of manifolds (or varieties) as correspondences is common to other contexts in mathematics, such as the geometric correspondences of KK-theory or the correspondences based on algebraic cycles in the theory of motives. As in these other theories, we will later introduce a suitable equivalence relation on the geometric correspondences, that reduces the size of the sets $\mathcal{C}(G, G')$.

2.2. Composition of correspondences

The first step, in order to show that we can use Definition 2.2 as a good notion of morphisms in a category where objects are embedded graphs in the 3-sphere, is to show that we have a well defined associative composition rule

$$\circ : \mathcal{C}(G, G') \times \mathcal{C}(G', G'') \to \mathcal{C}(G, G''). \tag{2.4}$$

Definition 2.5. Let $M \in \mathcal{C}(G, G')$ and $\tilde{M} \in \mathcal{C}(G', G'')$ be closed oriented PL 3-manifolds with proper PL branched covering maps

$$\begin{aligned} G \subset E \subset S^3 &\xleftarrow{\pi_G} M \xrightarrow{\pi_1} S^3 \supset E_1 \supset G' \\ G' \subset E_2 \subset S^3 &\xleftarrow{\pi_2} \tilde{M} \xrightarrow{\tilde{\pi}_{G''}} S^3 \supset E'' \supset G'', \end{aligned} \tag{2.5}$$

for some embedded graphs E, E_1, E_2 and E''. The composition $M \circ \tilde{M}$ is given by the fibered product

$$M \circ \tilde{M} := M \times_{G'} \tilde{M}, \tag{2.6}$$

with

$$M \times_{G'} \tilde{M} := \{(x, y) \in M \times \tilde{M} \mid \pi_1(x) = \pi_2(y)\}. \tag{2.7}$$

First we check that this indeed defines an element $M \circ \tilde{M} \in \mathcal{C}(G, G'')$.

Lemma 2.6. *The composition $\hat{M} = M \times_{G'} \tilde{M}$ is a branched cover*

$$E \cup \pi_G\pi_1^{-1}(E_2) \subset S^3 \overset{\hat{\pi}_G}{\leftarrow} \hat{M} \overset{\hat{\pi}_{G''}}{\rightarrow} S^3 \supset E'' \cup \pi_{G''}\pi_2^{-1}(E_1),$$

which defines an element in $\mathcal{C}(E, E'')$. If n and m are the generic multiplicities of the covering maps π_G and π_1 of M and $\tilde{n}$ and $\tilde{m}$ are the generic multiplicities for π_2 and $\pi_{G''}$, respectively, then the covering maps $\hat{\pi}_G$ and $\hat{\pi}_{G''}$ have generic multiplicities $n\tilde{n}$ and $m\tilde{m}$.

Proof. Consider the projections $P_1 : M \times_{G'} \tilde{M} \to M$ and $P_2 : M \times_{G'} \tilde{M} \to \tilde{M}$. They are branched covers, respectively branched over $\pi_1^{-1}(E_2)$ and $\pi_2^{-1}(E_1)$, of order $\tilde{n}$ and m, respectively. In fact, we have

$$P_1^{-1}(x) = \{y \in \tilde{M} \mid \pi_2(y) = \pi_1(x)\}.$$

Thus, the map P_1 is branched over the points $x \in M$ such that $\pi_1(x)$ lies in the branch locus of the map π_2, that is, the points of $\pi_1^{-1}(E_2) \subset M$. Similarly, the branch locus of the map P_2 is the set $\pi_2^{-1}(E_1) \subset \tilde{M}$. Under the PL covering maps the preimages $\pi_1^{-1}(E_2)$ and $\pi_2^{-1}(E_1)$ are embedded graphs in M and $\tilde{M}$, respectively.

The composite map $\hat{\pi}_G = \pi_G \circ P_1 : \hat{M} \to S^3$ is branched over the set $E \cup \pi_G\pi_1^{-1}(E_2)$ and the map $\hat{\pi}_{G''} = \pi_{G''} \circ P_2 : \hat{M} \to S^3$ is branched over $E'' \cup \pi_{G''}\pi_2^{-1}(E_1)$. Again, because the covering maps are PL maps, the sets $\pi_G\pi_1^{-1}(E_2)$ and $\pi_{G''}\pi_2^{-1}(E_1)$ are also 1-complexes (graphs) in S^3 and so are the resulting branch loci. Thus, the fibered product $\hat{M}$ defines an element of $\mathcal{C}(G, G'')$.

For the multiplicites at the generic point, one can just observe that the covering maps P_1 and P_2 have generic multiplicities respectively equal to $\tilde{n}$ and $\tilde{m}$ so that the composite maps $\pi_G \circ P_1$ and $\pi_{G''} \circ P_2$ have generic multiplicities $n\tilde{n}$ and $m\tilde{m}$. $\square$

One can similarly derive the formula for the multiplicities over the branch locus and the branching indices that count how many branches of the covering come together over components of the branch locus. A simple explicit example of the composition law, in the simplest case of branching over the unknot, is given by the following.

Example 2.7. Let $M(n)$ denote the n-fold branched cyclic cover of S^3 branched along the unknot $G = O$. The composition $M(m) \circ M(n)$ is the cyclic branched cover $M(mn)$, viewed as a correspondence in $\mathcal{C}(O, O)$.

We then show that the composition is associative. Consider elements $M_i \in \mathcal{C}(G_i, G_{i+1})$, $i = 1, 2, 3$, where we use the following notation for the embedded graphs and the branched covering maps:

$$\begin{array}{l} G_1 \subset E_1 \subset S^3 \stackrel{\pi_{11}}{\leftarrow} M_1 \stackrel{\pi_{12}}{\to} S^3 \supset E_2 \supset G_2 \\ G_2 \subset E_2' \subset S^3 \stackrel{\pi_{22}}{\leftarrow} M_2 \stackrel{\pi_{23}}{\to} S^3 \supset E_3 \supset G_3 \\ G_3 \subset E_3' \subset S^3 \stackrel{\pi_{33}}{\leftarrow} M_3 \stackrel{\pi_{34}}{\to} S^3 \supset E_4 \supset G_4. \end{array} \tag{2.8}$$

We then have the following result.

Proposition 2.8. *The composition is associative, namely*

$$M_1 \circ (M_2 \circ M_3) = (M_1 \circ M_2) \circ M_3. \tag{2.9}$$

Proof. Consider first the composition $\hat{M}_{23} := M_2 \circ M_3 = M_2 \times_{G_2} M_3$. By Lemma 2.6, it is a branched cover

$$\hat{E}_2 \subset S^3 \stackrel{\hat{\pi}_{232}}{\leftarrow} \hat{M}_{23} \stackrel{\hat{\pi}_{234}}{\to} S^3 \supset \hat{E}_4,$$

with branch loci

$$\hat{E}_2 = E_2' \cup \pi_{22}\pi_{23}^{-1}(E_3') \quad \text{and} \quad \hat{E}_4 = E_4 \cup \pi_{34}\pi_{33}^{-1}(E_3). \tag{2.10}$$

Then the composition $\hat{M}_{1(23)} := M_1 \circ \hat{M}_{23} = M_1 \circ (M_2 \circ M_3)$ is a covering

$$J_1 \subset S^3 \stackrel{\hat{\pi}_{J_1}}{\leftarrow} \hat{M}_{1(23)} \stackrel{\hat{\pi}_{J_4}}{\to} S^3 \supset J_4,$$

with branch loci

$$J_1 = E_1 \cup \pi_{11}\pi_{12}^{-1}(\hat{E}_2) \qquad J_4 = \hat{E}_4 \cup \hat{\pi}_{234}\hat{\pi}_{232}^{-1}(E_2). \tag{2.11}$$

Consider now the composition $\hat{M}_{12} := M_1 \circ M_2$. By Lemma 2.6, this is a branched cover

$$\hat{E}_1 \subset S^3 \stackrel{\hat{\pi}_{121}}{\leftarrow} \hat{M}_{12} \stackrel{\hat{\pi}_{123}}{\to} S^3 \supset \hat{E}_3$$

where $\hat{E}_1$ and $\hat{E}_3$ are given by

$$\hat{E}_1 = E_1 \cup \pi_{11}\pi_{12}^{-1}(E_2') \qquad \hat{E}_3 = E_3 \cup \pi_{23}\pi_{22}^{-1}(E_2). \tag{2.12}$$

Then the composition $\hat{M}_{(12)3} := \hat{M}_{12} \circ M_3 = (M_1 \circ M_2) \circ M_3$ is a branched covering

$$I_1 \subset S^3 \stackrel{\hat{\pi}_{I_1}}{\leftarrow} \hat{M}_{(12)3} \stackrel{\hat{\pi}_{I_4}}{\to} S^3 \supset I_4,$$

with branch locus

$$I_1 = \hat{E}_1 \cup \hat{\pi}_{121}\hat{\pi}_{123}^{-1}(E_3') \qquad I_4 = E_4 \cup \pi_{34}\pi_{33}^{-1}(\hat{E}_3). \tag{2.13}$$

We have

$$\hat{\pi}_{121}\hat{\pi}_{123}^{-1}(E_3') = \pi_{11}\pi_{12}^{-1}\pi_{22}\pi_{23}^{-1}(E_3'),$$

so that the branch loci $J_1 = I_1$ agree. Similarly, we have

$$\hat{\pi}_{234}\hat{\pi}_{232}^{-1}(E_2) = \pi_{34}\pi_{33}^{-1}\pi_{23}\pi_{22}^{-1}(E_2)$$

so that the branch loci $J_4 = I_4$ also coincide. A direct computation, using this same argument, shows that the multiplicities of the covering maps also agree, as well as the branching indices. Thus, the manifolds $M_1 \circ (M_2 \circ M_3)$ and $(M_1 \circ M_2) \circ M_3$ are the same as branched covers. $\square$

2.3. The unit of composition

Let $\mathbb{U}$ denote the trivial unbranched covering $id : S^3 \to S^3$, viewed as an element $\mathbb{U}_G \in \mathcal{C}(G, G)$ for any embedded graph G. We have the following result.

Lemma 2.9. *The trivial covering $\mathbb{U}$ is the identity element for composition.*

Proof. Consider the composition $M \circ \mathbb{U}_{G'}$. The fibered product satisfies

$$M \times_{G'} S^3 = \{(m, s) \in M \times S^3 \mid \pi_2(m) = s\} = \bigcup_{s \in S^3} \pi_2^{-1}(s) = M.$$

So the projection map $P_1 : M \times_{G'} S^3 \to M$ is just the identity map $id : M \to M$, with the composite map $\hat{\pi}_G = \pi_1 \circ P_1 = \pi_1$. The projection map $P_2 : M \times_{G'} S^3 \to S^3$, sending $(m, s) \mapsto s$ for $m \in \pi_2^{-1}(s)$, is just the map $P_2 = \pi_2$, hence $\hat{\pi}_G = \pi_4 \circ P_2 = \pi_2$. Thus, we see that $M \times_G S^3 = M$ with $\pi_G = \pi_1$ and $\pi_{G'} = \pi_2$. This shows that $M \circ \mathbb{U}_{G'} = M$. The argument for the composition $\mathbb{U}_G \circ M$ is analogous. □

3. Semigroupoids and additive categories

A semigroupoid (*cf.* [25]) is a collection $\mathcal{G}$ with a partially defined associative product. An element $\gamma \in \mathcal{G}$ is a unit if $\gamma\alpha = \alpha$ and $\beta\gamma = \beta$ for all α and β in $\mathcal{G}$ for which the product is defined. We denote by $\mathcal{U}(\mathcal{G})$ the set of units of $\mathcal{G}$. A semigroupoid is regular if, for all $\alpha \in \mathcal{G}$ there exist units γ and γ' such that $\gamma\alpha$ and $\alpha\gamma'$ are defined. Such units, if they exist, are unique. We denote them by $s(\alpha)$ (the source) and $r(\alpha)$ (the range). They satisfy $s(\alpha\beta) = s(\alpha)$ and $r(\alpha\beta) = r(\beta)$. To each unit $\gamma \in \mathcal{U}(\mathcal{G})$ in a regular semigroupoid one associates a subsemigroupoid $\mathcal{G}_\gamma = \{\alpha \in \mathcal{G} \mid r(\alpha) = \gamma\}$.

We can reformulate the results on embedded graphs and 3-manifolds obtained in the previous section in terms of semigroupoids in the following way.

Lemma 3.1. *The set of closed oriented* 3*-manifolds forms a regular semigroupoid, whose set of units is identified with the set of embedded graphs.*

Proof. We let $\mathcal{G}$ be the collection of data $\alpha = (M, G, G')$ with M a closed oriented 3-manifold with branched covering maps to S^3 of the form (2.1). We define a composition rule as in Definition 2.5, given by the fibered product. In the multi-connected case, for

$$M = M_1 \amalg M_2 \amalg \cdots \amalg M_k \tag{3.1}$$

with (M_i, G, G') as in (2.1) with M_i connected, we extend the composition $M \circ \tilde{M}$ to mean

$$M \circ \tilde{M} = M_1 \circ \tilde{M} \amalg M_2 \circ \tilde{M} \amalg \cdots \amalg M_k \circ \tilde{M}, \tag{3.2}$$

and similarly for $\tilde{M}$ multi-connected. It is necessary to include the multi-connected case since the fibered product of connected manifolds may consist of different connected components. We impose the condition that the composition of $\alpha = (M_1, G_1, G_1')$ and $\beta = (M_2, G_2, G_2')$ is only defined when the $G_1' = G_2$.

By Lemma 2.9, we know that, for each $\alpha = (M, G, G') \in \mathcal{G}$ the source and range are given by the trivial coverings $\gamma = \mathbb{U}_G = (\mathbb{U}, G, G)$ and $\gamma' = \mathbb{U}_{G'} = (\mathbb{U}, G', G')$. That is, we can identify them with $s(\alpha) = G$ and $r(\alpha) = G'$. Thus, the set of units $\mathcal{U}(\mathcal{G})$ is the set of embedded graphs in S^3. □

For a given embedded graph G, the subsemigroupoid $\mathcal{G}_G$ is given by the set of all 3-manifolds that are covering of S^3 branched along embedded graphs E containing G as a subgraph.

Given a semigroupoid $\mathcal{G}$, and a commutative ring R, one can define an associated semigroupoid ring $R[\mathcal{G}]$, whose elements are finitely supported functions $f : \mathcal{G} \to R$, with the associative product

$$(f_1 * f_2)(\alpha) = \sum_{\alpha_1, \alpha_2 \in \mathcal{G} : \alpha_1\alpha_2 = \alpha} f_1(\alpha_1) f_2(\alpha_2). \tag{3.3}$$

Elements of $R[\mathcal{G}]$ can be equivalently described as finite R-combinations of elements in $\mathcal{G}$, namely $f = \sum_{\alpha \in \mathcal{G}} a_\alpha \delta_\alpha$, where $a_\alpha = 0$ for all but finitely many $\alpha \in \mathcal{G}$ and $\delta_\alpha(\beta) = \delta_{\alpha,\beta}$, the Kronecker delta.

In the multi-connected case $\alpha = (M, G, G')$ with M as in (3.1) we impose the relation

$$\delta_\alpha = \sum_{i=1}^{k} \delta_{\alpha_i}, \tag{3.4}$$

where $\alpha_i = (M_i, G, G')$ with M_i connected.

The following statement is a semigroupoid version of the representations of groupoid algebras generalizing the regular representation of group rings.

Lemma 3.2. *Suppose given a unit $\gamma \in \mathcal{U}(\mathcal{G})$. Let $\mathcal{H}_\gamma$ denote the R-module of finitely supported functions $\xi : \mathcal{G}_\gamma \to R$. The action*

$$\rho_\gamma(f)(\xi)(\alpha) = \sum_{\alpha_1 \in \mathcal{G}, \alpha_2 \in \mathcal{G}_\gamma : \alpha = \alpha_1 \alpha_2} f(\alpha_1)\xi(\alpha_2), \tag{3.5}$$

for $f \in R[\mathcal{G}]$ and $\xi \in \mathcal{H}_\gamma$, defines a representation of $R[\mathcal{G}]$ on $\mathcal{H}_\gamma$.

Proof. We have

$$\rho_\gamma(f_1 * f_2)(\xi)(\alpha) = \sum (f_1 * f_2)(\alpha_1)\xi(\alpha_2) = \sum_{\beta_1\beta_2 = \alpha_1 \in \mathcal{G}} \sum_{\alpha_1\alpha_2 = \alpha} f_1(\beta_1) f_2(\beta_2)\xi(\alpha_2) = \sum_{\beta_1\beta = \alpha} f_1(\beta_1)\rho_\gamma(f_2)(\xi)(\beta),$$

hence $\rho_\gamma(f_1 * f_2) = \rho_\gamma(f_1)\rho_\gamma(f_2)$. Since for elements of a semi-groupoid the range satisfies $r(\alpha\beta) = r(\beta)$, the action is well defined on $\mathcal{H}_\gamma$. □

In the next section we see that in fact the difference in the representation (3.5) between the semigroupoid and the groupoid case manifests itself in the compatibility with the involutive structure.

A semigroupoid is just an equivalent formulation of a small category, so the result above simply states that embedded graphs form a small category with the sets $\mathcal{C}(G, G')$ as morphisms. Passing from the semigroupoid $\mathcal{G}$ to $R[\mathcal{G}]$ corresponds to passing from a small category to its additive envelope, as follows.

Let R be a commutative ring. We replace the sets $\mathcal{C}(G, G')$ of geometric correspondences by R-modules.

Definition 3.3. For given embedded graphs G and G' in S^3, let $\mathrm{Hom}_R(G, G')$ denote the free R-module generated by the elements of $\mathcal{C}(G, G')$.

Namely, elements $\phi \in \mathrm{Hom}_R(G, G')$ are finite R-combinations $\phi = \sum_M a_M M$, where the sum ranges over the set of all 3-manifolds that are branched covers as in (2.1) and the $a_M \in R$ satisfy $a_M = 0$ for all but finitely many M. Again, in the case of geometric correspondences given by multi-connected manifolds, we impose the relation $M = M_1 + M_2$ when $M = M_1 \amalg M_2$.

For $R = \mathbb{Z}$, we simply write $\mathrm{Hom}(G, G')$ for $\mathrm{Hom}_{\mathbb{Z}}(G, G')$. It then follows immediately that we obtain in this way a pre-additive category.

Lemma 3.4. *The category $\mathcal{K}$ whose objects are embedded graphs and with morphisms the* $\mathrm{Hom}(G, G')$ *is a pre-additive category.*

One can pass to its additive closure by considering the category $\mathrm{Mat}(\mathcal{K})$ whose objects are formal direct sums of objects of $\mathcal{K}$ and whose morphisms are matrices of morphisms in $\mathcal{K}$. In the following we continue to use the notation $\mathcal{K}$ for the additive closure. For $R = k$ a field, we obtain in this way a k-linear category, where morphism spaces are k-vector spaces.

4. Convolution algebra and time evolutions

Consider as above the semigroupoid ring $\mathbb{C}[\mathcal{G}]$ of complex valued functions with finite support on $\mathcal{G}$, with the associative convolution product (3.3),

$$(f_1 * f_2)(M) = \sum_{M_1, M_2 \in \mathcal{G} : M_1 \circ M_2 = M} f_1(M_1) f_2(M_2). \tag{4.1}$$

We define an involution on the semigroupoid $\mathcal{G}$ by setting

$$\mathcal{C}(G, G') \ni \alpha = (M, G, G') \mapsto \alpha^\vee = (M, G', G) \in \mathcal{C}(G', G), \tag{4.2}$$

where, if α corresponds to the 3-manifold M with branched covering maps

$$G \subset E \subset S^3 \stackrel{\pi_G}{\leftarrow} M \stackrel{\pi_{G'}}{\rightarrow} S^3 \supset E' \supset G'$$

then $\alpha^\vee$ corresponds to the same 3-manifold with maps

$$G' \subset E' \subset S^3 \stackrel{\pi_{G'}}{\leftarrow} M \stackrel{\pi_G}{\rightarrow} S^3 \supset E \supset G$$

taken in the opposite order. In the following, for simplicity of notation, we write $M^\vee$ instead of $\alpha^\vee = (M, G', G)$.

Lemma 4.1. *The algebra $\mathbb{C}[\mathcal{G}]$ is an involutive algebra with the involution*

$$f^\vee(M) = \overline{f(M^\vee)}. \tag{4.3}$$

Proof. We clearly have $(af_1 + bf_2)^\vee = \bar{a}f_1^\vee + \bar{b}f_2^\vee$ and $(f^\vee)^\vee = f$. We also have

$$(f_1 * f_2)^\vee(M) = \sum_{M^\vee = M_1^\vee \circ M_2^\vee} \overline{f_1}(M_1^\vee)\overline{f_2}(M_2^\vee) = \sum_{M = M_2 \circ M_1} f_2^\vee(M_2) f_1^\vee(M_1)$$

so that $(f_1 * f_2)^\vee = f_2^\vee * f_1^\vee$. □

4.1. Time evolutions

Given an algebra $\mathcal{A}$ over $\mathbb{C}$, a time evolution is a 1-parameter family of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$. There is a natural time evolution on the algebra $\mathbb{C}[\mathcal{G}]$ obtained as follows.

Lemma 4.2. *Suppose given a function $f \in \mathbb{C}[\mathcal{G}]$. Consider the action defined by*

$$\sigma_t(f)(M) := \left(\frac{n}{m}\right)^{it} f(M), \tag{4.4}$$

where M a covering as in (2.1), with the covering maps π_G and $\pi_{G'}$ respectively of generic multiplicity n and m. This defines a time evolution on $\mathbb{C}[\mathcal{G}]$.

Proof. Clearly $\sigma_{t+s} = \sigma_t \circ \sigma_s$. We check that $\sigma_t(f_1 * f_2) = \sigma_t(f_1) * \sigma_t(f_2)$. By (4.1), we have

$$\sigma_t(f_1 * f_2)(M) = \left(\frac{n}{m}\right)^{it} (f_1 * f_2)(M) = \sum_{M_1, M_2 \in \mathcal{G} : M_1 \circ M_2 = M} \left(\frac{n_1}{m_1}\right)^{it} f_1(M_1) \left(\frac{n_2}{m_2}\right)^{it} f_2(M_2) = (\sigma_t(f_1) * \sigma_t(f_2))(M),$$

where n_i, m_i are the generic multiplicities of the covering maps for M_i, with $i = 1, 2$. In fact, we know by Lemma 2.6 that $n = n_1 n_2$ and $m = m_1 m_2$. The time evolution is compatible with the involution (4.3), since we have

$$\sigma_t(f^\vee)(M) = \left(\frac{n}{m}\right)^{it} f^\vee(M) = \left(\frac{n}{m}\right)^{it} \overline{f(M^\vee)} = \overline{\left(\frac{m}{n}\right)^{it} f(M^\vee)} = \overline{\sigma_t(f)(M^\vee)} = (\sigma_t(f))^\vee(M). \quad \square$$

Similarly, we define the left and right time evolutions on $\mathcal{A}$ by setting

$$\sigma_t^L(f)(M) := n^{it} f(M), \qquad \sigma_t^R(f)(M) := m^{it} f(M), \tag{4.5}$$

where n and m are the multiplicities of the two covering maps as above. The same argument of Lemma 4.2 shows that the $\sigma_t^{L,R}$ are time evolutions. One sees by construction that they commute, *i.e.* that $[\sigma_t^L, \sigma_t^R] = 0$. The time evolution (4.4) is the composite

$$\sigma_t = \sigma_t^L \sigma_{-t}^R. \tag{4.6}$$

The involution exchanges the two time evolutions by

$$\sigma_t^L(f^\vee) = (\sigma_{-t}^R(f))^\vee. \tag{4.7}$$

4.2. Creation and annihilation operators

Given an embedded graph $G \subset S^3$, consider, as above, the set $\mathcal{G}_G$ of all 3-manifolds that are branched covers of S^3 branched along an embedded graph $E \supset G$.

On the vector space $\mathcal{H}_G$ of finitely supported complex valued functions on $\mathcal{G}_G$ we have a representation of $\mathbb{C}[\mathcal{G}]$ as in Lemma 3.2, defined by

$$(\rho_G(f)\xi)(M) = \sum_{M_1 \in \mathcal{G}, M_2 \in \mathcal{G}_G : M_1 \circ M_2 = M} f(M_1)\xi(M_2). \tag{4.8}$$

It is natural to consider on the space $\mathcal{H}_G$ the inner product

$$\langle \xi, \xi' \rangle = \sum_{M \in \mathcal{G}_G} \overline{\xi(M)} \xi'(M). \tag{4.9}$$

Notice however that, unlike the usual case of groupoids, the involution (4.3) given by the transposition of the correspondence does not agree with the adjoint in the inner product (4.9), namely $\rho_\gamma(f)^* \neq \rho_\gamma(f^\vee)$.

The reason behind this incompatibility is that semigroupoids behave like semigroup algebras implemented by isometries rather than like group algebras implemented by unitaries. The model case for an adjoint and involutive structure that is compatible with the representation (4.8) and the pairing (4.9) is therefore given by the algebra of creation and annihilation operators.

We need the following preliminary result.

Lemma 4.3. *Suppose given elements $\alpha = (M, G, G')$ and $\alpha_1 = (M_1, G_1, G_1')$ in $\mathcal{G}$. If there exists an element $\alpha_2 = (M_2, G_2, G_2')$ in $\mathcal{G}(G_2, G_2')$ such that $\alpha = \alpha_1 \circ \alpha_2 \in \mathcal{G}$, then α_2 is unique.*

Proof. We have $M = M_1 \circ M_2$. We denote by $E \supset G$, $E' \supset G'$ and $E_1 \supset G_1$ and $E'_1 \supset G'_1$ the embedded graphs that are the branching loci of the covering maps $\pi_G, \pi_{G'}$ and $\pi_{G_1}, \pi_{G'_1}$ of M and M_1, respectively.

By construction we know that for the composition $\alpha_1 \circ \alpha_2$ to be defined in $\mathcal{G}$ we need to have $G'_1 = G_2$. Moreover, by Lemma 2.6 we know that $E = E_1 \cup \pi_{G_1}\pi^{-1}_{G'_1}(E_2)$ and $E' = E'_2 \cup \pi_{G'_2}\pi^{-1}_{G_2}(E'_1)$, where E_2 and E'_2 are the branch loci of the two covering maps of M_2.

The manifold M_2 and the branched covering maps π_{G_2} and $\pi_{G'_2}$ can be reconstructed by determining the multiplicities, branch indices, and branch loci E_2, E'_2.

The n-fold branched covering $\pi_G : M \to S^3 \supset E \supset G$ is equivalently described by a representation of the fundamental group $\pi_1(S^3 \smallsetminus E) \to S_n$. Similarly, the n_1-fold branched covering $\pi_{G_1} : M_1 \to S^3 \supset E_1 \supset G_1$ is specified by a representation $\pi_1(S^3 \smallsetminus E_1) \to S_{n_1}$. Given these data, we obtain the branched covering $P_1 : M \to M_1$ such that $\pi_G = \pi_{G_1} \circ P_1$ in the following way. The restrictions $\pi_G : M \smallsetminus \pi_G^{-1}(E) \to S^3 \smallsetminus E$ and $\pi_{G_1} : M_1 \smallsetminus \pi^{-1}_{G_1}(E) \to S^3 \smallsetminus E$ are ordinary coverings, and we obtain from these the covering $P_1 : M \smallsetminus \pi_G^{-1}(E) \to M_1 \smallsetminus \pi^{-1}_{G_1}(E)$. Since this is defined on the complement of a set of codimension two, it extends uniquely to a branched covering $P_1 : M \to M_1$. The image under $\pi_{G'_1}$ of the branch locus of P_1 and the multiplicities and branch indices of P_1 then determine uniquely the manifold M_2 as a branched covering $\pi_{G_2} : M_2 \to S^3 \supset E_2$. Having determined the branched covering π_{G_2} we have the covering maps realizing M as the fibered product of M_1 and M_2, hence we also have the branched covering map $P_2 : M \to M_2$.

The knowledge of the branch loci, multiplicities and branch indices of $\pi_{G'}$ and P_2 then allows us to identify the part of the branch locus E' that constitutes E'_2 and the multiplicities and branch indices of the map $\pi_{G'_2}$. This completely determines also the second covering map $\pi_{G'_2} : M_2 \to S^3 \supset E'_2$. $\square$

We denote in the following by the same notation $\mathcal{H}_G$ the Hilbert space completion of the vector space $\mathcal{H}_G$ of finitely supported complex valued functions on $\mathcal{G}_G$ in the inner product (4.9). We denote by δ_M the standard orthonormal basis consisting of functions $\delta_M(M') = \delta_{M,M'}$, with $\delta_{M,M'}$ the Kronecker delta.

Given an element $M \in \mathcal{G}$, we define an associated bounded linear operator A_M on $\mathcal{H}_G$ of the form

$$(A_M\xi)(M') = \begin{cases} \xi(M'') & \text{if } M' = M \circ M'' \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Notice that (4.10) is well defined because of Lemma 4.3.

Lemma 4.4. *The adjoint of the operator (4.10) in the inner product (4.9) is given by the operator*

$$(A^*_M\xi)(M') = \begin{cases} \xi(M \circ M') & \text{if the composition is defined} \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Proof. We have

$$\langle \xi, A_M \zeta\rangle = \sum_{M'=M\circ M''} \overline{\xi(M')}\zeta(M'') = \sum_{M''} \overline{\xi(M\circ M'')}\zeta(M'') = \langle A^*_M\xi, \zeta\rangle. \quad \square$$

We regard the operators A_M and A^*_M as the annihilation and creation operators on $\mathcal{H}_G$ associated to the manifold M. They satisfy the following relations.

Lemma 4.5. *The products $A^*_M A_M = P_M$ and $A_M A^*_M = Q_M$ are given, respectively, by the projection P_M onto the subspace of $\mathcal{H}_G$ given by the range of composition by M, and the projection Q_M onto the subspace of $\mathcal{H}_G$ spanned by the M' with $s(M') = r(M)$.*

Proof. This follows directly from (4.10) and (4.11). $\square$

The following result shows the relation between the algebra $\mathbb{C}[\mathcal{G}]$ and the algebra of creation and annihilation operators A_M, A^*_M.

Lemma 4.6. *The algebra of linear operators on $\mathcal{H}_G$ generated by the A_M is the image $\rho_G(\mathbb{C}[\mathcal{G}])$ of $\mathbb{C}[\mathcal{G}]$ under the representation ρ_G of (4.8).*

Proof. Every function $f \in \mathbb{C}[\mathcal{G}]$ is by construction a finite linear combination $f = \sum_M a_M \delta_M$, with $a_M \in R$. Under the representation ρ_G we have

$$(\rho_G(\delta_M)\xi)(M') = \sum_{M'=M_1\circ M_2} \delta_M(M_1)\xi(M_2) = (A_M\xi)(M'). \quad \square \tag{4.12}$$

This shows that, when working with the representations ρ_G the correct way to obtain an involutive structure is by extending the algebra generated by the A_M to include the A^*_M, instead of using the involution (4.3) of $\mathbb{C}[\mathcal{G}]$.

4.3. Time evolutions and Hamiltonians

Given a representation $\rho : \mathcal{A} \to \mathrm{End}(\mathcal{H})$ of an algebra $\mathcal{A}$ with a time evolution σ, one says that the time evolution, in the representation ρ, is generated by a Hamiltonian H if for all $t \in \mathbb{R}$ one has

$$\rho(\sigma_t(f)) = \mathrm{e}^{-\mathrm{i}tH}\rho(f)\mathrm{e}^{\mathrm{i}tH}, \tag{4.13}$$

for an operator $H \in \mathrm{End}(\mathcal{H})$.

Lemma 4.7. *The time evolutions σ_t^L and σ_t^R of (4.5) and $\sigma_t = \sigma_t^L\sigma_{-t}^R$ of (4.4) extend to time evolutions of the involutive algebra generated by the operators A_M and A_M^* by*

$$\begin{aligned}
&\sigma_t^L(A_M) = n^{\mathrm{i}t}A_M \qquad \sigma_t^L(A_M^*) = n^{-\mathrm{i}t}A_M^* \\
&\sigma_t^R(A_M) = m^{\mathrm{i}t}A_M \qquad \sigma_t^R(A_M^*) = m^{-\mathrm{i}t}A_M^* \\
&\sigma_t(A_M) = \left(\frac{n}{m}\right)^{\mathrm{i}t}A_M \qquad \sigma_t(A_M^*) = \left(\frac{n}{m}\right)^{-\mathrm{i}t}A_M^*.
\end{aligned} \tag{4.14}$$

Proof. The result follows directly from (4.12) and the condition $\sigma_t(T^*) = (\sigma_t(T))^*$. $\square$

We then have immediately the following result. We state it for the time evolution σ_t^L, while the case of σ_t^R is analogous.

Lemma 4.8. *Consider the unbounded linear operators $H_{G'}^L$ and $H_{G'}^R$ on the space $\mathcal{H}_{G'}$ defined by*

$$(H_{G'}^L\,\xi)(M) = \log(n)\xi(M), \qquad (H_{G'}^R\,\xi)(M) = \log(m)\xi(M) \tag{4.15}$$

for M a geometric correspondence of the form

$$G \subset E \subset S^3 \xleftarrow{\pi_G} M \xrightarrow{\pi_{G'}} S^3 \supset E' \supset G'$$

with π_G and $\pi_{G'}$ branched coverings of order n and m, respectively. Then $H_{G'}^L$ and $H_{G'}^R$ are, respectively, Hamiltonians for the time evolutions σ_t^L and σ_t^R in the representation $\rho_{G'}$ of (4.8).

Proof. It is immediate to check that $\rho_{G'}(\sigma_t^L(f)) = \mathrm{e}^{-\mathrm{i}tH^L}\rho_{G'}(f)\mathrm{e}^{\mathrm{i}tH^L}$ and $\rho_{G'}(\sigma_t^R(f)) = \mathrm{e}^{-\mathrm{i}tH^R}\rho_{G'}(f)\mathrm{e}^{\mathrm{i}tH^R}$ for $f \in \mathbb{C}[\mathcal{G}]$. In fact, it suffices to use the explicit form of the time evolutions on the creation and annihilation operators given in Lemma 4.7 above to see that they are implemented by the Hamiltonians $H_{G'}^L$ and $H_{G'}^R$. $\square$

An obvious problem with these time evolutions is the fact that the corresponding Hamiltonians typically can have infinite multiplicities of the eigenvalues. For example, by the strong form of the Hinden–Montesinos theorem [33] and the existence of universal knots [23], there exist knots K such that all closed oriented 3-manifolds can be obtained as a 3-fold branched cover of S^3, branched along K. For this reason it is useful to consider time evolutions on a convolution algebra of geometric correspondences that takes into account the equivalence given by 4-dimensional cobordisms. We turn to this in Sections 5 and 7.

5. Cobordisms and equivalence of correspondences

Whenever one defines morphisms via correspondences, be it cycles in the product as in the case of motives or submersions as in the case of geometric correspondences of [14], the most delicate step is always deciding up to what equivalence relation correspondences should be considered. In the case of 3-manifolds with the structure of branched covers, there is a natural notion of equivalence, which is given by cobordisms of branched covers, *cf.* [22]. Adapted to our setting, this is formulated in the following way.

Definition 5.1. Suppose given two correspondences M_1 and M_2 in $\mathcal{C}(G, G')$, of the form

$$G \subset E_1 \subset S^3 \xleftarrow{\pi_{G,1}} M_1 \xrightarrow{\pi_{G',1}} S^3 \supset E_1' \supset G'$$

$$G \subset E_2 \subset S^3 \xleftarrow{\pi_{G,2}} M_2 \xrightarrow{\pi_{G',2}} S^3 \supset E_2' \supset G'.$$

Then a cobordism between M_1 and M_2 is a 4-dimensional PL manifold W with boundary $\partial W = M_1 \cup -M_2$, endowed with two branched covering maps

$$S \subset S^3 \times [0, 1] \xleftarrow{q} W \xrightarrow{q'} S^3 \times [0, 1] \supset S', \tag{5.1}$$

branched along surfaces (PL embedded 2-complexes) $S, S' \subset S^3 \times [0, 1]$. The maps q and q' have the properties that $M_1 = q^{-1}(S^3 \times \{0\}) = q'^{-1}(S^3 \times \{0\})$ and $M_2 = q^{-1}(S^3 \times \{1\}) = q'^{-1}(S^3 \times \{1\})$, with $q|_{M_1} = \pi_{G,1}$, $q'|_{M_1} = \pi_{G',1}$, $q|_{M_2} = \pi_{G,2}$ and $q'|_{M_2} = \pi_{G',2}$. The surfaces S and S' have boundary $\partial S = E_1 \cup -E_2$ and $\partial S' = E_1' \cup -E_2'$, with $E_1 = S \cap (S^3 \times \{0\})$, $E_2 = S \cap (S^3 \times \{1\})$, $E_1' = S' \cap (S^3 \times \{0\})$, and $E_2' = S' \cap (S^3 \times \{1\})$.

Lemma 5.2. *Setting $M_1 \sim M_2$ if there exists a cobordims W as in Definition 5.1 defines an equivalence relation. Moreover, if $M_1 \sim M_2$ in $\mathcal{C}(G, G')$ and $M_1' \sim M_2'$ in $\mathcal{C}(G', G'')$, then $M_1 \circ M_1' \sim M_2 \circ M_2'$.*

Proof. We have $M \sim M$ through the trivial cobordism $M \times [0, 1]$. The symmetric property is satisfied by taking the opposite orientation cobordims and transitivity is achieved by gluing cobordisms along their common boundary, $W = W_1 \cup_{M_2} W_2$. This can be done compatibly with the branched covering maps, since these match along the common boundary. Thus, we have a well defined equivalence relation. To check the compatibility with composition, let

$$S_{11} \subset S^3 \times [0, 1] \xleftarrow{q_1} W_1 \xrightarrow{q_1'} S^3 \times [0, 1] \supset S_{12} \tag{5.2}$$

be a cobordism realizing the equivalence $M_1 \sim M_2$, and

$$S_{21} \subset S^3 \times [0, 1] \xleftarrow{q_2} W_2 \xrightarrow{q_2'} S^3 \times [0, 1] \supset S_{22}, \tag{5.3}$$

be a cobordims realizing $M_1' \sim M_2'$. We check that the fibered product

$$W_1 \circ W_2 := \{(x, y) \in W_1 \times W_2 | q_1'(x) = q_2(y)\} \tag{5.4}$$

gives a branched covers cobordism realizing the desired equivalence $M_1 \circ M_1' \sim M_2 \circ M_2'$. First notice that we have

$$\partial(W_1 \circ W_2) = \partial W_1 \circ \partial W_2 = (M_1 \circ M_1') \cup -(M_2 \circ M_2').$$

Moreover, $W_1 \circ W_2$ is a branched cover

$$\hat{S}_1 \subset S^3 \times [0, 1] \xleftarrow{T_1} W_1 \circ W_2 \xrightarrow{T_2} S^3 \times [0, 1] \supset \hat{S}_2,$$

with branch loci $\hat{S}_1 = S_{11} \cup q_1(q_1'^{-1}(S_{21}))$ and $\hat{S}_2 = S_{22} \cup q_2'(q_1^{-1}(S_{12}))$, satisfying

$$\partial\hat{S_1} = \partial(S_{11} \cup q_1(q_1'^{-1}(S_{21}))) = E_{11} \cup \pi_{11}(\pi_{12}^{-1}(E_{11}')) \cup (-E_{21} \cup \pi_{21}(\pi_{22}^{-1}(E_{21}'))) = I_1 \cup -I_3,$$

with the notation of Lemma 2.6. Similarly, we have $\partial\hat{S_2} = I_2 \cup -I_4$. The sets $q_1(q_1'^{-1}(S_{21}))$, $q_2'(q_1^{-1}(S_{12}))$ are 2-complexes in $S^3 \times [0, 1]$ with boundary the 1-complexes $\pi_{11}(\pi_{12}^{-1}(E_{11}'))$ and $\pi_{21}(\pi_{22}^{-1}(E_{21}'))$. We then see that

$$T_1^{-1}(S^3 \times \{0\}) = M_1 \circ M_1' = T_2^{-1}(S^3 \times \{0\})$$
$$T_1^{-1}(S^3 \times \{1\}) = M_2 \circ M_2' = T_2^{-1}(S^3 \times \{1\}).$$

In fact, the first set is equal to

$$\{(x, y) \in q_1^{-1}(S^3 \times \{0\}) \times W_2 : q_1'(x) = q_2(y)\} = \{(x, y) \in q_1^{-1}(S^3 \times \{0\}) \times q_2^{-1}(S^3 \times \{0\}) : q_1'(x) = q_2(y)\}.$$

The other case is analogous. Thus, the resulting $W_1 \circ W_2$ is a branched cover cobordism with the desired properties. □

We can now consider the sets of geometric correspondences, up to the equivalence relation of cobordism. The result of Lemma 5.2 immediately implies the following.

Lemma 5.3. *Let G and G' be embedded graphs in S^3 and let $\mathcal{C}(G, G')$ be the set of geometric correspondences as in Definition 2.2. Let*

$$\mathcal{C}_\sim(G, G') := \mathcal{C}(G, G')/\sim \tag{5.5}$$

denote the quotient of $\mathcal{C}(G, G')$ by the equivalence relation of cobordism of Definition 5.1. There is an induced associative composition

$$\circ : \mathcal{C}_\sim(G, G') \times \mathcal{C}_\sim(G', G'') \to \mathcal{C}_\sim(G, G''). \tag{5.6}$$

As in Section 3, given a commutative ring R we define $\mathrm{Hom}_{R,\sim}(G, G')$ to be the free R-module generated by $\mathcal{C}_\sim(G, G')$, that is, the set of finite R-combinations $\phi = \sum_{[M]} a_{[M]}[M]$, with $[M] \in \mathcal{C}_\sim(G, G')$ and $a_{[M]} \in R$ with $a_{[M]} = 0$ for all but finitely many $[M]$. We write $\mathrm{Hom}_\sim(G, G')$ for $\mathrm{Hom}_{\mathbb{Z},\sim}(G, G')$.

We then construct a category $\mathcal{K}_{R,\sim}$ of embedded graphs and correspondences in the following way.

Definition 5.4. The category $\mathcal{K}_{R,\sim}$ has objects the embedded graphs G in S^3 and morphisms the $\mathrm{Hom}_{R,\sim}(G, G')$

After passing to $\mathrm{Mat}(\mathcal{K}_{R,\sim})$ one obtains an additive category of embedded graphs and correspondences, which one still denotes $\mathcal{K}_{R,\sim}$.

5.1. Time evolutions and equivalence

We now return to the time evolutions (4.5) and (4.4) on the convolution algebra $\mathbb{C}[\mathcal{G}]$. After passing to equivalence classes by the relation of cobordism, we can consider the semigroupoid $\bar{\mathcal{G}}$ which is given by the data $\alpha = ([M], G, G')$, where $[M]$ denotes the equivalence class of M under the equivalence relation of branched cover cobordism. Lemma 5.2 shows that the composition in the semigroupoid $\mathcal{G}$ induces a well defined composition law in $\bar{\mathcal{G}}$. We can then consider the algebra $\mathbb{C}[\bar{\mathcal{G}}]$ with the convolution product as in (4.1),

$$(f_1 * f_2)([M]) = \sum_{[M_1],[M_2]\in\bar{\mathcal{G}}:[M_1]\circ[M_2]=[M]} f_1([M_1])f_2([M_2]). \tag{5.7}$$

The involution $f \mapsto f^\vee$ is also compatible with the equivalence relation, as it extends to the involution on the cobordisms W that interchanges the two branched covering maps.

Lemma 5.5. *The time evolutions (4.4) and (4.5) descend to well defined time evolutions on the algebra* $\mathbb{C}[\bar{\mathcal{G}}]$.

Proof. The result follows from the fact that the generic multiplicity of a branched covering is invariant under branched cover cobordisms. Thus, we have an induced time evolution of the form

$$\sigma_t^L(f)[M] := n^{it} f[M], \qquad \sigma_t^R(f)[M] := m^{it} f[M], \qquad \sigma_t(f)[M] := \left(\frac{n}{m}\right)^{it} f[M], \tag{5.8}$$

where each representative in the class $[M]$ has branched covering maps with multiplicities

$$G \subset E \subset S^3 \overset{n:1}{\leftarrow} M \overset{m:1}{\rightarrow} S^3 \supset E' \supset G'.$$

We see that the time evolution is compatible with the involution as in Lemma 4.2. $\square$

5.2. Representations and Hamiltonian

Similarly, we can again consider representations of $\mathbb{C}[\bar{\mathcal{G}}]$ as in (4.8)

$$(\rho(f)\xi)[M] = \sum_{[M_1]\in\bar{\mathcal{G}},[M_2]\in\bar{\mathcal{G}}_G:[M_1]\circ[M_2]=[M]} f[M_1]\xi[M_2]. \tag{5.9}$$

As in the previous case, we define on the space $\bar{\mathcal{H}}_G$ of finitely supported functions $\xi : \bar{\mathcal{G}}_G \to \mathbb{C}$ the inner product

$$\langle \xi, \xi' \rangle = \sum_{[M]} \overline{\xi[M]}\xi'[M]. \tag{5.10}$$

Once again we see that, in this representation, the adjoint does not correspond to the involution $f^\vee$ but it is instead given by the involution in the algebra of creation and annihilation operators

$$(A_{[M]}\xi)[M'] = \begin{cases} \xi[M''] & \text{if } [M'] = [M] \circ [M''] \\ 0 & \text{otherwise} \end{cases} \tag{5.11}$$

$$(A^*_{[M]}\xi)[M'] = \begin{cases} \xi[M \circ M'] & \text{if the composition is possible} \\ 0 & \text{otherwise.} \end{cases} \tag{5.12}$$

Again we have $\rho_G(\delta_{[M]}) = A_{[M]}$ so that the algebra generated by the $A_{[M]}$ is the same as the image of $\mathbb{C}[\bar{\mathcal{G}}]$ in the representation ρ_G and the algebra of the creation and annihilation operators $A_{[M]}$ and $A^*_{[M]}$ is the involutive algebra in $\mathcal{B}(\bar{\mathcal{H}}_G)$ generated by $\mathbb{C}[\bar{\mathcal{G}}]$. In fact, the same argument we used before shows that $A^*_{[M]}$ defined as in (5.12) is the adjoint of $A_{[M]}$ in the inner product (5.10).

We then have the following result.

Theorem 5.6. *The Hamiltonian $H = H_G^R$ generating the time evolution σ_t^R in the representation (5.9) has discrete spectrum*

$$\mathrm{Spec}(H) = \{\log(n)\}_{n\in\mathbb{N}},$$

with finite multiplicities

$$1 \le N_n \le \#\pi_3(B_n), \tag{5.13}$$

where B_n is the classifying space for branched coverings of order n.

Proof. It was proven in [5] that the n-fold branched covering spaces of a manifold M, up to cobordism of branched coverings, are parameterized by the homotopy classes

$$B_n(M) = [M, B_n], \tag{5.14}$$

where the B_n are classifying spaces. In particular, cobordism equivalence classes of n-fold branched coverings of the 3-sphere are classified by the homotopy group

$$B_n(S^3) = \pi_3(B_n). \tag{5.15}$$

The rational homotopy type of the classifying spaces B_n is computed in [5] in terms of the fibration

$$K(\pi, j-1) \to \bigvee^{t-1} \Sigma K(\pi, j-1) \to \bigvee^{t} K(\pi, j), \tag{5.16}$$

which holds for any abelian group π and any positive integers $t, j \geq 2$, with Σ denoting the suspension. For the B_n one finds

$$B_n \otimes \mathbb{Q} = \bigvee^{p(n)} K(\mathbb{Q}, 4) \tag{5.17}$$

with the fibration

$$S^3 \otimes \mathbb{Q} \to \bigvee^{p(n)-1} S^4 \times \mathbb{Q} \to B_n \otimes \mathbb{Q}, \tag{5.18}$$

where $p(n)$ is the number of partitions of n. The rational homotopy groups of B_n are computed from the exact sequence of the fibration (5.18) (see [5]) and are of the form $\pi_k(B_n) \otimes \mathbb{Q} = \mathbb{Q}^D$ with

$$D = \begin{cases} p(n) & k = 4 \\ Q\left(\dfrac{k-1}{3}, p(n)-1\right) & k = 1, 4, 10 \mod 12, \text{ with } k \neq 1, 4 \\ Q\left(\dfrac{k-1}{3}, p(n)-1\right) + Q\left(\dfrac{k-1}{6}, p(n)-1\right) & k \equiv 7 \mod 12 \\ 0 & \text{otherwise} \end{cases} \tag{5.19}$$

where

$$Q(a, b) = \frac{1}{a} \sum_{d|a} \mu(d) b^{a/d}$$

with $\mu(d)$ the Möbius function. The result (5.19) then implies that the homotopy groups $\pi_3(B_n)$ satisfy $\pi_3(B_n) \otimes \mathbb{Q} = 0$.

Moreover, in [5] the classifying spaces B_n are constructed explicitly by fitting together the classifying space $BO(2)$, that carries the information on the branch locus, with the classifying space BS_k, for S_k the group of permutations of k elements. For example, in the case of normalized simple coverings of [6], the classifying space is a mapping cylinder $BO(2) \cup_{BD_k} BS_k$, with D_k the dihedral group, over the maps induced by the inclusion $D_k \hookrightarrow O(2)$ as the subgroup leaving the set of k-th roots of unity globally invariant, and $D_k \to S_k$ giving the permutation action on the k-th roots of unity. In the case of [5] that we consider here, where more general branched coverings are considered, the explicit form of B_k in terms of $BO(2)$ and BS_k is more complicated, as it also involves a union over partitions of k, which accounts for the different choices of branching indices, of data of disk bundles associated to each partition. The skeleta of the classifying space have finitely generated homology in each degree, *i.e.* they are spaces of finite type, and simply connected in the case of [5]. By a result of Serre it is known that, for simply connected spaces of finite type, the homotopy groups are also finitely generated (*cf.* also Section 0.a of [20]). The condition $\pi_3(B_n) \otimes \mathbb{Q} = 0$ then implies that the groups $\pi_3(B_n)$ are finite for all n.

By the same argument used in Lemma 4.8, the Hamiltonian $H = H_G^R$ generating the time evolution σ_t^R in the representation (5.9) is of the form

$$(H\,\xi)[M] = \log(n)\,\xi[M], \tag{5.20}$$

where M is a branched cover of S^3 of order n branched along $E \supset G$, for the given embedded graph G specifying the representation. Thus, the multiplicity of the eigenvalue $\log(n)$ is the number of cobordism classes $[M]$ branched along an embedded graph containing G as a subgraph. This number $N_n = N_n(G)$ is bounded by $1 \leq N_n(G) \leq \#\pi_3(B_n)$. □

The result can be improved by considering, instead of the Brand classifying spaces B_n of branched coverings, the more refined Tejada classifying spaces $B_n(\ell)$ introduced in [37,7]. In fact, the homotopy group $\pi_3(B_n)$ considered above parameterizes branched cobordism classes of branched coverings where the branch loci are embedded manifolds of codimension two. Since in each cobordism class there are representatives with such branch loci (*cf.* the discussion in Section 6 below) we can work with B_n and obtain the coarse estimate above. However, in our construction we are considering branch loci that are, more generally, embedded graphs and not just links. Similarly, our cobordisms are branched over 2-complexes, not just embedded surfaces. In this case, the appropriate classifying spaces are the generalizations $B_n(\ell)$ of [37,7]. These are such that $B_n(2) = B_n$ and $B_n(\ell)$, for $\ell > 2$, allows for branched coverings and cobordisms where the branch locus has strata of some codimension $2 \leq r \leq \ell$. We have then the following more refined result.

Corollary 5.7. *The multiplicity $N_n(G)$ of the eigenvalue $\log(n)$ of the Hamiltonian H_G satisfies the estimate*

$$1 \leq N_n(G) \leq \#\pi_3(B_n(4)). \tag{5.21}$$

Proof. In our construction, we are considering branched coverings of the 3-sphere with branch locus an embedded graph $E \supset G$, up to branched cover cobordism, where the cobordisms are branched over a 2-complex. Thus, the branch locus E has strata of codimension two and three and the branch locus for the cobordism has strata of codimension two, three, and four. Thus, we can consider, instead of the classifying space B_n, the more refined $B_n(4)$. The results of [7] show that $\pi_3(B_n) \cong \pi_3(B_n(3))$, while there is a surjection $\pi_3(B_n(3)) \to \pi_3(B_n(4))$, so that we have $\#\pi_3(B_n(4)) \leq \#\pi_3(B_n)$. Thus, the same argument of Theorem 5.6, using cobordisms with stratified branch loci, gives the finer estimate (5.21) for the multiplicities. □

We can then consider the partition function for the Hamiltonian of the time evolution (5.8). To stress the fact that we work in the representation $\rho = \rho_G$ associated to the subsemigroupoid $\bar{\mathcal{G}}_K$ for a given knot K, we write $H = H(G)$. We then have

$$Z_G(\beta) = \mathrm{Tr}(\mathrm{e}^{-\beta H}) = \sum_n \exp(-\beta \, \log(n)) \, N_n(G). \tag{5.22}$$

Thus, the question of whether the summability condition $\mathrm{Tr}(\mathrm{e}^{-\beta H}) < \infty$ holds depends on an estimate of the asymptotic growth of the cardinalities $\#\pi_3(B_n(4))$ for large $n \to \infty$, by the estimate

$$\zeta(\beta) = \sum_n n^{-\beta} \leq Z_G(\beta) \leq \sum_n \#\pi_3(B_n(4)) \, n^{-\beta}. \tag{5.23}$$

This corresponds to the question of studying a generating function for the numbers $\#\pi_3(B_n(4))$. We will not pursue this further in the present paper but we hope to return to this question in future work.

Notice that there is evidence in the results of [6] in favor of some strong constraints on the growth of the numbers $\#\pi_3(B_n)$ (hence of the $\#\pi_3(B_n(4))$), based on the periodicities along certain arithmetic progressions of the localizations at primes.

In fact, it is proved in [6] that, at least for the classifying spaces BR_n of normalized simple branched coverings, in the stable range $n > 4$ and for any given prime p, the localizations $\pi_3(BR_n)_{(p)}$ satisfy the periodicity

$$\pi_3(BR_n)_{(p)} = \pi_3(BR_{n+2^{a+i+1}p^{b+j}})_{(p)},$$

for $n = 2^a p^b m$ with $(2, m) = (p, m) = 1$. The number $2^i p^j$ is determined by homotopy theoretic data as described in Proposition 11 of [6]. Thus, one can consider associated zeta functions

$$Z_p(\beta) = \sum_n \#\pi_3(BR_n)_{(p)} n^{-\beta}. \tag{5.24}$$

If a finite summability $\mathrm{Tr}(\mathrm{e}^{-\beta H}) < \infty$ holds for sufficiently large $\beta \gg 0$, then one can recover invariants of embedded graphs as zero temperature KMS functionals, by considering functionals of the Gibbs form

$$\varphi_{G,\beta}(f) = \frac{\mathrm{Tr}(\rho_G(f)\mathrm{e}^{-\beta H})}{\mathrm{Tr}(\mathrm{e}^{-\beta H})}, \tag{5.25}$$

where, for instance, f is taken to be an invariant of embedded graphs in 3-manifolds and $f(M) := f(\pi_G^{-1}(G))$, for $\pi_G : M \to S^3$ the branched covering map. In this case, in the zero temperature limit, *i.e.* for $\beta \to \infty$, the weak limits of states of the form (5.25) would give back the invariant of embedded graphs in S^3 in the form

$$\lim_{\beta \to \infty} \varphi_{K,\beta}(f) = f(\mathbb{U}_G).$$

Notice that, to the purpose of studying KMS states for the algebra with time evolution, the convergence of the partition function $Z_G(\beta)$ is not needed, as KMS states need not necessarily be of the Gibbs form (5.25), *cf.* [18]. However, it is still useful to consider the question of the convergence of the partition function $Z_G(\beta)$, since Gibbs states of the form (5.25) may have applications to constructing interesting zeta functions for embedded graphs $G \subset S^3$.

For instance, suppose given an invariant F of cobordism classes of embedded graphs in S^3. Cobordism for embedded graphs can be defined, for connected graphs, as in [35], and in the multi-connected case using the same basic relation (attaching a 1-handle) as in the case of links, as in [24]. An example of such an invariant can be obtained, for instance, by considering the collection of links $T(G)$ constructed in [26] as an invariant of an embedded graph G and define a total linking number of $T(G)$ by adding the total linking numbers of all the links in the collection.

Given such an invariant F, one can then consider, for a set of representatives of the classes $[M] \in \pi_3(B_n)$, the values $F(\pi_{G'}\pi_G^{-1}(G))$ and form the series

$$\sum_n \sum_{[M]\in\pi_3(B_n)} F(\pi_{G'}\pi_G^{-1}(G))\, n^{-\beta}, \tag{5.26}$$

where the inner sum is over the classes $[M] \in \pi_3(B_n)$ such that M is a branched cover of S^3 branched along a graph $E \supset G$. Similarly, one can form variations of this same concept based on the zeta functions (5.24). When the function F on the set of the $\{\pi_{G'}\pi_G^{-1}(G)\}$ is either bounded or of some growth $\sim n^\alpha$, then the convergence of $Z_G(\beta)$ (or of the $Z_p(\beta)$ of (5.24)) would ensure the convergence of (5.26). Obviously such zeta functions are very complicated objects, even for very simple graphs G and it would be difficult to compute them explicitly, but it would be interesting to see whether some variant of this idea might have relevance in the context of spin networks, spin foams, and dynamical triangulations.

Finally, notice that, while the Hamiltonian H of the time evolution σ_t^L has finite multiplicities in the spectrum after passing to the quotient by the equivalence relation of cobordism (similarly for σ_t^R), the infinitesimal generator for the time evolution $\sigma_t = \sigma_t^L\sigma_{-t}^R$ still has infinite multiplicities. In fact, the time evolution (4.4) is generated by an unbounded operator D that acts on a densely defined domain in $\mathcal{H}_G$ by

$$D\,\delta_M = \log\left(\frac{n}{m}\right)\,\delta_M, \tag{5.27}$$

with n and m the multiplicities of the two covering maps, as above. This operator is not a good physical Hamiltonian since is does not have a lower bound on the spectrum. It has the following property.

Lemma 5.8. *The operator D of (5.27) has bounded commutators $[D, a]$ with the elements of the involutive algebra generated (algebraically) by the $A_{[M]}$ and $A^*_{[M]}$.*

Proof. It suffices to check that the commutators $[D, A_{[M]}]$ and $[D, A^*_{[M]}]$ are bounded. We have

$$[D, A^*_{[M]}]\delta_{[M']} = \left(\log\left(\frac{nn'}{mm'}\right) - \log\left(\frac{n'}{m'}\right)\right)\delta_{[M\circ M']} = \log\left(\frac{n}{m}\right)\delta_{[M\circ M']}.$$

The case of $[D, A_{[M]}]$ is analogous. □

Notice, however, that D fails to be a Dirac operator in the sense of spectral triples, because of the infinite multiplicities of the eigenvalues.

6. From graphs to knots

The Alexander branched covering theorem is greatly refined by the Hilden–Montesinos theorem, which ensures that all closed oriented 3-manifolds can be realized as branched covers of the 3-sphere, branched along a knot or a link (see [21,29], *cf.* also [33]).

One can see how to pass from a branch locus that is a multi-connected graph to one that is a link or a knot in the following way, [3]. One says that two branched coverings $\pi_0 : M \to S^3$ and $\pi_1 : M \to S^3$ are b-homotopic if there exists a homotopy $H_t : M \to S^3$ with $H_0 = \pi_0$, $H_1 = \pi_1$ and H_t a branched covering, for all $t \in [0, 1]$, with branch locus an embedded graph $G_t \subset S^3$.

The "Alexander trick" shows that two branched coverings of the 3-ball $D^3 \to D^3$ that agree on the boundary $S^2 = \partial D^3$ are b-homotopic. Using this trick, one can pass, by a b-homotopy, from an arbitrary branched covering to one that is *simple*, namely where all the fibers consist of al least $n-1$ points, n being the order of the covering. Simple coverings are *generic*. The same argument shows ([3], Corollary 6.6) that any branched covering $M \to S^3$ is b-homotopic to one where the branch set is a link.

We restrict to the case where the embedded graphs G and G' are knots K and K' and we consider geometric correspondences $\mathcal{C}(K, K')$ modulo the equivalence relation of b-homotopy. Namely, we say that two geometric correspondences $M_1, M_2 \in \mathcal{C}(K, K')$ are b-homotopic if there exist two homotopies Θ_t, Θ'_t relating the branched covering maps

$$S^3 \xleftarrow{\pi_{K,i}} M \xrightarrow{\pi_{K',i}} S^3.$$

Since we have the freedom to modify correspondences by b-homotopies, we can as well assume that the branch loci are links. Thus, we are considering geometric correspondences of the form

$$K \subset L \subset S^3 \xleftarrow{\pi_K} M \xrightarrow{\pi_{K'}} S^3 \supset L' \supset K', \tag{6.1}$$

where the branch loci are links L and L', containing the knots K and K', respectively. Notice also that, if we are allowed to modify the coverings by b-homotopy, we can arrange so that, in the composition $M_1 \circ M_2$, the branch loci $L \cup \pi_k\pi_1^{-1}(L'_2)$ and $L'' \cup \pi_{K''}\pi_2^{-1}(L'_1)$ are links in S^3.

We denote by $[M]_b$ the equivalence class of a geometric correspondence under the equivalence relation of b-homotopy. The equivalence relation of b-homotopy is a particular case of the relation of branched cover cobordism that we considered above. In fact, the homotopy Θ_t can be realized by a branched covering map $\Theta : M \times [0, 1] \to S^3 \times [0, 1]$, branched along a 2-complex $S = \cup_{t\in[0,1]} G_t$ in $S^3 \times [0, 1]$. Thus, by the same argument used to prove the compatibility of the composition of geometric correspondences with the equivalence relation of cobordism, we obtain the compatibility of composition

$$[M_1]_b \circ [M_2]_b = [M_1 \circ M_2]_b. \tag{6.2}$$

The b-homotopy is realized by the cobordims $(M_1 \circ M_2) \times [0, 1]$ with the branched covering maps $\hat{\Theta} = \Theta \circ P_1$ and $\hat{\Theta}' = \Theta' \circ P_2$.

While the knots K and K' are fixed in the construction of $\mathcal{C}(K, K')$, the other components of the links L and L', when we consider the correspondences up to b-homotopy, are only determined up to knot cobordism with trivial linking numbers (*i.e.* as classes in the knot cobordism subgroup of the link cobordism group, see [24]).

To make the role of the link components more symmetric, it is then more natural in this setting to consider a category where the objects are cobordism classes of knots $[K]$, $[K']$ and where the morphisms are given by the b-homotopy classes of geometric correspondences $\mathcal{C}([K], [K'])_b$.

The time evolution considered above still makes sense on the corresponding semigroupoid ring, since the order of the branched cover is well defined on the b-equivalence class and multiplicative under composition of morphisms.

7. Convolution algebras and 2-semigroupoids

In noncommutative geometry, it is customary to replace the operation of taking the quotient by an equivalence relation by forming a suitable convolution algebra of functions over the graph of the equivalence relation. This corresponds to replacing an equivalence relation by the corresponding groupoid and taking the convolution algebra of the groupoid, *cf.* [10].

In our setting, we can proceed in a similar way and, instead of taking the quotient by the equivalence relation of cobordism of branched cover, as we did above, keep the cobordisms explicitly and work with a 2-category.

Lemma 7.1. *The data of embedded graphs in the* 3*-sphere,* 3*-dimensional geometric correspondences, and* 4*-dimensional branched cover cobordisms form a* 2*-category* $\mathcal{G}^2$.

Proof. We already know that embedded graphs and geometric correspondences form a semigroupoid with associative composition of morphisms given by the fibered product of geometric correspondences. Suppose given geometric correspondences M_1 M_2 and M_3 in $\mathcal{C}(G, G')$, and suppose given cobordisms W_1 and W_2 with $\partial W_1 = M_1 \cup -M_2$ and $\partial W_2 = M_2 \cup -M_3$. As we have seen in Lemma 5.2, for the transitive property of the equivalence relation, the gluing of cobordisms $W_1 \cup_{M_2} W_2$ gives a cobordism between M_1 and M_3 and defines in this way a composition of 2-morphisms that has the right properties for being the vertical composition in the 2-category. Similarly, suppose given correspondences $M_1, \tilde{M}_1 \in \mathcal{C}(G, G')$, and $M_2, \tilde{M}_2 \in \mathcal{C}(G', G'')$, with cobordisms W_1 and W_2 with $\partial W_1 = M_1 \cup -\tilde{M}_1$ and $\partial W_2 = M_2 \cup -\tilde{M}_2$. Again by the argument of Lemma 5.2, we know that the fibered product $W_1 \circ W_2$ defines a cobordism between the compositions $M_1 \circ M_2$ and $\tilde{M}_1 \circ \tilde{M}_2$. This gives the horizontal composition of 2-morphisms. By the results of Lemma 5.2 and an argument like that of Proposition 2.8, one sees that both the vertical and horizontal compositions of 2-morphisms are associative. □

In the following, we denote the compositions of 2-morphisms by the notation

$$\text{horizontal (fibered product): } W_1 \circ W_2 \qquad \text{vertical (gluing): } W_1 \bullet W_2. \tag{7.1}$$

We obtain a convolution algebra associated to the 2-semigroupoid described above.

Consider the space of complex valued functions with finite support

$$f : \mathcal{U} \to \mathbb{C} \tag{7.2}$$

on the set

$$\mathcal{U} = \bigcup_{M_1, M_2 \in \mathcal{G}} \mathcal{U}_{(M_1,M_2)},$$

of branched cover cobordisms

$$\mathcal{U}_{(M_1,M_2)} = \{W \mid M_1 \overset{W}{\sim} M_2\}, \tag{7.3}$$

with

$$S \subset S^3 \times I \xleftarrow{q} W \xrightarrow{q'} S^3 \times I \supset S',$$

where $\sim$ denotes the equivalence relation given by branched cover cobordisms with $\partial W = M_1 \cup -M_2$, compatibly with the branched cover structures as in Section 5.

As in the case of the sets $\mathcal{C}(G, G')$ of geometric correspondences discussed in Section 2.1, the collection $\mathcal{U}_{(M_1,M_2)}$ of cobordisms can be identified with a set of branched covering data of a representation theoretic nature. In fact, as a PL manifold, one such cobordism W can be specified by assigning a representation

$$\sigma_W : \pi_1((S^3 \times I) \smallsetminus S) \to S_n, \tag{7.4}$$

which determines a covering space on the complement of the branch locus S [32].

This space of functions (7.2) can be made into an algebra $\mathcal{A}(\mathcal{G}^2)$ with the associative convolution product of the form

$$(f_1 \bullet f_2)(W) = \sum_{W=W_1 \bullet W_2} f_1(W_1) f_2(W_2), \tag{7.5}$$

which corresponds to the vertical composition of 2-morphisms, namely the one given by the gluing of cobordisms. Similarly, one also has on $\mathcal{A}(\mathcal{G}^2)$ an associative product which corresponds to the horizontal composition of 2-morphisms, given by the fibered product of cobordisms, of the form

$$(f_1 \circ f_2)(W) = \sum_{W=W_1 \circ W_2} f_1(W_1) f_2(W_2). \tag{7.6}$$

We also have an involution compatible with both the horizontal and vertical product structure. In fact, consider the two involutions on the cobordisms W

$$W \mapsto \bar{W} = -W, \qquad W \mapsto W^\vee, \tag{7.7}$$

where the first is the orientation reversal, so that if $\partial W = M_1 \cup -M_2$ then $\partial \bar{W} = M_2 \cup -M_1$, while the second extends the involution $M \mapsto M^\vee$ and exchanges the two branch covering maps, that is, if W has covering maps

$$S \subset S^3 \times I \overset{q}{\leftarrow} W \overset{q'}{\to} S^3 \times I \supset S'$$

then $W^\vee$ denotes the same 4-manifold but with covering maps

$$S' \subset S^3 \times I \overset{q'}{\leftarrow} W \overset{q}{\to} S^3 \times I \supset S.$$

We define an involution on the algebra $\mathcal{A}(\mathcal{G}^2)$ by setting

$$f^\dagger(W) = \bar{f}(\bar{W}^\vee). \tag{7.8}$$

Lemma 7.2. *The involution $f \mapsto f^\dagger$ makes $\mathcal{A}(\mathcal{G}^2)$ into an involutive algebra with respect to both the vertical and the horizontal product.*

Proof. We have $(f^\dagger)^\dagger = f$ since the two involutions $W \mapsto \bar{W}$ and $W \mapsto W^\vee$ commute. We also have $(af_1 + bf_2)^\dagger = \bar{a} f_1^\dagger + \bar{b} f_2^\dagger$. For the two product structures, we have

$$\bar{W} = \bar{W}_1 \circ \bar{W}_2 \quad \text{for } W = W_1 \circ W_2$$
$$W^\vee = W_1^\vee \bullet W_2^\vee \quad \text{for } W = W_1 \bullet W_2$$

which gives

$$(f_1 \circ f_2)^\dagger(W) = \sum_{\bar{W}^\vee = \bar{W}_1^\vee \circ \bar{W}_2^\vee} \overline{f_1}(\bar{W}_1^\vee) \overline{f_2}(\bar{W}_2^\vee) = (f_2^\dagger \circ f_1^\dagger)(W)$$
$$(f_1 \bullet f_2)^\dagger(W) = \sum_{\bar{W}^\vee = \bar{W}_1^\vee \bullet \bar{W}_2^\vee} \overline{f_1}(\bar{W}_1^\vee) \overline{f_2}(\bar{W}_2^\vee) = (f_2^\dagger \bullet f_1^\dagger)(W). \quad \square$$

8. Vertical and horizontal time evolutions

We say that σ_t is a *vertical time evolution* on $\mathcal{A}(\mathcal{G}^2)$ if it is a 1-parameter group of automorphisms of $\mathcal{A}(\mathcal{G}^2)$ with respect to the product structure given by the vertical composition of 2-morphisms as in (7.5), namely

$$\sigma_t(f_1 \bullet f_2) = \sigma_t(f_1) \bullet \sigma_t(f_2).$$

Similarly, a *horizontal time evolution* on $\mathcal{A}(\mathcal{G}^2)$ satisfies

$$\sigma_t(f_1 \circ f_2) = \sigma_t(f_1) \circ \sigma_t(f_2).$$

We give some simple examples of one type or the other first and then we move on to more subtle examples.

Lemma 8.1. *The time evolution by order of the coverings defined in (4.4) extends to a horizontal time evolution on $\mathcal{A}(\mathcal{G}^2)$.*

Proof. This clearly follows by taking the order of the cobordisms as branched coverings of $S^3 \times I$. It is not a time evolution with respect to the vertical composition. □

Lemma 8.2. *Any numerical invariant that satisfies an inclusion–exclusion principle*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \tag{8.1}$$

defines a vertical time evolution by

$$\sigma_t(f)(W) = \exp(\mathrm{i}t(\chi(W) - \chi(M_2)))f(W), \tag{8.2}$$

for $\partial W = M_1 \cup -M_2$.

Proof. This also follows immediately by direct verification, since

$$\begin{aligned}\sigma_t(f_1 * f_2)(W) &= \mathrm{e}^{\mathrm{i}t(\chi(W)-\chi(M_2))} \sum_{W=W_1 \cup_M W_2} f_1(W_1)f_2(W_2)\\ &= \mathrm{e}^{\mathrm{i}t(\chi(W_1)+\chi(W_2)-\chi(M)-\chi(M_2))} \sum_{W=W_1 \cup_M W_2} f_1(W_1)f_2(W_2)\\ &= \sum_{W=W_1 \cup_M W_2} \mathrm{e}^{\mathrm{i}t(\chi(W_1)-\chi(M))}f_1(W_1)\mathrm{e}^{\mathrm{i}t(\chi(W_2)-\chi(M_2))}f_2(W_2) = (\sigma_t(f_1) * \sigma_t(f_2))(W). \quad \square\end{aligned}$$

In particular, the following are two simple examples of this type of time evolution.

Example 8.3. Setting $\chi(W)$ to be the Euler characteristic gives a time evolution as in (8.2). Since the 4-dimensional volume of the boundary 3-manifold M is zero, also setting $\chi(W) = Vol(W)$ gives a time evolution.

A more elaborate example of this type is given in Section 10.

9. Vertical time evolution: Hartle–Hawking gravity

We describe here a first non-trivial example of a vertical time evolution, which is related to the Hartle–Hawking formalism of Euclidean quantum gravity [19].

The classical Euclidean action for gravity on a 4-manifold W with boundary is of the form

$$S(W, g) = -\frac{1}{16\pi}\int_W R\,\mathrm{d}v - \frac{1}{8\pi}\int_{\partial W} K\,\mathrm{d}v, \tag{9.1}$$

where R is the scalar curvature and K is the trace of the II fundamental form.

In the Hartle–Hawking approach to quantum gravity, the transition amplitude between two 3-dimensional geometries M_1 and M_2, endowed with Riemannian structures g_{M_1} and g_{M_2} is given by

$$\langle (M_1, g_1), (M_2, g_2)\rangle = \int \mathrm{e}^{\mathrm{i}S(g)}D[g], \tag{9.2}$$

in the Lorentzian signature, where the formal functional integration on the right hand side involves also a summation over topologies, meaning a sum over all cobordisms W with $\partial W = M_1 \cup -M_2$. In the Euclidean setting the probability amplitude $\mathrm{e}^{\mathrm{i}S(g)}$ is replaced by $\mathrm{e}^{-S(g)}$, with $S(g)$ the Euclidean action (9.1). We have suppressed the dependence of the probability amplitude on a quantization parameter $\hbar$.

This suggests setting

$$\sigma_t(f)(W, g) := \mathrm{e}^{\mathrm{i}tS(W,g)}f(W, g), \tag{9.3}$$

with $S(W, g)$ as in (9.1). For (9.3) to define a vertical time evolution, *i.e.* for it to satisfy the compatibility $\sigma_t(f_1 \bullet f_2) = \sigma_t(f_1) \bullet \sigma_t(f_2)$ with the vertical composition, we need to impose conditions on the metrics g on W so that the gluing of the Riemannian data near the boundary is possible when composing cobordisms $W_1 \bullet W_2 = W_1 \cup_M W_2$ by gluing them along a common boundary M.

For instance, one can assume cylindrical metrics near the boundary, though this is does not correspond to the physically interesting case of more general space-like hypersurfaces. Also, one needs to restrict here to cobordisms that are smooth manifolds, or to allow for weaker forms of the Riemannian structure adapted to PL manifolds, as is done in the context of Regge calculus of dynamical triangulations.

Then, formally, one obtains states for this vertical time evolution that can be expressed in the form of a functional integration as

$$\varphi_\beta(f) = \frac{\int f(W, g)\mathrm{e}^{-\beta S(g)}D[g]}{\int \mathrm{e}^{-\beta S(g)}D[g]}. \tag{9.4}$$

We give in the next section a more mathematically rigorous example of vertical time evolution.

10. Vertical time evolution: Index splitting and gauge moduli

Consider again the vertical composition $W_1 \bullet W_2 = W_1 \cup_{M_2} W_2$ given by gluing two cobordisms along their common boundary. In order to construct interesting time evolutions on the corresponding convolution algebra, we consider the spectral theory of Dirac type operators on these 4-dimensional manifolds with boundary, *cf.* [4].

Consider first the simpler case where X is a closed connected 4-manifold and M is a hypersurface that partitions $X \smallsetminus M$ in two components $X = X_1 \cup_M X_2$ with boundary $\partial X_1 = M = -\partial X_2$. We assume that X is endowed with a cylindrical metric on a collar neighborhood $M \times [-1, 1]$ of the hypersurface M. Let $\mathcal{D}$ be an elliptic differential operator on X of Dirac type. We take it to be the Dirac operator assuming that X is a spin 4-manifold. The restriction $\mathcal{D}|_{M\times[-1,1]}$ has the form

$$\mathcal{D}|_{M\times[-1,1]} = c\left(\frac{\partial}{\partial s} + \mathcal{B}\right),$$

where c denotes Clifford multiplication by ds and $\mathcal{B}$ is the self-adjoint tangential Dirac operator on M. We let $P_{\geq}$ denote the spectral Atiyah–Patodi–Singer boundary conditions, *i.e.* the projection onto the subspace of the Hilbert space of square integrable spinors $L^2(M, \mathcal{S}^+|M)$ spanned by the eigenvectors of $\mathcal{B}$ with non-negative eigenvalues. Here $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is the spinor bundle on X, with $\mathcal{D}^+ : C^\infty(X, \mathcal{S}^+) \to C^\infty(X, \mathcal{S}^-)$. The projection $P_{\leq}$ is defined similarly. Let $\mathcal{D}_i$ denote the Dirac operator on X_i with APS boundary conditions, that is,

$$\mathcal{D}_1^+ : C^\infty(X_1, \mathcal{S}^+, P_{\leq}) \to C^\infty(X_1, \mathcal{S}^-), \qquad \mathcal{D}_2^+ : C^\infty(X_1, \mathcal{S}^+, P_{\geq}) \to C^\infty(X_2, \mathcal{S}^-),$$

where

$$C^\infty(X_1, \mathcal{S}^+, P_{\leq}) = \{\psi \in C^\infty(X_1, \mathcal{S}^+) \mid P_{\leq}(\psi|_M) = 0\},$$
$$C^\infty(X_2, \mathcal{S}^+, P_{\geq}) = \{\psi \in C^\infty(X_2, \mathcal{S}^+) \mid P_{\geq}(\psi|_M) = 0\}.$$

The index of the Dirac operator $\mathcal{D}$ is computed by the Atiyah–Singer index theorem and is given by a local formula, while the index of $\mathcal{D}_i$ is given by the Atiyah–Patodi–Singer index theorem and consists of a local formula, together with a correction given by an eta invariant of the boundary manifold M. Moreover, one has the following splitting formula for the index (*cf.* [4], p. 77)

$$\operatorname{Ind}(\mathcal{D}) = \operatorname{Ind}(\mathcal{D}_1) + \operatorname{Ind}(\mathcal{D}_2) - \dim \operatorname{Ker}(\mathcal{B}). \tag{10.1}$$

In the case of 4-manifolds $W = W_1 \cup_M W_2$, where $\partial W = M_1 \cup -M_3$, $\partial W_1 = M_1 \cup -M_2$, and $W_2 = M_2 \cup -M_3$, one can modify the above setting by imposing APS boundary conditions at both ends of the cobordims. Namely, we assume that W is a smooth manifold with boundary endowed with a Riemannian metric with cylindrical ends $M_1 \times [0, 1]$ and $M_3 \times [-1, 0]$, as well as a cylindrical metric on a collar neighborhood $M \times [-1, 1]$.

Thus, the operator $\mathcal{D}$ will be the Dirac operator with APS boundary conditions $P_{\geq}$ and $P_{\leq}$ at M_1 and M_3, and similarly for the operators $\mathcal{D}_1$ and $\mathcal{D}_2$. We also denote by $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ the tangential Dirac operators on M, M_1 and M_2, respectively. We then obtain a time evolution on the algebra $\mathcal{A}(\mathcal{G}^2)$ with the product (7.5) associated to the splitting of the index, in the following way.

Lemma 10.1. *Let $W = W_1 \cup_M W_2$ be a composition of 4-dimensional cobordisms with metrics as above, and with $\mathcal{D}$, $\mathcal{D}_i$ the corresponding Dirac operators with APS boundary conditions. We let*

$$\delta(W) := \operatorname{Ind}(\mathcal{D}) - \dim \operatorname{Ker}(\mathcal{B}_2). \tag{10.2}$$

Then setting

$$\sigma_t(f)(W) = \exp(\mathrm{i}t\delta(W))f(W) \tag{10.3}$$

defines a time evolution on $\mathcal{A}(\mathcal{G}^2)$ with the product (7.5) of vertical composition.

Proof. Using the splitting formula (10.1) for the index one sees immediately that

$$\begin{aligned}\sigma_t(f_1 \bullet f_2)(W) &= \sum_{W=W_1\bullet W_2} \mathrm{e}^{\mathrm{i}t\delta(W)} f_1(W_1) f_2(W_2) \\ &= \sum_{W=W_1\bullet W_2} \mathrm{e}^{\mathrm{i}t(\operatorname{Ind}\mathcal{D}_1 + \operatorname{Ind}\mathcal{D}_2 - \dim\operatorname{Ker}\mathcal{B} - \dim\operatorname{Ker}\mathcal{B}_2)} f_1(W_1) f_2(W_2) \\ &= \sum_{W=W_1\bullet W_2} \mathrm{e}^{\mathrm{i}t\delta(W_1)} f_1(W_1) \mathrm{e}^{\mathrm{i}t\delta(W_2)} f_2(W_2) = \sigma_t(f_1) \bullet \sigma_t(f_2)(W). \quad \square\end{aligned}$$

The type of spectral problem described above arises typically in the context of invariants of 4-dimensional geometries that behave well under gluing. A typical such setting is given by the topological quantum field theories, as outlined in [2], where to every 3-dimensional manifolds one assigns functorially a vector space and to every cobordism between 3-manifolds a linear map between the vector spaces.

In the case of Yang–Mills gauge theory, the gluing theory for moduli spaces of anti-self-dual $SO(3)$-connections on smooth 4-manifolds (see [36] for an overview) shows that if M is a closed oriented smooth 3-manifold that separates a closed smooth 4-manifold X in two connected pieces

$$X = X_+ \cup_M X_- \tag{10.4}$$

glued along the common boundary $M = \partial X_+ = -\partial X_-$, then the moduli space $\mathcal{M}(X)$ of gauge equivalence classes of framed anti-self-dual $SO(3)$-connections on X decomposes as a fibered product

$$\mathcal{M}(X) = \mathcal{M}(X_+) \times_{\mathcal{M}(M)} \mathcal{M}(X_-), \tag{10.5}$$

where $\mathcal{M}(X_\pm)$ are moduli spaces of anti-self-dual $SO(3)$-connections on the 4-manifolds with boundary and $\mathcal{M}(M)$ is a a neighborhood of the moduli space of gauge classes of flat connections on the 3-manifold M. The fibered product is over the restriction maps induced by the inclusion of M in $X_\pm$. Setting up the appropriate analytical theory to compute the virtual dimensions is a technically very demanding task a detailed discussion of which is beyond the scope of this short paper. We only mention the fact that virtual dimensions are given by indices of elliptic operators. These operators arise via deformation complexes $\Omega^0 \to \Omega^1 \to \Omega^2$, where the first map correspond to the infinitesimal gauge action and the second to the linearization of the nonlinear elliptic equations at a solution. The counting of virtual dimensions that corresponds to the fibered product formula (10.5) of Donaldson–Floer theory is given by a splitting formula for the index of the type discussed above in (10.1). One finds similar fibered product formulae in the gluing theory of other gauge theoretic moduli spaces, such as Seiberg–Witten (*cf. e.g.* [8]). It would be interesting to see if invariants of 4-manifolds constructed from various gauge theories and topological quantum field theories can give rise to interesting dynamics and equilibrium states on the algebra $\mathcal{A}(\mathcal{G}^2)$ of geometric correspondences and cobordisms, in a way that uses more information than just the virtual dimension of the moduli spaces.

11. Horizontal time evolution: Bivariant Chern character

The time evolution of Lemma 10.1, however, does not detect the structure of W as a branched cover of $S^3 \times I$ branched along an embedded surface $S \subset S^3 \times I$. Thus, there is no reason why a time evolution defined in this way should also be compatible with the other product given by the horizontal composition of 2-morphisms. However, the time evolution (10.3) obtained using the splitting formula (10.1) for the index suggests a possible way to define other time evolutions, also related to properties of an index, which would be compatible with the horizontal composition.

Although we are working here in the commutative context, in view of the extension to noncommutative spectral correspondences outlined in the next section, we give here a formulation using the language of KK-theory and cyclic cohomology that carries over naturally to the noncommutative cases.

In noncommutative geometry, one thinks of the index theorem as a pairing of K-theory and K-homology, or equivalently as the pairing $\langle ch_n(e), ch_n(x)\rangle$ of Connes–Chern characters

$$ch_n : K_i(\mathcal{A}) \to HC_{2n+i}(\mathcal{A}) \quad \text{and} \quad ch_n : K^i(\mathcal{A}) \to HC^{2n+i}(\mathcal{A}), \tag{11.1}$$

under the natural pairing of cyclic homology and cohomology, *cf.* [10].

Recall that cyclic (co)homology has a natural description in terms of the derived functors Ext and Tor in the abelian category of cyclic modules (*cf.* [11]), namely

$$HC^n(\mathcal{A}) = \mathrm{Ext}^n_\Lambda(\mathcal{A}^\natural, \mathbb{C}^\natural) \quad \text{and} \quad HC_n(\mathcal{A}) = \mathrm{Tor}^\Lambda_n(\mathbb{C}^\natural, \mathcal{A}^\natural), \tag{11.2}$$

where Λ denotes the cyclic category and $\mathcal{A}^\natural$ is the cyclic module associated to an associative algebra $\mathcal{A}$.

It was shown in [30] that the characters (11.1) extend to a bivariant Connes–Chern character

$$ch_n : KK^i(\mathcal{A}, \mathcal{B}) \to \mathrm{Ext}^{2n+i}_\Lambda(\mathcal{A}^\natural, \mathcal{B}^\natural) \tag{11.3}$$

defined on KK-theory, with the natural cap product pairings

$$\mathrm{Tor}^\Lambda_m(\mathbb{C}^\natural, \mathcal{A}^\natural) \otimes \mathrm{Ext}^n_\Lambda(\mathcal{A}^\natural, \mathcal{B}^\natural) \to \mathrm{Tor}^\Lambda_{m-n}(\mathbb{C}^\natural, \mathcal{B}^\natural) \tag{11.4}$$

corresponding to an index theorem

$$\psi = ch(x)\phi, \quad \text{with } \phi(e \circ x) = \psi(e). \tag{11.5}$$

The construction of a bivariant Connes–Chern character that is fully compatible with the composition products, namely that sends the Kasparov product

$$\circ : KK^i(\mathcal{A}, \mathcal{C}) \times KK^j(\mathcal{C}, \mathcal{B}) \to KK^{i+j}(\mathcal{A}, \mathcal{B})$$

to the Yoneda products

$$\mathrm{Ext}^{2n+i}_\Lambda(\mathcal{A}^\natural, \mathcal{C}^\natural) \times \mathrm{Ext}^{2m+j}_\Lambda(\mathcal{C}^\natural, \mathcal{B}^\natural) \to \mathrm{Ext}^{2(n+m)+i+j}(\mathcal{A}^\natural, \mathcal{B}^\natural), \tag{11.6}$$

requires a modification of both KK-theory and cyclic cohomology. Such a general form of the bivariant Connes–Chern character is given in [15].

The construction of [14] of geometric correspondences realizing KK-theory classes shows that, given manifolds X_1 and X_2, classes in $KK(X_1, X_2)$ are realized by geometric data (Z, E) of a manifold Z with submersions $X_1 \leftarrow Z \rightarrow X_2$ and a vector bundle E on Z. The Kasparov product $x \circ y \in KK(X_1, X_3)$, for $x = kk(Z, E) \in KK(X_1, X_2)$ and $y = kk(Z', E') \in KK(X_2, X_3)$, is given by the fibered product $x \circ y = kk(Z \circ Z', E \circ E')$, where

$$Z \circ Z' = Z \times_{X_2} Z' \quad \text{and} \quad E \circ E' = \pi_1^* E \times \pi_2^* E'.$$

To avoid momentarily the complication caused by working with manifolds with boundary, we consider the simpler situation where W is a 4-manifold endowed with branched covering maps to a closed 4-manifold X (for instance $S^3 \times S^1$ or S^4) instead of $S^3 \times [0, 1]$,

$$S \subset X \xleftarrow{q} W \xrightarrow{q'} X \supset S' \tag{11.7}$$

branched along surfaces S and S' in X.

We can then think of an elliptic operator $\mathcal{D}_W$ on a 4-manifold W, which has branched covering maps as in (11.7), as defining an unbounded Kasparov bimodule, *i.e.* as defining a KK-class $[\mathcal{D}_W] \in KK(X, X)$. We can think of this class as being realized by a geometric correspondence in the sense of [14]

$$[\mathcal{D}_W] = kk(W, E_W),$$

with the property that, for the horizontal composition $W = W_1 \circ W_2 = W_1 \times_X W_2$ we have

$$[\mathcal{D}_{W_1}] \circ [\mathcal{D}_{W_2}] = kk(W_1, E_{W_1}) \circ kk(W_2, E_{W_2}) = kk(W, E_W) = [\mathcal{D}_W].$$

Under a bivariant Chern character that is compatible with the composition products these classes will map to elements in the Yoneda algebra

$$\begin{aligned} &ch_n([\mathcal{D}_W]) \in \mathcal{Y} := \oplus_j \operatorname{Ext}^{2n+j}(\mathcal{A}^\natural, \mathcal{A}^\natural) \\ &ch_n([\mathcal{D}_{W_1}]) ch_m([\mathcal{D}_{W_2}]) = ch_{n+m}([\mathcal{D}_{W_1}] \circ [\mathcal{D}_{W_2}]). \end{aligned} \tag{11.8}$$

Let $\chi : \mathcal{Y} \to \mathbb{C}$ be a character of the Yoneda algebra. Then by composing $\chi \circ ch$ we obtain

$$\chi ch([\mathcal{D}_{W_1}] \circ [\mathcal{D}_{W_2}]) = \chi ch([\mathcal{D}_{W_1}]) \chi ch([\mathcal{D}_{W_2}]) \in \mathbb{C}.$$

This can be used to define a time evolution for the horizontal product of the form

$$\sigma_t(f)(W) = |\chi ch([\mathcal{D}_W])|^{it} f(W).$$

12. Noncommutative spaces and spectral correspondences

We return now briefly to the problem of spectral correspondences of [13], mentioned in the introduction.

Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of the data of a unital involutive algebra $\mathcal{A}$, a representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ as bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D on $\mathcal{H}$ with compact resolvent, such that $[D, \rho(a)]$ is a bounded operator for all $a \in \mathcal{A}$. We extend this notion to a correspondence in the following way, following [13].

Definition 12.1. A spectral correspondence is a set of data $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{H}, D)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital involutive algebras, with representations $\rho_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H})$, $i = 1, 2$, as bounded operators on a Hilbert space $\mathcal{H}$, such that

$$[\rho_1(a_1), \rho_2(a_2)] = 0, \quad \forall a_1 \in \mathcal{A}_1,\ \forall a_2 \in \mathcal{A}_2, \tag{12.1}$$

and with a self-adjoint operator D with compact resolvent, such that

$$[[D, \rho_1(a_1)], \rho_2(a_2)] = 0, \quad \forall a_1 \in \mathcal{A}_1,\ \forall a_2 \in \mathcal{A}_2, \tag{12.2}$$

and such that $[D, \rho_1(a_1)]$ and $[D, \rho_2(a_2)]$ are bounded operators for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. A spectral correspondence is even if there exists an operator γ on $\mathcal{H}$ with $\gamma^2 = 1$ and such that D anticommutes with γ and $[\gamma, \rho_i(a_i)] = 0$ for all $a_i \in \mathcal{A}_i$, $i = 1, 2$. A spectral correspondence is odd if it is not even.

One might relax the condition of compact resolvent on the operator D if one wants to allow more degenerate types of operators in the correspondences, including possibly $D \equiv 0$, as seems desirable in view of the considerations of [13]. For our purposes here, we consider this more restrictive definition. Notice also that the condition (12.2) also implies $[[D, \rho_2(a_2)], \rho_1(a_1)] = 0$ because of (12.1).

A more refined notion of spectral correspondences as morphisms between spectral triples, in a setting for families, is being developed by Mesland, [28].

We first show that our geometric correspondences define commutative spectral correspondences and then we give a noncommutative example based on taking products with finite geometries as in [13].

Lemma 12.2. *Suppose given a closed connected oriented smooth 3-manifold with two branched covering maps $S^3 \xleftarrow{\pi_1} M \xrightarrow{\pi_2} S^3$. Given a choice of a Riemannian metric and a spin structure on M, this defines a spectral correspondence for $\mathcal{A}_1 = \mathcal{A}_2 = C^\infty(S^3)$.*

Proof. We consider the Hilbert space $\mathcal{H} = L^2(M, S)$, where S is the spinor bundle on M for the chosen spin structure. Let $\partial\!\!\!/_M$ be the corresponding Dirac operator. The covering maps π_i, for $i = 1, 2$, determine representations $\rho_i : C^\infty(S^3) \to \mathcal{B}(\mathcal{H})$, by $\rho_i(f) = c(f \circ \pi_i)$, where c denotes the usual action of $C^\infty(M)$ on $\mathcal{H}$ by Clifford multiplication on spinors. Then we have $[\partial\!\!\!/_M, \rho_i(f)] = c(d(f \circ \pi_i))$, which is a bounded operator on $\mathcal{H}$. All the commutativity conditions are satisfied in this case. □

Let $\mathcal{A}$ and $\mathcal{B}$ be finite dimensional unital (noncommutative) involutive algebras. Let V be a finite dimensional vector space with commuting actions of $\mathcal{A}$ and $\mathcal{B}$. Let $T \in \mathrm{End}(V)$ be a linear map such that $[[T, a], b] = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then we obtain noncommutative spectral correspondences of the type described in the last section of [13] in the following way.

Lemma 12.3. *The cup product $S_M \cup S_F$ of $S_M = (C^\infty(S^3), C^\infty(S^3), L^2(M, S), /\partial_M)$ and $S_F = (\mathcal{A}, \mathcal{B}, V, T)$ defines a noncommutative spectral correspondence for the algebras $C^\infty(S^3) \otimes \mathcal{A}$ and $C^\infty(S^3) \otimes \mathcal{B}$.*

Proof. We simply adapt the usual notion of cup product for spectral triples to the case of correspondences. If the correspondence $(\mathcal{A}, \mathcal{B}, V, T)$ is even, with grading γ, then we consider the Hilbert space $\mathcal{H} = L^2(M, S) \otimes V$ and the operator $D = T \otimes 1 + \gamma \otimes \partial\!\!\!/_M$. Then the usual argument for cup products of spectral triples show that $(C^\infty(S^3) \otimes \mathcal{A}, C^\infty(S^3) \otimes \mathcal{B}, \mathcal{H}, D)$ is an odd spectral correspondence. Similarly, if $(\mathcal{A}, \mathcal{B}, V, T)$ is odd, then take $\mathcal{H} = L^2(M, S) \otimes V \oplus L^2(M, S) \otimes V$, with the diagonal actions of $C^\infty(S^3) \otimes \mathcal{A}$ and $C^\infty(S^3) \otimes \mathcal{B}$. Consider then the operator

$$D = \begin{pmatrix} 0 & \delta^* \\ \delta & 0 \end{pmatrix},$$

for $\delta = T \otimes 1 + i \otimes \partial\!\!\!/_M$. Then, by the same standard argument that holds for spectral triples, the data $(C^\infty(S^3) \otimes \mathcal{A}, C^\infty(S^3) \otimes \mathcal{B}, \mathcal{H}, D)$ form an even spectral correspondence with respect to the $\mathbb{Z}/2\mathbb{Z}$ grading

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In either case, we denote the resulting correspondence $(C^\infty(S^3) \otimes A, C^\infty(S^3) \otimes B, \mathcal{H}, D)$ as the cup product $S_M \cup S_F$. □

We can then form a convolution algebra on the space of correspondences, using the equivalence relation given by cobordism of branched covering spaces of Section 5, as in Section 7. This requires extending the equivalence relation defined by cobordisms of branched coverings to the case of the product by a finite geometry. We propose the following construction.

The existence of a cobordism W of branched coverings between two geometric correspondences M_1 and M_2 in $\mathcal{C}(K, K')$ implies the existence of a spectral correspondence *with boundary* of the form

$$S_W = (C^\infty(M_1), C^\infty(M_2), L^2(W, S), \partial\!\!\!/_W).$$

We will not discuss here the setting of spectral triples with boundary. A satisfactory theory was recently developed by Chamseddine and Connes (*cf.* [9]). We only recall here briefly the following notions, from [12]. A spectral triple with boundary $(\mathcal{A}, \mathcal{H}, D)$ is *boundary even* if there is a $\mathbb{Z}/2\mathbb{Z}$-grading γ on $\mathcal{H}$ such that $[a, \gamma] = 0$ for all $a \in \mathcal{A}$ and $\mathrm{Dom}(D) \cap \gamma \mathrm{Dom}(D)$ is dense in $\mathcal{H}$. The boundary algebra $\partial\mathcal{A}$ is the quotient $\mathcal{A}/(J \cap J^*)$ by the two-sided ideal $J = \{a \in \mathcal{A} | a \,\mathrm{Dom}(D) \subset \gamma \mathrm{Dom}(D)\}$. The boundary Hilbert space $\partial\mathcal{H}$ is the closure in $\mathcal{H}$ of $D^{-1} \operatorname{Ker} D_0^*$, where D_0 is the symmetric operator obtained by restricting D to $\mathrm{Dom}(D) \cap \gamma \mathrm{Dom}(D)$. The boundary algebra acts on the boundary Hilbert space by $a - D^{-2}[D^2, a]$. The boundary Dirac operator ∂D is defined on $D^{-1} \operatorname{Ker} D_0^*$ and satisfies $\langle \xi, \partial D \eta \rangle = \langle \xi, D\eta \rangle$ for $\xi \in \partial\mathcal{H}$ and $\eta \in D^{-1} \operatorname{Ker} D_0^*$. It has bounded commutators with $\partial\mathcal{A}$.

One can extend from spectral triples to correspondences, by having two commuting representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ on $\mathcal{H}$ with the properties above and such that the resulting boundary data $(\partial\mathcal{A}_1, \partial\mathcal{A}_2, \partial\mathcal{H}, \partial D)$ define a spectral correspondence.

If one wants to extend to the product geometries the condition of cobordism of geometric correspondences, it seems that one is inevitably faced with the problem of defining spectral triples with corners. In fact, if S_W and S_F are both spectral triples with boundary, then their cup product $S_W \cup S_F$ would not longer give rise to a spectral triple with boundary but to one with corners. At present there isn't a well defined theory of spectral triples with corners. However, we can still propose a way of dealing with products of cobordisms by finite noncommutative geometries, which remains within the theory of spectral triples with boundary. To this purpose, we assume that the finite part S_F is an ordinary spectral triple, while only the cobordims part is a spectral triple with boundary. We then relate the cup product $S_W \cup S_F$ to the spectral correspondences $S_{M_i} \cup S_{F_i}$ via the boundary ∂S_W and bimodules relating the S_{F_i} to S_F. More precisely, we consider the following data.

Suppose given $M_i \in \mathcal{C}(K, K')$, $i = 1, 2$ as above and finite spectral correspondences $S_{F_i} = (A_i, B_i, V_i, T_i)$. Then we say that the cup products $S_{M_i} \cup S_{F_i}$ are related by a spectral cobordism if the following conditions hold. The geometric correspondences are equivalent $M_1 \sim M_2$ via a cobordism W. There exist finite dimensional (noncommutative) algebras R_i, $i = 1, 2$ together with R_i–A_i bimodules E_i and B_i–R_i bimodules F_i, with connections. There exists a finite spectral correspondence

$S_F = (R_1, R_2, V_F, D_F)$ such that $S_W \cup S_F = (\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with boundary in the sense of Chamseddine–Connes with

$$\partial\mathcal{A} = \bigoplus_{i=1,2} C^\infty(M_i) \otimes R_i$$
$$\partial\mathcal{H} = \bigoplus_{i=1,2} L^2(M_i, S) \oplus (E_i \otimes_{A_i} V_i \otimes_{B_i} F_i)$$

and $\partial\mathcal{D}$ gives the cup product of the Dirac operators ∂_{M_i} with the T_i, with the latter twisted by the connections on E_i and F_i.

We do not give more details here. In fact, in order to use this notion to extend the equivalence relation of cobordims of branched coverings and the 2-category we considered in Section 7 above to the noncommutative case, one needs a gluing theory for spectral triples with boundary that makes it possible to define the horizontal and vertical compositions of 2-morphisms as in the case of $W_1 \circ W_2$ and $W_1 \bullet W_2$. The analysis necessary to develop such gluing results is beyond the scope of this paper and the problem will be considered elsewhere.

13. Questions and future work

We sketch briefly an outline of ongoing work where the construction presented in this paper is applied to other constructions related to noncommutative geometry and knot invariants.

13.1. Time evolutions and moduli spaces

We have constructed vertical time evolutions from virtual dimensions of moduli spaces. It would be more interesting to construct time evolutions on the algebra of correspondences, in such a way that the actual gauge theoretic invariants obtained by integrating certain differential forms over the moduli spaces can be recovered as low temperature equilibrium states. The formal path integral formulations of gauge theoretic invariants of 4-manifolds suggests that something of this sort may be possible, by analogy to the case we described of Hartle–Hawking gravity. In the case of the horizontal time evolution, it would be interesting to see if that can also be related to gauge theoretic invariants. The closest model available would be the gauge theory on embedded surfaces developed in [27].

13.2. Categorification and homology invariants

We have constructed a category of knots and links, or more generally of embedded graphs, where it is possible to use homological algebra to construct complexes and cohomological invariants. The process of categorifications in knot theory, applied to a different category of knots, has already proved very successful in deriving new knot invariants such as Khovanov homology. We intend to investigate possible constructions of cohomological invariants using the category defined in this paper.

13.3. Noncommutative spaces and dynamical systems

Another way to construct noncommutative spaces out of the geometric correspondences considered here is via the subshifts of finite type constructed in [34] out of the representations $\sigma : \pi_1(S^3 \setminus L) \to S_m$ describing branched coverings. A subshift of finite type naturally determines a noncommutative space in the form of associated Cuntz–Krieger algebras. The covering moves (or colored Reidemeister moves) of [31] will determine correspondences between these noncommutative spaces.

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