Cosmology and the Poisson summation formula

Matilde Marcolli

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This talk is based on:

- MPT M. Marcolli, E. Pierpaoli, K. Teh, *The spectral action and cosmic topology*, arXiv:1005.2256 (to appear in CMP)
- MPT2 M. Marcolli, E. Pierpaoli, K. Teh, *The coupling of topology and inflation in Noncommutative Cosmology*, arXiv:1012.0780
- CMT B. Ćaćić, M. Marcolli, K. Teh, *Topological coupling of gravity to matter, spectral action and cosmic topology*, in preparation

The NCG standard model and cosmology

- CCM A. Chamseddine, A. Connes, M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007), no. 6, 991–1089.
- MP M. Marcolli, E. Pierpaoli, *Early universe models from noncommutative geometry*, arXiv:0908.3683
- KM D. Kolodrubetz, M. Marcolli, *Boundary conditions of the RGE flow in noncommutative cosmology*, arXiv:1006.4000

Two topics of current interest to cosmologists:

• Modified Gravity models in cosmology:
Einstein-Hilbert action (+cosmological term) replaced or extended with other gravity terms (conformal gravity, higher derivative terms) $\Rightarrow$ cosmological predictions

• The question of Cosmic Topology:
Nontrivial (non-simply-connected) spatial sections of spacetime, homogeneous spherical or flat spaces: how can this be detected from cosmological observations?

Our approach:

- NCG provides a modified gravity model through the spectral action
- The nonperturbative form of the spectral action determines a slow-roll inflation potential
- The underlying geometry (spherical/flat) affects the shape of the potential (possible models of inflation)
- Different inflation scenarios depending on geometry and topology of the cosmos
- More refined topological properties from coupling to matter

The noncommutative space $X \times F$ extra dimensions
product of 4-dim spacetime and finite NC space
The spectral action functional

$$\mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

$D_A = D + A + \varepsilon' J A J^{-1}$ Dirac operator with inner fluctuations
$A = A^* = \sum_k a_k[D, b_k]$

- Action functional for gravity on X (modified gravity)
- Gravity on $X \times F$ = gravity coupled to matter on X

Spectral triples $(\mathcal{A}, \mathcal{H}, D)$:

- involutive algebra $\mathcal{A}$
- representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
- self adjoint operator D on $\mathcal{H}$
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$
- even $\mathbb{Z}/2$-grading $[\gamma, a] = 0$ and $D\gamma = -\gamma D$
- real structure: antilinear isom $J : \mathcal{H} \to \mathcal{H}$ with $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, and $J\gamma = \varepsilon'' \gamma J$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- bimodule: $[a, b^0] = 0$ for $b^0 = Jb^* J^{-1}$
- order one condition: $[[D, a], b^0] = 0$

Asymptotic formula for the spectral action (Chamseddine–Connes)

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{k \in \mathrm{DimSp}} f_k \Lambda^k \fint |D|^{-k} + f(0)\zeta_D(0) + o(1)$$

for large Λ with $f_k = \int_0^\infty f(v) v^{k-1} dv$ and integration given by residues of zeta function $\zeta_D(s) = \mathrm{Tr}(|D|^{-s})$; DimSp poles of zeta functions

Asymptotic expansion $\Rightarrow$ Effective Lagrangian
(modified gravity + matter)

At low energies: only nonperturbative form of the spectral action

$$\mathrm{Tr}(f(D_A/\Lambda))$$

Need explicit information on the Dirac spectrum!

Product geometry $(C^\infty(X), L^2(X,S), D_X) \cup (\mathcal{A}_F, \mathcal{H}_F, D_F)$

- $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_F = C^\infty(X, \mathcal{A}_F)$
- $\mathcal{H} = L^2(X,S) \otimes \mathcal{H}_F = L^2(X, S \otimes \mathcal{H}_F)$
- $D = D_X \otimes 1 + \gamma_5 \otimes D_F$

Inner fluctuations of the Dirac operator

$$D \to D_A = D + A + \varepsilon' J A J^{-1}$$

A self-adjoint operator

$$A = \sum a_j [D, b_j], \quad a_j, b_j \in \mathcal{A}$$

$\Rightarrow$ boson fields from inner fluctuations, fermions from $\mathcal{H}_F$

Get realistic particle physics models [CCM]
Need Ansatz for the NC space F

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

$\Rightarrow$ everything else follows by *computation*

- Representation: $\mathcal{M}_F$ sum of all inequiv irred odd $\mathcal{A}_{LR}$-bimodules (fix N generations) $\mathcal{H}_F = \oplus^N \mathcal{M}_F$ fermions
- Algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$: order one condition
- F zero dimensional but KO-dim 6
- J_F = matter/antimatter, γ_F = L/R chirality
- Classification of Dirac operators (moduli spaces)

Dirac operators and Majorana mass terms

$$D(Y) = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}, \qquad S = S_1 \oplus (S_3 \otimes 1_3), \qquad T = Y_R : |\nu_R\rangle \to J_F\, |\nu_R\rangle$$

$$S_1 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 1)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 1)} \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 3)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 3)} \\ Y_{(\uparrow 3)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 3)} & 0 & 0 \end{pmatrix}$$

Yukawa matrices: Dirac masses and mixing angles in $\mathrm{GL}_{N=3}(\mathbb{C})$

$Y_e = Y_{(\downarrow 1)}$ (charged leptons)

$Y_\nu = Y_{(\uparrow 1)}$ (neutrinos)

$Y_d = Y_{(\downarrow 3)}$ (d/s/b quarks)

$Y_u = Y_{(\uparrow 3)}$ (u/c/t quarks)

$M = Y_R^t$ Majorana mass terms symm matrix

Moduli space of Dirac operators on finite NC space F

$$\mathcal{C}_3 \times \mathcal{C}_1$$

- $\mathcal{C}_3 =$ pairs $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ modulo W_j unitary matrices:

$$Y'_{(\downarrow 3)} = W_1 \, Y_{(\downarrow 3)} \, W_3^* \, , \quad Y'_{(\uparrow 3)} = W_2 \, Y_{(\uparrow 3)} \, W_3^*$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$: $\quad \mathcal{C}_3 = (K \times K) \backslash (G \times G)/K$
$\dim_{\mathbb{R}} \mathcal{C}_3 = 10 = 3 + 3 + 4$ (eigenval, coset 3 angles 1 phase)

- $\mathcal{C}_1 =$ triplets $(Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_R)$ with Y_R symmetric modulo

$$Y'_{(\downarrow 1)} = V_1 \, Y_{(\downarrow 1)} V_3^* \, , \quad Y'_{(\uparrow 1)} = V_2 \, Y_{(\uparrow 1)} \, V_3^* \, ,$$

$$Y'_R = V_2 \, Y_R \, \bar{V}_2^*$$

$\pi : \mathcal{C}_1 \to \mathcal{C}_3$ surjection forgets Y_R fiber symm matrices mod $Y_R \mapsto \lambda^2 Y_R$
$\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$ (dim fiber 12-1=11)

Parameters of νMSM

- three coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses, 3 lepton mixing angles, 1 complex phase
- 3 neutrino masses
- 11 Majorana mass matrix parameters
- QCD vacuum angle

Moduli space of Dirac operators on F $\Rightarrow$ geometric form of all the Yukawa and Majorana parameters

Fields content of the model

- Bosons: inner fluctuations $A = \sum_j a_j[D, b_j]$

- In M direction: $U(1)$, $SU(2)$, and $SU(3)$ gauge bosons

- In F direction: Higgs field $H = \varphi_1 + \varphi_2 j$

- Fermions: basis of $\mathcal{H}_F$

$$|\uparrow\rangle \otimes \mathbf{3}^0, \quad |\downarrow\rangle \otimes \mathbf{3}^0, \quad |\uparrow\rangle \otimes \mathbf{1}^0, \quad |\downarrow\rangle \otimes \mathbf{1}^0$$

Gauge group $SU(\mathcal{A}_F) = U(1) \times SU(2) \times SU(3)$
(up to fin abelian group)

- Hypercharges: adjoint action of $\mathrm{U}(1)$ (in powers of $\lambda \in \mathrm{U}(1)$)

	$\uparrow \otimes \mathbf{1}^0$	$\downarrow \otimes \mathbf{1}^0$	$\uparrow \otimes \mathbf{3}^0$	$\downarrow \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

$\Rightarrow$ Correct hypercharges to the fermions

Action functional

$$\mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

Fermion part: antisymmetric bilinear form $\mathfrak{A}(\tilde{\xi})$ on

$$\mathcal{H}^+ = \{\xi \in \mathcal{H} \,|\, \gamma\xi = \xi\}$$

$\Rightarrow$ nonzero on Grassmann variables
Euclidean functional integral $\Rightarrow$ Pfaffian

$$Pf(\mathfrak{A}) = \int e^{-\frac{1}{2}\mathfrak{A}(\tilde{\xi})} D[\tilde{\xi}]$$

(avoids Fermion doubling problem of previous models based on symmetric $\langle \xi, D_A\xi\rangle$ for NC space with KO-dim=0)

Explicit computation gives part of SM Larangian with

- $\mathcal{L}_{Hf}$ = coupling of Higgs to fermions
- $\mathcal{L}_{gf}$ = coupling of gauge bosons to fermions
- $\mathcal{L}_{f}$ = fermion terms

The asymptotic expansion of the spectral action from [CCM]

$$
\begin{aligned}
S = \quad & \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, \mathfrak{c} + \frac{f_0}{4}\, \mathfrak{d}) \int \sqrt{g}\, d^4x \\
+ \quad & \frac{96\, f_2\, \Lambda^2 - f_0\, \mathfrak{c}}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0}{10\,\pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma})\, \sqrt{g}\, d^4x \\
+ \quad & \frac{(-2\, \mathfrak{a}\, f_2\, \Lambda^2 + \mathfrak{e}\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0 \mathfrak{a}}{2\,\pi^2} \int |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
- \quad & \frac{f_0 \mathfrak{a}}{12\,\pi^2} \int R\, |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0 \mathfrak{b}}{2\,\pi^2} \int |\varphi|^4\, \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0}{2\,\pi^2} \int (g_3^2\, G^i_{\mu\nu}\, G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu})\, \sqrt{g}\, d^4x,
\end{aligned}
$$

Parameters:

- f_0, f_2, f_4 free parameters, $f_0 = f(0)$ and, for $k > 0$,

$$f_k = \int_0^\infty f(v) v^{k-1} dv.$$

- $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ functions of Yukawa parameters of SM+r.h.ν

$$\begin{aligned}
\mathfrak{a} &= \operatorname{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d)) \\
\mathfrak{b} &= \operatorname{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2) \\
\mathfrak{c} &= \operatorname{Tr}(MM^\dagger) \\
\mathfrak{d} &= \operatorname{Tr}((MM^\dagger)^2) \\
\mathfrak{e} &= \operatorname{Tr}(MM^\dagger Y_\nu^\dagger Y_\nu).
\end{aligned}$$

Normalization and coefficients

$$
\begin{aligned}
S = \;& \frac{1}{2\kappa_0^2} \int R \sqrt{g}\, d^4x + \gamma_0 \int \sqrt{g}\, d^4x \\
+ \;& \alpha_0 \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g}\, d^4x + \tau_0 \int R^* R^* \sqrt{g}\, d^4x \\
+ \;& \frac{1}{2} \int |DH|^2 \sqrt{g}\, d^4x - \mu_0^2 \int |H|^2 \sqrt{g}\, d^4x \\
- \;& \xi_0 \int R\, |H|^2 \sqrt{g}\, d^4x + \lambda_0 \int |H|^4 \sqrt{g}\, d^4x \\
+ \;& \frac{1}{4} \int (G^i_{\mu\nu} G^{\mu\nu i} + F^\alpha_{\mu\nu} F^{\mu\nu\alpha} + B_{\mu\nu} B^{\mu\nu}) \sqrt{g}\, d^4x,
\end{aligned}
$$

Energy scale: Unification ($10^{15} - 10^{17}$ GeV)

$$\frac{g^2 f_0}{2\pi^2} = \frac{1}{4}$$

Preferred energy scale, unification of coupling constants

Coefficients

$$\frac{1}{2\kappa_0^2} = \frac{96 f_2 \Lambda^2 - f_0 \mathfrak{c}}{24\pi^2} \qquad \gamma_0 = \frac{1}{\pi^2}(48 f_4 \Lambda^4 - f_2 \Lambda^2 \mathfrak{c} + \frac{f_0}{4}\mathfrak{d})$$

$$\alpha_0 = -\frac{3 f_0}{10\pi^2} \qquad \tau_0 = \frac{11 f_0}{60\pi^2}$$

$$\mu_0^2 = 2\frac{f_2 \Lambda^2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} \qquad \xi_0 = \frac{1}{12}$$

$$\lambda_0 = \frac{\pi^2 \mathfrak{b}}{2 f_0 \mathfrak{a}^2}$$

In [MP] [KM]: running coefficients with RGE flow of particle physics content from unification energy down to electroweak.
$\Rightarrow$ Very early universe models! $(10^{-36}s < t < 10^{-12}s)$

Effective gravitational constant

$$G_{\rm eff} = \frac{\kappa_0^2}{8\pi} = \frac{3\pi}{192 f_2 \Lambda^2 - 2 f_0 \mathfrak{c}(\Lambda)}$$

Effective cosmological constant

$$\gamma_0 = \frac{1}{4\pi^2}(192 f_4 \Lambda^4 - 4 f_2 \Lambda^2 \mathfrak{c}(\Lambda) + f_0 \mathfrak{d}(\Lambda))$$

Conformal non-minimal coupling of Higgs and gravity

$$\frac{1}{16\pi G_{\rm eff}} \int R \sqrt{g} d^4x - \frac{1}{12} \int R |H|^2 \sqrt{g} d^4x$$

Conformal gravity

$$\frac{-3 f_0}{10\pi^2} \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g} d^4x$$

$C^{\mu\nu\rho\sigma}$ = Weyl curvature tensor (trace free part of Riemann tensor)

Cosmological implications of the NCG SM [MP]

- Linde's hypothesis (antigravity in the early universe)
- Primordial black holes and gravitational memory
- Gravitational waves in modified gravity
- Gravity balls
- Varying effective cosmological constant
- Higgs based slow-roll inflation
- Spontaneously arising Hoyle-Narlikar in EH backgrounds

Effects in the very early universe: inflation mechanisms
- Remark: Cannot extrapolate to modern universe, nonperturbative effects in the spectral action: requires nonperturbative spectral action

Cosmological models for the not-so-early-universe?

Need to work with non-perturbative form of the spectral action

Can to for specially symmetric geometries!

Concentrate on pure gravity part: X instead of $X \times F$

The spectral action and the question of cosmic topology

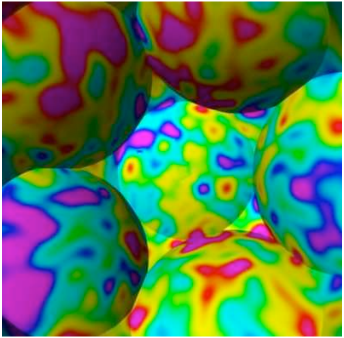
(with E. Pierpaoli and K. Teh)

Spatial sections of spacetime closed 3-manifolds $\neq S^3$?

- Cosmologists search for signatures of topology in the CMB
- Model based on NCG distinguishes cosmic topologies?

Yes! the non-perturbative spectral action predicts different models of slow-roll inflation

Cosmic topology



(Luminet, Lehoucq, Riazuelo, Weeks, et al.: simulated CMB sky)
Best candidates: Poincaré homology 3-sphere and other spherical forms (quaternionic space), flat tori
Testable Cosmological predictions? (in various gravity models)

What to look for? (in the background radiation)
Friedmann metric (expanding universe)

$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

Separate tensor and scalar perturbation h_{ij} of metric $\Rightarrow$ Fourier modes: power spectra for scalar and tensor fluctuations, $\mathcal{P}_s(k)$ and $\mathcal{P}_t(k)$ satisfy power law

$$\mathcal{P}_s(k) \sim \mathcal{P}_s(k_0) \left(\frac{k}{k_0}\right)^{1-n_s+\frac{\alpha_s}{2}\log(k/k_0)}$$

$$\mathcal{P}_t(k) \sim \mathcal{P}_t(k_0) \left(\frac{k}{k_0}\right)^{n_t+\frac{\alpha_t}{2}\log(k/k_0)}$$

Amplitudes and exponents: constrained by observational parameters and predicted by models of *slow roll inflation* (slow roll potential)

Main Question: Can get predictions of power spectra from slow roll inflation via NCG model, so that distinguish topologies?

Slow roll parameters Minkowskian Friedmann metric on $Y \times \mathbb{R}$

$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

accelerated expansion $\frac{\ddot{a}}{a} = H^2(1-\epsilon)$ Hubble parameter

$$H^2(\phi)\left(1 - \frac{1}{3}\epsilon(\phi)\right) = \frac{8\pi}{3m_{Pl}^2} V(\phi)$$

m_{Pl} Planck mass

$$\epsilon(\phi) = \frac{m_{Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

inflation phase $\epsilon(\phi) < 1$

$$\eta(\phi) = \frac{m_{Pl}^2}{8\pi}\left(\frac{V''(\phi)}{V(\phi)}\right) - \frac{m_{Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

second slow-roll parameter $\Rightarrow$ measurable quantities

$$n_s = 1 - 6\epsilon + 2\eta \quad r = 16\epsilon$$

spectral index and tensor-to-scalar ratio (n_t, α_s, α_t also from slow-roll parameters)

Spectral action and Poisson summation formula

$$\sum_{n\in\mathbb{Z}} h(x+\lambda n) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} \exp\left(\frac{2\pi i n x}{\lambda}\right)\ \widehat{h}(\frac{n}{\lambda})$$

$\lambda \in \mathbb{R}_+^*$ and $x \in \mathbb{R}$ with

$$\widehat{h}(x) = \int_{\mathbb{R}} h(u)\, e^{-2\pi i u x}\, du$$

Idea: write $\mathrm{Tr}(f(D/\Lambda))$ as sums over lattices

- Need explicit spectrum of D with multiplicities
- Need to write as a union of arithmetic progressions $\lambda_{n,i}$, $n \in \mathbb{Z}$
- Multiplicities polynomial functions $m_{\lambda_{n,i}} = P_i(\lambda_{n,i})$

$$\mathrm{Tr}(f(D/\Lambda)) = \sum_i \sum_{n\in\mathbb{Z}} P_i(\lambda_{n,i}) f(\lambda_{n,i}/\Lambda)$$

The standard topology S^3 (Chamseddine–Connes)

Dirac spectrum $\pm a^{-1}(\frac{1}{2}+n)$ for $n \in \mathbb{Z}$, with multiplicity $n(n+1)$

$$\mathrm{Tr}(f(D/\Lambda)) = (\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{4}(\Lambda a)\widehat{f}(0) + O((\Lambda a)^{-k})$$

with $\widehat{f}^{(2)}$ Fourier transform of $v^2 f(v)$ 4-dimensional Euclidean $S^3 \times S^1$

$$\mathrm{Tr}(h(D^2/\Lambda^2)) = \pi \Lambda^4 a^3 \beta \int_0^\infty u\, h(u)\, du - \frac{1}{2}\pi \Lambda a \beta \int_0^\infty h(u)\, du + O(\Lambda^{-k})$$

$$g(u,v) = 2P(u)\, h(u^2(\Lambda a)^{-2} + v^2(\Lambda\beta)^{-2})$$

$$\widehat{g}(n,m) = \int_{\mathbb{R}^2} g(u,v) e^{-2\pi i(xu+yv)}\, du\, dv$$

A slow roll potential from non-perturbative effects
perturbation $D^2 \mapsto D^2 + \phi^2$ gives potential $V(\phi)$ scalar field coupled to gravity

$$\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2)) = \pi\Lambda^4\beta a^3 \int_0^\infty u h(u) du - \frac{\pi}{2}\Lambda^2\beta a \int_0^\infty h(u) du$$

$$+\pi\Lambda^4\beta a^3\, \mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a\, \mathcal{W}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u)) du, \qquad \mathcal{W}(x) = \int_0^x h(u) du$$

Slow-roll parameters from spectral action $S = S^3$

$$\epsilon(x) = \frac{m_{Pl}^2}{16\pi} \left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta(x) = \frac{m_{Pl}^2}{8\pi} \frac{h'(x) + 2\pi(\Lambda a)^2 h(x)}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}$$

$$-\frac{m_{Pl}^2}{16\pi} \left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

In Minkowskian Friedmann metric $\Lambda(t) \sim 1/a(t)$
Also independent of β (artificial Euclidean compactification)

The quaternionic space $SU(2)/Q8$ (quaternion units $\pm 1, \pm\sigma_k$)

Dirac spectrum (Ginoux)

$$\frac{3}{2} + 4k \quad \text{with multiplicity} \quad 2(k+1)(2k+1)$$

$$\frac{3}{2} + 4k + 2 \quad \text{with multiplicity} \quad 4k(k+1)$$

Polynomial interpolation of multiplicities

$$P_1(u) = \frac{1}{4}u^2 + \frac{3}{4}u + \frac{5}{16}$$

$$P_2(u) = \frac{1}{4}u^2 - \frac{3}{4}u - \frac{7}{16}$$

Spectral action

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{8}(\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{32}(\Lambda a)\widehat{f}(0) + O(\Lambda^{-k})$$

(1/8 of action for S^3) with $g_i(u) = P_i(u) f(u/\Lambda)$:

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{4}\left(\widehat{g}_1(0) + \widehat{g}_2(0)\right) + O(\Lambda^{-k})$$

from Poisson summation $\Rightarrow$ Same slow-roll parameters

The dodecahedral space Poincaré homology sphere S^3/Γ
binary icosahedral group 120 elements
Dirac spectrum: eigenvalues of S^3 different multiplicities $\Rightarrow$
generating function (Bär)

$$F_+(z) = \sum_{k=0}^{\infty} m(\frac{3}{2}+k, D) z^k \quad F_-(z) = \sum_{k=0}^{\infty} m(-(\frac{3}{2}+k), D) z^k$$

$$F_+(z) = -\frac{16(710647+317811\sqrt{5})G^+(z)}{(7+3\sqrt{5})^3(2207+987\sqrt{5})H^+(z)}$$

$$G^+(z) = 6z^{11}+18z^{13}+24z^{15}+12z^{17}-2z^{19}-6z^{21}-2z^{23}+2z^{25}+4z^{27}+3z^{29}+z^{31}$$

$$H^+(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20} -9z^{22}-6z^{24}-2z^{26}+2z^{28}+4z^{30}+3z^{32}+z^{34}$$

$$F_-(z) = -\frac{1024(5374978561+2403763488\sqrt{5})G^-(z)}{(7+3\sqrt{5})^8(2207+987\sqrt{5})H^-(z)}$$

$$G^-(z) = 1+3z^2+4z^4+2z^6-2z^8-6z^{10}-2z^{12}+12z^{14}+24z^{16}+18z^{18}+6z^{20}$$

$$H^-(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20} -9z^{22}-6z^{24}-2z^{26}+2z^{28}+4z^{30}+3z^{32}+z^{34}$$

Polynomial interpolation of multiplicities: 60 polynomials $P_i(u)$

$$\sum_{j=0}^{59} P_j(u) = \frac{1}{2}u^2 - \frac{1}{8}$$

Spectral action: functions $g_j(u) = P_j(u) f(u/\Lambda)$

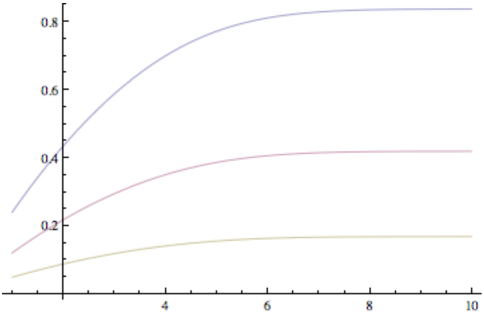
$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{60} \sum_{j=0}^{59} \widehat{g}_j(0) + O(\Lambda^{-k})$$

$$= \frac{1}{60} \int_{\mathbb{R}} \sum_j P_j(u) f(u/\Lambda) du + O(\Lambda^{-k})$$

by Poisson summation $\Rightarrow$ 1/120 of action for S^3

Same slow-roll parameters

But ... different amplitudes of power spectra:
multiplicative factor of potential $V(\phi)$



$$\mathcal{P}_s(k) \sim \frac{V^3}{(V')^2}, \quad \mathcal{P}_t(k) \sim V$$

$$V \mapsto \lambda V \quad \Rightarrow \mathcal{P}_s(k_0) \mapsto \lambda \mathcal{P}_s(k_0), \quad \mathcal{P}_t(k_0) \mapsto \lambda \mathcal{P}_t(k_0)$$

$\Rightarrow$ distinguish different spherical topologies

Topological factors (spherical cases):

Theorem (K.Teh): spherical forms $Y = S^3/\Gamma$ spectral action

$$\mathrm{Tr}(f(D_Y/\Lambda)) = \frac{1}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) = \frac{1}{\#\Gamma}\mathrm{Tr}(f(D_{S^3}/\Lambda))$$

up to order $O(\Lambda^{-\infty})$ with

Y spherical	λ_Y
sphere	1
lens N	$1/N$
binary dihedral $4N$	$1/(4N)$
binary tetrahedral	$1/24$
binary octahedral	$1/48$
binary icosahedral	$1/120$

Note: λ_Y does not distinguish all of them

The flat tori

Dirac spectrum (Bär)

$$\pm 2\pi \parallel (m,n,p) + (m_0,n_0,p_0) \parallel, \qquad (1)$$

$(m,n,p) \in \mathbb{Z}^3$ multiplicity 1 and constant vector (m_0,n_0,p_0) depending on spin structure

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \sum_{(m,n,p)\in\mathbb{Z}^3} 2f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Poisson summation

$$\sum_{\mathbb{Z}^3} g(m,n,p) = \sum_{\mathbb{Z}^3} \widehat{g}(m,n,p)$$

$$\widehat{g}(m,n,p) = \int_{\mathbb{R}^3} g(u,v,w) e^{-2\pi i(mu+nv+pw)} du dv dw$$

$$g(m,n,p) = f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Spectral action for the flat tori

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2)du\,dv\,dw + O(\Lambda^{-k})$$

$X = T^3 \times S^1_\beta$:

$$\mathrm{Tr}(h(D_X^2/\Lambda^2)) = \frac{\Lambda^4\beta\ell^3}{4\pi}\int_0^\infty uh(u)du + O(\Lambda^{-k})$$

using

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} 2\ h\left(\frac{4\pi^2}{(\Lambda\ell)^2}((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)+\frac{1}{(\Lambda\beta)^2}(r+\frac{1}{2})^2\right)$$

$$g(u,v,w,y) = 2\ h\left(\frac{4\pi^2}{\Lambda^2}(u^2+v^2+w^2)+\frac{y^2}{(\Lambda\beta)^2}\right)$$

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} g(m+m_0,n+n_0,p+p_0,r+\frac{1}{2}) = \sum_{(m,n,p,r)\in\mathbb{Z}^4}(-1)^r\,\widehat{g}(m,n,p,r)$$

Different slow-roll potential and parameters Introducing the perturbation $D^2 \mapsto D^2 + \phi^2$:

$$\mathrm{Tr}(h((D_X^2 + \phi^2)/\Lambda^2)) = \mathrm{Tr}(h(D_X^2/\Lambda^2)) + \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

slow-roll potential

$$V(\phi) = \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u\,(h(u+x) - h(u))\,du$$

Slow-roll parameters (different from spherical cases)

$$\epsilon = \frac{m_{Pl}^2}{16\pi} \left(\frac{\int_x^\infty h(u)du}{\int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta = \frac{m_{Pl}^2}{8\pi} \left(\frac{h(x)}{\int_0^\infty u(h(u+x) - h(u))du} - \frac{1}{2} \left(\frac{\int_x^\infty h(u)du}{\int_0^\infty u(h(u+x) - h(u))du} \right)^2 \right)$$

Bieberbach manifolds

Quotients of T^3 by group actions: $G2$, $G3$, $G4$, $G5$, $G6$

spin structures

	δ_1	δ_2	δ_3
(a)	± 1	1	1
(b)	± 1	-1	1
(c)	± 1	1	-1
(d)	± 1	-1	-1

$G2(a)$, $G2(b)$, $G2(c)$, $G2(d)$, etc.

Dirac spectra known (Pfäffle):

spectra often different for different spin structures

but spectral action same!

Bieberbach cosmic topologies (t_i = translations by a_i)

- $G2$ = half turn space

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, and $a_3 = (T, S, 0)$, with $H, L, S \in \mathbb{R}_+^*$ and $T \in \mathbb{R}$

$$\alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}$$

- $G3$ = third turn space

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$ and $a_3 = (-\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, for H and L in $\mathbb{R}_+^*$

$$\alpha^3 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1}$$

- $G4$ = quarter turn space

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, and $a_3 = (0, L, 0)$, with $H, L > 0$

$$\alpha^4 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1}$$

• $G5$ = sixth turn space

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$ and $a_3 = (\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, $H, L > 0$

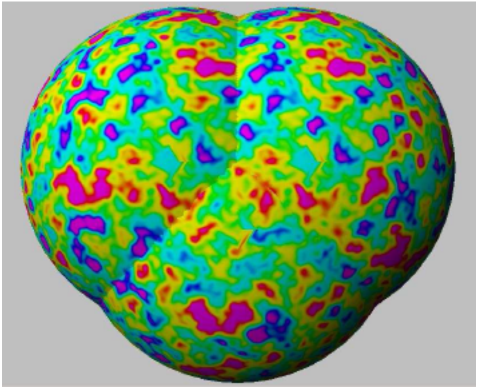
$$\alpha^6 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3$$

• $G6$ = Hantzsche–Wendt space (π-twist along each coordinate axis)

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, and $a_3 = (0, S, 0)$, with $H, L, S > 0$

$$\begin{array}{c} \alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ \beta^2 = t_2, \quad \beta t_1 \beta^{-1} = t_1^{-1}, \quad \beta t_3 \beta^{-1} = t_3^{-1}, \\ \gamma^2 = t_3, \quad \gamma t_1 \gamma^{-1} = t_1^{-1}, \quad \gamma t_2 \gamma^{-1} = t_2^{-1}, \\ \gamma\beta\alpha = t_1 t_3. \end{array}$$

Simulated CMB sky for a Bieberbach G6-cosmology



(from Riazuelo, Weeks, Uzan, Lehoucq, Luminet, 2003)

Topological factors (flat cases):

Theorem [MPT2]: Bieberbach manifolds spectral action

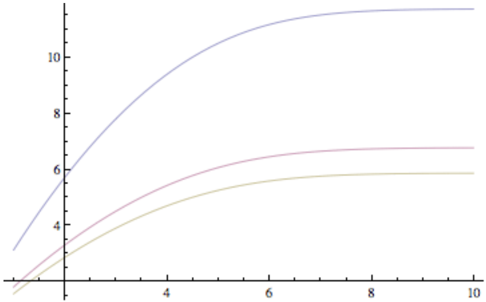
$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{\lambda_Y \Lambda^3}{4\pi^3} \int_{\mathbb{R}^3} f(u^2+v^2+w^2)dudvdw$$

up to oder $O(\Lambda^{-\infty})$ with factors

$$\lambda_Y = \begin{cases} \frac{HSL}{2} & G2 \\ \frac{HL^2}{2\sqrt{3}} & G3 \\ \frac{HL^2}{4} & G4 \\ \frac{HLS}{4} & G6 \end{cases}$$

Note lattice summation technique not immediately suitable for $G5$, but expect like $G3$ up to factor of 2

Topological factors and inflation slow-roll potential



⇒ Multiplicative factor in amplitude of power spectra

Adding the coupling to matter $Y \times F$

Not only product but nontrivial fibration

Vector bundle V over 3-manifold Y, fiber $\mathcal{H}_F$ (fermion content)

Dirac operator D_Y twisted with connection on V (bosons)

Spectra of twisted Dirac operators on spherical manifolds (Cisneros–Molina)

Similar computation with Poisson summation formula [CMT]

$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{N}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right)$$

up to order $O(\Lambda^{-\infty})$

representation V dimension N; spherical form $Y = S^3/\Gamma$

$\Rightarrow$ topological factor $\lambda_Y \mapsto N\lambda_Y$

Conclusion (for now)

A modified gravity model based on the spectral action can distinguish between the different cosmic topology in terms of the slow-roll parameters (distinguish spherical and flat cases) and the amplitudes of the power spectral (distinguish different spherical space forms and different Bieberbach manifolds).
Different inflation scenarios in different topologies