Noncommutative geometry and cosmology

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Beijing, August 2013

This talk is based on:

MPT M. Marcolli, E. Pierpaoli, K. Teh, *The spectral action and cosmic topology*, arXiv:1005.2256, CMP 304 (2011) 125-174

MPT2 M. Marcolli, E. Pierpaoli, K. Teh, *The coupling of topology and inflation in Noncommutative Cosmology*, arXiv:1012.0780, CMP 309 (2012) N.2, 341-369

CMT B. Ćaćić, M. Marcolli, K. Teh, *Topological coupling of gravity to matter, spectral action and cosmic topology*, arXiv:1106.5473, to appear in JNCG

and refs on NCG standard model and cosmology:

CCM A. Chamseddine, A. Connes, M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007), no. 6, 991–1089.

MP M. Marcolli, E. Pierpaoli, *Early universe models from noncommutative geometry*, arXiv:0908.3683, Adv. Theor. Math. Phys. 14 (2010) 1373-1432

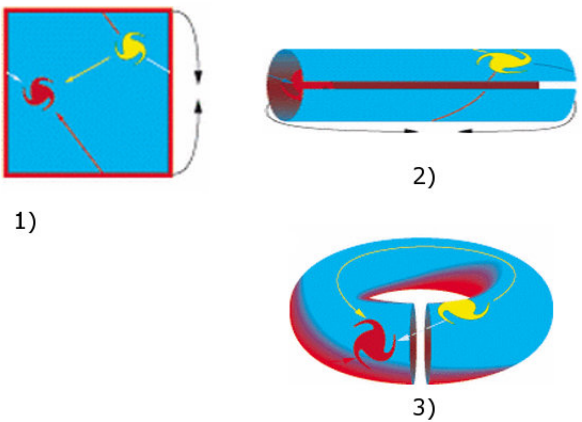
KM D. Kolodrubetz, M. Marcolli, *Boundary conditions of the RGE flow in noncommutative cosmology*, arXiv:1006.4000, Phys. Lett. B 693 (2010) 166-174.

Two topics of current interest to cosmologists:

• Modified Gravity models in cosmology:
Einstein-Hilbert action (+cosmological term) replaced or extended with other gravity terms (conformal gravity, higher derivative terms) $\Rightarrow$ cosmological predictions

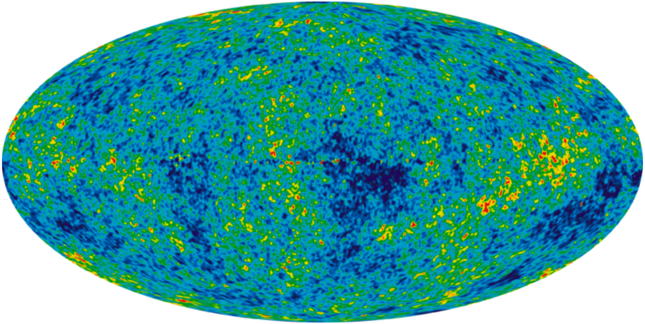
• The question of Cosmic Topology:
Nontrivial (non-simply-connected) spatial sections of spacetime, homogeneous spherical or flat spaces: how can this be detected from cosmological observations?

The question of Cosmic Topology:



Nontrivial (non-simply-connected) spatial sections of spacetime, homogeneous spherical or flat spaces: how can this be detected from cosmological observations?

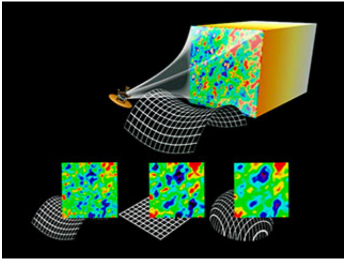
Cosmic Microwave Background best source of cosmological data on which to test theoretical models (modified gravity models, cosmic topology hypothesis, particle physics models)



- COBE satellite (1989)
- WMAP satellite (2001)
- Planck satellite (2009): new data available now!

Cosmic topology and the CMB

- Einstein equations determine geometry not topology (don't distinguish S^3 from S^3/Γ with round metric)
- Cosmological data (BOOMERanG experiment 1998, WMAP data 2003): spatial geometry of the universe is flat or slightly positively curved
- Homogeneous and isotropic compact case: spherical space forms S^3/Γ or Bieberbach manifolds T^3/Γ



Is cosmic topology detected by the Cosmic Microwave Background (CMB)? Search for signatures of multiconnected topologies

CMB sky and spherical harmonics temperature fluctuations

$$\frac{\Delta T}{T} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}$$

$Y_{\ell m}$ spherical harmonics

Methods to address cosmic topology problem

- Statistical search for matching circles in the CMB sky: identify a nontrivial fundamental domain
- Anomalies of the CMB: quadrupole suppression, the small value of the two- point temperature correlation function at angles above 60 degrees, and the anomalous alignment of the quadrupole and octupole
- Residual gravity acceleration: gravitational effects from other fundamental domains
- Bayesian analysis of different models of CMB sky for different candidate topologies

Results: no conclusive evidence of a non-simply connected topology

Our approach:

- NCG provides a modified gravity model through the spectral action
- The nonperturbative form of the spectral action determines a slow-roll inflation potential
- The underlying geometry (spherical/flat) affects the shape of the potential (possible models of inflation)
- Different inflation scenarios depending on geometry and topology of the cosmos
- More refined topological properties from coupling to matter

The noncommutative space $X \times F$ extra dimensions
product of 4-dim spacetime and finite NC space
The spectral action functional

$$\mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

$D_A = D + A + \varepsilon' J A J^{-1}$ Dirac operator with inner fluctuations
$A = A^* = \sum_k a_k[D, b_k]$

- Action functional for gravity on X (modified gravity)
- Gravity on $X \times F$ = gravity coupled to matter on X

Spectral triples $(\mathcal{A}, \mathcal{H}, D)$:

- involutive algebra $\mathcal{A}$
- representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
- self adjoint operator D on $\mathcal{H}$
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$
- even $\mathbb{Z}/2$-grading $[\gamma, a] = 0$ and $D\gamma = -\gamma D$
- real structure: antilinear isom $J : \mathcal{H} \to \mathcal{H}$ with $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, and $J\gamma = \varepsilon'' \gamma J$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- bimodule: $[a, b^0] = 0$ for $b^0 = Jb^* J^{-1}$
- order one condition: $[[D, a], b^0] = 0$

Asymptotic formula for the spectral action (Chamseddine–Connes)

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{k \in \mathrm{DimSp}} f_k \Lambda^k \int\!\!\!\!\!- |D|^{-k} + f(0)\zeta_D(0) + o(1)$$

for large Λ with $f_k = \int_0^\infty f(v) v^{k-1} dv$ and integration given by residues of zeta function $\zeta_D(s) = \mathrm{Tr}(|D|^{-s})$; DimSp poles of zeta functions

Asymptotic expansion $\Rightarrow$ Effective Lagrangian
(modified gravity + matter)

At low energies: only nonperturbative form of the spectral action

$$\mathrm{Tr}(f(D_A/\Lambda))$$

Need explicit information on the Dirac spectrum!

Product geometry $(C^\infty(X), L^2(X,S), D_X) \cup (\mathcal{A}_F, \mathcal{H}_F, D_F)$

- $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_F = C^\infty(X, \mathcal{A}_F)$
- $\mathcal{H} = L^2(X,S) \otimes \mathcal{H}_F = L^2(X, S \otimes \mathcal{H}_F)$
- $D = D_X \otimes 1 + \gamma_5 \otimes D_F$

Inner fluctuations of the Dirac operator

$$D \to D_A = D + A + \varepsilon' J A J^{-1}$$

A self-adjoint operator

$$A = \sum a_j [D, b_j], \quad a_j, b_j \in \mathcal{A}$$

$\Rightarrow$ boson fields from inner fluctuations, fermions from $\mathcal{H}_F$

Get realistic particle physics models [CCM]
Need Ansatz for the NC space F

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

$\Rightarrow$ everything else follows by *computation*

- Representation: $\mathcal{M}_F$ sum of all inequiv irred odd $\mathcal{A}_{LR}$-bimodules (fix N generations) $\mathcal{H}_F = \oplus^N \mathcal{M}_F$ fermions
- Algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$: order one condition
- F zero dimensional but KO-dim 6
- J_F = matter/antimatter, γ_F = L/R chirality
- Classification of Dirac operators (moduli spaces)

Parameters of νMSM

Get an *extension* of minimal Standard Model with right handed neutrinos, Majorana mass terms, and lepton mixing

- three coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses, 3 lepton mixing angles, 1 complex phase
- 3 neutrino masses
- 11 Majorana mass matrix parameters

• *Bosons* from inner fluctuations of D (gauge: manifold direction, Higgs: NC direction)

• *Fermions* from basis of representation space $\mathcal{H}_F$

• Correct gauge group and hypercharges of particles

• *Action functional*: spectral action on $X \times F$, asymptotic expansion gives full Lagrangian of νMSM with non-minimal coupling to (modified) gravity.

The asymptotic expansion of the spectral action from [CCM]

$$
\begin{aligned}
S = \;& \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, \mathfrak{c} + \frac{f_0}{4}\, \mathfrak{d}) \int \sqrt{g}\, d^4x \\
+ \;& \frac{96\, f_2\, \Lambda^2 - f_0\, \mathfrak{c}}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0}{10\, \pi^2} \int \big(\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma}\big) \sqrt{g}\, d^4x \\
+ \;& \frac{(-2\, \mathfrak{a}\, f_2\, \Lambda^2 \;+\; \mathfrak{e}\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0 \mathfrak{a}}{2\, \pi^2} \int |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
- \;& \frac{f_0 \mathfrak{a}}{12\, \pi^2} \int R\, |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0 \mathfrak{b}}{2\, \pi^2} \int |\varphi|^4\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0}{2\, \pi^2} \int (g_3^2\, G^i_{\mu\nu}\, G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu})\, \sqrt{g}\, d^4x,
\end{aligned}
$$

Parameters:

- f_0, f_2, f_4 free parameters, $f_0 = f(0)$ and, for $k > 0$,

$$f_k = \int_0^\infty f(v) v^{k-1} dv.$$

- $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ functions of Yukawa parameters of SM+r.h.ν

$$\begin{aligned}
\mathfrak{a} &= \mathrm{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d)) \\
\mathfrak{b} &= \mathrm{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2) \\
\mathfrak{c} &= \mathrm{Tr}(MM^\dagger) \\
\mathfrak{d} &= \mathrm{Tr}((MM^\dagger)^2) \\
\mathfrak{e} &= \mathrm{Tr}(MM^\dagger Y_\nu^\dagger Y_\nu).
\end{aligned}$$

Normalization and coefficients

$$
\begin{aligned}
S = \;& \frac{1}{2\kappa_0^2} \int R \sqrt{g}\, d^4x + \gamma_0 \int \sqrt{g}\, d^4x \\
+ \;& \alpha_0 \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g}\, d^4x + \tau_0 \int R^* R^* \sqrt{g}\, d^4x \\
+ \;& \frac{1}{2} \int |DH|^2 \sqrt{g}\, d^4x - \mu_0^2 \int |H|^2 \sqrt{g}\, d^4x \\
- \;& \xi_0 \int R\, |H|^2 \sqrt{g}\, d^4x + \lambda_0 \int |H|^4 \sqrt{g}\, d^4x \\
+ \;& \frac{1}{4} \int (G^i_{\mu\nu} G^{\mu\nu i} + F^\alpha_{\mu\nu} F^{\mu\nu\alpha} + B_{\mu\nu} B^{\mu\nu}) \sqrt{g}\, d^4x,
\end{aligned}
$$

Energy scale: Unification ($10^{15} - 10^{17}$ GeV)

$$\frac{g^2 f_0}{2\pi^2} = \frac{1}{4}$$

Preferred energy scale, unification of coupling constants

Coefficients

$$\frac{1}{2\kappa_0^2} = \frac{96 f_2 \Lambda^2 - f_0 \mathfrak{c}}{24\pi^2} \qquad \gamma_0 = \frac{1}{\pi^2}\left(48 f_4 \Lambda^4 - f_2 \Lambda^2 \mathfrak{c} + \frac{f_0}{4}\mathfrak{d}\right)$$

$$\alpha_0 = -\frac{3 f_0}{10\pi^2} \qquad \tau_0 = \frac{11 f_0}{60\pi^2}$$

$$\mu_0^2 = 2\frac{f_2 \Lambda^2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} \qquad \xi_0 = \frac{1}{12}$$

$$\lambda_0 = \frac{\pi^2 \mathfrak{b}}{2 f_0 \mathfrak{a}^2}$$

Renormalization group running of parameters

Used to obtain physical predictions from the particle physics sector of the model:

- [CCM] RGE for particle sector to estimate Higgs mass: 170 GeV... too large!

- Recent methods for correcting to realistic Higgs mass ∼125 GeV:

• A. Chamseddine, A. Connes, arXiv:1208.1030 (additional scalar field coupled to Higgs)

• C. Estrada, M. Marcolli, arXiv:1208.5023 (gravity correction terms, asymptotic safety, anomalous dimensions)

- In [MP] [KM]: running coefficients with RGE flow of particle physics content from unification energy down to electroweak.

$\Rightarrow$ Very early universe models! ($10^{-36}s < t < 10^{-12}s$)

Effective gravitational constant

$$G_{\text{eff}} = \frac{\kappa_0^2}{8\pi} = \frac{3\pi}{192 f_2 \Lambda^2 - 2 f_0 \mathfrak{c}(\Lambda)}$$

Effective cosmological constant

$$\gamma_0 = \frac{1}{4\pi^2}(192 f_4 \Lambda^4 - 4 f_2 \Lambda^2 \mathfrak{c}(\Lambda) + f_0 \mathfrak{d}(\Lambda))$$

Conformal non-minimal coupling of Higgs and gravity

$$\frac{1}{16\pi G_{\text{eff}}} \int R \sqrt{g} d^4x - \frac{1}{12} \int R\, |H|^2 \sqrt{g} d^4x$$

Conformal gravity

$$\frac{-3 f_0}{10\pi^2} \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g} d^4x$$

$C^{\mu\nu\rho\sigma}$ = Weyl curvature tensor (trace free part of Riemann tensor)

Cosmological implications of the NCG SM [MP]

- Linde's hypothesis (antigravity in the early universe)
- Primordial black holes and gravitational memory
- Gravitational waves in modified gravity
- Gravity balls
- Varying effective cosmological constant
- Higgs based slow-roll inflation
- Spontaneously arising Hoyle-Narlikar in EH backgrounds

Effects in the very early universe: inflation mechanisms
- Remark: Cannot extrapolate to modern universe, nonperturbative effects in the spectral action: requires nonperturbative spectral action

Cosmological models for the not-so-early-universe?

Need to work with non-perturbative form of the spectral action

Can to for specially symmetric geometries!

Concentrate on pure gravity part: X instead of $X \times F$

The spectral action and the question of cosmic topology

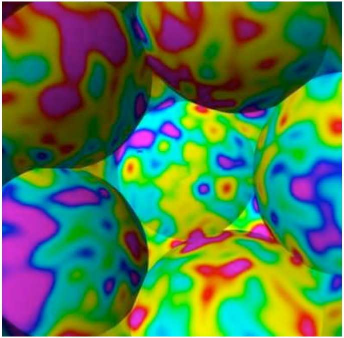
(with E. Pierpaoli and K. Teh)

Spatial sections of spacetime closed 3-manifolds $\neq S^3$?

- Cosmologists search for signatures of topology in the CMB
- Model based on NCG distinguishes cosmic topologies?

Yes! the non-perturbative spectral action predicts different models of slow-roll inflation

Cosmic topology



(Luminet, Lehoucq, Riazuelo, Weeks, et al.: simulated CMB sky)
Best candidates: Poincaré homology 3-sphere and other spherical forms (quaternionic space), flat tori
Testable Cosmological predictions? (in various gravity models)

What to look for? (in the background radiation)
Friedmann metric (expanding universe)

$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

Separate tensor and scalar perturbation h_{ij} of metric $\Rightarrow$ Fourier modes: power spectra for scalar and tensor fluctuations, $\mathcal{P}_s(k)$ and $\mathcal{P}_t(k)$ satisfy power law

$$\mathcal{P}_s(k) \sim \mathcal{P}_s(k_0) \left(\frac{k}{k_0}\right)^{1-n_s+\frac{\alpha_s}{2}\log(k/k_0)}$$

$$\mathcal{P}_t(k) \sim \mathcal{P}_t(k_0) \left(\frac{k}{k_0}\right)^{n_t+\frac{\alpha_t}{2}\log(k/k_0)}$$

Amplitudes and exponents: constrained by observational parameters and predicted by models of *slow roll inflation* (slow roll potential)

Main Question: Can get predictions of power spectra from slow roll inflation via NCG model, so that distinguish topologies?

Slow-roll models of inflation in the early universe

Minkowskian Friedmann metric on $Y \times \mathbb{R}$

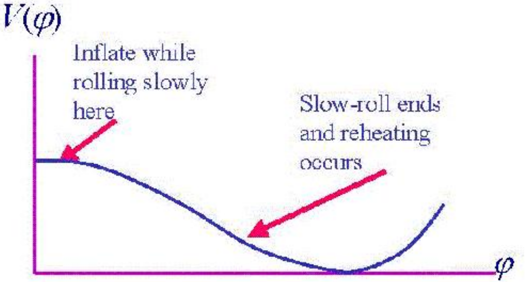
$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

accelerated expansion $\frac{\ddot{a}}{a} = H^2(1-\epsilon)$ Hubble parameter

$$H^2(\phi)\left(1 - \frac{1}{3}\epsilon(\phi)\right) = \frac{8\pi}{3m_{Pl}^2} V(\phi)$$

m_{Pl} Planck mass, inflation phase $\epsilon(\phi) < 1$

A potential $V(\phi)$ for a scalar field ϕ that runs the inflation



Slow roll parameters Minkowskian Friedmann metric on $Y \times \mathbb{R}$

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m_{Pl} Planck mass, inflation phase $\epsilon(\phi) < 1$

$$\epsilon(\phi) = \frac{m_{Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

$$\eta(\phi) = \frac{m_{Pl}^2}{8\pi}\frac{V''(\phi)}{V(\phi)}$$

$$\xi(\phi) = \frac{m_{Pl}^4}{64\pi^2}\frac{V'(\phi)V'''(\phi)}{V^2(\phi)}$$

$\Rightarrow$ measurable quantities

$$n_s \simeq 1 - 6\epsilon + 2\eta, \quad n_t \simeq -2\epsilon, \quad r = 16\epsilon,$$

$$\alpha_s \simeq 16\epsilon\eta - 24\epsilon^2 - 2\xi, \quad \alpha_t \simeq 4\epsilon\eta - 8\epsilon^2$$

Spectral action and Poisson summation formula

$$\sum_{n\in\mathbb{Z}} h(x+\lambda n) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} \exp\left(\frac{2\pi i n x}{\lambda}\right)\ \widehat{h}(\frac{n}{\lambda})$$

$\lambda \in \mathbb{R}_+^*$ and $x \in \mathbb{R}$ with

$$\widehat{h}(x) = \int_{\mathbb{R}} h(u)\, e^{-2\pi i u x}\, du$$

Idea: write $\mathrm{Tr}(f(D/\Lambda))$ as sums over lattices
- Need explicit spectrum of D with multiplicities
- Need to write as a union of arithmetic progressions $\lambda_{n,i}$, $n \in \mathbb{Z}$
- Multiplicities polynomial functions $m_{\lambda_{n,i}} = P_i(\lambda_{n,i})$

$$\mathrm{Tr}(f(D/\Lambda)) = \sum_i \sum_{n\in\mathbb{Z}} P_i(\lambda_{n,i}) f(\lambda_{n,i}/\Lambda)$$

The standard topology S^3 (Chamseddine–Connes)
Dirac spectrum $\pm a^{-1}(\frac{1}{2}+n)$ for $n \in \mathbb{Z}$, with multiplicity $n(n+1)$

$$\mathrm{Tr}(f(D/\Lambda)) = (\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{4}(\Lambda a)\widehat{f}(0) + O((\Lambda a)^{-k})$$

with $\widehat{f}^{(2)}$ Fourier transform of $v^2 f(v)$ 4-dimensional Euclidean $S^3 \times S^1$

$$\mathrm{Tr}(h(D^2/\Lambda^2)) = \pi \Lambda^4 a^3 \beta \int_0^\infty u\, h(u)\, du - \frac{1}{2}\pi \Lambda a \beta \int_0^\infty h(u)\, du + O(\Lambda^{-k})$$

$$g(u,v) = 2P(u)\, h(u^2(\Lambda a)^{-2} + v^2(\Lambda\beta)^{-2})$$

$$\widehat{g}(n,m) = \int_{\mathbb{R}^2} g(u,v) e^{-2\pi i(xu+yv)}\, du\, dv$$

A slow roll potential from non-perturbative effects

perturbation $D^2 \mapsto D^2 + \phi^2$ gives potential $V(\phi)$ scalar field coupled to gravity

$$\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2)) = \pi\Lambda^4\beta a^3 \int_0^\infty uh(u)du - \frac{\pi}{2}\Lambda^2\beta a \int_0^\infty h(u)du$$

$$+\pi\Lambda^4\beta a^3\,\mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a\,\mathcal{W}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))du, \qquad \mathcal{W}(x) = \int_0^x h(u)du$$

Slow-roll parameters from spectral action $S = S^3$

$$\epsilon(x) = \frac{m_{Pl}^2}{16\pi} \left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta(x) = \frac{m_{Pl}^2}{8\pi} \frac{h'(x) + 2\pi(\Lambda a)^2 h(x)}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}$$

In Minkowskian Friedmann metric $\Lambda(t) \sim 1/a(t)$
Also independent of β (artificial Euclidean compactification)

The quaternionic space $SU(2)/Q8$ (quaternion units $\pm 1, \pm\sigma_k$)

Dirac spectrum (Ginoux)

$$\frac{3}{2} + 4k \quad \text{with multiplicity} \quad 2(k+1)(2k+1)$$

$$\frac{3}{2} + 4k + 2 \quad \text{with multiplicity} \quad 4k(k+1)$$

Polynomial interpolation of multiplicities

$$P_1(u) = \frac{1}{4}u^2 + \frac{3}{4}u + \frac{5}{16}$$

$$P_2(u) = \frac{1}{4}u^2 - \frac{3}{4}u - \frac{7}{16}$$

Spectral action

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{8}(\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{32}(\Lambda a)\widehat{f}(0) + O(\Lambda^{-k})$$

(1/8 of action for S^3) with $g_i(u) = P_i(u) f(u/\Lambda)$:

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{4}\left(\widehat{g}_1(0) + \widehat{g}_2(0)\right) + O(\Lambda^{-k})$$

from Poisson summation $\Rightarrow$ Same slow-roll parameters

The dodecahedral space Poincaré homology sphere S^3/Γ
binary icosahedral group 120 elements
Dirac spectrum: eigenvalues of S^3 different multiplicities $\Rightarrow$
generating function (Bär)

$$F_+(z) = \sum_{k=0}^{\infty} m(\frac{3}{2}+k, D) z^k \quad F_-(z) = \sum_{k=0}^{\infty} m(-(\frac{3}{2}+k), D) z^k$$

$$F_+(z) = -\frac{16(710647 + 317811\sqrt{5}) G^+(z)}{(7+3\sqrt{5})^3 (2207 + 987\sqrt{5}) H^+(z)}$$

$$G^+(z) = 6z^{11}+18z^{13}+24z^{15}+12z^{17}-2z^{19}-6z^{21}-2z^{23}+2z^{25}+4z^{27}+3z^{29}+z^{31}$$

$$H^+(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20}$$
$$-9z^{22} - 6z^{24} - 2z^{26} + 2z^{28} + 4z^{30} + 3z^{32} + z^{34}$$

$$F_-(z) = -\frac{1024(5374978561 + 2403763488\sqrt{5}) G^-(z)}{(7+3\sqrt{5})^8 (2207 + 987\sqrt{5}) H^-(z)}$$

$$G^-(z) = 1+3z^2+4z^4+2z^6-2z^8-6z^{10}-2z^{12}+12z^{14}+24z^{16}+18z^{18}+6z^{20}$$

$$H^-(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20}$$
$$-9z^{22} - 6z^{24} - 2z^{26} + 2z^{28} + 4z^{30} + 3z^{32} + z^{34}$$

Polynomial interpolation of multiplicities: 60 polynomials $P_i(u)$

$$\sum_{j=0}^{59} P_j(u) = \frac{1}{2}u^2 - \frac{1}{8}$$

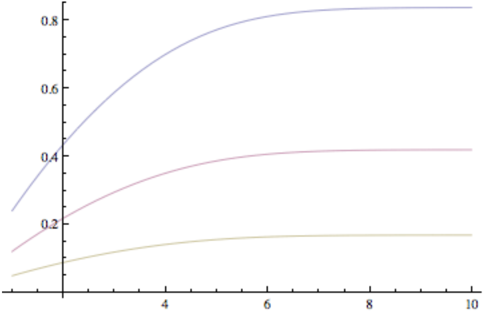
Spectral action: functions $g_j(u) = P_j(u) f(u/\Lambda)$

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{60} \sum_{j=0}^{59} \widehat{g}_j(0) + O(\Lambda^{-k})$$

$$= \frac{1}{60} \int_{\mathbb{R}} \sum_j P_j(u) f(u/\Lambda) du + O(\Lambda^{-k})$$

by Poisson summation $\Rightarrow 1/120$ of action for S^3
Same slow-roll parameters

But ... different amplitudes of power spectra:
multiplicative factor of potential $V(\phi)$



$$\mathcal{P}_s(k) \sim \frac{V^3}{(V')^2}, \quad \mathcal{P}_t(k) \sim V$$

$$V \mapsto \lambda V \quad \Rightarrow \mathcal{P}_s(k_0) \mapsto \lambda \mathcal{P}_s(k_0), \quad \mathcal{P}_t(k_0) \mapsto \lambda \mathcal{P}_t(k_0)$$

$\Rightarrow$ distinguish different spherical topologies

Topological factors (spherical cases):

Theorem (K.Teh): spherical forms $Y = S^3/\Gamma$ spectral action

$$\mathrm{Tr}(f(D_Y/\Lambda)) = \frac{1}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) = \frac{1}{\#\Gamma}\mathrm{Tr}(f(D_{S^3}/\Lambda))$$

up to order $O(\Lambda^{-\infty})$ with

Y spherical	λ_Y
sphere	1
lens N	$1/N$
binary dihedral $4N$	$1/(4N)$
binary tetrahedral	$1/24$
binary octahedral	$1/48$
binary icosahedral	$1/120$

Note: λ_Y does not distinguish all of them

The flat tori

Dirac spectrum (Bär)

$$\pm 2\pi \parallel (m,n,p) + (m_0,n_0,p_0) \parallel, \tag{1}$$

$(m,n,p) \in \mathbb{Z}^3$ multiplicity 1 and constant vector (m_0,n_0,p_0) depending on spin structure

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \sum_{(m,n,p)\in\mathbb{Z}^3} 2f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Poisson summation

$$\sum_{\mathbb{Z}^3} g(m,n,p) = \sum_{\mathbb{Z}^3} \widehat{g}(m,n,p)$$

$$\widehat{g}(m,n,p) = \int_{\mathbb{R}^3} g(u,v,w) e^{-2\pi i(mu+nv+pw)} dudvdw$$

$$g(m,n,p) = f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Spectral action for the flat tori

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2)du\,dv\,dw + O(\Lambda^{-k})$$

$X = T^3 \times S^1_\beta$:

$$\mathrm{Tr}(h(D_X^2/\Lambda^2)) = \frac{\Lambda^4\beta\ell^3}{4\pi}\int_0^\infty uh(u)du + O(\Lambda^{-k})$$

using

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} 2\ h\left(\frac{4\pi^2}{(\Lambda\ell)^2}((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)+\frac{1}{(\Lambda\beta)^2}(r+\frac{1}{2})^2\right)$$

$$g(u,v,w,y) = 2\ h\left(\frac{4\pi^2}{\Lambda^2}(u^2+v^2+w^2)+\frac{y^2}{(\Lambda\beta)^2}\right)$$

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} g(m+m_0, n+n_0, p+p_0, r+\frac{1}{2}) = \sum_{(m,n,p,r)\in\mathbb{Z}^4} (-1)^r\,\widehat{g}(m,n,p,r)$$

Different slow-roll potential and parameters Introducing the perturbation $D^2 \mapsto D^2 + \phi^2$:

$$\mathrm{Tr}(h((D_X^2 + \phi^2)/\Lambda^2)) = \mathrm{Tr}(h(D_X^2/\Lambda^2)) + \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

slow-roll potential

$$V(\phi) = \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u\,(h(u+x) - h(u))\,du$$

Slow-roll parameters (different from spherical cases)

$$\epsilon = \frac{m_{Pl}^2}{16\pi} \left(\frac{\int_x^\infty h(u)du}{\int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta = \frac{m_{Pl}^2}{8\pi} \left(\frac{h(x)}{\int_0^\infty u(h(u+x) - h(u))du} \right)$$

Bieberbach manifolds

Quotients of T^3 by group actions: $G2$, $G3$, $G4$, $G5$, $G6$

spin structures

	δ_1	δ_2	δ_3
(a)	± 1	1	1
(b)	± 1	-1	1
(c)	± 1	1	-1
(d)	± 1	-1	-1

$G2(a)$, $G2(b)$, $G2(c)$, $G2(d)$, etc.

Dirac spectra known (Pfäffle):

spectra often different for different spin structures

but spectral action same!

Bieberbach cosmic topologies (t_i = translations by a_i)

- $G2$ = half turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (T,S,0)$, with $H, L, S \in \mathbb{R}_+^*$ and $T \in \mathbb{R}$

$$\alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}$$

- $G3$ = third turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$ and $a_3 = (-\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, for H and L in $\mathbb{R}_+^*$

$$\alpha^3 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1}$$

- $G4$ = quarter turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (0,L,0)$, with $H, L > 0$

$$\alpha^4 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1}$$

• $G5$ = sixth turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$ and $a_3 = (\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, $H, L > 0$

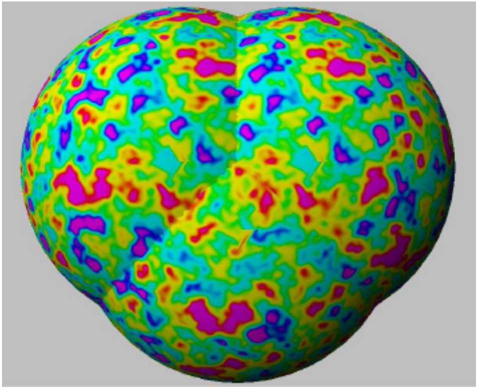
$$\alpha^6 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3$$

• $G6$ = Hantzsche–Wendt space (π-twist along each coordinate axis)

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (0,S,0)$, with $H, L, S > 0$

$$\begin{array}{ccc} \alpha^2 = t_1, & \alpha t_2 \alpha^{-1} = t_2^{-1}, & \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ \beta^2 = t_2, & \beta t_1 \beta^{-1} = t_1^{-1}, & \beta t_3 \beta^{-1} = t_3^{-1}, \\ \gamma^2 = t_3, & \gamma t_1 \gamma^{-1} = t_1^{-1}, & \gamma t_2 \gamma^{-1} = t_2^{-1}, \\ & \gamma\beta\alpha = t_1 t_3. & \end{array}$$

Simulated CMB sky for a Bieberbach G6-cosmology



(from Riazuelo, Weeks, Uzan, Lehoucq, Luminet, 2003)

Topological factors (flat cases):

Theorem [MPT2]: Bieberbach manifolds spectral action

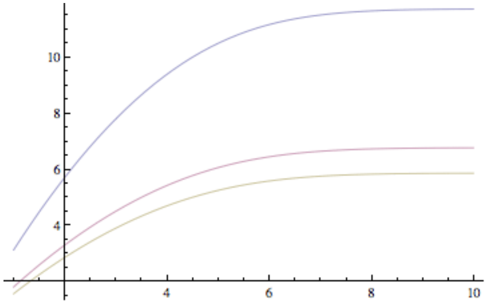
$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{\lambda_Y \Lambda^3}{4\pi^3} \int_{\mathbb{R}^3} f(u^2+v^2+w^2) du dv dw$$

up to oder $O(\Lambda^{-\infty})$ with factors

$$\lambda_Y = \begin{cases} \frac{HSL}{2} & G2 \\ \frac{HL^2}{2\sqrt{3}} & G3 \\ \frac{HL^2}{4} & G4 \\ \frac{HLS}{4} & G6 \end{cases}$$

Note lattice summation technique not immediately suitable for $G5$, but expect like $G3$ up to factor of 2

Topological factors and inflation slow-roll potential



$\Rightarrow$ Multiplicative factor in amplitude of power spectra

Adding the coupling to matter $Y \times F$

Not only product but nontrivial fibration

Vector bundle V over 3-manifold Y, fiber $\mathcal{H}_F$ (fermion content)

Dirac operator D_Y twisted with connection on V (bosons)

Spectra of twisted Dirac operators on spherical manifolds (Cisneros–Molina)

Similar computation with Poisson summation formula [CMT]

$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{N}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right)$$

up to order $O(\Lambda^{-\infty})$

representation V dimension N; spherical form $Y = S^3/\Gamma$

$\Rightarrow$ topological factor $\lambda_Y \mapsto N\lambda_Y$

Conclusion (for now)

A modified gravity model based on the spectral action can distinguish between the different cosmic topology in terms of the slow-roll parameters (distinguish spherical and flat cases) and the amplitudes of the power spectral (distinguish different spherical space forms and different Bieberbach manifolds).
Different inflation scenarios in different topologies