Contact Geometry of the Visual Cortex

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Ma191b: Geometry of Neuroscience

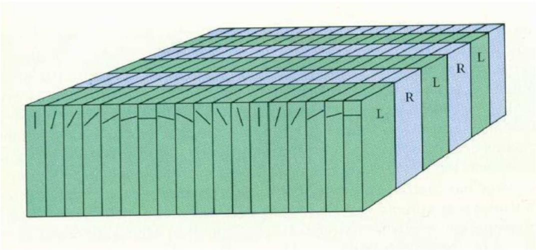
References for this lecture

- Jean Petitot, *Neurogéométrie de la Vision*, Les Éditions de l'École Polytechnique, 2008
- John B. Entyre, *Introductory Lectures on Contact Geometry*, arXiv:math/0111118
- W.C. Hoffman, *The visual cortex is a contact bundle*, Applied Mathematics and Computation, 32 (1989) 137–167.
- O. Ben-Shahar, S. Zucker, *Geometrical Computations Explain Projection Patterns of Long-Range Horizontal Connections in Visual Cortex*, Neural Computation, 16, 3 (2004) 445–476
- Alessandro Sarti, Giovanna Citti, Jean Petitot, *Functional geometry of the horizontal connectivity in the primary visual cortex*, Journal of Physiology - Paris 103 (2009) 3–45
- V. Liontou, M. Marcolli, *Gabor frames from contact geometry in models of the primary visual cortex*, Math. Neuro. and Appl. 3 (2023), article no. 2, 1–28.

Columnar Structure

- another type of geometric structure present in visual cortex V1
- Hubel–Wiesel: columnar structures in V1: neurons sensitive to orientation record data (z,ℓ)
  - $z =$ a position on the retina
  - $\ell =$ a line in the plane
- local product structure

$$\pi : \mathcal{R} \times \mathbb{P}^1 \twoheadrightarrow \mathcal{R}$$



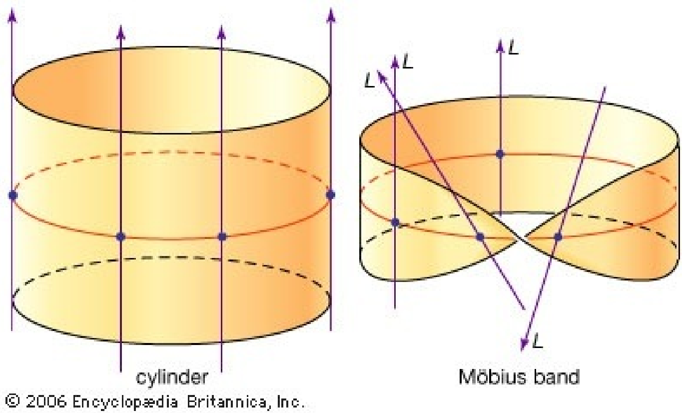
Fiber bundles

• topological space (or smooth differentiable manifold) E with base B and fiber F with

- surjection $\pi : E \twoheadrightarrow B$
- fibers $E_x = \pi^{-1}(x) \simeq F$ for all $x \in B$
- open covering $\mathcal{U} = \{U_\alpha\}$ of B such that $\pi^{-1}(U_\alpha) \simeq U_\alpha \times F$ with π restricted to $\pi^{-1}(U_\alpha)$ projection $(x, s) \mapsto x$ on $U_\alpha \times F$

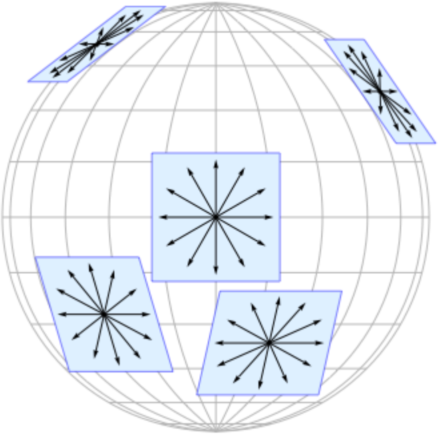
• sections $s : B \to E$ with $\pi \circ s = id$; locally on U_α

$$s|_{U_\alpha}(x) = (x, s_\alpha(x)), \quad \text{with } s_\alpha : U_\alpha \to F$$



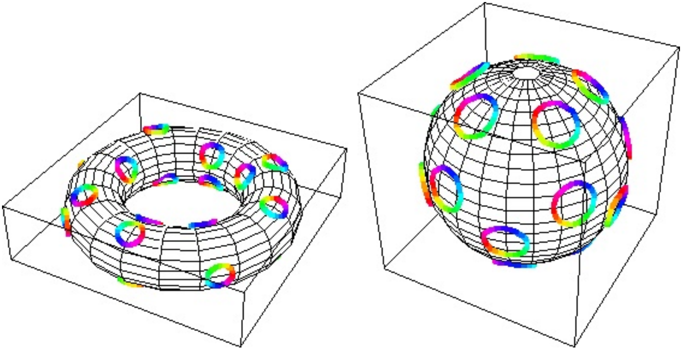
trivial and nontrivial $\mathbb{R}$-bundles over S^1

Tangent bundle TM of a smooth manifold M



Tangent bundle on a 2-sphere

- model of V1: bundle $\mathcal{E}$ with base $\mathcal{R}$ the retinal surface, fiber $\mathbb{P}^1$ the set of lines in the plane
- topologically $\mathbb{P}^1(\mathbb{R}) = S^1$ (circle) so locally V1 product $\mathbb{R}^2 \times S^1$
- circle bundle over a 2-dimensional surface



- We will see this leads to a geometric models of V1 based on Contact Geometry

Contact Geometry on 3-dimensional manifolds

- plane field ξ on 3-manifold M: subbundle of tangent bundle TM such that $\xi_x = T_x M \cap \xi$ is 2-dimensional subspace for all $x \in M$
- Example: $M = \Sigma \times S^1$ product of a 2-dimensional surface Σ and a circle S^1, then $\xi_{(x,\theta)} = T_x \Sigma \subset T_{(x,\theta)} M$ is a plane field
- real 1-form α on M determines at each point $x \in M$ a linear map

$$\alpha_x : T_x M \to \mathbb{R}$$

Kernel $\ker(\alpha_x)$ is either a plane or all of $T_x M$
if $\ker(\alpha_x) \neq T_x M$ for all $x \in M$ then $\xi = \ker(\alpha)$ is a plane field

- all plane fields locally given by $\xi = \ker(\alpha)$ for some 1-form α
- Example: $M = \Sigma \times S^1$ as above: $\xi = \ker(\alpha)$ with $\alpha = d\theta$
- contact 1-form α also determines a Reeb vector field $R = R_\alpha$ transverse to ξ preserving α:

$$\alpha(R_\alpha) = 1 \quad \text{and} \quad d\alpha(R_\alpha, \cdot) = 0$$

• plane field $\xi = \ker(\alpha)$ on 3-manifold M is contact structure iff

$$\alpha \wedge d\alpha \neq 0$$

equivalent condition $d\alpha|_\xi \neq 0$

• Standard Example: $M = \mathbb{R}^3$ with

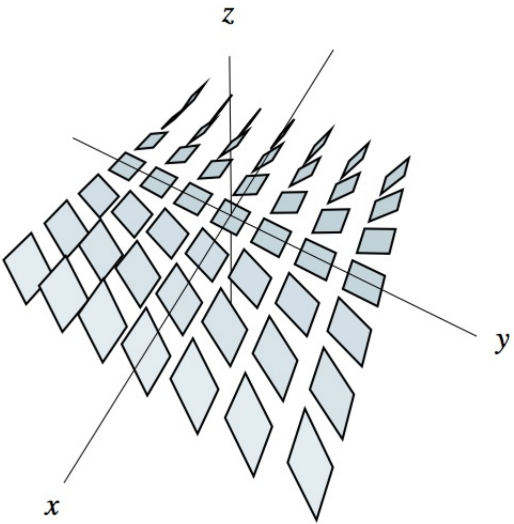
$$\alpha = dz + xdy$$

so $d\alpha = dx \wedge dy$ and $\alpha \wedge d\alpha = dz \wedge dx \wedge dy \neq 0$

• at a point (x, y, z) contact plane $\xi_{(x,y,z)}$ spanned by basis

$$\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\}$$

• geometry of contact plane field ξ: when $x = 0$ (yz-plane) contact plane horizontal; at $(1, 0, 0)$ spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial z} - \frac{\partial}{\partial y}$, tangent to x-axis, but tilted 45% clockwise, etc. start at origin and move along x-axis, plane keeps twisting clockwise



the standard contact structure on $\mathbb{R}^3$

Darboux's Theorem

• locally all contact structures look like the standard one

• (M, ξ) and (N, η) contact 3-manifolds, contactomorphism is diffeomorphism $f : M \to N$ such that $f_*(\xi) = \eta$; in terms of 1-forms $f^*(\alpha_\eta) = h\, \alpha_\xi$ for some non-zero $h : M \to \mathbb{R}$

• (M, ξ) contact 3-manifold, point $x \in M$, there are neighborhoods $\mathcal{N}$ of x and $\mathcal{U}$ of $(0,0,0)$ in $\mathbb{R}^3$ and contactomorphism

$$f : (\mathcal{N}, \xi|_{\mathcal{N}}) \to (\mathcal{U}, \xi_0|_{\mathcal{U}})$$

with ξ_0 the standard contact structure on $\mathbb{R}^3$

Example: contact structure on sphere S^3

- $f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2$ with $S^3 = f^{-1}(1) \subset \mathbb{R}^4$
- tangent spaces $T_{(x_1,y_1,x_2,y_2)} S^3 = \ker df_{(x_1,y_1,x_2,y_2)} = \ker(2x_1 dx_1 + 2y_1 dy_1 + 2x_2 dx_2 + 2y_2 dy_2)$
- identify $\mathbb{R}^4 = \mathbb{C}^2$ with complex structure $Jx_i = y_i$ and $Jy_i = -x_i$

$$J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \qquad J \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$$

- contact structure on S^3

$$\alpha = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3}$$

$$\alpha \wedge d\alpha \neq 0$$

$\xi = \ker(\alpha)$ contact planes

- contact planes $\xi = \ker(\alpha)$ on S^3 are set of complex tangencies

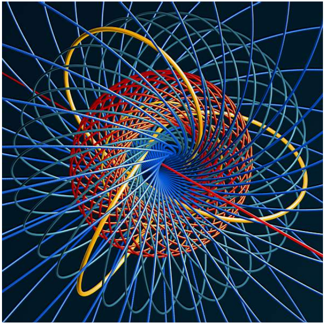
$$\xi = T_{(x_1,y_1,x_2,y_2)} S^3 \cap J(T_{(x_1,y_1,x_2,y_2)} S^3)$$

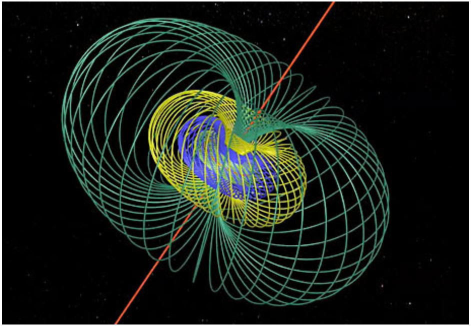
- 1-form α and complex structure:

$$\alpha = (df \circ J)|_{S^3}$$

- plane field $\xi = \ker(\alpha)$ orthogonal to the Hopf vector field

$$\dot{x}_1 = -y_1, \quad \dot{y}_1 = x_1, \quad \dot{x}_2 = -y_2, \quad \dot{y}_2 = x_2$$





Hopf vector field and Hopf fibration of S^3

Contact Structures and Complex Manifolds

• X complex manifold $\dim_{\mathbb{C}}(X) = 2$ with boundary ∂X, with $\dim_{\mathbb{R}} \partial X = 3$, and complex structure J on TX; function ϕ near boundary with $\partial X = \phi^{-1}(0)$ (collar neighborhood of boundary)

• complex tangencies

$$\ker(d\phi \circ J)$$

contact structure iff $d(d\phi \circ J)$ non-degenerate 2-form on planes ξ

• contact structure is fillable if obtained in this way

• Lutz–Martinet theorem: all 3-manifolds admit a contact structure (not always fillable)

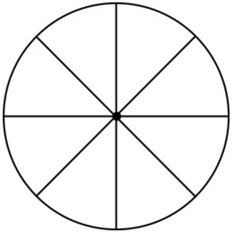
Contact Geometry and Symplectic Geometry

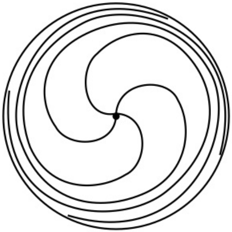
- X real 4-dimensional manifold (or more generally even dimensional); symplectic structure on X: closed 2-form ω such that $\omega \wedge \omega \neq 0$ (or in dimension $2n$ form $\wedge^n \omega \neq 0$)
- Darboux's Theorem for symplectic forms: locally $\omega = dp \wedge dq$ (like a cotangent bundle)
- (X, ω) symplectic filling of contact 3-manifold (M, ξ) if $\partial X = M$ and $\omega|_\xi \neq 0$ area form on contact planes
- fillability by complex manifold special case: $\omega = d(d\phi \circ J)$ is symplectic
- not all contact structures are fillable by symplectic structures: if a contact structure is symplectically fillable then it is tight

[Note: can always extend to symplectic on cylinder $X = M \times \mathbb{R}$ with $\omega = d\alpha + \alpha \wedge dt$ but not $M = \partial X$]

Tight and Overtwisted Contact Structures

- characteristic foliation: embedded oriented surface Σ in contact 3-manifold (M, ξ), lines $\ell_x = \xi_x \cap T_x\Sigma$ except at singular points where intersection is all $T_x\Sigma$; obtain foliation $\mathcal{F}_{\xi,\Sigma}$ of Σ with singular points
- overtwisted contact structure if $\exists$ embedded disk D with characteristic foliation $\mathcal{F}_{\xi,D}$ homeomorphic to either

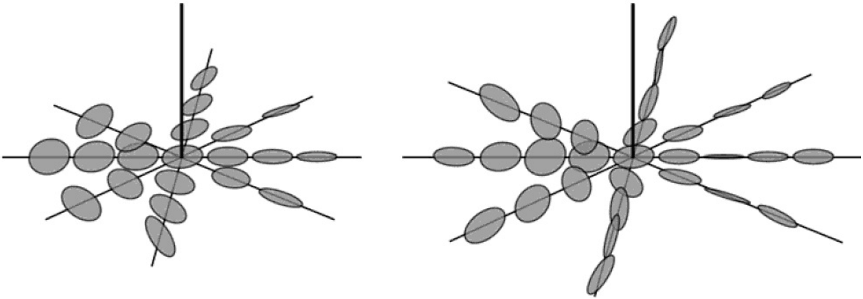




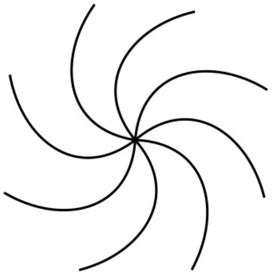
- tight contact structure: contains no overtwisted disk

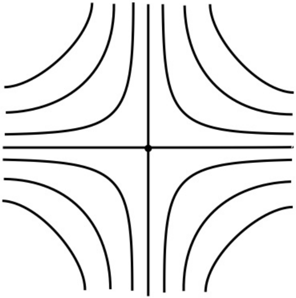
Examples

- tight: standard polar coordinates (r, θ, z) contact structure $\xi = \ker(dz + r^2 d\theta)$
- overtwisted: $\xi = \ker(\cos(r)\, dz + r \sin(r) d\theta)$, the overtwisted property sees the fact that contact planes $dz/d\theta = -r \tan(r)$ become vertical and twist over periodically (fig on the right)
- overtwisted disk $\{z = r^2 \,:\, 0 \leq r \leq \pi/2\}$



Generic singularities of the characteristic foliation





Some facts about contact structures and 3-manifolds

(Eliashberg, Gromov, Entyre, Honda, Bennequin, etc.)

- All 3-manifolds admit contact structures
- Some 3-manifolds do not admit any tight contact structure (though most of them do)
- If a contact structure is symplectically fillable then it is tight
- contact plane field ξ has an Euler class $e(\xi) \in H^2(M, \mathbb{Z})$: if tight then genus bound

$$|e(\xi)[\Sigma]| \leq -\chi(\Sigma)$$

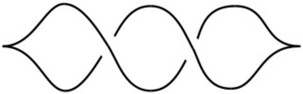
  if $\Sigma \neq S^2$ and zero otherwise (key idea: express in terms of singular points of the characteristic foliation, Poincaré–Hopf)

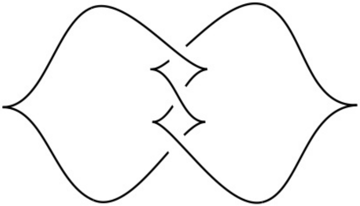
Legendrian knots

• knots $S^1 \hookrightarrow M$ in contact 3-manifold (M, ξ) such that curve always tangent to contact planes ξ

• every knot in a contact manifold can be continuously approximated by a Legendrian knot

• in standard contact structure in $\mathbb{R}^3$ with $\xi = \ker(dz + xdy)$ front projection (in yz-plane) looks like these





• invariants of Legendrian knots used to study contact manifolds (see Bennequin invariants, etc.)

Transverse knots

• knots $S^1 \hookrightarrow M$ in contact 3-manifold (M, ξ) such that curve always *transverse* to the contact planes ξ

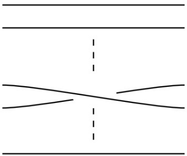
• for standard contact structure projections of transverse knots in the *xz*-planes cannot have segments like

because $z'(t) - y(t)x'(t) > 0$ along a tranverse knot and vertical tangency would have $x' = 0$ and $z' < 0$, while second case $y(t)$ bounded by slope $z'(t)/x'(t)$ in *xz*-plane

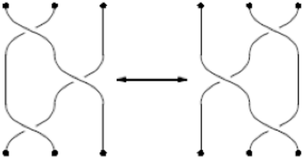
• any transverse knot in the standard contact structure is transversely isotopic to a closed braid

Braids: braid group

$$B_n = \langle \sigma_1, \ldots, \sigma_n \,|\, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i, \ |i-j| \geq 2 \rangle$$



A generator σ_i for the braid group B_n



braid group relations

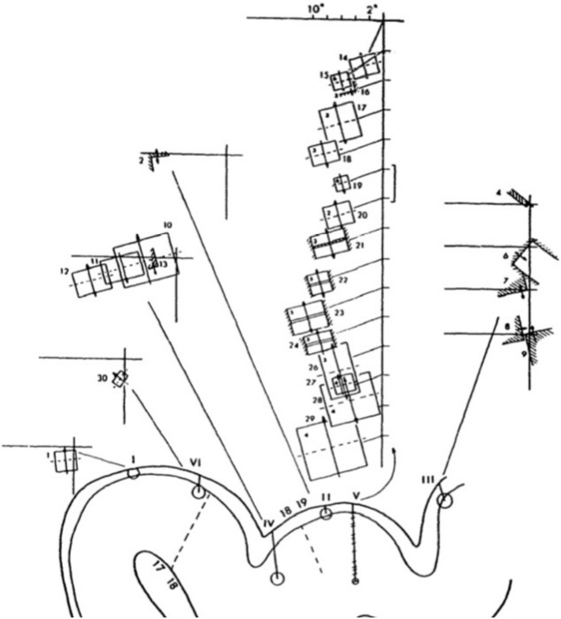
Visual Cortex as Contact Bundle

• W.C. Hoffman, *The visual cortex is a contact bundle*, Applied Mathematics and Computation, 32 (1989) 137–167

• Hubel–Wiesel microcolumns in columnar structure of V1 cortex exhibit both directional and areal response: model directional-areal response fields as contact planes directions

• "orientation response" refers to directionally sensitive response field of a single cortical neuron

• microelectrodes penetration measurements of directional and area response of neurons in the cat visual cortex show contact planes (Hubel, Wiesel)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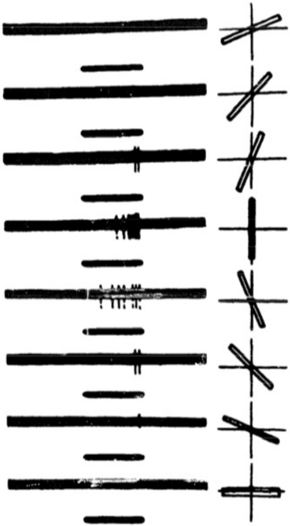


Fig. 8. A typical orientation response field (ORF) in the visual cortex: the neuronal firing rate response to shining a rectangular $1° \times 8°$ slit of light on the receptive field of a neuron whose "orientation" (i.e., directional) response is maximal in the vertical direction.

Relation to shape of Gabor filters

• From: Alessandro Sarti, Giovanna Citti, Jean Petitot, *Functional geometry of the horizontal connectivity in the primary visual cortex*, Journal of Physiology - Paris 103 (2009) 3–45

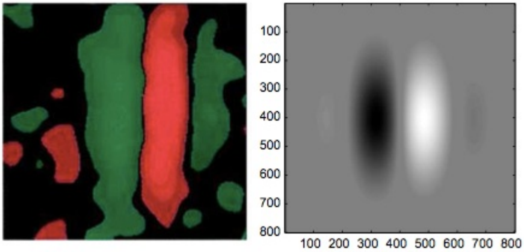


Fig. 2. In vivo registered odd receptive field (left, from (De Angelis et al., 1995)) and a schematic representation of it as a Gabor filter (right).

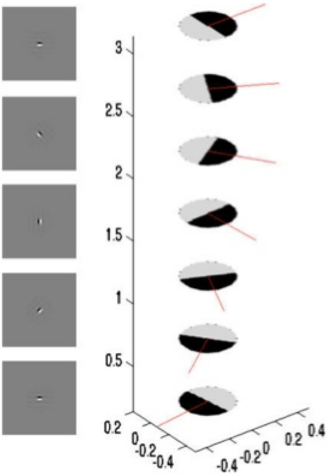
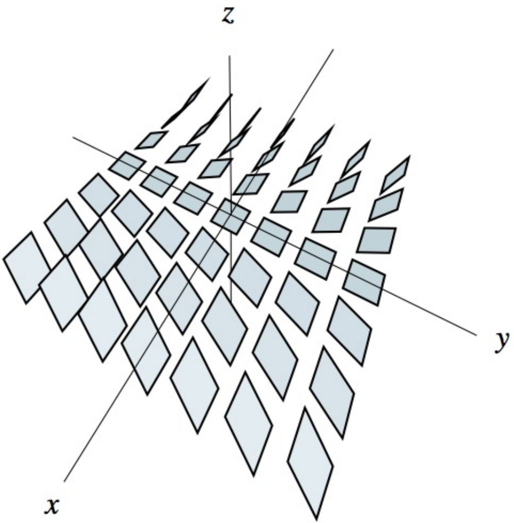


Fig. 3. Odd part of Gabor filters with different orientations (left) and schemata of odd simple cells arranged in a hypercolumn of orientations.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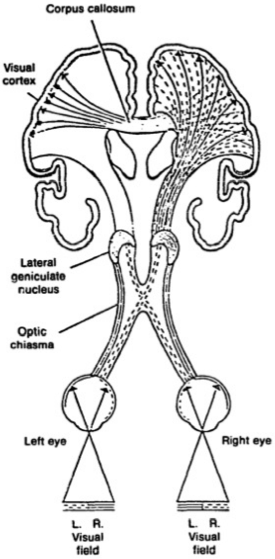
Visual Pathways



visual pathways from the retina to the visual cortex

Visual pathways and Connections on Fiber Bundles

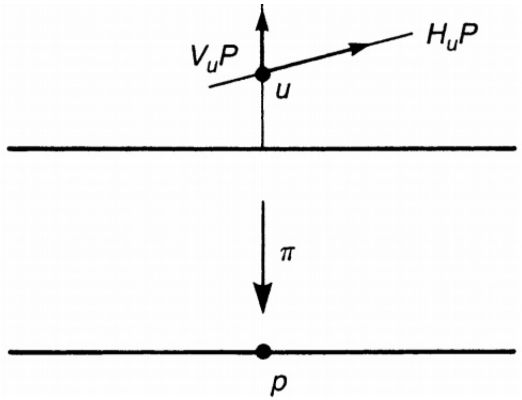
• *paths* (visual contours) are *lifted* along visual pathways from the retina to the visual cortex

• patterns of "constancies" are detected (shape, size, motion, color, etc.), then higher forms (areas 18 and 19 of the human visual cortex)

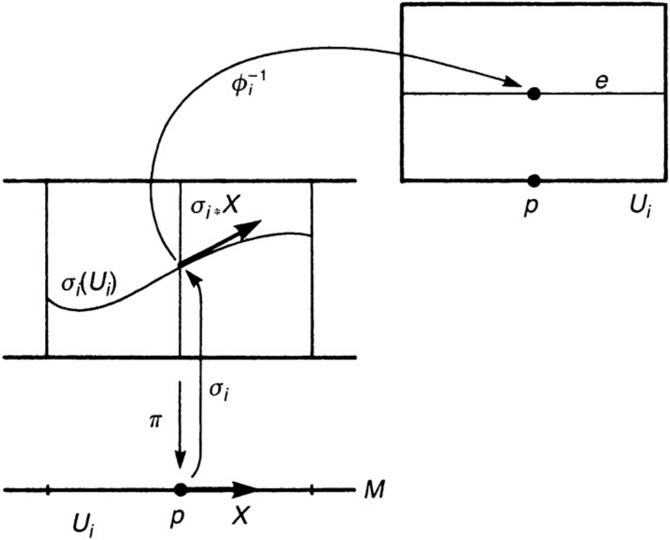
• *path lifting property* (from retina to cortex); geometrically path lifting from base $\mathcal{R}$ to total space of fibration $\mathcal{F}$ with fiber $\mathbb{P}^1(\mathbb{R})$

$$\mathbb{P}^1(\mathbb{R}) \hookrightarrow \mathcal{F} \stackrel{\pi}{\twoheadrightarrow} \mathcal{R}$$

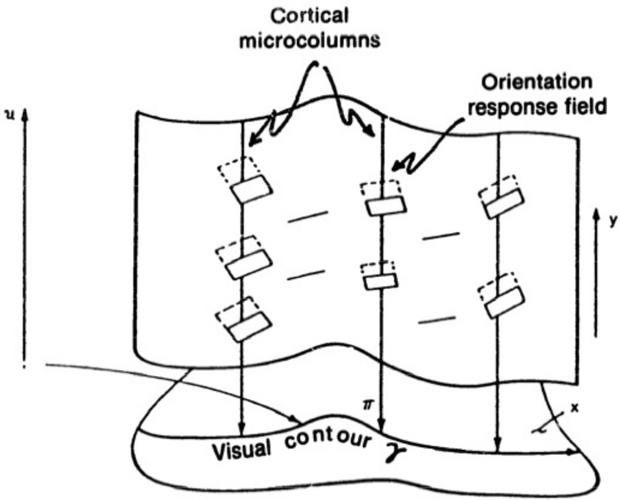
• lifting a path along projection of a fibration: need to choose a horizontal direction at each point in the total space of the fibration (there is always a well defined vertical direction): a connection determines the choice of a horizontal direction



horizontal and vertical subspaces in the tangent space of a fibration



trivialization defined by local sections
(from Nakahara, *Geometry, Topology, and Physics*, CRC Press, 2003)



path lifting to the visual cortex (Hoffman)

Connection 1-form and Contact Planes

• connections and 1-forms: view a connection as a splitting of exact sequence

$$T\mathbb{P}^1 \to T\mathcal{F} \xrightarrow{\pi_*} T\mathcal{R}$$

of tangent spaces of fibration: choice of horizontal direction at each point; achieved by a 1-form α (scalar valued because circle bundle $\mathbb{P}^1(\mathbb{R}) \simeq S^1$) while vertical direction is $V = \ker(\pi_*)$

• Geometric Model: orientation response fields (ORFs) are contact planes $\xi = \ker(\alpha)$ determined by the connection 1-form α that performs the path lifting from the retina to the visual cortex

Question

- when lifting a path from retina to visual cortex get a path everywhere transversal to contact planes
- lift of a closed path in general not a closed path: endpoints lie on the same fiber of the fibration, but not necessarily the same point
- if obtain closed path, this can be knotted in the contact 3-manifold (transverse knot)
- when does this happen? what is the significance of knottedness? role of transverse and Legendrian knots in the visual cortex contact bundle?

Horizontal Connectivity in the Primary Visual Cortex

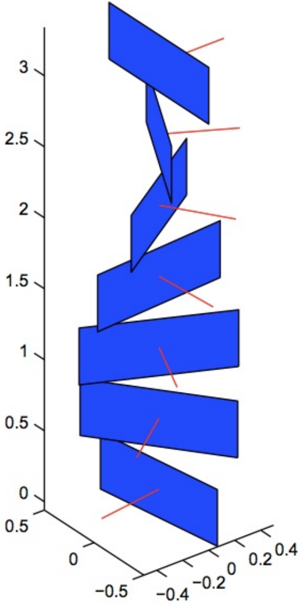
- Alessandro Sarti, Giovanna Citti, Jean Petitot, *Functional geometry of the horizontal connectivity in the primary visual cortex*, Journal of Physiology - Paris 103 (2009) 3–45
- on product $\mathcal{F} = \mathcal{R} \times \mathbb{P}^1(\mathbb{R})$ where $\mathcal{R} \simeq \mathbb{R}^2$ coordinates (x, y) and $\mathbb{P}^1(\mathbb{R}) \simeq S^1$ coordinate θ

$$\alpha = -\sin(\theta) dx + \cos(\theta) dy$$

is a contact form

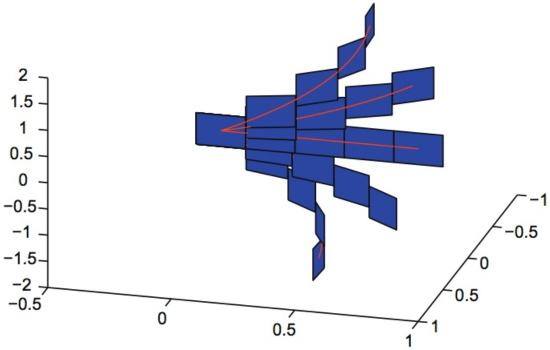
$$d\alpha = (\cos(\theta) dx + \sin(\theta) dy) \wedge d\theta, \quad \alpha \wedge d\alpha = -dx \wedge dy \wedge d\theta \neq 0$$

- contact planes spanned by $(\cos(\theta), \sin(\theta), 0)$ and $(0, 0, 1)$



The contact planes at every point, and the orthogonal vector X_3.

• the 1-form α relates local tangent vectors (in lift of retinal image) and forms integral curves, either along contact planes (Legendrian) or transverse: mechanism responsible for creating regular and illusory contours



integral curves along the contact planes

Scale Variable

- an additional scale variable $\sigma \in \mathbb{R}_+$: think of the visual field information recorded in the lift to the visual cortex not as a delta function but as a smeared distribution with Gaussian parameter σ (Gabor frames)
- when $\sigma \to 0$ recover geometric picture described above with integral curves
- geometric space $\mathcal{X} = \mathbb{R}^2 \times S^1 \times \mathbb{R}_+$, coordinates (x, y, θ, σ)
- 2-form on $\mathcal{X}$: scale $\alpha \mapsto \sigma^{-1}\alpha$

$$\omega = d(\sigma^{-1}\alpha) = \sigma^{-1}d\alpha + \sigma^{-2}\alpha \wedge d\sigma$$

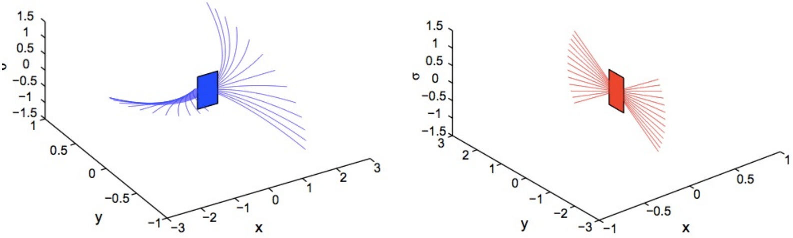
symplectic $\omega \wedge \omega = 2\sigma^{-3}d\alpha \wedge \alpha \wedge d\sigma = 2\sigma^{-3}dx \wedge dy \wedge d\theta \wedge d\sigma$

- not symplectically filling: blowing up at $\sigma \to 0$, don't have $\omega|_\xi$ at boundary, but $d\alpha + \alpha \wedge d\sigma$ would be

• $\omega = \sigma^{-1}\omega_1 \wedge \omega_2 + \sigma^{-2}\omega_3 \wedge \omega_4$ with ω_i 1-form dual to vector field X_i, corresponding vector fields

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \qquad X_2 = \partial_\theta,$$
$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y, \quad X_4 = \partial_\sigma$$

• for small σ predominant X_1X_2 contact planes; for large σ predominant X_3X_4-planes



integral curves in the X_1X_2-planes and in the X_3X_4-planes

Geometry of Gabor filters

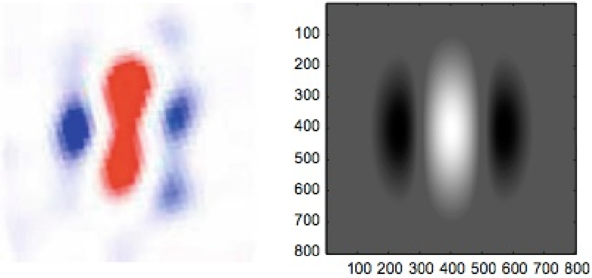


Fig. 7. In vivo registered even receptive field (left – from Niell and Stryker (2008)) and a schematic representation of it as a Gabor filter (right)). Positive sign is in white and negative in black.

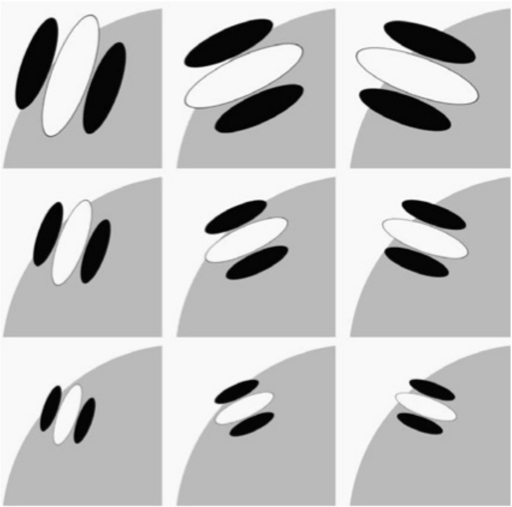


Fig. 8. The two-dimensional fiber of even simple cells, obtained via rotation and dilation of the mother filter.

Geometry of Gabor filters with both rotation and dilation

Projection in the retinal plane

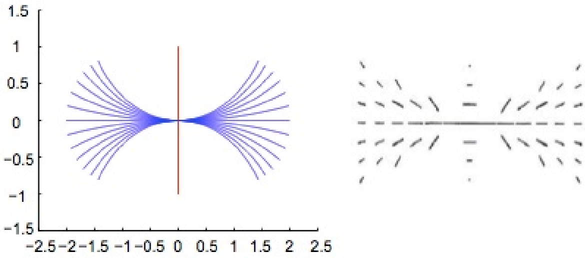
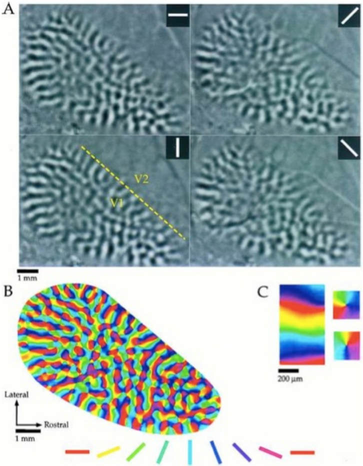


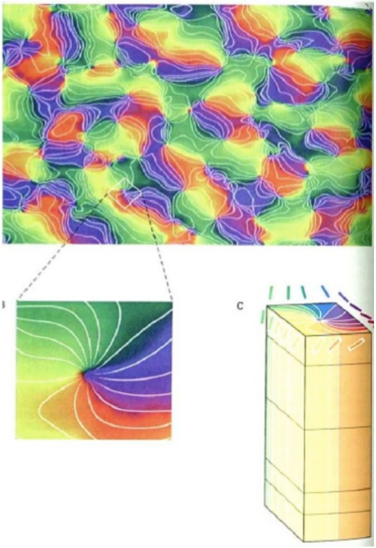
Fig. 12. The projection of the integral curves of the symplectic structure on the retinal plane (left) reveals the pattern of co-axial and trans-axial connections found by neurophysiological experiments (right, from Yen and Finkel (1998)).

- S.C. Yen, L.H. Finkel, *Extraction of perceptually salient contours by striate cortical networks*, Vision Res. 38 (1998) N.5, 719–741.

Pinwheel Structure in the Visual Cortex



V1 cortex of tupaya tree shrew: different orientations coded by colors
zoom in on regular and singular points (Petitot)



isoorientation (isochromatic) lines in the V1 cortex (Petitot)

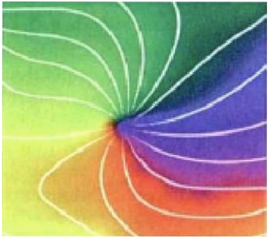
• given a section $\sigma : \mathcal{R} \to \mathcal{F}$ of the fibration

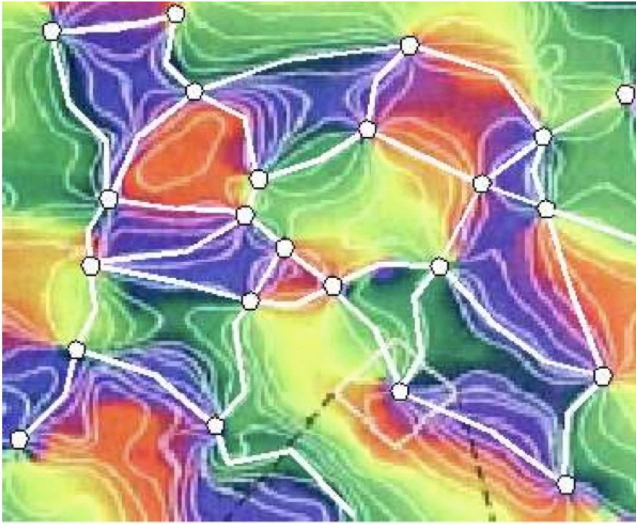
$$\mathbb{P}^1(\mathbb{R}) \hookrightarrow \mathcal{F} \stackrel{\pi}{\twoheadrightarrow} \mathcal{R}$$

determines a surface $\Sigma = \sigma(\mathcal{R}) \subset \mathcal{F}$

• isoorientation curves are canonical foliation $\ell_x = \xi_x \cap T_x \Sigma$ for this surface

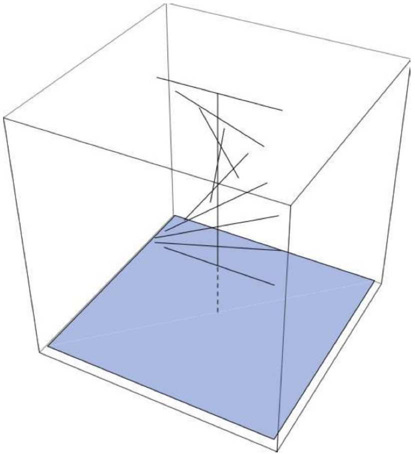
• pinwheels in Σ are overtwisted disks on the canonical foliation





networks of pinwheels (Petitot)

- projected down to $\mathcal{R}$ with $\pi : \mathcal{F} \to \mathcal{R}$ have network of pinwheels on $\mathcal{R}$ via $\pi \circ \sigma = 1$ identification of Σ and $\mathcal{R}$
- fiber over each pinwheel point is $\mathbb{P}^1(\mathbb{R})$
- can view these fibers as (real) *blowup* of $\mathcal{R}$ at pinwheel points

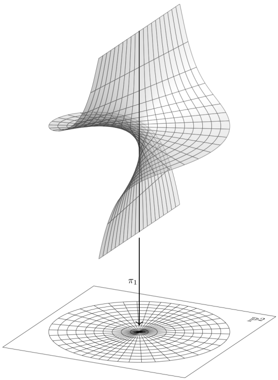


real blowup of $\mathbb{R}^2$ at a point (Petitot)

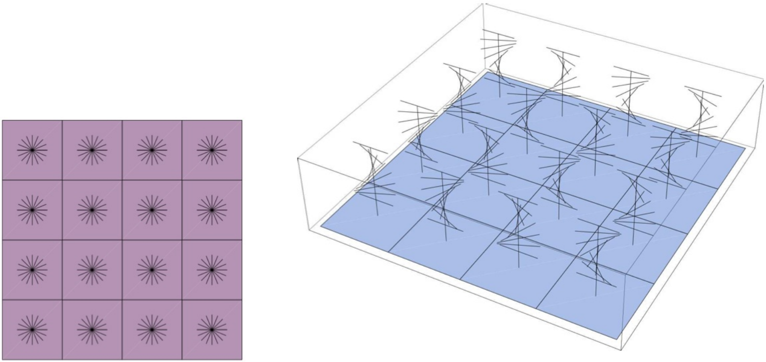
$$\mathrm{Bl}_p\mathbb{A}^2 = \{(x,y),[z:w]\,|\,xz+yw=0\} \subset \mathbb{A}^2\times\mathbb{P}^1$$

$$\mathrm{Bl}_p\mathbb{A}^2 = \{(q,\ell)\,|\,p,q\in\ell\}$$

for $p\neq q$ projection $\pi_1:\mathrm{Bl}_p\mathbb{A}^2\to\mathbb{A}^2$, $(q,\ell)\mapsto q$ isomorphism, because unique line ℓ through p and q, but over $p=q$ fiber is $\mathbb{P}^1$ set of all lines ℓ

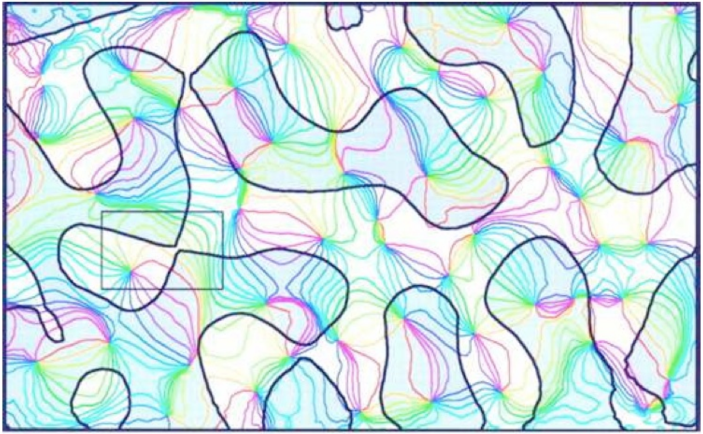


real blowup of $\mathbb{R}^2$ at a point (image by Charles Staats)



pinwheels in the base $\mathcal{R}$ and fibers (Petitot)

Observed relation between pinwheel structure and *ocular dominance domains*



pinwheels cut boundaries of ocular dominance domains transversely and nearly orthogonally (Petitot)

Contact Geometry Invariants

• there are interesting invariants of contact manifolds, such as contact homology, a Morse or Morse–Bott theory for the functional

$$\mathcal{A}(\gamma) = \int_\gamma \alpha$$

on loops in (M, ξ) with $\xi = \ker(\alpha)$

• in the neuroscience setting we are especially interested in transverse and Legendrian knots that occur via the path lifting property from the retinal surface to the contact bundle $\mathcal{R} \times \mathbb{P}^1(\mathbb{R})$

• want some nice invariants of contact geometry built from transverse and Legendrian knots

Heegard Floer invariants of Legendrian and transverse knots in contact 3-manifolds

- Paolo Lisca, Peter Ozsváth, András I. Stipsicz, Zoltán Szabó, *Heegaard Floer invariants of Legendrian knots in contact three-manifolds*, J. Eur. Math. Soc. 11 (2009) 1307–1363.

• datum: null-homologous Legendrian and transverse knots in a closed contact 3-manifold (M, ξ)

• Note: the null-homologous condition is meaningful in the context of neuroscience applications, since one wants closed curves to be (lifted) contours of two-dimensional surfaces

A very sketchy overview:

• given a null-homologous knot K in a 3-manifold M, associated homology theory (Heegard Floer theory) $HFK(M, K, \mathfrak{s})$ (in different versions), so that isotopic K_1, K_2 have same Floer homology ($\mathfrak{s}$ spin$_c$-structure)

• given null-homologous Legendrian knot L construction of a class $\alpha(L) \in HFK(M, L, \mathfrak{s})$ (with $\mathfrak{s}$ determined by ξ)

$$\mathcal{L}(L) = [HFK(M, L, \mathfrak{s}), \alpha(L)]$$

equivalence relation $\mathbb{Z}/2\mathbb{Z}[t]$-module structure of $HFK(M, L, \mathfrak{s})$, isom class under $\mathbb{Z}/2\mathbb{Z}[t]$-module isomorphisms

• there is a global invariant $c(M, \xi)$ of contact manifolds (also a class in a Floer homology $HFK(M, \mathfrak{s})$, up to a filp of orientation), Ozsváth–Szabó invariant: if $c(M, \xi) \neq 0$ then all $\mathcal{L}(L) \neq 0$, if $c(M, \xi) = 0$ then $t^d \cdot \mathcal{L}(L) = 0$ for some large d

• (M, ξ) symplectically fillable if $M = \partial X$ symplectic (X, ω) with $\omega|_\xi > 0$; strongly symplectically fillable if $M = \partial X$ symplectic (X, ω) and $\omega|_Y = d\alpha$

• (M, ξ) strongly symplectically fillable $\Rightarrow \mathcal{L}(L) \neq 0$ for all null-homologous Legendrian knots

• (M, ξ) overtwisted $\Rightarrow$ for any null-homologous Legendrian knot L there is a $d \geq 0$ with $t^d \cdot \mathcal{L}(L) = 0$

• transverse knot K in (M, ξ) contact 3-manifold: $\exists$ "approximation by Legendrian knot" L_K, then $\mathcal{L}(K) := \mathcal{L}(L_K)$

• these provide invariants to study null-homologous transverse and Legendrian knots in contact manifolds

• example of properties of such knots: L is *loose* if the complement contains an overtwisted disk; *non-loose* if (M, ξ) overtwisted but complement of L tight (position of overtwisted disks with respect to knot) ... if L is loose then $\mathcal{L}(L) = 0$

• General Problem: is there a use for invariants of contact manifolds and of Legendrian and transverse knots in contact manifolds to understand the geometry of path lifting from the retina to V1 and its relation to ocular domimance regions in V1?

Contact Geometry and Gabor Frames

- V. Liontou, M. Marcolli, *Gabor frames from contact geometry in models of the primary visual cortex*, Math. Neuro. and Appl. 3 (2023), article no. 2, 1–28.

- there is an interesting interplay in the V1 cortex between the geometry describing the connectivity in terms of a contact structure and the signal analysis carried out by Gabor filters associated to the orientation-sensitive receptor fields
- what is this mathematical structure that gives rise to signal analysis from contact geometry?
- they seem to live in different places: Gabor analysis is a linear structure, contact geometry is a non-linear manifold structure

Circle bundles of contact elements

- contact 3-manifold (M, α) *Legendrian circle bundle* over surface S
$$S^1 \hookrightarrow M \to S$$
fiber directions TS^1 inside tangent bundle TM contained in the contact planes distribution $\xi = \mathrm{Ker}(\alpha) \subset TM$
- classification (R.Lutz) of Legendrian circle bundles: either unit cosphere bundle $M = \mathbb{S}(T^*S)$ or d-fold cyclic cover for $d|(2g-2)$ with $g = g(S)$ genus
- for V1 model $S = S^2 = \mathbb{P}^1(\mathbb{C})$ genus zero and $M = \mathbb{S}(T^*S)$
- contact form α on $M = \mathbb{S}(T^*S)$ determined by tautological Liouville form λ on T^*S

tautological Liouville form

- on any manifold Y cotangent bundle T^*Y has canonical 1-form

$$\lambda_{(x,p)}(v) = p(d\pi(v))$$

  for $v \in T_x Y$ and $\pi : T^*Y \to Y$ bundle projection

- local coordinates (x,p) with $x = (x_i) \in Y$ and $p = (p_i) \in T_x Y$: Liouville 1-form is

$$\lambda = \sum_i p_i \, dx^i$$

- this 1-form determines canonical symplectic form $\omega = d\lambda$ of the cotangent bundle T^*Y
- restriction of Liouville 1-form λ to unit spheres in $T^*_x Y$ induces contact 1-form α on cosphere bundle $\mathbb{S}(T^*Y)$

local coordinates

- (x, y) local coordinates on surface S and (u, v) local coordinates of cotangent fiber $T^*_{(x,y)}S$
- tautological Liouville 1-form $\lambda = u\,dx + v\,dy$
- symplectic form $\omega = du \wedge dx + dv \wedge dy$
- contact 1-form induced on cosphere bundle $M_w = \mathbb{S}_w(T^*S)$ (radius w)
$$\alpha = w\cos(\theta)dx + w\sin(\theta)dy$$
polar coordinates (w, θ) in cotangent planes
- contact planes distribution generated by two vector fields ∂_θ and $-\sin(\theta)\partial_x + \cos(\theta)\partial_y$
- Reeb vector field (transversal)

$$R_\alpha = w^{-1}\cos(\theta)\partial_x + w^{-1}\sin(\theta)\partial_y$$

complex structure and twisted tautological form

- surface $S = \mathbb{P}^1(\mathbb{C})$ Riemann sphere has complex structure (also responsible for conformal retinotopic mapping)
- complex coordinate $z = x + iy$ on S and induced complex structure $J : (u, v) \mapsto (-v, u)$ on $T^*_{(x,y)}S$
- *twisted tautological form* with respect to J

$$\lambda_J = -v\, dx + u\, dy$$

- associated symplectic form $\omega_J = -dv \wedge dx + du \wedge dy$
- two contact structures α and α_J on $M = \mathbb{S}(T^*S)$
- contact planes of these two contact structures intersect along the fiber direction, spanned by ∂_θ, so *fibers are Legendrian circles with respect to both*
- Reeb field R_α of α is Legendrian for α_J and the Reeb field R_{α_J} is Legendrian for α
- complex structure J fixes ∂_θ and exchanges R_{α_J} and R_α

bundle of signal planes

- $\mathcal{E}$ real 2-plane bundle $\mathbb{R}^2 \hookrightarrow \mathcal{E} \to M$ over contact 3-manifold $M = \mathbb{S}(T^*S)$

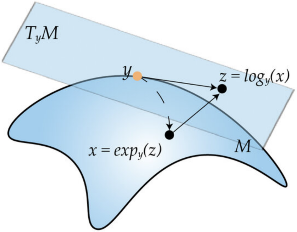
$$\mathcal{E} = \pi^* TS$$

pullback of tangent bundle TS of surface S to M along projection $\pi : \mathbb{S}(T^*S) \to S$ of unit cosphere bundle

- at a point $(x, y, \theta) \in M$, with $z = x + iy$ on S and θ on circle fiber at z, the fiber $\mathcal{E}_{(x,y,\theta)}$ is same as the fiber $T_z S$
- meaning: S curved surface, TS collection of its local linear approximations $T_z S$, bundle $\mathcal{E}$ places such linear approximations at each location (x, y, θ) in M
- think of as a retinotopy designed to transfer linear signal analysis filters in linear spaces $T_z S$ (for signals on S) to the 3-dim columnal structure (with information on orientation sensitivity θ) of the V1 cortex
- signal on the retina $S \Rightarrow$ lifted to signals on linear spaces $T_z S$ $\Rightarrow$ lifted to signal on $\mathcal{E}$ signal $\mathcal{I} \in L^2(\mathcal{E}, \mathbb{R})$

exponential map and injectivity radius

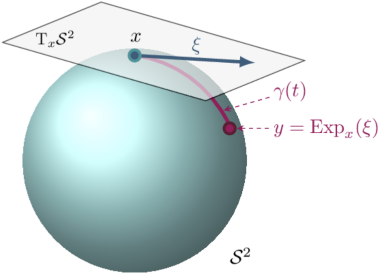
- on a Riemannian manifold (Y, g) can *locally* map flat linear tangent space $T_x Y$ to curved coordinates neighborhood of point $x \in Y$ through *exponential map*



- using (locally) unique geodesic $\gamma_{x,v}(t)$ with $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$ with exponential map defined as

$$\exp_x : \mathcal{U} \to Y \quad \text{with} \quad \exp_x(v) = \gamma_{x,v}(1)$$

- manifold is geodesically complete if exponential map is defined globally $\exp_x : T_x Y \to Y$ (compact manifolds)
- in general not a diffeomorphism (geodesics on Y can close up: injectivity fail): injectivty radius



- at point $x \in Y$ take $R_{inj}(x) > 0$ supremum of all radii $R > 0$ such that exponential map $\exp_x : T_x Y \to Y$ is diffeomorphism on ball $B(0, R)$ of radius R in $T_x Y$
- on compact manifold: continuous *injectivity radius function* $R_{inj} : Y \to \mathbb{R}_+^*$ given by $x \mapsto R_{inj}(x)$

pathway of signals: faithfulness

- signal as a real valued function $\mathcal{J} : S \to \mathbb{R}$ on the retinal surface S
- lifted to collection of functions $\mathcal{J}_z : T_z S \to \mathbb{R}$ on the linear tangent spaces $T_z S$ pulling back through the exponential map, $\mathcal{J}_z = \mathcal{J} \circ \exp_z$
- a single function $\tilde{\mathcal{I}} : TS \to \mathbb{R}$ with $\tilde{\mathcal{I}}(z, v) = \mathcal{J}_z(v)$ hence by pullback a function $\mathcal{I} : \mathcal{E} \to \mathbb{R}$ with $\mathcal{I}((z, \theta), v)) = \tilde{\mathcal{I}}(z, v)$
- injectivity radius function $R_{inj} : S \to \mathbb{R}^*_+$ of the exponential map $\exp_z : T_z S \to S$ describes a region where signal $\mathcal{I}$ on the bundle $\mathcal{E}$ faithfully represents retinal signal $\mathcal{J}$

Lattices and Gabor frames

- dual vector bundle (bundle of linear functionals on $\mathcal{E}$)

$$\mathcal{E}^\vee = \mathrm{Hom}(\mathcal{E}, \mathbb{R})$$

- symmetry of Reeb fields under complex structure twist gives dual local bases $\{\alpha, \alpha_J\}$ and $\{R_\alpha, R_{\alpha_J}\}$ on dense open $U \subset S$
- bundles of framed lattices (lattices with an assigned basis) over a local chart in M

$$\Lambda_{\alpha,J} := \mathbb{Z}\, R_\alpha + \mathbb{Z}\, R_{\alpha_J} \quad \text{and} \quad \Lambda^\vee_{\alpha,J} := \mathbb{Z}\,\alpha + \mathbb{Z}\,\alpha_J$$

bundle of framed lattices in $\mathcal{E} \oplus \mathcal{E}^\vee$

- equivalently $\Lambda_{\alpha,J} \oplus \Lambda^\vee_{\alpha,J}$

$$\Lambda = \mathbb{Z}\, R_\alpha \oplus \mathbb{Z}\,\alpha, \quad \Lambda_J = \mathbb{Z}\, R_{\alpha_J} \oplus \mathbb{Z}\,\alpha_J$$

- window function: on $TS \oplus T^*S$

$$\Phi_{0,(x,y)}(V,\eta) := \exp\left(-V^t A_{(x,y)} V - i\langle \eta, V\rangle_{(x,y)}\right)$$

- A smooth section of $T^*S \otimes T^*S$ symmetric and positive definite quadratic form on fibers of TS
- at all points (x,y) in local chart U in S the matrix $A_{(x,y)}$ has eigenvalues uniformly bounded away from zero, $\mathrm{Spec}(A_{(x,y)}) \subset [\lambda, \infty)$ for some $\lambda > 0$
- restriction of Φ_0 to $TS \oplus \mathbb{S}_w(T^*S)$ gives function on $\mathcal{E}$

$$\Psi_{0,(x,y,\theta)}(V) := \exp\left(-V^t A_{(x,y)} V - i\langle \eta_\theta, V\rangle_{(x,y)}\right)$$

  with $\eta_\theta = (w\cos(\theta), w\sin(\theta))$
- so $\mathcal{E} \oplus \mathcal{E}^\vee$ where window function lives

Gabor system

$$\mathcal{G}(\Psi_0, \Lambda_{\alpha,J} \oplus \Lambda^\vee_{\alpha,J})$$

- collection at each point $(x, y, \theta) \in M$ of the Gabor systems

$$\mathcal{G}(\Psi_{0,(x,y,\theta)}, \Lambda_{\alpha,J,(x,y,\theta)} \oplus \Lambda^\vee_{\alpha,J,(x,y,\theta)})$$

consisting of the functions $\rho(\lambda)\Psi_0 = \rho(\xi)\rho(W)$, for $\lambda = (W, \xi) \in \Lambda + \Lambda_J$

$$\rho(\lambda)\Psi_0(V) := e^{2\pi i \langle \xi, V \rangle}\Psi_0(V - W),$$

in the function spaces $L^2(\mathcal{E}_{(x,y,\theta)})$

injectivity radius and scaling function

- empirical cutoff scale: finite (though large) number of receptor profiles involved in signal analysis in V1, hence only filters $\rho(\lambda)\Psi_0$ with λ contained in some ball of radius $R_{\max}$
- geometric cutoff scale: injectivity radius where moving signals between curved S and linear $T_z S$ fails fiathfulness
- scaling factor that compares these two cutoffs

$$b_M(x,y,\theta) := \frac{R_{inj}(x,y)}{R_{\max}} \quad < 1$$

the intrinsic scale of the Gabor system construction for signal analysis in V1 :

rescaled Gabor system

- ensures empirically available filters cover desired signal region

$$\mathcal{G}(\Psi_0, \Lambda_{b,\alpha,J} \oplus \Lambda^{\vee}_{b,\alpha,J})$$

with

$$\Lambda_{b,\alpha,J} := b_M \Lambda_{\alpha,J} = \mathbb{Z}\, b_M\, R_\alpha + \mathbb{Z}\, b_M\, R_{\alpha_J}\,,$$

$$\begin{cases} \Lambda_b = \mathbb{Z}\, b_M\, R_\alpha \oplus \mathbb{Z}\,\alpha, \\ \Lambda_{b,J} = \mathbb{Z}\, b_M\, R_{\alpha_J} \oplus \mathbb{Z}\,\alpha_J \end{cases} .$$

- condition

$$b_M(x,y,\theta) < 1, \quad \forall (x,y,\theta) \in M$$

ensures rescaled Gabor frame satisfies the frame condition (pointwise)

$$C_{(x,y,\theta)}\, \|f\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})} \leq \sum_{\lambda_{(x,y,\theta)}} |\langle f, \rho(\lambda_{(x,y,\theta)})\Psi_0\rangle|^2 \leq C'_{(x,y,\theta)}\, \|f\|^2_{L^2(\mathcal{E}_{(x,y,\theta)})}$$

for $\lambda_{(x,y,\theta)} \in (\Lambda_{b,\alpha,J} \oplus \Lambda^{\vee}_{b,\alpha,J})_{(x,y,\theta)}$

- so performing good signal analysis

Symplectization and contactization

1. contact manifold $(M, \alpha) \Rightarrow$ symplectic manifold $(M \times \mathbb{R}, \omega)$ (symplectization)
2. symplectic manifold $(Y, \omega) \Rightarrow$ contact manifold $(Y \times S^1, \alpha)$ (contactization)

symplectization (always possible)

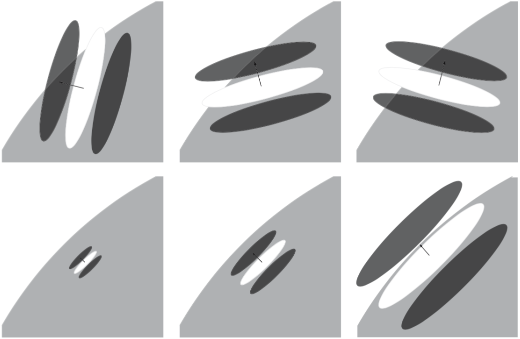
- $(M \times \mathbb{R}, \omega)$ with $\omega = d(e^s \cdot \alpha)$ with $s \in \mathbb{R}$ cylinder coordinate
- equivalently with $w = e^s \in \mathbb{R}^*_+$ symplectic form $\omega = dw \wedge \alpha + w\, d\alpha$
- contact manifold $M = \mathbb{S}(T^*S) \Rightarrow$ symplectization $=$ complement of zero section $T^*S_0 := T^*S \smallsetminus \{0\}$ with the symplectic form

$$\omega = dw \wedge \alpha + w\, d\alpha = du \wedge dx + dv \wedge dy$$

and J-twisted symplectic form

$$\omega_J = dw \wedge \alpha_J + w\, d\alpha_J = -dv \wedge dx + du \wedge dy$$

with $(u, v) = (w \cos\theta, w \sin\theta)$



variables $\theta \in S^1$ and $s \in \mathbb{R}^*_+$ in the symplectic form $\omega = d(e^s \cdot \alpha)$ account for orientation dependence (through α) and *scale* contactization (more constrained)

- symplectic manifold (Y, ω): if the symplectic form is exact $\omega = d\lambda$ then contactization $(Y \times S^1, \alpha)$ with $\alpha = \lambda - d\phi$, where ϕ is the angle coordinate on S^1
- the symplectization of a contact manifold is an exact symplectic manifold $\omega = d(e^s \cdot \alpha) = d(\lambda)$, so there is always a contactization of the symplectization
- when ω not exact sometimes still possible via "prequantization bundles"
- ontactization of symplectization of contact 3-manifold $M = \mathbb{S}(T^*S)$ is 5-manifold $T^*S_0 \times S^1$ with contact form

$$\tilde{\alpha} = \lambda - d\phi = w\alpha - d\phi,$$

  and J-twist $\tilde{\alpha}_J = w\alpha_J - d\phi$
- additional angle coordinate $\phi \in S^1$ (in addition to orientation variable $\theta \in S^1$ and scale $s \in \mathbb{R}^*_+$)

- 5-dimensional contact manifold $\mathcal{CS}(M) := T^*S_0 \times S^1$
- corresponds to model with additional pair of dual variables is introduced describing phase and velocity of spatial wave propagation (orientation dependence in 3=manifold $(M = \mathbb{S}(T^*S), \alpha)$ and additional direction/phase sensitivity in 5-manifold $(\mathcal{CS}(M), \tilde{\alpha})$

model proposed in

- E. Baspinar, A. Sarti, G. Citti, *A sub-Riemannian model of the visual cortex with frequency and phase*, The Journal of Mathematical Neuroscience, Vol. 10 (2020) Article 11, 31pp